Superconductivity in an Exactly Solvable Model of the Pseudogap State: Absence of Self-Averaging.

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Abstract

We analyze the anomalies of superconducting state within a simple exactly solvable model of the pseudogap state, induced by fluctuations of “dielectric” short range order, for the model of the Fermi surface with “hot” patches. The analysis is performed for the arbitrary values of the correlation length ξ_{corr} of this short range order. It is shown that superconducting energy gap averaged over these fluctuations is non zero in a wide temperature range above T_c — the temperature of homogeneous superconducting transition. This follows from the absence of self averaging of the gap over the random field of fluctuations. For temperatures $T > T_c$ superconductivity apparently appears in separate regions of space (“drops”). These effects become weaker for shorter correlation lengths ξ_{corr} and the region of “drops” on the phase diagram becomes narrower and disappears for $\xi_{corr} \to 0$, however, for the finite values of ξ_{corr} the complete self averaging is absent.

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I. INTRODUCTION

Among the anomalies of electronic properties of high – temperature superconducting cuprates (HTSC) especially interesting are the properties of the pseudogap state, observed in a wide region of the phase diagram [1,2]. From our point of view, the preferential scenario for the formation of the pseudogap state is based on the existence (mainly in the underdoped region of the phase diagram) of strong scattering of current carriers on well developed fluctuations of "dielectric" short range order (of antiferromagnetic (AFM) or charge density wave (CDW) type) [2]. This scattering leads to essentially non – Fermi liquid like renormalization of electronic spectrum in certain parts of momentum space close to the Fermi surface and near the so called "hot" spots or "hot" (flat) patches [2]. The preferential nature of "dielectric" and not of "superconducting" pseudogap formation [3] follows from the number experiments, the appropriate discussion can be found in Ref. [2].

The major part of existing theoretical papers is dealing with pseudogap formation and its influence on the properties of the system in normal (non superconducting) state, while only few are consider the anomalies of superconductivity in the pseudogap state [4–6]. In particular, in Ref. [5] we have analyzed superconductivity in a simple exactly solvable model of the pseudogap state, based on the model Fermi surface (in two dimensions) with "hot" patches [4]. For the description of the pseudogap state we have used an exactly solvable model first developed for the one dimensional case in Ref. [7] and for the limit of very large correlation lengths of "dielectric" short range order. It was shown that the superconducting energy gap averaged over the fluctuations of short range order is, in general, non zero also in the temperature region above the "mean-field" temperature of superconducting transition T_c, corresponding (according to Ref. [5]) to the appearance of homogeneous superconducting state in a sample as a whole. This lead us to the conclusion [5], that for temperatures $T > T_c$ there appear superconducting "drops", which exist up to the temperature T_{c0} of superconducting transition in the absence of the pseudogap of "dielectric" nature. This effect was attributed in Ref. [5] to the absence of self averaging of superconducting order parameter (gap) in situation when the correlation length of short range order fluctuations is larger than the coherence length of superconductivity (the size of Cooper pairs).

Under the assumption of self averaging superconducting energy gap the effects of finiteness of the correlation length of short range order fluctuations were analyzed in Ref. [6], where we have considered the pseudogap influence on T_c, calculated the temperature dependence of the gap for $T < T_c$, and derived the Ginzburg – Landau expansion for $T \sim T_c$. In this case we have used our "nearly exact" solution for the pseudogap state induced by Gaussian fluctuations of short range order, proposed first in Refs. [8,9] for one dimensional case and generalized for two dimensions in Refs. [10,11]. Within this approach it seems rather difficult to find the solution without the assumption of the self averaging superconducting energy gap. Note that the problem of the presence or absence of self averaging of superconducting gap was not studied well enough in most of the previous work on disordered superconductors. In most studies the self averaging property was just assumed on "physical grounds", with the reference on significantly different length scales on which the superconducting gap changes (coherence length ξ_0) and characteristic scales for the electronic subsystem (interatomic distance or the inverse Fermi momentum), e.g. in the impurity problem [12–14], or the correlation length ξ_{corr} of short range order in our problem [2,5,6]), where it may be expected that the complete self averaging of the superconducting energy gap appears for $\xi_{corr} \ll \xi_0$ [2,6]. We are unaware of any work, where the problem of self averageness was analyzed within some exactly solvable model of disorder.

One of the main aims of the present work is to perform precisely this type of analysis within very simple (though probably not realistic enough) one dimensional model of the pseudogap state, induced by fluctuations of "dielectric" short range order with finite correlation length, proposed in a recent paper of Bartosch and Kopietz [15]. An exact solution found in this work is very similar to that of our previous studies [7–9] and allows to perform a complete analysis of the self averaging properties of superconducting energy gap for two dimensional model of "hot" patches, proposed in Refs. [4,6,11]. Besides we shall study the full temperature dependences of superconducting energy gap for a superconductor with the pseudogap of "dielectric" nature.

II. SIMPLIFIED MODEL OF THE PSEUDOGAP STATE.

Let us consider an exactly solvable model of the pseudogap state proposed in Ref. [15] using slightly different approach. Consider an electron in one dimension in a periodic potential of the following form:

$$V(x) = 2D\cos(Qx+\phi) \tag{1}$$

with $Q = 2p_F - k$, where p_F – is Fermi momentum, and $k \ll p_F$ – is some small deviation from the scattering vector $2p_F$ [1]. Electronic spectrum close to the Fermi level is taken in the usual linearized form:

$$\begin{aligned} \xi_1 \equiv \xi_p = v_F(|p|-p_F) \qquad \xi_{p-2p_F} &= -\xi_p \quad ("") \\ \xi_2 \equiv \xi_{p-Q} = -\xi_p - v_F k &\equiv -\xi_p - \eta \end{aligned} \tag{2}$$

where we introduced the notation $\eta = v_F k$ (v_F – Fermi velocity), which will be widely used in the following. The potential (1) can be rewritten as:

$$V(x) = De^{i2p_F x - ikx} + D^* e^{-i2p_F x + ikx} \tag{3}$$

where we have introduced the complex amplitude $D \to De^{i\phi}$.

The solution of this problem is elementary. In the "two-wave" approximation of the usual band theory the one-electron (normal) Green's function, corresponding to the diagonal transition $p \to p$, takes the following form (in Matsubara's representation):

$$\begin{aligned} g_{11}(i\varepsilon_n pp) &= \frac{1}{i\varepsilon_n - \xi_1} + \frac{1}{i\varepsilon_n - \xi_1} D^* \frac{1}{i\varepsilon_n - \xi_2} D \frac{1}{i\varepsilon_n - \xi_1} + \ldots = \\ &= \frac{i\varepsilon_n - \xi_2}{(i\varepsilon_n - \xi_1)(i\varepsilon_n - \xi_2) - |D|^2} = \frac{i\varepsilon + \xi + \eta}{(i\varepsilon - \xi)(i\varepsilon + \xi + \eta) - |D|^2} \end{aligned} \tag{4}$$

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[1] This choice of the vector of AFM or CDW superstructure corresponds, in general, to the case of incommensurate ordering and fluctuations.

