Combinatorics of Feynman Diagrams for the Problems with Gaussian Random Field

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Abstract

The algorithm to calculate the generating function for the number of "skeleton" diagrams for the irreducible self-energy and vertex parts is derived for the problems with Gaussian random fields. We find an exact recurrence relation determining the number of diagrams for any given order of perturbation theory, as well as its asymptotics for the large order limit. These results are applied to the analysis of the problem of an electron in the Gaussian random field with the "white-noise" correlation function. Assuming the equality of all "skeleton" diagrams for the self-energy part in the given order of perturbation theory, we construct the closed integral equation for the one-particle Green's function, with its kernel defined by the previously introduced generating function. Our analysis demonstrate that this approximation gives the qualitatively correct form of the localized states "tail" in the density of states in the region of negative energies and is apparently quite satisfactory in the most interesting region of strong scattering close to the former band-edge,

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where we can derive the asymptotics of the Green's function and density of

states in the limit of very strong scattering.

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I. INTRODUCTION

Different methods of summation of Feynman diagrams are widely used in rather wide class of problems of theoretical physics dealing with elementary excitations propagation in systems with static random fields, created by different types of inhomogeneites. A simplest example is the problem of an electron in a metal with impurities. Apparently this was the first problem in which the diagram technique considered in this paper was formulated for the first time. [1,2]. Analogous formalism is being widely used in problems of statistical radiophysics and optics, dealing with the propagation of electromagnetic waves in disordered media [3]. Equivalent mathematical approach is used in a number of problems of critical phenomena in disordered systems [4], in the problem of polymer chain with "excluded volume" and in some other problems of polymer physics [5]. The same diagram technique describes the regular model of critical phenomena with "zero-component" order parameter [4].

In any problem, dealing with summation of Feynman diagrams, any kind of information on combinatorics of graphs, i.e. on the number of diagrams of different types for the given order of perturbation theory, is quite useful. In this paper we present a detailed study of Feynman graphs combinatorics for the problems with the Gaussian random field.

II. GENERATING FUNCTION FOR THE NUMBER OF "SKELETON" DIAGRAMS. RECURRENCE RELATION.

For definiteness we shall always discuss the problem of an electron with energy E and momentum $\mathbf{p}$, propagating in the Gaussian random field (random impurity system) [1,2]. The averaged Green's function is determined by the perturbation series shown in Fig.1(a). In a standard way this series is reduced to Dyson's equation:

$$G(E, p) = \frac{1}{E - \varepsilon_p - \Sigma(E, p)} \tag{1}$$

where $\varepsilon_p = \frac{p^2}{2m}$ – is the free electron spectrum, while the self-energy part $\Sigma(E, p)$ is defined by the "skeleton" graphs of Fig.1(b), where the internal electronic line represents the fully renormalized ("dressed") Green's function $G(E, p)$.

The total number of diagrams in the N-th order of perturbation expansion shown in Fig.1(a) is easily demonstrated to be equal to:

$$G_N = (2N - 1)!! = \frac{(2N - 1)!}{2^{N-1}(N - 1)!} \tag{2}$$

which is just the total number of ways in which we can connect $2N$ vertices by N impurity lines. The analogous number of graphs for Σ_N in the expansion of Fig.1(b) is much more difficult to find and the exact answer for this problem is, as far as we know, absent in the literature. Only in Ref. [6] a simple inequality was shown to hold:

$$(2N - 1)!! > \Sigma_N > (2N - 3)!! \tag{3}$$

This inequality gives only a very rough estimate of Σ_N. As we shall show below this problem can be solved exactly. This follows immediately from an exact solution for the electron in the random potential $V(\mathbf{r}) = V$, where the value of V is independent of the spatial coordinate $\mathbf{r}$, though Gaussian with distribution width $< V^2 >= W^2$. Naturally, in this case the diagram technique is of the standard form like in Fig.1, and each impurity line transfers just the zero value of momentum, i.e. it is associated (in momentum space) with correlator of the form $(2\pi)^d W^2 \delta(\mathbf{q})$ (d – is spatial dimensionality) [7,8]. All contributions of the same order in perturbation expansion of Fig.1(a) are equal and the series for the Green's function can be expressed as [7]:

$$G(E, p) = G_0(E, p) \left\{ 1 + \sum_{N=1}^{\infty} (2N - 1)!! G_0^{2N}(E, p) W^{2N} \right\} \tag{4}$$

Then using the representation:

$$(2N - 1)!! = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt t^{2N-2} e^{-\frac{t^2}{2}} \tag{5}$$

the series (4) is summed directly and we obtain: [1]

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[1]Mathematically it is equivalent to Borel summation.

$$G(E,p) = \frac{1}{W}\Psi\left(\frac{1}{WG_0(E,p)}\right) \tag{6}$$

where we introduced the function:

$$\Psi(z) = -\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} dt e^{-\frac{t^2}{2}}\frac{1}{t-z} \tag{7}$$

Consider the self-energy part corresponding to the Green's function (6). Addition of a new impurity line to any diagram in this problem leads just to the appearance of an additional factor W^2G^2, and the self-energy part, defined by the expansion shown in Fig.1(b) can be written as:

$$\Sigma = Q(W^2G^2)W^2G \tag{8}$$

where $Q(x)$ is some function. We shall see, that this function is the generating function for the number of "skeleton" graphs for the self-energy part, i.e. the coefficients of its Taylor expansion determine the required numbers Σ_N.

