

On Planckian limit for inelastic relaxation in metals

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We consider the simplest model for T – linear growth of resistivity in metals. It is shown that the so called “Planckian” limit for the temperature dependent relaxation rate of electrons follows from a certain procedure for representation of experimental data on resistivity and, in this sense, is a kind of delusion.

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Linear with temperature growth of electrical resistivity in cuprates and some other correlated systems in a wide region from very low to pretty high temperatures for many years remains one of the major puzzles of the physics of high – temperature superconductors. In recent years, a number of interesting papers appeared [1, 2], where by the analysis of experiments on rather wide range of compounds, it was shown that in the T – linear region of resistivity growth, the scattering rate of electrons (inverse relaxation time) with rather high accuracy is described as $\Gamma = \frac{1}{\tau} = \alpha \frac{k_B T}{\hbar}$, where $\alpha \sim 1$ and is weakly depending on the choice of the material. In particular, for systems being close to a quantum critical point (on the phase diagram of cuprates and some other similar systems) α belongs to the interval 0.7 – 1.1. More so, the similar dependence is describes rather well the data for a number of usual metals (like Cu, Au, Al, Ag, Pb, Nb etc.) in the region of T – linear growth of resistivity (which is usually realized at temperatures $T > \Theta_D/5$, where Θ_D is Debye temperature). in this case α covers significantly wider interval from 0.7 to 2.7 [1, 2]. In connection with these (and some similar) results the notion of the universal (independent of interaction strength) “Planckian” *upper* limit of scattering rate was introduced $\frac{1}{\tau_P} = \Gamma_P = \frac{k_B T}{\hbar}$ [3]. To explain this temperature behavior of resistivity for such different systems, also starting from very low temperatures, up to now a number of relatively complicated theoretical models were proposed [4, 5, 6, 7], including some rather exotic, based on the analogies taken from black hole physics, cosmology and superstring theory (e.g. see Refs. [8, 9, 10, 11]). In usual metals the temperature dependence of resistivity (conductivity) is almost completely related to inelastic scattering of electrons by phonons. In usual metals at high enough temperatures $T > \Theta_D/5$ it dominates and leads to T – linear growth of resistivity:

$$\rho(T) - \rho_0 = AT \quad (1)$$

where ρ_0 is the residual resistivity at $T = 0$ due to the scattering by random impurities.

In terms of conductivity we may write the simple Drude expression:

$$\sigma(T) = \sigma_0 + \sigma(T) \quad (2)$$

where σ_0 is the residual conductivity at $T = 0$ and

$$\sigma(T) = \frac{ne^2}{m} \tau(T) = \frac{ne^2}{m} \Gamma^{-1}(T) \quad (3)$$

Here and below m is understood to be the *band* effective mass, while $\Gamma(T) = \frac{1}{\tau(T)}$ is the temperature dependent relaxation (scattering) rate due to inelastic scattering of electrons by phonons, which grows linearly with temperature for $T > 0.2\Theta_D$. Correspondingly we get the resistivity as:

$$\rho(T) - \rho_0 = \frac{m}{ne^2} \Gamma(T) \quad (4)$$

The concept of “Planckian” relaxation rate can be introduced via elementary estimates [9]. At $T > 0$ the processes of inelastic scattering appear due to different interactions (electron – phonon, spin fluctuations, quantum fluctuation of arbitrary origin). In particular, these processes are responsible for thermodynamic equilibrium of electronic subsystem leading to Fermi distribution. Conductivity of a metal (degenerate case) is determined by electrons distributed in a layer of the width $\sim k_B T$ around the Fermi level (chemical potential).