where in the last equality we have introduced the following short notations: $\xi_p = \xi$, $\varepsilon_n = \varepsilon$. We can also introduce the non diagonal (anomalous) Green's function, corresponding to the Umklapp process $p \to p - Q$:

$$g_{12}(i\varepsilon_n pp - Q) = \frac{1}{i\varepsilon_n - \xi_1} D^* \frac{1}{i\varepsilon_n - \xi_2} + \ldots =$$

$$= \frac{D^*}{(i\varepsilon_n - \xi_1)(i\varepsilon_n - \xi_2) - |D|^2} = \frac{D^*}{(i\varepsilon - \xi)(i\varepsilon + \xi + \eta) - |D|^2} \quad (5)$$

Let now (1) be random field. Following Ref. [15] we shall consider the very special model of disorder, when we shall assume the randomness of the scattering vector deviation k, with Lorentzian distribution [2]:

$$\mathcal{P}_k(k) = \frac{1}{\pi}\frac{\kappa}{k^2 + \kappa^2} \quad (6)$$

where $\kappa \equiv \xi_{corr}^{-1}$ and ξ_{corr} – is the correlation length of short range order. Phase ϕ in (1) is also considered as random with uniform distribution between 0 and 2π:

$$\mathcal{P}_\phi(\phi) = \begin{cases} \frac{1}{2\pi} & \text{for} \quad 0 \leq \phi \leq 2\pi \\ 0 & \text{for other values} \end{cases} \quad (7)$$

Correlation function of random fields $V(x)$ at different points can be calculated directly and is given by:

$$< V(x)V(x') >= 2D^2 \cos[2p_F(x - x')] \exp[-\kappa|x - x'|] \quad (8)$$

where the angular brackets denote averaging over (6) and (7). The random field with precisely this form of pair correlator was treated first in Ref. [16], as well as in Refs. [7–9], though in these works the Gaussian statistics of the random field was also assumed [3]. In the model under consideration the random field $V(x)$ is, in general, non Gaussian [15]. The Fourier transformation of (8) takes the form of characteristic Lorentzian, which defines the effective interaction of an electron with fluctuations of short range order [2]:

$$V_{eff}(q) = 2D^2 \left\{ \frac{\kappa}{(q - 2p_F)^2 + \kappa^2} + \frac{\kappa}{(q + 2p_F)^2 + \kappa^2} \right\} \quad (9)$$

This type of effective interaction was introduced in all papers on the pseudogap of "dielectric" nature cited above.

Green's functions averaged over the ensemble of random fields (1) with distributions (6) and (7) are calculated by elementary integration. The average value of the anomalous Green's function (5) is simply zero (after the averaging over (7)), which corresponds to the

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[2] In fact this corresponds to a specific model of phase fluctuations of the potential (1).

[3] For the Gaussian random field all higher correlators of the random field $V(x)$ are factorized "a'la Wick" into products of pair correlators (8).

absence of "dielectric" long range order. The average of the Green's function (4) is obtained by term by term integration of the perturbation series (4) with (6), so that:

$$G(i\varepsilon_n p) = \frac{1}{i\varepsilon_n - \xi_p} + \frac{1}{i\varepsilon_n - \xi_p} D^* \frac{1}{i\varepsilon_n + \xi_p + iv_F\kappa} D \frac{1}{i\varepsilon_n - \xi_p} +$$
$$+ \frac{1}{i\varepsilon_n - \xi_p} D^* \frac{1}{i\varepsilon_n + \xi_p + iv_F\kappa} D \frac{1}{i\varepsilon_n - \xi_p} D^* \frac{1}{i\varepsilon_n + \xi_p + iv_F\kappa} D \frac{1}{i\varepsilon_n - \xi_p} + \ldots =$$
$$= \frac{i\varepsilon_n + \xi_p + iv_F\kappa}{(i\varepsilon_n - \xi_p)(i\varepsilon_n + \xi_p + iv_F\kappa) - |D|^2} \tag{10}$$

This is precisely the exact solution for the Green's function given in Ref. [15].

In the following we shall also consider the model with fluctuating amplitude D of the field (1), so that the appropriate Green's function can be obtained by the averaging of (10) with some amplitude distribution $\mathcal{P}_D(D)$. In particular, we can take the amplitude distribution in the Rayleigh form [7,8,15]:

$$\mathcal{P}_D(D) = \frac{2D}{W^2} \exp\left(-\frac{D^2}{W^2}\right) \tag{11}$$

In this case the additional averaging of correlators (8) and (9) leads to the simple replacement $D \to W$. The average Green's function of an electron in this case becomes:

$$G(i\varepsilon_n p) = \int_0^\infty dD \mathcal{P}_D(D) \frac{i\varepsilon_n + \xi_p + iv_F\kappa}{(i\varepsilon_n - \xi_p)(i\varepsilon_n + \xi_p + iv_F\kappa) - |D|^2} =$$
$$= \int_0^\infty d\zeta e^{-\zeta} \frac{i\varepsilon_n + \xi_p + iv_F\kappa}{(i\varepsilon_n - \xi_p)(i\varepsilon_n + \xi_p + iv_F\kappa) - \zeta W^2} \tag{12}$$

where W determines now the effective width of the pseudogap. In the limit of the large correlation length of fluctuations of (1), i.e. for $\xi_{corr} \to \infty$ ($\kappa \to 0$), the solution (12) coincides with that found earlier in Refs. [7] for the case of the Gaussian random field. For finite κ this solution coincides with that proposed in Ref. [11] during the formal analysis of approximations, used in Refs. [8,9] in the analysis of the general problem of an electron in the Gaussian random field with pair correlator given by (8). In Refs. [11,15] it was shown that the density of states, corresponding to the Green's function (12), possesses a characteristic smooth pseudogap in the vicinity of the Fermi level, and the values of the density of states are quantitatively very close [11,15,17] (practically for all energies in the incommensurate case) to the values obtained in Ref. [8], as well as to the results of exact numerical calculation for the Gaussian random field, performed in Refs. [18–20] [4].

If the random field (1) is created by fluctuations of some kind of "dielectric" order parameter (e.g. CDW or AFM), distribution (11) may describe Gaussian fluctuations, existing at

[4] Using the approach of Ref. [7] in the present model it is easy to find also the two particle Green's function and accordingly the frequency dependence of conductivity [15]. Unfortunately, the specific form of "disorder" (random field) leads to unphysical behavior at zero frequency, corresponding to an "ideal" conductor.

high enough temperatures [10,11]. For lower temperatures, even before the appropriate long range order appears, the amplitude fluctuations of the order parameter are "frozen out" (cf. [3,21]) and we may assume the amplitude to be non random and put $D = W$, while phase fluctuations remain important up to much lower temperatures. Accordingly, the solution of the form of (10), leading to sharp enough pseudogap for large correlation lengths ξ_{corr} [16], can be used to describe the low temperature region of fluctuations of short range order. As we do not analyze the microscopics of "dielectric" fluctuations, all the parameters, characterizing these fluctuations, such as correlation length $\xi_{corr} = \kappa^{-1}$ and amplitudes D and W (the energy width of the pseudogap) are considered as phenomenological. The "low temperature" and "high temperature" regime of fluctuations of short range order can, in principle, be realized at different temperatures in comparison to the temperature of superconducting transition.