The Dyson's equation for the problem under consideration is written as:

$$G = G_0 + G_0\Sigma G = G_0\left(1 + Q(W^2G^2)W^2G^2\right) \tag{9}$$

Introducing $z = (WG_0)^{-1}$ and $y = W^2G^2$, from Eqs. (6) and (9) we obtain the following parametric representation for $Q(y)$:

$$1 + yQ(y) = z\Psi(z) = z\sqrt{y}$$
$$y = \Psi^2(z) \tag{10}$$

This representation of Q is rather inconvenient. Below we show that this function obeys certain differential equation. It is easily seen that previously introduced function $\Psi(z)$, satisfies the usual dispersion relation:[2]

$$Re\Psi(z) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} dt\frac{Im\Psi(t)}{t-z}; \qquad \frac{1}{\pi}Im\Psi(t) = \mp\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} \tag{11}$$

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[2]The sign of imaginary part is defined as we consider either retarded or advanced Green's function.

from which it follows immediately that $\Psi(z)$ satisfies the differential equation:

$$\frac{d\Psi}{dz} = 1 - z\Psi \tag{12}$$

with the initial condition:

$$\Psi(z = \pm i0) = \mp i\sqrt{\frac{\pi}{2}} \tag{13}$$

Differentiating the first equation in (10) over y, we obtain:

$$\frac{dz}{dy} = \frac{1}{2}y^{-\frac{3}{2}}\left\{2y^2\frac{dQ(y)}{dy} + yQ(y) - 1\right\} \tag{14}$$

Differentiating the second-equation in (10) over z and using (12) we obtain:

$$\frac{dy}{dz} = 2\Psi(z)\frac{d\Psi(z)}{dz} = 2\Psi(z)(1 - z\Psi(z)) = -2y^{\frac{3}{2}}Q(y) \tag{15}$$

Comparing (14) and (15) we obtain the following non-linear differential equation for $Q(y)$:

$$\frac{dQ(y)}{dy} = \frac{1}{2y^2}\left\{1 - Q^{-1}(y) + yQ(y)\right\} \tag{16}$$

Using (10) and (13) we get $y = \Psi^2(z)|_{z=\pm i0} = -\frac{\pi}{2}$, so that

$$Q(-\frac{\pi}{2}) = \left.\frac{z\Psi(z) - 1}{y}\right|_{z=\pm i0} = \frac{2}{\pi} \tag{17}$$

which gives the initial condition for Eq.(16). Note that the condition $Q(0) = 1$, which obviously follows from diagram expansion for Σ, is a special one for Eq.(16) and can not be used as an initial condition.

Eq.(16) can be expressed in more convenient form:

$$Q(y) = 1 + y\frac{d}{dy}yQ^2(y) \tag{18}$$

We are interested in Taylor expansion for $Q(y)$:

$$Q(y) = \sum_{n=0}^{\infty} a_n y^n \tag{19}$$

As the number of "skeleton" diagrams of the N-th order for the self-energy part is just the coefficient of W^{2N} in the expansion of Σ over the powers of W^2, it is easily seen that Eq.(8) gives the required value of Σ_N as:

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$$\Sigma_N = a_{N-1} \tag{20}$$

This means that the function $Q(y)$ is the generating function for combinatorial factors Σ_N.

Substitution of Eq.(19) into Eq.(18) leads to the following recurrence relation for the coefficients a_n:

$$a_n = n \sum_{m=0}^{n-1} a_m a_{n-1-m} \tag{21}$$

where $a_0 = 1$. From $a_0 = 1$ it follows that $Q(0) = 1$. It is due to this fact this point is special – equation $Q(0) = 1$ is satisfied for any initial conditions, for which Eq.(18) has a solution.

From Eq.(21) it is easy to find the values of a_n for small n, the appropriate results are presented in Table 1.

The knowledge of combinatorics for the self-energy part allows to find also the combinatorics for the two-particle Green's function – both for the full vertex-part Γ and for the irreducible vertex U. The appropriate diagram representations of these vertices is shown in Fig.2. The self-energy part Σ is connected with the vertex-part Γ by the equation shown graphically in Fig.3. For the problem with zero transferred momentum [7,8] this equation has the following form:

$$\Sigma = W^2 G(1 + G^2 \Gamma) \tag{22}$$

Thus, for the number of diagrams in the N-th order of the full vertex Γ_N, we obtain immediately:

$$\Gamma_N = \Sigma_{N+1} = a_N \tag{23}$$

In this sense $Q(y)$ is also the generating function for the number of diagrams for the full vertex-part.

The number of diagrams of the N-th order for the irreducible vertex-part U_N can be easily obtained if we note, that the cut of any of $2N - 1$ internal Green's function lines in

the diagram for the self-energy part of N-th order produces the appropriate diagram for the N-th order contribution to U (Fig.4). Thus:

$$U_N = (2N - 1)\Sigma_N = (2N - 1)a_{N-1} \tag{24}$$

In Appendix A we once again derive the differential equation (18) for the generating function $Q(y)$, using only the Bethe-Salpeter equation, connecting U and Γ, as well as the Ward-type identity, with no use of the explicit solution for the Green's function (6).

III. ASYMTOTICS FOR THE NUMBER OF DIAGRAMS IN THE LIMIT OF LARGE N.

In the limit of large orders of perturbation theory $N \gg 1$ the use of the recurrence relation (21) becomes inconvenient due to the factorial growth of the number of diagrams [6]. At the same time the fact of this factorial growth itself can be used for significant simplification of the problem. Let us rewrite (21) as:

$$a_n = 2na_0 a_{n-1} + 2na_1 a_{n-2} + 2na_2 a_{n-3} + \cdots \tag{25}$$

where $a_0 = 1$, $a_1 = 1$, $a_2 = 4$. It is natural to assume that in the limit of large n we have $a_n \approx (2n + \beta)a_{n-1}$, then $a_{n-2} \approx \frac{a_{n-1}}{2n-2+\beta}$ etc. Substituting these expressions into (25) immediately leads to $\beta = 1$ and

$$a_n = (2n + 1 + O(\frac{1}{n}))a_{n-1} \tag{26}$$

This means that in the limit of large n we have $a_n \sim (2n + 1)!!$. Let us define b_n as:

$$b_n = \frac{a_n}{(2n + 1)!!} \tag{27}$$

Substituting (27) into (21), we obtain the recurrence relation for b_n:

$$b_n = n \sum_{m=0}^{n-1} \frac{(2m + 1)!!(2n - 2m - 1)!!}{(2n + 1)!!} b_m b_{n-1-m} \tag{28}$$

with $b_0 = 1$. In the limit of large n, taking into account $b_1 = \frac{1}{3}$, $b_2 = \frac{4}{15}$, and limiting ourselves to the accuracy of the order of b/n^2 (where $b \sim b_n \sim b_{n-1} \sim b_{n-2} \sim b_{n-3}$), we get:

$$\Delta b_n = b_n - b_{n-1} = \frac{5}{4}\frac{b_{n-1}}{n^2} + O(\frac{b}{n^3}) \tag{29}$$