Let us perform an elementary estimates using the energy – time uncertainty relation:

$$\Delta E \tau > \hbar \quad (5)$$

where τ is the lifetime of a quantum state and ΔE is it energy uncertainty. Naturally, in our case $\tau = \tau(T)$, while $\Delta E \sim k_B T$, which immediately leads to an estimate

$$\Gamma(T) = \frac{1}{\tau(T)} < \alpha \frac{k_B T}{\hbar} \equiv \alpha \Gamma_P = \frac{\alpha}{\tau_P} \quad (6)$$

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where $\alpha \sim 1$. We conclude that according to such elementary estimate the ‘‘Planckian’’ relaxation rate determines precisely the *upper* limit for resistivity due to inelastic scatterings:

$$\rho(T) - \rho_0 = \frac{m}{ne^2} \Gamma(T) < \frac{m}{ne^2} \alpha \frac{k_B T}{\hbar} \equiv \alpha \rho_P(T) \quad (7)$$

However, it is obvious that this estimate is of rather speculative nature for the system of many interacting particles.

Consider the following Hamiltonian for interaction of metallic electrons with arbitrary quantum Bose – type fluctuations²⁾:

$$H_{int} = \frac{1}{N} \sum_{\mathbf{p}\mathbf{q}} g_{\mathbf{q}} a_{\mathbf{p}+\mathbf{q}}^+ a_{\mathbf{p}} \rho_{\mathbf{q}} \quad (8)$$

Here we use the standard notations for creation and annihilation operators of electrons, $\rho_{\mathbf{q}}$ is the quantum fluctuation operator of ‘‘any kind’’ (e.g. ion density in a lattice), $g_{\mathbf{q}}$ is the appropriate coupling constant (matrix element of interaction potential) [12, 13]. Let us introduce the appropriate (Matsubara) Green’s function as:

$$F(\mathbf{q}, \tau) = - \langle T_{\tau} \rho_{\mathbf{q}}(\tau) \rho_{\mathbf{q}}^+(0) \rangle \quad (9)$$

For this function we can write down the standard (Bose) spectral representation [14]:

$$F(\mathbf{q}, i\omega_m) = \int_{-\infty}^{\infty} d\omega \frac{A(\mathbf{q}, \omega)}{i\omega_m - \omega} \quad (10)$$

where $\omega_m = 2\pi mT$ and spectral density is defined as:

$$A(\mathbf{q}, \omega) = Z^{-1} \sum_{mn} e^{-\frac{E_n}{T}} |(\rho_{\mathbf{q}})_{nm}|^2 \left(1 - e^{-\frac{\omega_{mn}}{T}}\right) \delta(\omega - \omega_{mn}) \quad (11)$$

where $\omega_{mn} = E_m - E_n$, $(\rho_{\mathbf{q}})_{nm} = \langle n | \rho_{\mathbf{q}} | m \rangle = (\rho_{\mathbf{q}}^+)_{mn}$.

Dynamic structure factors of fluctuations is defined as [12, 13]:

$$S(\mathbf{q}, \omega) = Z^{-1} \sum_{mn} e^{-\frac{E_n}{T}} |(\rho_{\mathbf{q}})_{nm}|^2 \delta(\omega - \omega_{mn}) \quad (12)$$

Comparing (11) and (12) we obtain:

$$A(\mathbf{q}, \omega) = S(\mathbf{q}, \omega) [1 - e^{-\frac{\omega}{T}}] \quad (13)$$

Electronic Greens’s function in Matsubara representation is written as:

$$G(\varepsilon_n, \mathbf{p}) = \frac{1}{i\varepsilon_n - \xi_{\mathbf{p}} - \Sigma(\varepsilon_n, \mathbf{p})} \quad (14)$$

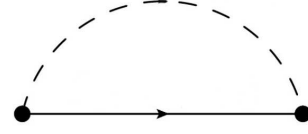


Diagram of the second order for electron self – energy. Dashed line – Green’s function of quantum fluctuations F , continuous line – electron Green’s function G .

where $\varepsilon_n = (2n + 1)\pi T$, $\xi_{\mathbf{p}}$ is electronic spectrum counted from the chemical potential. Assuming the validity of Migdal’s theorem [15] we can take the electron self – energy in the simplest approximation, shown in Fig. 1:

$$\begin{aligned} \Sigma(\varepsilon_n, \mathbf{p}) &= \frac{T}{N} \sum_{\mathbf{q}} g_{\mathbf{q}}^2 \sum_m F(\mathbf{q}, i\omega_m) G(\varepsilon_n + \omega_m, \mathbf{p} + \mathbf{q}) \\ &= \frac{T}{N} \sum_{\mathbf{q}} g_{\mathbf{q}}^2 \sum_m \int_{-\infty}^{\infty} d\omega \frac{S(\mathbf{q}, \omega)}{i\omega_m - \omega} (1 - e^{-\frac{\omega}{T}}) \\ &\quad \frac{1}{i\varepsilon_n + i\omega_m - \xi_{\mathbf{p}+\mathbf{q}}} \end{aligned} \quad (15)$$