Generalization to the case of two dimensional system of electrons, typical for HTSC – cuprates, can be done in the spirit of "hot patches" model for the Fermi surface, proposed in Refs. [4–6]. In this case we shall assume the existence of two independent types of fluctuations of the type of (1) [5], oriented along the orthogonal axes x and y, strongly interacting only with electrons from flat parts of the two dimensional Fermi surface, orthogonal to these axes. Accordingly, we assume the factorized form of two dimensional (random) potential for electrons: $V(x,y) = V(x)V(y)$ [4–6]. The size of these flat ("hot") parts of the Fermi surface is determined by an additional parameter α, so that 2α gives the angular size of the flat part, as seen from the center of the Brillouin zone [2,4–6]. In particular, the value of $\alpha = \pi/4$ corresponds to the square like Fermi surface (complete nesting), when all Fermi surface becomes "hot". For $\alpha < \pi/4$ the "cold" parts of the Fermi surface appear, where we neglect the scattering on fluctuations of the "dielectric" order parameter and electrons are considered as free. In this model all the physical characteristics, determined by the integrals over the Fermi surface, consist of additive contributions from "hot" and "cold" parts. Pseudogap renormalization of electronic spectrum takes place only on "hot" parts (and close to these parts), while on "cold" part the usual Fermi liquid (gas) behavior remains [2].

This picture is in qualitative agreement with a numerous ARPES experiments on underdoped cuprates [1,2], which show that pseudogap anomalies appear in the vicinity of $(0,\pi)$ point of the Brillouin zone, and vanish as we move to the diagonal direction. The presence of flat parts on the Fermi surface of HTSC – cuprates was also reliably observed in the ARPES experiments by several independent groups [2].

III. GOR'KOV'S EQUATIONS IN THE PSEUDOGAP STATE.

To study superconductivity for the system with pseudogap due to fluctuations of "dielectric" short range order we shall assume the simplest BCS form of pairing interaction, characterized by the attraction constant V, which is non zero in some energy interval $2\omega_c$

[5]Note the crude analogy of this picture to the concept of phase separation in HTSC – cuprates (stripes) [22], if we associate the correlation length ξ_{corr} with characteristic size (period) of stripes [2].

in the vicinity of the Fermi level (ω_c - is the characteristic frequency of the quanta, responsible for electron attraction). We have already used the same approach in Refs. [4–6]. In the present work we shall limit ourselves only to the analysis of s-wave pairing. There are no serious obstacles within our approach for the analysis of d-wave pairing, typical for HTSC – cuprates, though in this case the presence of the angular dependence (anisotropy) of superconducting gap leads [4,5] to the additional integration over the angle and to a significant growth of time necessary for numerical computations. At the same time, it was shown in Refs. [4–6] , that the pseudogap influence on superconductivity is essentially similar both for s and d-wave cases, differing only by the scale of parameters leading to the same changes of basic characteristics of superconducting state (d-wave pairing is less stable to the dielectrization of electronic spectrum in comparison to s-wave pairing).

On "cold" parts of the Fermi surface superconductivity is described by standard BCS equations. In the following we shall concentrate on the derivation of Gor'kov's equations for the one dimensional model, which is equivalent to the analysis of the situation at "hot" parts of the Fermi surface for two dimensional case [5,6]. Green's functions (4), (5) for one dimensional system in periodic field (1), can be written as matrix elements:

$$
\begin{aligned}
g_{11} &= \frac{i\varepsilon_n - \xi_2}{(i\varepsilon_n - \xi_1)(i\varepsilon_n - \xi_2) - |D|^2} \qquad & g_{12} &= \frac{D^*}{(i\varepsilon_n - \xi_1)(i\varepsilon_n - \xi_2) - |D|^2} \\
g_{21} &= \frac{D}{(i\varepsilon_n - \xi_1)(i\varepsilon_n - \xi_1) - |D|^2} \qquad & g_{22} &= \frac{i\varepsilon_n - \xi_1}{(i\varepsilon_n - \xi_1)(i\varepsilon_n - \xi_2) - |D|^2}
\end{aligned} \tag{13}
$$

In the presence of Cooper pairing the Gor'kov's equations, constructed on "free" Green's functions of the type of (13), are shown in diagrammatic form in Fig. 1. In analytic form we have:

$$
\begin{aligned}
G_{11} &= g_{11} - g_{11}\Delta F^+_{11} - g_{12}\Delta F^+_{21} \\
F^+_{11} &= g^*_{11}\Delta^* G_{11} + g^*_{12}\Delta^* G_{12} \\
G_{21} &= g_{21} - g_{21}\Delta F^+_{11} - g_{22}\Delta F^+_{21} \\
F^+_{21} &= g^*_{21}\Delta^* G_{11} + g^*_{22}\Delta^* G_{21}
\end{aligned} \tag{14}
$$

where superconducting energy gap is determined, as usual, from:

$$
\Delta^* = VT\sum_{np} F^+_{11}(\varepsilon_n p) = \lambda T\sum_n \int_{-\infty}^{\infty} d\xi_p F^+_{11}(\varepsilon_n \xi_p) \equiv \lambda T \sum_n \overline{F^+_{11}(\varepsilon_n)} \tag{15}
$$

where we have introduced the dimensionless pairing constant $\lambda = N_0(0)V$ ($N_0(0)$ – is the free electron density of states at the Fermi level.).

The solution of Eqs. (14) gives:

$$
\begin{aligned}
G_{11} &= -\frac{1}{Det}[(i\varepsilon + \xi_1)(\varepsilon^2 + \xi_2^2 + D^2 + \Delta^2) - D^2(\xi_1 + \xi_2)] = \\
&= -\frac{1}{Det}\{(i\varepsilon + \xi)[\varepsilon^2 + (\xi + \eta)^2 + D^2 + \Delta^2] + D^2\eta\} \\
F^+_{11} &= -\frac{1}{Det}\Delta^*(\varepsilon^2 + \xi_2^2 + D^2 + \Delta^2) = \\
&= -\frac{1}{Det}\Delta^*[\varepsilon^2 + (\xi + \eta)^2 + D^2 + \Delta^2]
\end{aligned} \tag{16}
$$

where

$$Det = (\varepsilon^2 + \xi_1^2 + D^2 + \Delta^2)(\varepsilon^2 + \xi_2^2 + D^2 + \Delta^2) - (\xi_1 + \xi_2)^2 D^2 =$$
$$= (\varepsilon^2 + \xi^2 + D^2 + \Delta^2)(\varepsilon^2 + (\xi + \eta)^2 + D^2 + \Delta^2) - \eta^2 D^2 \quad (17)$$

where D denotes the real amplitude of fluctuation field (1). In accordance with Eq. (15) Gor'kov's Green's function F_{11}^{+} determines the superconducting energy gap. Taking into account the random nature of "dielectric" fluctuations Eq. (15) should be averaged over the fluctuations of both "phase" $\eta = v_F k$ and amplitude D, using distributions (6) and (for high temperature fluctuations) (11).