Thus, in the limit of large n we can write down the following differential equation for b_n:

$$\frac{db_n}{dn} = \frac{5}{4}\frac{b_n}{n^2} + O(\frac{b}{n^3}) \tag{30}$$

from which it follows immediately:

$$b_n = b \cdot exp\left(-\frac{5}{4}\frac{1}{n} + O(\frac{1}{n^2})\right) = b\left\{1 - \frac{5}{4}\frac{1}{n} + O(\frac{1}{n^2})\right\} \tag{31}$$

It is natural, that this analysis can not provide us with the value of the constant $b = \lim_{n\to\infty} b_n$. Numerical study of b_n using the recurrence relation (28) completely supports the dependence defined by Eq.(31) (Cf.Fig.5) and leads to the value of $b = \frac{1}{e} = 0.36787944\cdots$ (calculations were done up to $n = 5000$, which guarantees the claimed accuracy). We do not know any analytical way to obtain this rather curious result.

Finally, the asymptotic expressions for the number of diagrams of different types for large N have the following form: [3]

$$\Sigma_N = a_{N-1} = b_{N-1}(2N-1)!! = \frac{1}{e}\left\{1 - \frac{5}{4}\frac{1}{N} + O(\frac{1}{N^2})\right\}(2N-1)!! =$$
$$= \frac{1}{\sqrt{\pi e}}\left\{1 - \frac{5}{4}\frac{1}{N} + O(\frac{1}{N^2})\right\}2^N\Gamma(N+\frac{1}{2}) \tag{32}$$

$$\Gamma_N = a_N = \frac{1}{e}\left\{1 - \frac{5}{4}\frac{1}{N} + O(\frac{1}{N^2})\right\}(2N+1)!! =$$
$$= \frac{1}{\sqrt{\pi e}}\left\{1 - \frac{5}{4}\frac{1}{N} + O(\frac{1}{N^2})\right\}2^{N+1}\Gamma(N+\frac{3}{2}) \tag{33}$$

$$U_N = (2N-1)a_{N-1} = \frac{1}{e}\left\{1 - \frac{5}{4}\frac{1}{N} + O(\frac{1}{N^2})\right\}(2N-1)(2N-1)!! =$$
$$= \frac{1}{e}\left\{1 - \frac{9}{4}\frac{1}{N} + O(\frac{1}{N^2})\right\}(2N+1)!! = \frac{1}{\sqrt{\pi e}}\left\{1 - \frac{9}{4}\frac{1}{N} + O(\frac{1}{N^2})\right\}2^{N+1}\Gamma(N+\frac{3}{2}) \tag{34}$$

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[3]Asymptotic dependence of the type of Eq.(32) $\Sigma_N \approx c2^N\Gamma(N+\beta)$ was obtained in Ref. [6]. However, the values of the coefficients c and β were not determined.

It is interesting to note that:

$$\frac{\Sigma_N}{G_N} = b_{N-1} = \frac{1}{e}\left\{1 - \frac{5}{4}\frac{1}{N} + O(\frac{1}{N^2})\right\} \to \frac{1}{e} \tag{35}$$

$$\frac{U_N}{\Gamma_N} = 1 - \frac{1}{N} + O(\frac{1}{N^2}) \to 1 \tag{36}$$

In Table 1 we present the summary of the main results for the number of diagrams of different types.

IV. ELECTRON IN THE GAUSSIAN RANDOM FIELD WITH THE "WHITE-NOISE" CORRELATOR.

As an example of the practical use of the results obtained above, let us consider the problem of an electron in the Gaussian random field with "white-noise" correlator, when the expression associated with impurity interaction line is [1,2,9]:

$$w(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = W^2\delta(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) \tag{37}$$

where $W^2 = \rho V^2$, ρ – density of impurities, V – Born scattering amplitude of the point-like impurity. It is well known that the basic difficulties in this problem appear in the energy region, defined by the condition [9]:

$$|E| \overset{<}{\sim} \gamma(E) \quad |E| \overset{<}{\sim} E_{sc} \tag{38}$$

where $\gamma(E) = \pi\rho V^2 N(E)$ – is Born scattering rate ($N(E)$ – density of states at the energy E), $E_{sc} \sim m^{\frac{d}{4-d}}(\rho V^2)^{\frac{2}{4-d}}$ – characteristic size of the "critical" region around the band-edge, where the strong scattering appears. These difficulties are mainly due to the impossibility to sum any kind of dominating series of Feynman diagrams, analogous to that being summed in the limit of weak scattering $E \gg \gamma(E)$, $E \gg E_{sc}$ [1,2] [4]. In fact all the diagrams for the

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[4]In this case diagrams with noncrossing impurity lines dominate and we have only to take into account the first diagram in Fig.1(b)

self-energy part become of the same order of magnitude in the energy region $|E| \overset{<}{\sim} E_{sc}$ and should be taken into account.

Perturbation expansion for the self-energy part in terms of "skeleton" graphs is shown in Fig.1(b). All the graphs of 3-rd order in this expansion are in fact equal to each other (Appendix B). Despite the fact that this property is lost already in the next order, it seems reasonable to formulate an approximation *assuming the equality* of all diagrams of this type in every order of perturbation theory. Apparently this approximation can be good enough especially in the "critical region" $|E| \overset{<}{\sim} E_{sc}$ where all contributions are of the same order of magnitude. Let us take as a "basic" graph in any given order the "maximally crossed" diagram like shown in Fig.6(a). For systems invariant to time reversal this graph can be transformed into a "ladder"–like, as shown in Fig.6(b). Then the full series for the self-energy part in our approximation can be expressed like:

$$\Sigma(p) = \sum_{n=1}^{\infty} W^2 \Sigma_n \sum_{\mathbf{p}_1} \sum_{\mathbf{p}_2} \left[W^2 G(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}) G(-\mathbf{p}_2) \right]^{n-1} G(\mathbf{p}_1) =$$
$$= \sum_{\mathbf{p}_1} W^2 Q \left[W^2 \sum_{\mathbf{p}_2} G(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}) G(\mathbf{p}_2) \right] G(\mathbf{p}_1) \qquad (39)$$