Consider now the case when the average frequency of fluctuations $\langle \Omega \rangle$ is significantly lower than the temperature T . Then in Eq. (15) we can limit ourselves only by term with $m = 0$ and thus to the picture of *quasielastic* scattering by fluctuations:

$$\begin{aligned} \Sigma(\varepsilon_n, \mathbf{p}) &= \frac{1}{N} \sum_{\mathbf{q}} g_{\mathbf{q}}^2 \int_{-\infty}^{\infty} d\omega \frac{S(\mathbf{q}, \omega)}{i\varepsilon_n - \xi_{\mathbf{p}+\mathbf{q}}} = \\ &= \sum_{\mathbf{q}} g_{\mathbf{q}}^2 S(\mathbf{q}) \frac{1}{i\varepsilon_n - \xi_{\mathbf{p}+\mathbf{q}}} \end{aligned} \quad (16)$$

where we have introduced the structure factor of fluctuations as [13]:

$$S(\mathbf{q}) = \frac{1}{N} \int_{-\infty}^{\infty} d\omega S(\mathbf{q}, \omega) \quad (17)$$

In fact this is a direct analog the well-known Ziman – Edwards approximation in the theory of liquid metals. The case of $S(\mathbf{q}) = const$ corresponds to chaotic distribution of static scattering centers [15].

Fluctuation operator $\rho_{\mathbf{q}}$ for the case of some collective mode can be expressed via (Boson) annihilation and creation operators for corresponding quanta (e.g. phonons) [13]:

$$\rho_{\mathbf{q}} = \frac{1}{\sqrt{2}} (b_{\mathbf{q}}^+ + b_{-\mathbf{q}}) \quad (18)$$

Then we have:

²⁾In the following we assume $\hbar = k_B = 1$

$$S(\mathbf{q}, \omega) = Z^{-1} \sum_m e^{-\beta E_m} [\langle m | b_{\mathbf{q}} b_{\mathbf{q}}^+ | m \rangle \delta(\omega - \omega_{\mathbf{q}}) + \langle m | b_{-\mathbf{q}}^+ b_{-\mathbf{q}} | m \rangle \delta(\omega + \omega_{\mathbf{q}})] \quad (19)$$

where $\omega_{\mathbf{q}}$ is the spectrum of corresponding excitations. Introducing the usual Bose distribution:

$$n_{\mathbf{q}} = Z^{-1} \sum_m e^{-\beta E_m} \langle m | b_{\mathbf{q}}^+ b_{\mathbf{q}} | m \rangle = \frac{1}{e^{\beta \omega_{\mathbf{q}}} - 1} \quad (20)$$

we get [13]

$$S(\mathbf{q}, \omega) = [(n_{\mathbf{q}} + 1)\delta(\omega - \omega_{\mathbf{q}}) + n_{\mathbf{q}}\delta(\omega + \omega_{\mathbf{q}})] = \delta(\omega - \omega_{\mathbf{q}}) + n_{\mathbf{q}} [\delta(\omega - \omega_{\mathbf{q}}) + \delta(\omega + \omega_{\mathbf{q}})] \quad (21)$$

In case of $T \gg \omega_{\mathbf{q}}$, we have:

$$n_{\mathbf{q}} = \frac{T}{\omega_{\mathbf{q}}} \quad (22)$$

and correspondingly:

$$S(\mathbf{q}) = \frac{2T}{\omega_{\mathbf{q}}} \quad (23)$$

i.e. we obtain the structure factor linear in T and its momentum dependence is determined simply by excitation spectrum of the appropriate collective mode (fluctuation). Then:

$$\Sigma(\varepsilon_n, \mathbf{p}) = T \sum_{\mathbf{q}} \frac{2g^2}{\omega_{\mathbf{q}}} \frac{1}{i\varepsilon_n - \xi_{\mathbf{p}+\mathbf{q}}} \quad (24)$$