Rather long, though direct, calculation of the integral in (15) via residues gives:

$$\overline{F_{11}^{+}(\varepsilon)} = \frac{\pi\Delta^*}{\sqrt{2}} \frac{1}{\sqrt{\sqrt{(\tilde{\varepsilon}^2 + D^2 + \frac{\eta^2}{4})^2 - \eta^2 D^2} + \tilde{\varepsilon}^2 + D^2 - \frac{\eta^2}{4}}} \left\{ 1 + \frac{\tilde{\varepsilon}^2 + D^2 + \frac{\eta^2}{4}}{\sqrt{(\tilde{\varepsilon}^2 + D^2 + \frac{\eta^2}{4})^2 - \eta^2 D^2}} \right\} \equiv$$
$$\equiv \pi\Delta^* \mathcal{F}(\varepsilon, \Delta, \eta, D) \quad (18)$$

where we have introduced:

$$\tilde{\varepsilon} = \sqrt{\varepsilon^2 + \Delta^2} \quad (19)$$

Then from Eq. (15) we can immediately obtain the equation for superconducting energy gap in two dimensional "hot patches" model [4–6]:

$$1 = 2\pi\lambda T \sum_{n=0}^{\left[\frac{\omega_c}{2\pi T}\right]} \left\{ \tilde{\alpha}\mathcal{F}(\varepsilon, \Delta, \eta, D) + \frac{1 - \tilde{\alpha}}{\tilde{\varepsilon}} \right\} \quad (20)$$

where we have introduced the relative fraction of "hot" parts on the Fermi surface $\tilde{\alpha} = \frac{4}{\pi}\alpha$. The second term in (20) gives the standard BCS-like contribution from "cold" parts, occupying $(1 - \tilde{\alpha})$ part of the Fermi surface. Summation over n in (20) is performed up to some maximal value determined by the integer part of the ratio $\frac{\omega_c}{2\pi T}$.

Numerical solution of (20) allows to find the value of the gap $\Delta(\eta, D)$ for fixed η and D (i.e. for the given value of the random field of fluctuations (1)) for any given temperature. After that we can perform averaging over (6) and (11) and find in this way temperature dependence of the average energy gap. In particular, for the "low temperature" regime of "dielectric" fluctuations it is sufficient to average only over the "phase" η, so that the superconducting gap is given by:

$$< \Delta >= \frac{1}{\pi} \int_{-\infty}^{\infty} d\eta \frac{v_F \kappa}{\eta^2 + v_F^2 \kappa^2} \Delta(\eta, D) \quad (21)$$

In "high temperature" regime we have to add also the averaging over the amplitude D with distribution function (11):

$$< \Delta >= \frac{2}{W^2} \int_0^{\infty} dD D \exp\left(-\frac{D^2}{W^2}\right) \frac{1}{\pi} \int_{-\infty}^{\infty} d\eta \frac{v_F \kappa}{\eta^2 + v_F^2 \kappa^2} \Delta(\eta, D) \quad (22)$$

As a result we shall find the temperature dependences of the average superconducting gap $< \Delta >$ without any statistical assumptions like the self averaging nature of the order parameter.

Analogously we can calculate the temperature dependence of dispersion $<\Delta^2> - <\Delta>^2$, allowing to estimate the randomness of Δ, i.e. the presence or absence of self averaging. Results of these calculations will be presented in the next section.

As we have already noted in the Introduction, most papers on superconductivity in disordered systems actually assume the self averaging nature of the superconducting gap Δ. In this case Δ is treated, in fact, as non random and independent of parameters of the random field in which the electrons forming the Cooper pairs actually propagate. In our case these parameters are the amplitude D and "phase" η of (1), accordingly the self averaging over these characteristics may be studied separately.

Let Δ be self averaging over the fluctuations of η. In this case we can treat Δ in Eq. (16) is independent of η. Then the anomalous Gor'kov's function averaged over the fluctuations of η can be written as:

$$< F^{+}_{11} >= \frac{\Delta^*}{\pi}\int_{-\infty}^{\infty} d\eta \frac{v_F\kappa}{\eta^2+v_F^2\kappa^2}\frac{\varepsilon^2+(\xi+\eta)^2+D^2+\Delta^2}{(\varepsilon^2+\xi^2+D^2+\Delta^2)[\varepsilon^2+(\xi+\eta)^2+D^2+\Delta^2]-\eta^2 D^2} \tag{23}$$

This integral can be directly calculated, so that after long calculations we get:

$$< F^{+}_{11} >= \Delta^* \frac{\tilde\varepsilon^2\left(1+\frac{v_F\kappa}{\tilde\varepsilon}\right)^2 + D^2\left(1+\frac{v_F\kappa}{\tilde\varepsilon}\right)+\xi^2}{\left[\left(1+\frac{v_F\kappa}{\tilde\varepsilon}\right)\tilde\varepsilon^2+\xi^2+D^2\right]^2+v_F^2\kappa^2\xi^2} \tag{24}$$

where $\tilde\varepsilon$ was introduced in (19). Accordingly we can calculate the integral of (24), entering the gap equation:

$$\overline{< F^{+}_{11} >} \equiv \int_{-\infty}^{\infty} d\xi < F^{+}_{11} >= \frac{\pi\Delta^*\left(1+\frac{v_F\kappa}{2\tilde\varepsilon}\right)}{\sqrt{D^2+\tilde\varepsilon^2\left(1+\frac{v_F\kappa}{2\tilde\varepsilon}\right)^2}} \tag{25}$$

So, despite rather complicated form of the anomalous Green's function (24), in the gap equation the account of the interaction with fluctuations on "hot" (flat) parts of the Fermi surface reduces to more or less "standard" renormalization:

$$\begin{aligned} \varepsilon \to \varepsilon\left(1+\frac{v_F\kappa}{2\tilde\varepsilon}\right) &= \varepsilon\left(1+\frac{v_F\kappa}{2\sqrt{\varepsilon^2+\Delta^2}}\right) \\ \Delta \to \Delta\left(1+\frac{v_F\kappa}{2\tilde\varepsilon}\right) &= \Delta\left(1+\frac{v_F\kappa}{2\sqrt{\varepsilon^2+\Delta^2}}\right) \end{aligned} \tag{26}$$

analogous to that appearing in the problem of impurity scattering in superconductors [23]. Similar renormalization was already noted for the variant of the present problem in Ref. [6]. The analogy with impurity problem here is rather natural as our parameter $v_F\kappa = v_F\xi_{corr}^{-1}$ actually determines the characteristic inverse time of flight of an electron through the region of short range order of the size of $\sim \xi_{corr}$. Of course, the additional pseudogap influence is also connected with the appearance in Eqs. (24), (25) of the square of dielectric gap D^2.

Finally, the gap equation determining superconductivity in the "hot patches" model with the assumption of self averaging over "phase" fluctuations takes the following form:

$$1 = 2\pi\lambda T \sum_{n=0}^{\left[\frac{\omega_c}{2\pi T}\right]} \left\{ \tilde{\alpha} \frac{1 + \frac{v_F\kappa}{2\tilde{\varepsilon}}}{\sqrt{D^2 + \tilde{\varepsilon}^2 \left(1 + \frac{v_F\kappa}{2\tilde{\varepsilon}}\right)^2}} + \frac{1-\tilde{\alpha}}{\tilde{\varepsilon}} \right\} \quad (27)$$

The solution of this equations, naturally, is simpler than that of Eq. (20) with later averaging over (21). In the absence of fluctuations of the amplitude of "dielectric" field D, which is valid for the "low temperature" region of fluctuations of short range order, it is Eq. (27) that determines "mean field" (in terms of Ref. [5]) behavior of $\Delta(T)$ with respect to fluctuations of the random field (1).