where we used notations defined in (19) and (20), as well as the property $G(\mathbf{p}) = G(-\mathbf{p})$, valid for the isotropic system. Accordingly we obtain *closed* equation for the averaged one-particle Green's function:

$$G^{-1}(p) = G_0^{-1}(p) - W^2 \sum_{\mathbf{q}} Q \left[W^2 \sum_{\mathbf{p}_1} G(\mathbf{p}_1 - \mathbf{q}) G(\mathbf{p}_1) \right] G(\mathbf{p} + \mathbf{q}) \qquad (40)$$

where $G_0^{-1}(p) = E - \frac{p^2}{2m}$. All the non-trivial part of the problem is contained now in our generating function $Q(y)$, which defines the "kernel" of the complicated nonlinear integral equation (40). Naturally, if we limit ourselves to the first term in the expansion (19), we get $Q = 1$ and Eq.(40) reduces to the standard sum of "noncrossing" diagrams [1,2]. The obvious advantage of Eq.(40) in comparison with standard approach [1,2], based upon the summation of dominating diagrams (e.g. accounting only the first graph of Fig.1(b)) is the formal account of *all* diagrams, which is made, however, in the approximation, assuming the

equality of all the "skeleton" graphs for the self-energy part in a given order of perturbation theory.

Equation (40) is quite complicated nonlinear integral equation and can not be solved in general case, more so due to the fact that we do not know the general form of $Q(y)$ (which enters Eq.(40) as a function of a complex argument). Below we shall limit ourselves with some qualitative analysis of Eq.(40). Let us write Eq.(40) in the following compact form:

$$G^{-1}(p) = G_0^{-1}(p) - W^2 Q \left[W^2 G \otimes G \right] \otimes G \tag{41}$$

where we have introduced the generalized product (convolution) of functions as:

$$F \otimes \Phi = \sum_{\mathbf{p}} F(\mathbf{p} - \mathbf{q}) \Phi(\mathbf{p}) \tag{42}$$

Let us return to Eqs.(10), defining Q parametrically. The second equation in (10) can be written now as:

$$G \otimes G = \frac{1}{W^2} \Psi^2(z) \tag{43}$$

We have seen that in the problem with zero transferred momentum $z = W^{-1} G_0^{-1}$. Consider now the limit of $W \to 0$ in (43). Then the left-hand side of (43) reduces to $G_0 \otimes G_0$, while in the right-hand side we can assume $z \sim W^{-1}$ in analogy with the problem with zero transferred momentum and use the easily established asymptotics $\Psi(z) \approx \frac{1}{z}$ for $|z| \gg 1$. Here we are slightly inaccurate, since the exact form of $\Psi(z)$ is:

$$\Psi(z) = R(z) \mp i\sqrt{\frac{\pi}{2}} e^{-\frac{z^2}{2}} \tag{44}$$

where for $R(z)$ we have asymptotic expansion:

$$R(z) = e^{-\frac{z^2}{2}} \int_0^z e^{\frac{t^2}{2}} dt = \frac{1}{z} + \frac{1}{z^3} + \frac{3}{z^5} + \cdots \qquad (-\frac{\pi}{4} < arg z < \frac{\pi}{4}) \tag{45}$$

We are using the asymptotics $\Psi(z) \approx \frac{1}{z}$, which is not rigorous, however, the results obtained below using this approximation are confirmed by more accurate, but rather long analysis. Thus, in the limit of $W \to 0$ Eq.(43) reduces to:

$$G_0 \otimes G_0 = \frac{1}{W^2 z^2} \quad z = \frac{1 + O(W^2)}{W\sqrt{G_0 \otimes G_0}} \tag{46}$$

Accordingly, in the limit of $W \to 0$ instead of (43) we can write:

$$G \otimes G = \frac{1}{W^2} \Psi^2 \left(\frac{1}{W\sqrt{G_0 \otimes G_0}} \right) \tag{47}$$

Consider the energy region $E < 0$, where the fluctuation "tail" in the density of states appears [9,10]. In this case from (46) we obviously have $z \in Re$. With the help of (44) and (46) we obtain from (47):

$$G \otimes G \approx G_0 \otimes G_0 - i\frac{2}{W}\sqrt{\frac{\pi}{2}}\sqrt{G_0 \otimes G_0}exp\left\{-\frac{1}{2W^2 G_0 \otimes G_0}\right\} \tag{48}$$

where the second term, as we shall see now, produces the fluctuation "tail" in the density of states. using $\sum_{\mathbf{q}} G \otimes G = \sum_{\mathbf{p}} \sum_{\mathbf{q}} G(\mathbf{p} - \mathbf{q})G(\mathbf{p}) = \left(\sum_{\mathbf{p}} G(p)\right)^2$, we immediately obtain from (48) the density of states in the form:

$$N(E) = -\frac{1}{\pi}\sum_{\mathbf{p}} ImG^R(E,p) = \frac{1}{\sqrt{2\pi}W}\frac{\sum_{\mathbf{q}}\sqrt{G_0 \otimes G_0}exp\left\{-\frac{1}{2W^2 G_0 \otimes G_0}\right\}}{\left|\sum_{\mathbf{p}} G_0(E,p)\right|} \tag{49}$$

Now everything is defined by the concrete form of $G_0 \otimes G_0$ for different spatial dimensions. The denominator of (49) is easily calculated in general case as:

$$\left|\sum_{\mathbf{p}} G_0(E,p)\right| = S_d \int_0^{p_0} dp p^{d-1} \frac{1}{|E| + \frac{p^2}{2m}} \tag{50}$$

where $S_d = 2^{-(d-1)}\pi^{d/2}\frac{1}{\Gamma(d/2)}$ while the value of p_0 – represents the cutoff of the order of inverse interatomic distance [9], which is necessary for $d \geq 2$ (for $d = 1$ in (50) we can extend integration up to infinity). Now E in Eq.(50) denotes the renormalized energy, calculated with respect to the band-edge, determined in "one-loop" approximation [9], which takes into account the infinite (in the limit of $p_0 \to \infty$ (for $d \geq 2$)) shift of this edge. The integral in (50) is easily calculated for any value of d.