To simplify the model further let us assume the spectrum of fluctuations to be dispersionless (like Einstein phonon or optical phonon with weak dispersion) $\omega_{\mathbf{q}} = \Omega_0$. Then performing all calculations similarly to the problem of an electron in the field of random impurities [15], we get:

$$\Sigma(\varepsilon_n, \mathbf{p}) = -i\pi \text{sign} \varepsilon_n \frac{2g_0^2}{\Omega_0} N(0)T \quad (25)$$

where $N(0)$ is the density of states at the Fermi level. Correspondingly, the scattering rate (damping) is written as:

$$\frac{\Gamma(T)}{2} = \pi \frac{2g_0^2}{\Omega_0} N(0)T = \pi \lambda_0 T \quad (26)$$

where we introduced in a standard way the dimensionless coupling constant:

$$\lambda_0 = \frac{2g_0^2 N(0)}{\Omega_0} \quad (27)$$

Now the electronic Green's function takes the usual form [15]:

$$G(i\varepsilon_n, \mathbf{p}) = \frac{1}{i\varepsilon_n - \varepsilon_{\mathbf{p}} + \frac{i}{2}\Gamma(T)\text{sign}\varepsilon_n} \quad (28)$$

where are no renormalization factors of any kind (the residue at the pole of Green's function $Z = 1$), which is natural for temperatures much exceeding the frequencies of quantum fluctuations.

After the standard calculations [15] we obtain the resistivity as:

$$\rho(T) = \frac{m}{ne^2} \Gamma(T) = 2\pi \lambda_0 \rho_P(T) \quad (29)$$

which in essence just coincide with high – temperature limit of Eliashberg – McMillan theory [16]. The constant α used in writing down the ‘‘Planckian’’ relaxation as (6) is expressed via the the parameters of the theory as;

$$\alpha = 2\pi \lambda_0 \quad (30)$$

Naturally, it is not universal and is just proportional to the coupling constant.

All this is known actually for a long time and trivially explains the T – linear growth of resistivity in accordance with many experiments. To make such resistivity growth starting from low temperatures it is sufficient to demand that $\Omega_0 \ll T$. In the vicinity of the quantum critical point (of any nature) we can expect the typical ‘‘softening’’ of fluctuation modes like [17]:

$$\Omega_0 \sim |x - x_c|^{z\nu} \quad (31)$$

where x , for example, may denote the carrier concentration close to the critical x_c , while ν and z are the standard critical exponents of the theory of quantum phase transitions, determining the critical behavior of characteristic lengths:

$$\xi \sim |x - x_c|^{-\nu}, \quad \xi_{\tau} \sim |x - x_c|^{-z\nu} \quad (32)$$

where τ is the imaginary (Matsubara) time, so that above we may just assume $\Omega_0 \sim \xi_{\tau}^{-1}$. This may be responsible for T – linear behavior in systems like cuprates.

We can avoid the explicit introduction of phonons (or any other quasiparticles related to fluctuations). From Eq. (13) for $\omega \ll T$ we have:

$$A(\mathbf{q}, \omega) \approx \frac{\omega}{T} S(\mathbf{q}, \omega) \quad (33)$$

or

$$S(\mathbf{q}, \omega) \approx \frac{T}{\omega} A(\mathbf{q}, \omega) \quad (34)$$

Substituting this expression into Eq. (16) we get the following expression for self – energy:

$$\Sigma(\varepsilon_n, \mathbf{p}) = \frac{T}{N} \sum_{\mathbf{p}'} g_{\mathbf{p}\mathbf{p}'}^2 \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{A(\mathbf{p} - \mathbf{p}', \omega)}{i\varepsilon_n - \xi_{\mathbf{p}'}} \quad (35)$$

where everything is determined by the spectral density of fluctuations $A(\mathbf{q}, \omega)$, which is not necessarily of quasi-particle form. Obviously, for the simplest model with $A(\mathbf{q}, \omega) = \delta(\omega - \Omega_0)$ (Einstein model of fluctuations) from Eq. (35) we immediately obtain the previous results of Eqs. (25) – (27).