In "high temperature" region of fluctuations of short range order, assuming for D the distribution function (11) and self averaging over the fluctuations of D, we obtain the following equation for the average superconducting gap:

$$1 = 2\pi\lambda T \sum_{n=0}^{\left[\frac{\omega_c}{2\pi T}\right]} \left\{ \frac{2\tilde{\alpha}}{W^2} \int_0^\infty dD D \exp\left(-\frac{D^2}{W^2}\right) \frac{1 + \frac{v_F\kappa}{2\tilde{\varepsilon}}}{\sqrt{D^2 + \tilde{\varepsilon}^2 \left(1 + \frac{v_F\kappa}{2\tilde{\varepsilon}}\right)^2}} + \frac{1-\tilde{\alpha}}{\tilde{\varepsilon}} \right\} \quad (28)$$

describing the situation analogous to that studied in our previous work [6], where we have analyzed the influence of Gaussian fluctuations of "dielectric" short range order using the methods of Refs. [8,9]. Here the fluctuations of (1) are treated exactly, but D is assumed to be self averaging. We shall see below that the results following from the solution of Eq. (28) are very close to that obtained in Ref. [6]. For $\kappa \to 0(\xi_{corr} \to \infty)$ Eq. (28) reduces to the similar "mean field" equation of Ref. [5]. The temperature of superconducting transition, determined by Eq. (27) or Eq. (28), can apparently be identified as the temperature of the appearance of infinitesimally small superconducting gap, homogeneous over the whole sample [5].

In the next section we shall present the results of numerical solution of Eqs. (27), (28) in comparison with the results of an exact analysis, based upon the approach using Eqs. (20), (21), (22).

IV. MAIN RESULTS AND DISCUSSION.

Let us discuss the results of numerical analysis of gap equations derived in the previous section[6].

In Figs. 2 and 3 we show the dependences of the critical temperature T_c of superconducting transition for the "low temperature" regime of "dielectric" fluctuations (at this temperature the "mean field" gap, determined by Eq. (27) becomes zero) on the width of the pseudogap W (which in this case coincide with the amplitude of dielectric gap D) and correlation length of short range order. These results are in qualitative agreement with similar dependences for the "high temperature" regime of "dielectric" fluctuations (when

[6]During our calculations we have assumed for the relative part of flat patches on the Fermi the value of $\tilde{\alpha} = 2/3$

T_c is determined by Eq. (28)), as well as with dependences, obtained by us earlier for the different model of "dielectric" fluctuations with finite correlation length in Ref. [6]. With the growth of the pseudogap width W the "mean field" value of T_c is suppressed. Diminishing correlation length leads to the "filling" of the pseudogap [2,8,15] and T_c suppression becomes less effective.

In Fig. 4 the curves show the temperature dependences of superconducting gap $<\Delta>$, averaged over both amplitude D and "phase" η ("high temperature" region of fluctuations of short range order, where $<\Delta>$ is given by (22)), for different values of $v_F\kappa$. Dashed curves give the appropriate "mean field" temperature dependences of superconducting gap, obtained assuming the self averaging of superconducting order parameter over both the fluctuations of amplitude and phase, as determined by Eq. (28).

Superconducting gap, averaged over the fluctuations of short range order, is non zero in the temperature region above T_c, corresponding to the zero of the "mean field" gap (i.e. the gap homogeneous over the sample). More so, it is seen that the average gap is non zero also in a narrow region of temperatures larger than the transition temperature in the absence of short range order fluctuations (pseudogap) T_{c0}. This effect is due to the existence of fluctuations of the "phase" η, when the Fermi level is in the energy interval, corresponding to the peaks of the density of states due to formation of dielectric gap. To understand this, note that for the given realization of "phase" η and of amplitude D, the density of states has the form:

$$\frac{N(E)}{N_0(0)} = -\frac{1}{\pi N_0(0)} Im \sum_{\mathbf{p}} g^R_{11}(Epp) = \begin{cases} \frac{|E+\frac{\eta}{2}|}{\sqrt{(E+\frac{\eta}{2})^2-D^2}} & \text{for} \quad |E+\frac{\eta}{2}| > D \\ 0 & \text{for other values} \end{cases} \tag{29}$$

where $g^R_{11}(Epp)$ – is the retarded Green's function, obtained from (4) via standard analytical continuation $i\varepsilon_n \to E + i0$, and $N_0(0)$ – is the density of states at the Fermi level in the absence of short range order fluctuations. Thus for $\frac{\eta}{2} \approx D$ the Fermi level position approximately coincides with the peak of the density of states, leading to larger values of superconducting gap $\Delta(\eta, D)$. The growth of the dielectric gap D leads also to the larger width of peaks in the density of states (29), so that for $\frac{\eta}{2} \approx D$ superconducting gap $\Delta(\eta, D)$ grows with D. Then, for any temperature above T_{c0} and for large enough amplitudes of dielectric gap $D > D^*(T)$, on the phase diagram in the variables η and D there is always a narrow region close to the line $\frac{\eta}{2} = D$, where superconducting gap $\Delta(\eta, D)$ is different from zero (cf. Fig.5). This leads to the appearance in the temperature dependence of the averaged (over random field fluctuations) superconducting gap $<\Delta>$ of an infinite, though exponentially small "tail" in the temperature region above T_{c0}.

At the insert in Fig. 4 we show the temperature dependence of the relative mean square fluctuation of the superconducting gap $\delta\Delta/\Delta = \sqrt{<\Delta^2> - <\Delta>^2}/<\Delta>$ for the "high temperature" regime of "dielectric" fluctuations. For large correlation lengths of short range order ($\xi_0/\xi_{corr} \ll 1$) these fluctuations of superconducting order parameter are very strong for all temperatures, signifying the obvious absence of self averaging. Surprisingly, these fluctuations of superconducting energy gap are strong enough also for small enough correlation lengths, at least in the region of $T > T_c$. In particular, the "tail" in the temperature dependence of $<\Delta>$ for $T > T_c$ is observed even for $v_F\kappa/T_{c0} = 100$, when $\xi_0/\xi_{corr} \approx 30 \gg 1$.