In one-dimensional $(d = 1)$ case all the integrals, contributing to (49), are calculated exactly. After rather tedious, but elementary, calculations we obtain (Appendix C):

$$N(E) = \frac{1}{2\pi}\sqrt{\frac{2m}{|E|}}exp\left\{-\sqrt{2}\frac{|E|^{\frac{3}{2}}}{m^{\frac{1}{2}}W^2}\right\} \tag{51}$$

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The exponential factor in (51) differs from the well known exact result of Halperin [11] (Cf. also Ch.II in [10]) by the absence of the factor of $\frac{4}{3}$. Preexponential factor in (51) is also different from the exact value which is of the order of $\sim \frac{|E|}{W^2}$ [11]. However, the qualitative form of the "tail" in the density of states is reproduced more or less satisfactorily, despite rather common opinion, that this "tail" can not be obtained using the usual perturbation theory.

Similar, but approximate, calculation of the density of states via Eq.(49) fro $d = 3$ (Appendix D) gives:

$$N(E) = \frac{\pi^2}{12}(2m)^{\frac{3}{4}}\frac{|E|^{\frac{5}{4}}}{E_0^{\frac{1}{2}}}\frac{1}{W}exp\left\{-\sqrt{2}\pi\frac{|E|^{\frac{1}{2}}}{m^{\frac{3}{2}}W^2}\right\} \tag{52}$$

where we have introduced the cutoff energy $E_0 = \frac{p_0^2}{2m}$ [9]. Here again the exponential factor coincides (up to a constant) with known results of non-perturbative instanton approach [9,12–14], while the preexponential in (52) does not agree with any of the variants obtained in these papers. However, in general, the result of Eq.(52) is again satisfactory enough, taking into account the approximate nature of our Eq.(40). [5]

Especially interesting is the analysis of Eq.(40) in the "strong coupling" region [9], which is defined by (38), i.e. in the vicinity of the initial band-edge, where the transition from extended to localized states takes place. In this region it is reasonable to assume, that the approximation of equal "skeleton" graphs contributions to the self-energy in the given order of perturbation theory can be rather good, simply due to the known fact that they are of the same order of magnitude. Strong condition of the type of Eq.(38) is obviously equivalent to the limit of $W \to \infty$. In this limit, in "zero-order" approximation we can neglect the first term in the r.h.s. of Eq.(41) and write:

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[5]For $d > 4$ the use of asymptotic expression of Eq.(32) and statistical analysis of Ref. [6] allows to find the *correct* power of W^{-1} in the preexponential factor of the density of states. In this case our approximation is equivalent to the hypothesis of stationarity of higher-order contributions used in Ref. [6], which is valid for $d > 4$.

$$G^{-1}(p) = -W^2 Q \left[W^2 G \otimes G \right] \otimes G \qquad (53)$$

It is easy to convince oneself that this is equivalent to the limit of $z = \pm i0$ in (43) or $y = -\frac{\pi}{2}$ in (10). In this case (43) reduces to:

$$W^2 G \otimes G = \Psi(z = \pm i0) = -\frac{\pi}{2} \qquad (54)$$

and from (17) we get:

$$Q \left[W^2 G \otimes G \right] = \frac{2}{\pi} \qquad (55)$$

Formal solution of Eq.(54) has the form:

$$G = \pm i \sqrt{\frac{\pi}{2}} \frac{1}{W \sqrt{\aleph}} \qquad (56)$$

where $\aleph = \sum_{\mathbf{p}} 1$ – is the number of states in the band. Direct substitution of (56) and (55) into (53) shows that this equation is satisfied. Thus in the "first-order" approximation, in the limit of $W \to \infty$ we can write down the Green's function (41) as:

$$G(p) = \frac{1}{G_0^{-1}(p) - \frac{2}{\pi} W^2 \sum_{\mathbf{p}} G(p)} \qquad (57)$$

which is surprisingly coincides with the result of the self-consistent Born approximation (equivalent to the first diagram in Fig.1(b) or Fig.3) [1,2], only with an extra factor of $\frac{2}{\pi}$. Obviously Eq.(57) leads to the density of states of the Born approximation $N_0(E)$, which practically coincides for $d = 3$ with the density of states of free electrons (with the account of the shift of the band-edge in one-loop approximation). In Fig.7 we compare the results following from Eq.(57) for the density of states in one-dimensional ($d = 1$) system with the exact result of Halperin [11], which demonstrates rather satisfactory agreement in the region of "strong coupling" $|E| < E_{sc} \sim m^{\frac{1}{3}} W^{\frac{4}{3}}$. The width of this region grows with the growth of W. Note that while the "tail" of the density of states is suppressed with the growth of W (Cf.(51)), the crossover region, where $|E| \sim E_{sc}$, becomes wider.

It is possible, that Eq.(57) provides the qualitative justification of the use of the simplest Born–like approximation for the Green's function in the approaches similar to the self-consistent theory of localization [9,15] – the mobility edge appears in the "strong coupling"

region $|E| \stackrel{<}{\sim} E_{sc}$ (38), where the approximations leading to (57) becomes rather satisfactory and the Green's function really takes the simple Born–like form.

These results demonstrate the effectiveness of the use of diagram combinatorics in formulation of new approximations for realistic physical problems.

APPENDIX A:

Let us derive Eq.(18) for generating function $Q(y)$ without the use of the explicit form of one-particle Green's function (6). In the problem with zero transferred momentum the Bethe-Salpeter equation shown in Fug.2(c) reduces to:

$$\Gamma = U + UG^2\Gamma \tag{A1}$$

so that

$$\Gamma = \frac{U}{1 - UG^2} \tag{A2}$$

Using (A2) and (22) we obtain an equation, connecting the self-energy part with irreducible vertex U:

$$\Sigma = \frac{W^2 G}{1 - UG^2} \tag{A3}$$

Use now the "Ward identity":

$$W^2 \left.\frac{\partial}{\partial W}\right|_G \frac{\Sigma}{W} = UG \tag{A4}$$

which can be derived via (8) and (24), and Eq.(A3) to obtain:

$$W^2 \left.\frac{\partial}{\partial W}\right|_G \frac{\Sigma}{W} = UG = \frac{1}{G}\left\{1 - W^2\frac{G}{\Sigma}\right\}$$

$$\Sigma = W^2 G + W^2 G\Sigma \left.\frac{\partial}{\partial W}\right|_G \frac{\Sigma}{W} \tag{A5}$$

Now from (8), we obtain our differential equation for Q:

$$Q(W^2G^2) = 1 + W^2 GQ(W^2G^2) \left.\frac{\partial}{\partial W}\right|_G WGQ(W^2G^2) =$$

$$= 1 + W^2 G^2 \frac{d}{d(W^2G^2)} W^2 G^2 Q^2(W^2G^2)$$

which can be rewritten as:

$$Q(y) = 1 + y\frac{d}{dy}yQ^2(y) \tag{A6}$$

Note, however, that from this analysis it is impossible to find the correct initial condition (17), which is closely related to (11), reflecting the causality principle.