Further, let us assume that fluctuations scatter electrons in some pretty narrow layer around the Fermi surface with width determined by their characteristic frequencies ($\langle \Omega \rangle \ll T$). Then, in the spirit of Eliashberg – McMillan theory we can introduce the self – energy averaged over momenta at the Fermi surface:

$$\Sigma(\varepsilon_n) = \frac{1}{N(0)} \sum_{\mathbf{p}} \delta(\xi_{\mathbf{p}}) \Sigma(\varepsilon_n, \mathbf{p}), \quad (36)$$

and also an effective interaction averaged over the initial and final momenta at the Fermi surface:

$$\begin{aligned} g_{\mathbf{p}\mathbf{p}'}^2 A(\mathbf{p} - \mathbf{p}', \omega) &\implies \\ \frac{1}{N(0)} \sum_{\mathbf{p}} \frac{1}{N(0)} \sum_{\mathbf{p}'} g_{\mathbf{p}\mathbf{p}'}^2 A(\mathbf{p} - \mathbf{p}', \omega) \delta(\xi_{\mathbf{p}}) \delta(\xi_{\mathbf{p}'}) & \\ \equiv \frac{1}{N(0)} \alpha^2(\omega) F(\omega) & \quad (37) \end{aligned}$$

where

$$F(\omega) = \sum_{\mathbf{q}} A(\mathbf{q}, \omega) \quad (38)$$

is the density of states of fluctuations. Then for (36) from (35) we get:

$$\begin{aligned} \Sigma(\varepsilon_n) &= \frac{T}{N(0)} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha^2(\omega) F(\omega) N(0) \int_{-\infty}^{\infty} d\xi \frac{1}{i\varepsilon_n - \xi} \\ &= -i\pi \text{sign} \varepsilon_n T \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha^2(\omega) F(\omega) \\ &= -i\pi \text{sign} \varepsilon_n \lambda T \equiv -i \frac{\Gamma(T)}{2} \text{sign} \varepsilon_n \end{aligned} \quad (39)$$

where we have introduced the dimensionless coupling constant similar to that in Eliashberg – McMillan theory:

$$\lambda = 2 \int_0^{\infty} \frac{d\omega}{\omega} \alpha^2(\omega) F(\omega) \quad (40)$$

which is in fact determined by (averaged according to (37)) the spectral density of fluctuations $A(\mathbf{q}, \omega)$.

Then we obtain:

$$\Gamma(T) = 2\pi\lambda T \quad (41)$$

which is formally of the same form as (26) and immediately leads to (29).

In Refs. [1, 2] experimental data on resistivity were fitted to standard Drude expression (4), where the effective mass was determined from low temperature measurements (electronic specific heat and oscillation effects in high magnetic fields) which is actually related to band structure effective mass by a simple replacement $m \rightarrow \tilde{m} = m(1 + \lambda)$, which explicitly takes into account renormalization due to phonons. The deficiency of such approach was already noted in Ref. [18]. Let us show that precisely this kind of representation of data creates a *delusion* of universal “Planckian” relaxation. In fact, the expression (29) for *high – temperature* limit of resistivity can be rewritten as:

$$\rho(T) = \frac{m(1 + \lambda)}{ne^2} \frac{\Gamma(T)}{1 + \lambda} = \frac{\tilde{m}}{ne^2} \tilde{\Gamma}(T) \quad (42)$$

where

$$\tilde{\Gamma}(T) = 2\pi \frac{\lambda}{1 + \lambda} T \quad (43)$$

which for $\lambda \gg 1$ reduces to:

$$\tilde{\Gamma}(T) = 2\pi T \quad (44)$$

and simply imitates the universal “Planckian” behavior of relaxation rate (6) with $\alpha = 2\pi$, which is independent of coupling constant of electrons with fluctuations (phonons). The replacement $m \rightarrow \tilde{m} = m(1 + \lambda)$ in Eq. (42) is correct despite the fact, that here we are dealing with high – temperature limit as fitting the experimental data in Refs. [1, 2] was performed using the effective mass \tilde{m} , obtained from *low temperature* measurements, which contains renormalization effects. For quantitative estimates it is also quite important to take into account mass renormalization due to interelectron interactions. Correspondingly, Eq. (43) should be rewritten as:

$$\tilde{\Gamma}(T) = 2\pi \frac{\lambda}{1 + \lambda + \lambda_{ee}} T \quad (45)$$

where λ_{ee} is dimensionless parameter, determining mass renormalization due to electron – electron interactions. In Landau – Silin theory of Fermi liquids $\lambda_{ee} = \frac{F_1^s}{3}$, where F_1^s is the appropriate coefficient in Landau function expansion [13]. Direct DMFT calculations for the Hubbard model produce the values of renormalization factor $Z = (1 + \lambda_{ee})^{-1}$ in metallic phase monotonously changing with Hubbard interaction U in the interval between 1 and 0 [19]. Thus, for rough estimates for typical metal we can safely take $\lambda_{ee} \sim 1$. Then:

$$\alpha = \frac{2\pi\lambda}{1 + \lambda + \lambda_{ee}} \quad (46)$$

Then the interval of $\alpha = 0.7 - 2.7$ [1, 2] for $\lambda_{ee} = 1$ corresponds to $\lambda = 0.25 - 1.5$, which seems quite reasonable. For example for Al we have the calculated value

$\lambda = 0.44$ [16], which immediately gives $\alpha = 1.03$ in nice correspondence with “experimental” value $\alpha = 1$ from Ref. [1]. For Pb, taking $\lambda = 1.68$ [16] we get $\alpha \sim 2.86$ in reasonable agreement with $\alpha = 2.7$ [1]. Similarly, for Nb we have $\lambda = 1.26$ [16] and $\alpha \sim 2.42$, also in good agreement with $\alpha = 2.3$ of Ref. [1]. In general, taking into account the roughness of our estimate of $\lambda_{ee} \sim 1$ this agreement seems rather convincing³⁾.

Thus, the “experimentally” observed universal “Planckian” relaxation rate in metals, independent of interaction strength, is nothing more than *delusion*, connected with the procedure used in Refs. [1, 2] to represent the experimental data. At low temperatures ($T < \langle \Omega \rangle$) Green’s function takes the form:

$$G(\varepsilon_n, \mathbf{p}) = \frac{Z}{i\varepsilon_n - Z\xi_{\mathbf{p}} + \frac{i}{2}Z\Gamma(T)\text{sign}\varepsilon_n} \quad (47)$$

where the renormalization factor $Z < 1$ is assumed for simplicity a constant. The term $Z\Gamma(T) = \tilde{\Gamma}(T)$ in the denominator describes quasiparticle damping for which it may seem we have the “universal” high – temperature limit of Eq. (44). However, it is wrong – at high temperatures ($T > \langle \Omega \rangle$) the renormalization factor $Z \rightarrow 1$, as can be seen from our results above. Also for the low temperatures, when $Z < 1$, the term $Z\xi_{\mathbf{p}}$ in the denominator of (47) describes the renormalized spectrum of electrons with mass $\tilde{m} = m(1 + \lambda)$, so that electron velocity at the Fermi surface $v_F = p_F/m \rightarrow \tilde{v}_F = p_F/\tilde{m} = v_F/(1 + \lambda)$. Renormalization of damping corresponds to renormalization of mean free time $\tilde{\Gamma}^{-1} = \Gamma^{-1}(1 + \lambda)$. Now we see, that the mean free path is not renormalized: $l = \tilde{v}_F\tilde{\Gamma}^{-1} = v_F\Gamma^{-1}$ and renormalization due to many particle effects, important at low temperatures, actually do not affect conductivity (resistivity) at all [20]. In fact, this follows from the general Ward identity [21].

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³⁾We neglect rather insignificant [16] difference between λ and λ_{tr} .