The full curves in Fig. 6 show the temperature dependence of superconducting gap $<\Delta>$, averaged over the "phase" η (cf. (21)), for the "low temperature regime" of "dielectric" fluctuations, when the amplitude fluctuations of dielectric gap are frozen out and $D = W$. Dashed curves show the appropriate temperature dependences of the "mean field" superconducting gap, obtained with the assumption of self averaging superconducting order parameter over fluctuations of the "phase" η, and defined by Eq. (27). For large enough correlation lengths of the short range order the average gap for $T < T_c$ is very close to its "mean field" values and its "tail" in the region of $T > T_c$ is relatively small. This behavior for the "low temperature" region of "dielectric" fluctuations is due to the fact, that for $\xi_{corr} \to \infty$ there is no randomness in this model at all ($\eta = 0$, $D = W$). Accordingly, the mean square fluctuation of the gap, shown at the insert in Fig. 6, is rather small for large correlation lengths and $T < T_c$, but grows sharply for $T > T_c$. As correlation length becomes smaller, fluctuations of superconducting gap $\delta\Delta$ for $T < T_c$ at first grow , mainly due to the growth of randomness (parameter $v_F\kappa$ determines the width of the distribution of fluctuations of η), but afterwards diminish in the region of $\xi_0/\xi_{corr} \gg 1$. In the "tail" region of the averaged superconducting gap ($T > T_c$) fluctuations of superconducting gap are quite large. Though these fluctuations diminish for smaller correlation lengths of the short range order ξ_{corr}, they still remain significant even for small enough correlation lengths, i.e. for $\xi_0/\xi_{corr} \gg 1$.

As well as in the "high temperature" region of "dielectric" fluctuations the "tail" in the temperature dependence of the average gap is observed here also in the region of $T > T_{c0}$. It is explained by the same reasons as discussed above. However, for the "low temperature" region the amplitude of the dielectric gap is non random and fixed at $D = W$. Thus, for $T_{c0} < T < T_c^*$, where T_c^* is defined by $D^*(T_c^*) = W$, there exists a narrow region of "phases" close to $\eta = 2W$, where the superconducting gap $\Delta(\eta, W)$ is different from zero, while for $T > T_c^*$ such region is absent (cf. Fig.5). The value of T_c^* determines the temperature up to which exists the "tail" of the average gap, i.e. the critical temperature for the average gap $< \Delta >$. From the definition of T_c^* it is obvious that this temperature is independent of correlation lengths and depends only on W. As the width of the peaks in the density of states (29), as well as $\Delta(\eta, D)$, grows with the growth of D if condition $\frac{\eta}{2} \approx D$ is fulfilled, the value of T_c^* grows with W. The dependence of T_c^* on W is shown at the insert in Fig. 6.

V. CONCLUSION

In this work we have studied superconductivity within very simple model of the pseudogap in two dimensional electronic system which allows an exact solution. The central result is the explicit demonstration of the absence[21 e of complete self averaging of superconducting order parameter (energy gap) over the random field of "dielectric" fluctuations, leading to the formation of the pseudogap. This is rather surprising from the point of view of the standard theory of superconductivity in disordered systems [12–14]. The absence of self averaging is reflected in the appearance of fluctuations of the gap, especially strong for the temperatures larger than the "mean field" temperature of superconducting transition T_c, following from the standard equations, assuming the the self averaging nature of the order parameter. We identify this temperature with the critical temperature for the appearance of homogeneous superconducting state in the whole sample, while for the real disordered system the superconducting state is inhomogeneous and for $T > T_c$ superconductivity exists in separate regions ("drops"), appearing due to random fluctuations of the local density of electronic states. The difference with our previous work [5], where this picture was discussed in the limit of very large correlation lengths of short range order $\xi_{corr} \to \infty$, the use of the model of Ref. [15] allowed to find the complete solution for arbitrary values of ξ_{corr}. This solution demonstrates the absence of complete self averaging of superconducting gap even for $\xi_{corr} < \xi_0$, which contradicts the naive expectations of the standard approach [2]. As we noted above, we are unaware of works, where the problem of self averaging of Δ was analyzed within any exactly solvable model of disorder. Here we presented precisely this type of analysis. It is unclear at present whether the results obtained will be conserved in more realistic models of disorder.

It will be very interesting to analyze also the behavior of the spectral density and tunneling density of states, analogous to that done in our previous work [5] in the limit of $\xi_{corr} \to \infty$. It will be especially important to consider the self averaging properties of the density of states, which is the standard assumption in the theory of disordered systems.

As to comparison with experiments on high – temperature superconductors, we note Refs. [24,25], where the scanning tunneling microscopy (measuring the local density of states) on films of $Bi_2Sr_2CaCu_2O_{8+\delta}$ clearly demonstrated the existence in this system of microscopic superconducting regions along with dominant semiconducting regions with typical pseudogap in the electronic spectrum. These observations are in qualitative agreement with the main conclusions of the present study.

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FIGURES

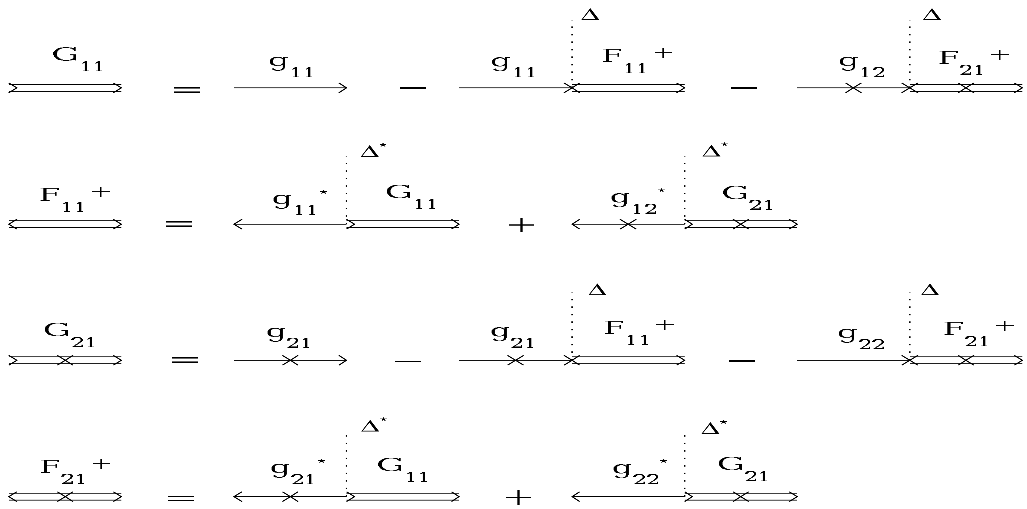

FIG. 1. Gor'kov's equations in one dimensional periodic field.