APPENDIX B:

Let us show that all diagrams of the 3-rd order, shown in Fig.1(b) are equal. Consider the "maximally crossed" graph of Fig.1(b.1). Its analytical form is:

$$\text{Fig.1(b.1)} = W^6 \sum_{\mathbf{p_1,p_2,p_3,p_4,p_5}} G(p_1)G(p_2)G(p_3)G(p_4)G(p_5) \times$$

$$\times \delta(\mathbf{p} - \mathbf{p_1} + \mathbf{p_3} - \mathbf{p_4})\delta(\mathbf{p_1} - \mathbf{p_2} + \mathbf{p_4} - \mathbf{p_5})\delta(\mathbf{p_2} - \mathbf{p_3} + \mathbf{p_5} - \mathbf{p}) \qquad \text{(B1)}$$

In isotropic system the Green's function depends only on the absolute magnitude of the momentum and $G(\mathbf{p}) = G(-\mathbf{p})$. In (B1) we can make the following change of variables of integration (which obviously does not change the value of integrals): $\mathbf{p_1} \rightleftharpoons -\mathbf{p_3}$; $\mathbf{p_2} \rightleftharpoons -\mathbf{p_2}$. Correspondingly:

$$\delta(\mathbf{p} - \mathbf{p_1} + \mathbf{p_3} - \mathbf{p_4})\delta(\mathbf{p_1} - \mathbf{p_2} + \mathbf{p_4} - \mathbf{p_5})\delta(\mathbf{p_2} - \mathbf{p_3} + \mathbf{p_5} - \mathbf{p}) \rightarrow$$

$$\rightarrow \delta(\mathbf{p} - \mathbf{p_1} + \mathbf{p_3} - \mathbf{p_4})\delta(\mathbf{p_2} - \mathbf{p_3} + \mathbf{p_4} - \mathbf{p_5})\delta(\mathbf{p_1} - \mathbf{p_2} + \mathbf{p_5} - \mathbf{p}) \qquad \text{(B2)}$$

and we obtain the contribution of diagram shown in Fig.1(b.2). Analogously, the change of variables $\mathbf{p_2} \rightleftharpoons -\mathbf{p_4}$; $\mathbf{p_3} \rightleftharpoons -\mathbf{p_3}$ produces the contribution of diagram of Fig.1(b.4), while $\mathbf{p_3} \rightleftharpoons -\mathbf{p_5}$; $\mathbf{p_4} \rightleftharpoons -\mathbf{p_4}$ gives the contribution of Fig.1(b.3). Thus, all the contributions of "skeleton" diagrams of the 3-rd order are equal to each other.

Analogous change of integration variables allows to show that under any impurity in the given "skeleton" diagram we can actually "rotate" the electronic line with all incoming impurity lines, so that the new diagram will give the same contribution as an initial one. This symmetry produces rather wide classes of equal diagrams in any order of perturbation theory.

APPENDIX C:

Here we present some details of calculations, leading from (49) to (51) for one-dimensional $(d = 1)$ case. For $E < 0$ the convolution of Green's functions, entering (49) is calculated by usual contour integration:

$$G_0 \otimes G_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \frac{1}{|E| + p^2 - 2pq + q^2} \frac{1}{|E| + p^2} = \frac{1}{\sqrt{|E|}(q^2 + 4|E|)} \qquad (C1)$$

(Here and below for brevity we put $2m = 1$.) Then:

$$\sum_{\mathbf{q}} \sqrt{G_0 \otimes G_0} exp\left\{-\frac{1}{2W^2 G_0 \otimes G_0}\right\} = \frac{1}{\pi|E|^{\frac{1}{4}}} \int_0^{\infty} \frac{dq}{\sqrt{q^2 + 4|E|}} exp\left\{-\frac{\sqrt{|E|}}{2W^2}(q^2 + 4|E|)\right\} =$$

$$= \frac{1}{2\pi|E|^{\frac{1}{4}}} exp\left\{-\frac{|E|^{\frac{3}{2}}}{W^2}\right\} K_0\left(\frac{|E|^{\frac{3}{2}}}{W^2}\right) = \frac{W}{2\sqrt{2\pi}|E|} exp\left\{-\frac{2|E|^{\frac{3}{2}}}{W^2}\right\} \qquad (C2)$$

After that by elementary calculations we get:

$$\left|\sum_{\mathbf{P}} G_0(p)\right| = \frac{1}{\pi} \int_0^{\infty} dp \frac{1}{|E| + p^2} = \frac{1}{2\sqrt{|E|}} \qquad (C3)$$

Putting all these expressions into (49) and restoring $2m$, we obtain (51).