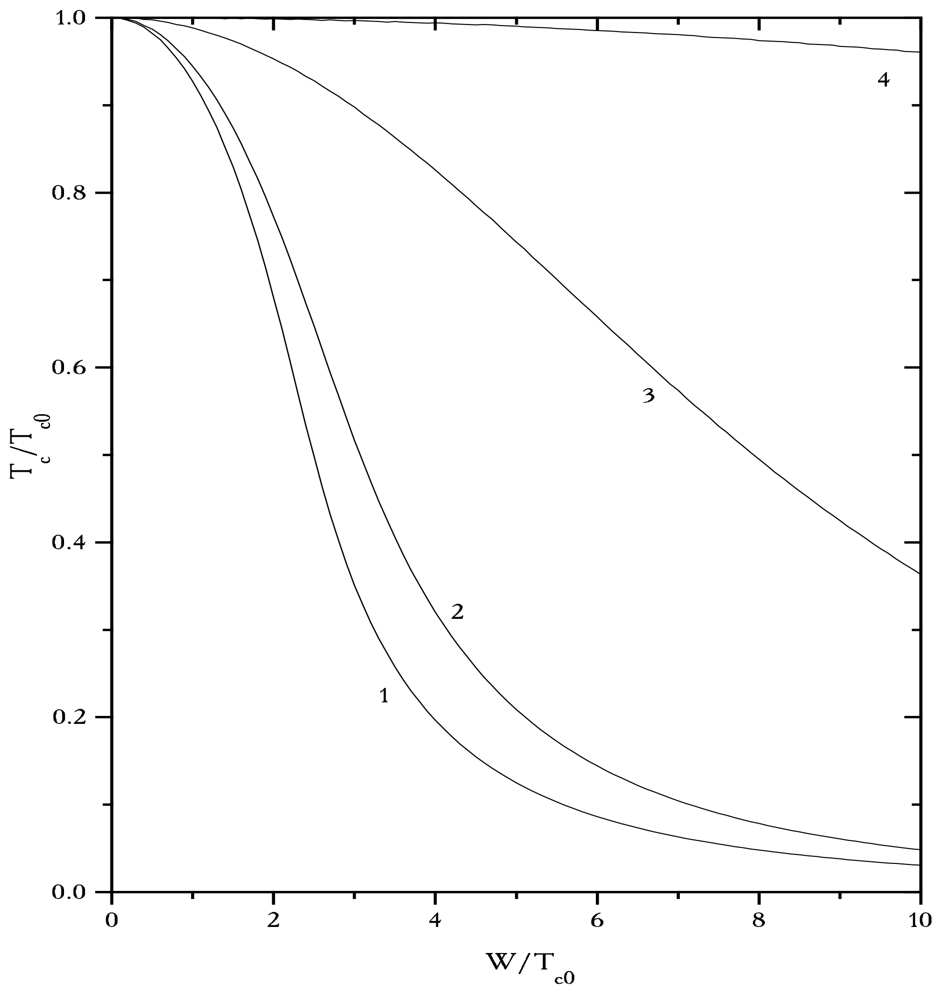

FIG. 2. Dependence of the critical temperature of superconducting transition for the “low temperature” region of “dielectric” fluctuations on the width of the pseudogap W for different values of correlation length of short range order $\frac{v_F\kappa}{T_{c0}}$ =: 1. 0.1; 2. 1; 3. 10; 4. 100.

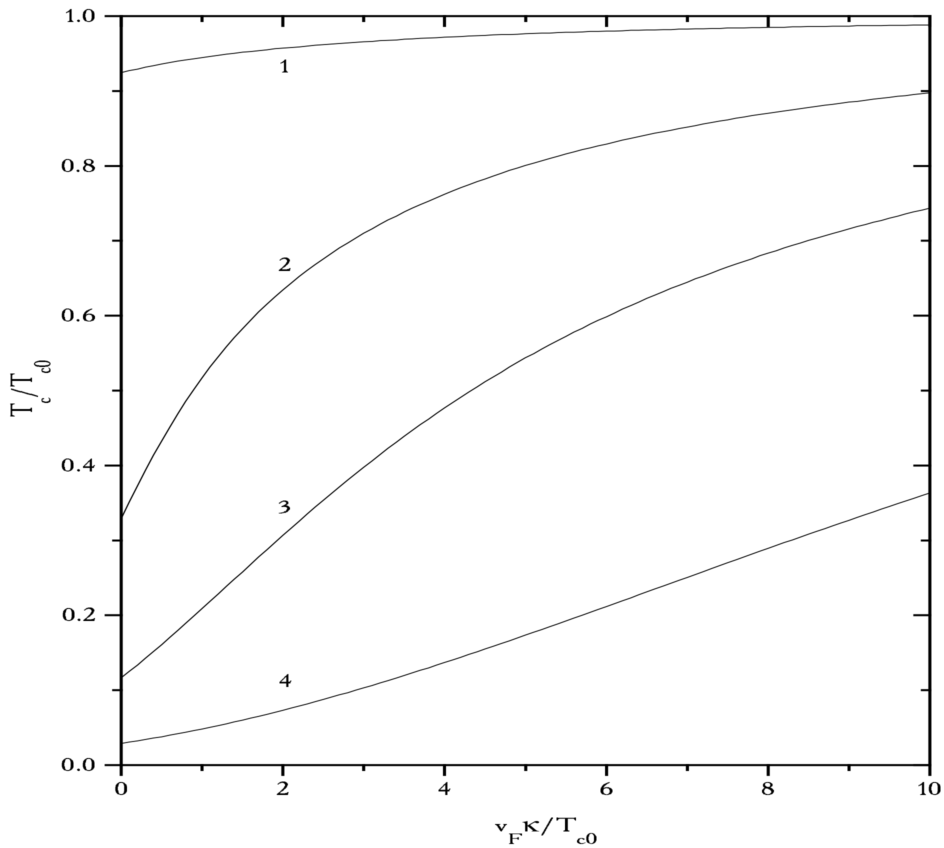

FIG. 3. Dependence of the critical temperature of superconducting transition for the "low temperature" region of "dielectric" fluctuations on the correlation length of short range order for different values of the pseudogap width $\frac{W}{T_{c0}}$ =: 1. 1; 2. 3; 3. 5; 4. 10.

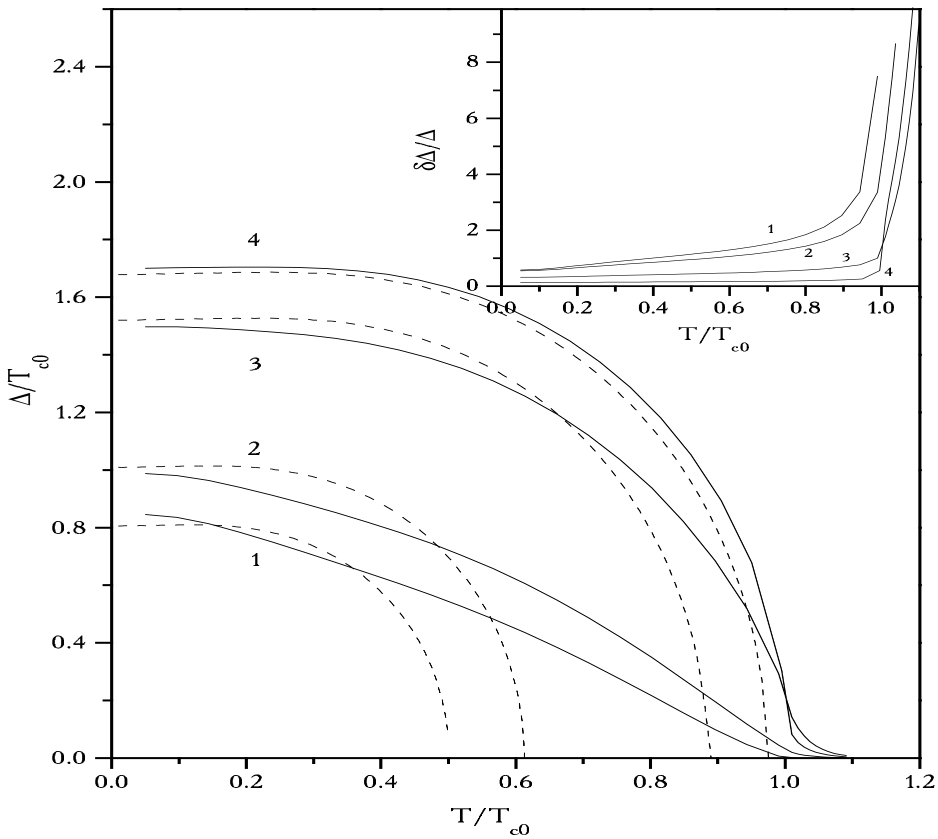

FIG. 4. Temperature dependence of superconducting energy gap for the "high temperature" region of "dielectric" fluctuations. Full curves – superconducting gap $<\Delta>$ averaged over the amplitude D and over the "phase" η, as determined by Eq. (22). Dashed curves – "mean field" superconducting gap determined by Eq. (28). At the insert – temperature dependence of the relative mean square fluctuation of superconducting gap. Curves are given for $\frac{W}{T_{c0}}=3$ and $\frac{v_F\kappa}{T_{c0}}=$: 1. 0.1; 2. 1; 3. 10; 4. 100.

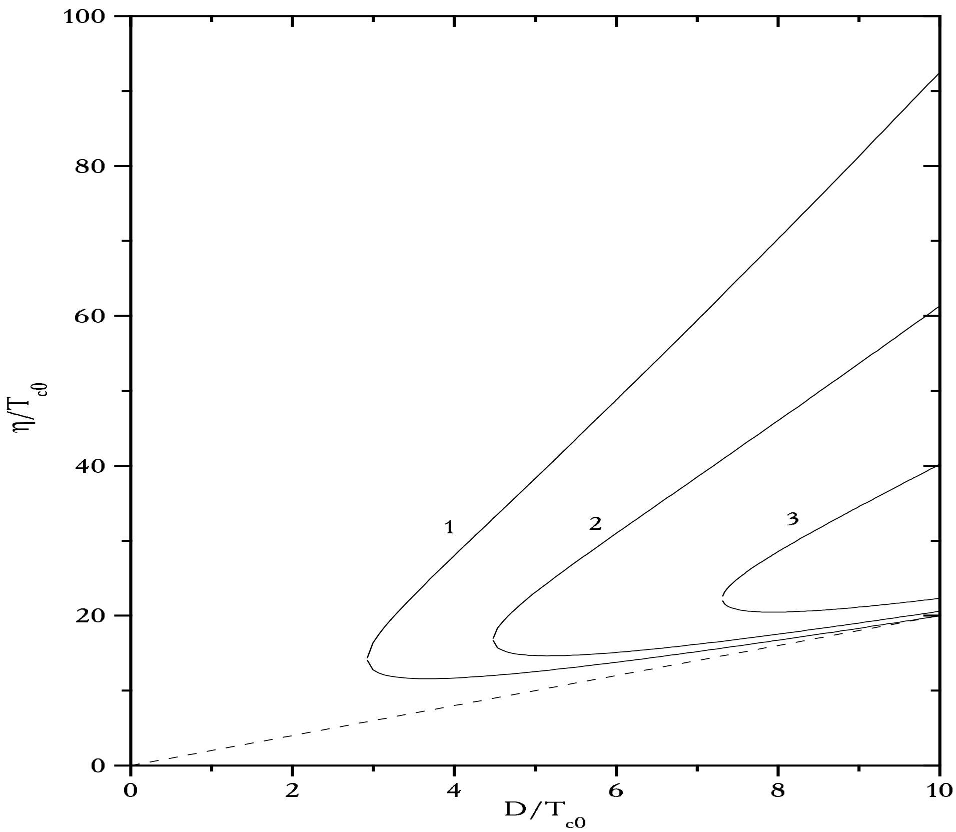

FIG. 5. The regions of the phase diagram with non zero superconducting gap for different temperatures above T_{c0}, shown for the values of T/T_{c0}: 1. 1.05; 2. 1.1; 3. 1.2. Dashed line corresponds to $D = \eta/2$.

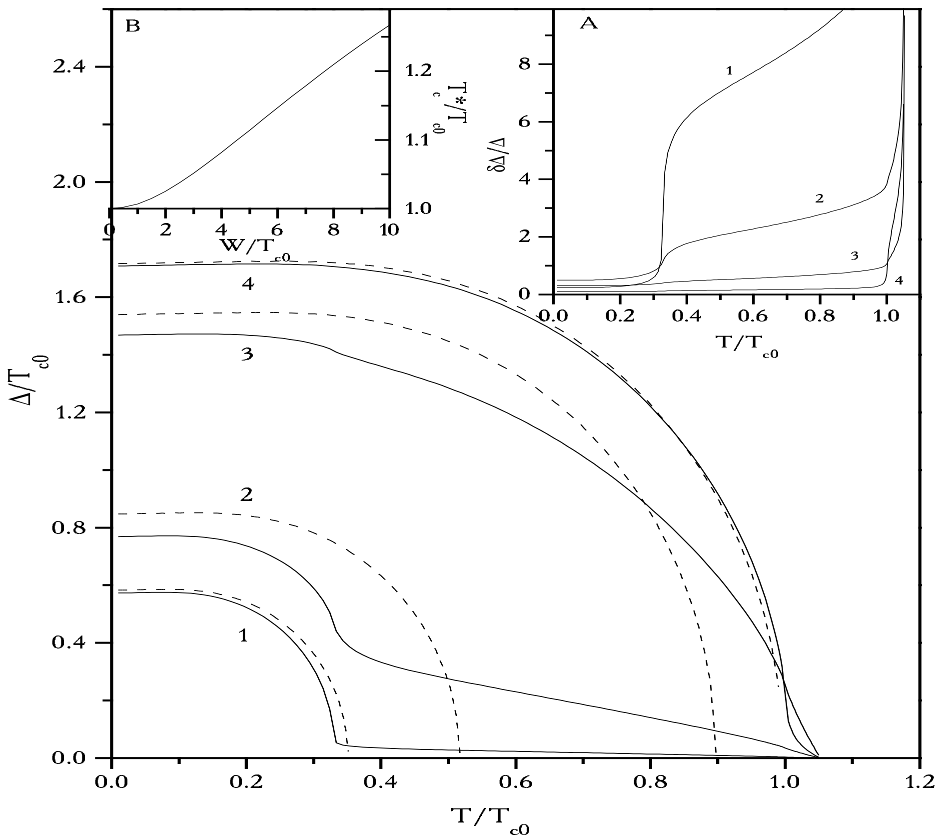

FIG. 6. Temperature dependence of superconducting gap for the "low temperature" region of "dielectric" fluctuations. Full curves – gaps Δ averaged over "phase" η for fixed values of the amplitude $D = W$, as determined by Eq. (27). Dashed curves – "mean field" gap, determined by Eq. (27). At the insert A – temperature dependence of the relative fluctuation of superconducting gap. Curves are given for $\frac{W}{T_{c0}} = 3$ and $\frac{v_F\kappa}{T_{c0}}$ =: 1. 0.1; 2. 1; 3. 10; 4. 100. At the insert B – dependence of the critical temperature T_c^* on the width of the pseudogap.

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