APPENDIX D:

Here we present the details of calculations leading from (49) to (52) for $d = 3$. Calculations in this case are rather tedious and it is convenient to use the coordinate representation for the Green's functions. In the region of $E < 0$ we finally obtain:

$$
G_0 \otimes G_0 = \begin{cases} \frac{1}{8\pi\sqrt{|E|}} & q \ll 2\sqrt{|E|} \\ \frac{1}{8q} & q \gg 2\sqrt{|E|} \end{cases} \tag{D1}
$$

In inequalities in (D1) it is convenient to replace 2 by π, which guarantees the smooth crossover between two limiting expressions of Eq.(D1) in the region of intermediate q. Using (D1) we can write:

$$
\sum_{\mathbf{q}} \sqrt{G_0 \otimes G_0}\, exp\left\{-\frac{1}{2W^2 G_0 \otimes G_0}\right\} =
$$

$$
= \frac{1}{2\pi^2} \int_0^{\pi\sqrt{|E|}} dq q^2 \sqrt{\frac{1}{8\pi\sqrt{|E|}}}\, exp\left\{-\frac{4\pi\sqrt{|E|}}{W^2}\right\} + \frac{1}{2\pi^2} \int_{\pi\sqrt{|E|}}^{\infty} dq q^2 \sqrt{\frac{1}{8q}}\, exp\left\{-\frac{4q}{W^2}\right\} \approx
$$

$$
\approx \frac{\sqrt{\pi}}{12\sqrt{2}} |E|^{\frac{5}{4}}\{1 + O(W^2)\} exp\left\{-\frac{4\pi\sqrt{|E|}}{W^2}\right\} \tag{D2}
$$

In fact the main contribution here is from the first integral. Then:

$$
\left|\sum_{\mathbf{p}} G_0(p)\right| = \frac{1}{\pi^2} \int_0^{p_0} dp \frac{p^2}{|E| + p^2} \approx \frac{1}{2\pi^2}\left(p_0 - \frac{\pi}{2}\sqrt{|E|}\right) \approx \frac{p_0}{2\pi^2} \tag{D3}
$$

Putting all these expressions into Eq.(49) and restoring $2m$, we obtain (52).

REFERENCES

[1]   S.F.Edwards. Phil.Mag. **3**, 1020 (1958)

[2]   A.A.Abrikosov, L.P.Gorkov, I.E.Dzyaloshinskii. Methods of Quantum Field Theory in Statistical Physics. Fizmatgiz, Moscow, 1962, Ch.7.

[3]   S.M.Ryitov, Yu.A.Kravtsov, V.N.Tatarskii. Introduction to Statistical Radiophysics. Part II. Nauka, Moscow, 1978, Ch.VIII.

[4]   S.Ma. Modern Theory of Critical Phenomena . W.A.Benjamin, NY, 1976, Ch.10.

[5]   A.Yu.Grosberg, A.R.Khokhlov. Statistical Physics of Macromolecules. Nauka, Moscow, 1989, Ch.2.

[6]   I.M.Suslov. Zh.Eksp.Teor.Fiz.(JETP),**102**, 1951 (1992)

[7]   L.V.Keldysh. Thesis. P.N.Lebedev Physical Institute. 1965.

[8]   A.L.Efros, B.I.Shklovskii. Electronic Properties of Doped Semiconductors. Nauka, Moscow, 1979, Ch.11.

[9]   M.V.Sadovskii. Sov.Sci.Rev.A-Phys.Rev. **7**, 1 (1986)

[10]   I.M.Lifshits, S.A.Gredeskul, L.A.Pastur. Introduction to the Theory of Disordered Systems. Nauka, Moscow, 1982, Ch.IV.

[11]   B.I.Halperin. Phys.Rev. **139**, A104 (1965)

[12]   J.Cardy. J.Phys. C**11**, L321 (1978)

[13]   M.V.Sadovskii. Fiz.Tverd.Tela (Sov.Phys.–Solid State) **21**, 743 (1979)

[14]   I.M.Suslov. Zh.Eksp.Teor.Fiz.(JETP),**111**, 1896 (1997)

[15]   D.Vollhardt, P.Wölfle. In "Electronic Phase Transition." Ed.by W.Hanke and Yu.V.Kopaev. North-Holland, Amsterdam, 1992, p.1.

Table I

N	$\Gamma_N = a_N$	$b_N = a_N/(2N+1)!!$	$\Sigma_N = a_{N-1}$	$U_N = (2N-1)a_{N-1}$
1	1	0.3333	1	1
2	4	0.2667	1	3
3	27	0.2571	4	20
4	248	0.2624	27	189
5	2830	0.2722	248	2232
6	38232	0.2829	2830	3130
7	593859	0.2930	38232	497016
8	10401712	0.3019	593859	8907885
9	202601898	0.3158	10401712	176829104
10	4342263000	0.3211	202601898	3849436062
$N \gg 1$	$\frac{1}{e}[1 - \frac{5}{4N}](2N+1)!!$	$\frac{1}{e}[1 - \frac{5}{4N}]$	$\frac{1}{e}[1 - \frac{5}{4N}](2N-1)!!$	$\frac{1}{e}[1 - \frac{9}{4N}](2N+1)!!$

Figure Captions.

Fig.1. Diagrammatic expansion for the averaged one-particle Green's function (a) and self-energy part (b). Dashed line corresponds to the quadratic correlator of the random field. G_0 – free particle Green's function.

Fig.2. Diagrammatic expansion for the full vertex-part Γ (a), for the irreducible vertex-part U (b) and Bethe-Salpeter equation, connecting Γ and U (c).

Fig.3. Equation connecting the self-energy part with the full vertex-part.

Fig.4. The cut of any of $2N - 1$ internal Green's function lines in the "skeleton" diagram of N-th order for the self-energy part produces the appropriate diagram for U.

Fig.5. The behavior of b_n with the growth of n. Dots represent the values of b_n, obtained from the recurrence relation (28), the curve represents the asymptotic dependence $\frac{1}{e}\left(1 - \frac{5}{4}\frac{1}{n}\right)$, dashed line — the asymptote $\frac{1}{e}$.

Fig.6. (a) – "basic" diagram, used to construct the approximation for the self-energy part, (b) – "rotated maximally crossed" diagrams produce the "ladder" in case of time reversal invariant system.

Fig.7. Density of states in one-dimensional system for the different values of the averaged square of the random field $\frac{W^2(2m)^{\frac{1}{2}}}{E_0^{\frac{3}{2}}}$: 1. 0.25; 2. 2; 3. 16. Continuous curves — the exact solution, dashed lines — self-consistent Born approximation (57). Energy is in units of E_0, density of states – in units of $\frac{\sqrt{2m}}{\sqrt{E_0}}$, E_0 – some arbitrary unit of energy.
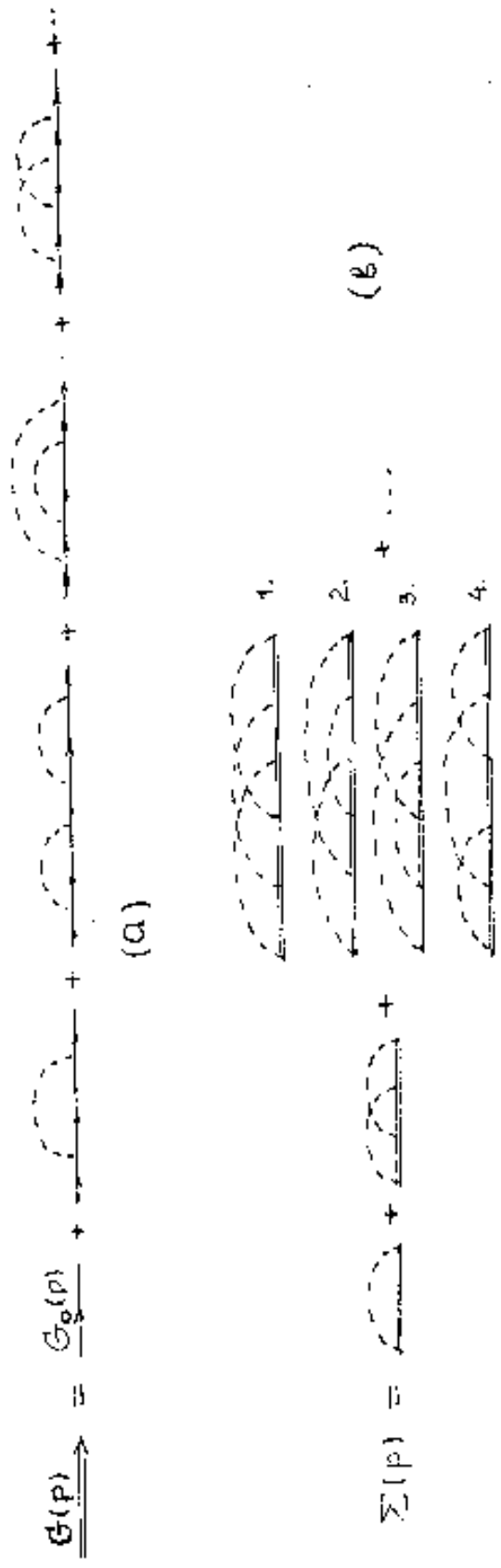
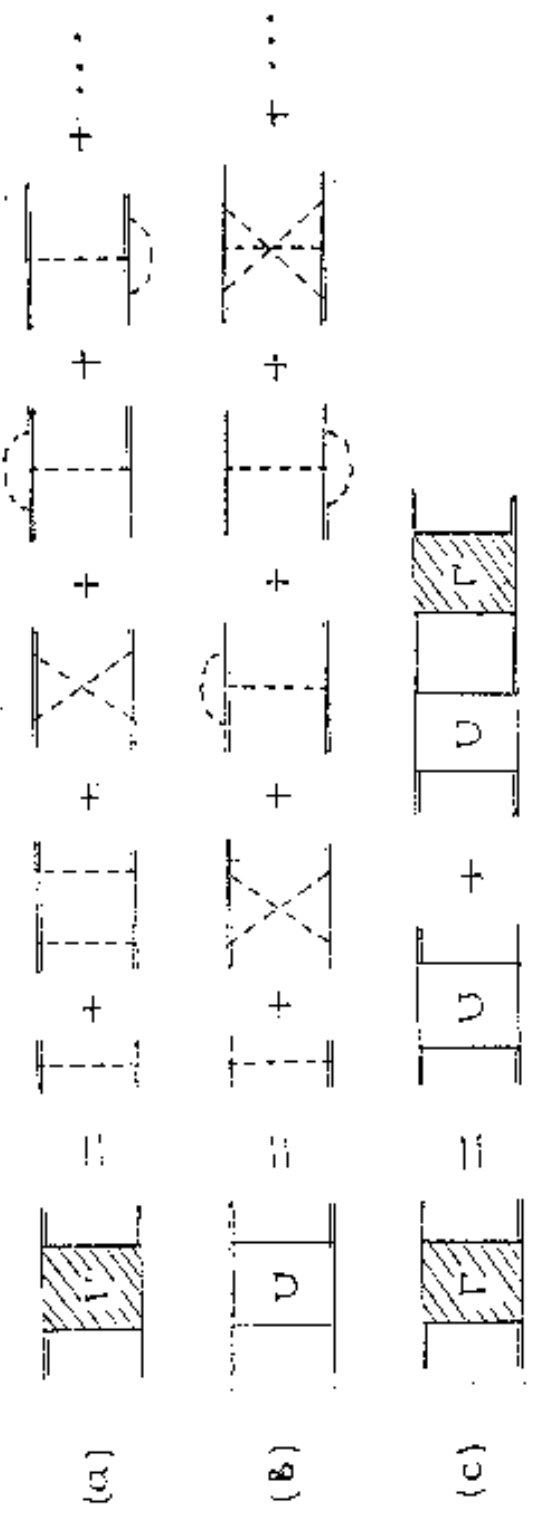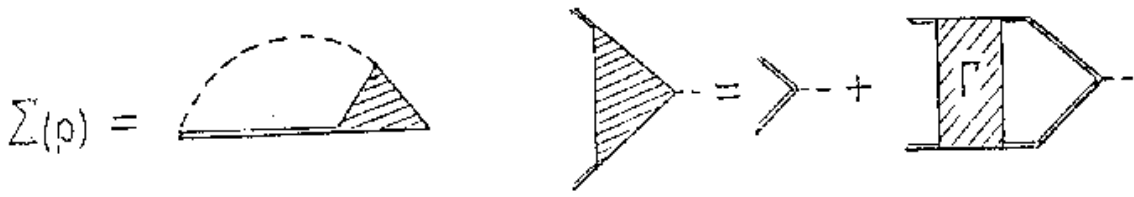$$\underline{G(p)} = \underline{G_0(p)} + \dots + \dots + \dots +$$

(a)

$$\Sigma(p) = \dots + \dots +$$

(b)

Fig. 1

Fig. 2

$$\Sigma(p) =$$

Fig.3

Fig.4