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ORDER, DISORDER, AND PHASE TRANSITION IN CONDENSED SYSTEM

Ginzburg–Landau Expansion in Strongly Disordered Attractive Anderson–Hubbard Model[1]

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Abstract—We have studied disordering effects on the coefficients of Ginzburg–Landau expansion in powers of superconducting order parameter in the attractive Anderson–Hubbard model within the generalized DMFT+Σ approximation. We consider the wide region of attractive potentials U from the weak coupling region, where superconductivity is described by BCS model, to the strong coupling region, where the superconducting transition is related with Bose–Einstein condensation (BEC) of compact Cooper pairs formed at temperatures essentially larger than the temperature of superconducting transition, and a wide range of disorder—from weak to strong, where the system is in the vicinity of Anderson transition. In the case of semielliptic bare density of states, disorder's influence upon the coefficients A and B of the square and the fourth power of the order parameter is universal for any value of electron correlation and is related only to the general disorder widening of the bare band (generalized Anderson theorem). Such universality is absent for the gradient term expansion coefficient C. In the usual theory of "dirty" superconductors, the C coefficient drops with the growth of disorder. In the limit of strong disorder in BCS limit, the coefficient C is very sensitive to the effects of Anderson localization, which lead to its further drop with disorder growth up to the region of the Anderson insulator. In the region of BCS–BEC crossover and in BEC limit, the coefficient C and all related physical properties are weakly dependent on disorder. In particular, this leads to relatively weak disorder dependence of both penetration depth and coherence lengths, as well as of related slope of the upper critical magnetic field at superconducting transition, in the region of very strong coupling.

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1. INTRODUCTION

The studies of disorder influence on superconductivity have a rather long history. The pioneer works by Abrikosov and Gor'kov [1–4] considered the limit of weak disorder ($p_F l \gg 1$, where p_F is the Fermi momentum and l is the mean free path) and weak coupling superconductivity well described by BCS theory. The notorious "Anderson theorem" on superconducting critical temperature T_c of superconductors with "normal" (nonmagnetic) disorder [5, 6] is usually also referred to these limits.

The generalization of the theory of "dirty" superconductors to the case of strong enough disorder ($p_F l \sim 1$) (and further up to the region of Anderson transition) was made in [7–9], where superconductivity was also considered in the weak coupling limit.

The problem of BCS theory generalization to the strong coupling region has also been studied for a long time. Significant progress in this direction was achieved by Nozieres and Schmitt-Rink [10], who proposed an effective method to study the crossover from BCS-type behavior in the weak coupling region to Bose–Einstein condensation (BEC) in the strong coupling region. At the same time, the problem of superconductivity of disordered systems in the limit of strong coupling and in the BCS–BEC crossover region remains relatively undeveloped.

One of the simplest models to study the BCS–BEC crossover is the attractive Hubbard model. The most successful approach to the studies of Hubbard model, both to describe strongly correlated systems in case of repulsive interactions and to study BCS–BEC crossover in case of attraction, is the dynamical mean-field theory (DMFT) [11–13].

In recent years, we have developed the generalized DMFT+Σ approach to the Hubbard model [14–19], which is very convenient to the description of different additional "external" (as compared to DMFT) inter-

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actions. In particular, this approach is well suited to describe also the two-particle properties, such as optical (dynamic) conductivity [18, 20].

In [21], we have used this approach to analyze single-particle properties of the normal phase and optical conductivity in the attractive Hubbard model. Further on, we used the DMFT+Σ method in [22] to study disorder effects on superconducting critical temperature, which was calculated within the Nozieres–Schmitt-Rink approach. In particular, for the case of the semielliptic model of the bare density of states, which is adequate to describe three-dimensional systems, we have demonstrated numerically that disorder influence upon the critical temperature (for the whole range of interaction parameters) is related only to the general widening of the bare band (density of states) by disorder. In [23], we have presented an analytic derivation of such disorder influence (in DMFT+Σ approximation) on all single-particle properties and the temperature of superconducting transition for the case of the semielliptic band.

Starting with the classic paper by Gor'kov [3] it is well known that Ginzburg–Landau expansion plays the fundamental role in the theory of "dirty" superconductors, allowing the effective treatment of disorder dependence of different physical properties close to superconducting critical temperature [6]. The generalization of this theory to the region of strong disorder (up to Anderson metal–insulator transition) was also based upon microscopic derivation of the coefficients of this expansion [7–9]. However, as noted above, all these derivations were performed in the weak coupling limit of BCS theory.

In [24], we have combined the Nozieres–Schmitt-Rink and DMFT+Σ approximations within the attractive Hubbard model to derive coefficients of homogeneous Ginzburg–Landau expansion A and B before the square and the fourth power of superconducting order parameter, demonstrating the universal disorder influence on coefficients A and B and the related discontinuity of specific heat at the transition temperature. After that, in [25], we have studied the behavior of coefficient C before the gradient term of Ginzburg–Landau expansion, where such universality is absent. In this work, we have only considered this coefficient in the region of weak disorder ($p_F l \gg 1$) in the "ladder" approximation for impurity scattering, as it is usually done in the standard theory of "dirty" superconductors [3], though for the whole range of pairing interactions including the BCS–BEC crossover region and the limit of very strong coupling. In fact, here we have neglected the effects of Anderson localization, which can significantly change the behavior of the coefficient C in the limit of strong disorder ($p_F l \sim 1$) [7–9].

In this work, we shall concentrate mainly on the study of the coefficient C in the region of strong disorder, when Anderson localization effects become relevant.

2. HUBBARD MODEL WITHIN DMFT+Σ APPROACH AND THE NOZIERES–SCHMITT-RINK APPROXIMATION

We consider the disordered nonmagnetic attractive Anderson–Hubbard model, described by the Hamiltonian:

$$H = -t\sum_{\langle ij\rangle\sigma} a_{i\sigma}^\dagger a_{j\sigma} + \sum_{i\sigma}\epsilon_i n_{i\sigma} - U\sum_i n_{i\uparrow}n_{i\downarrow}, \qquad (1)$$

where $t > 0$ is transfer amplitude between nearest neighbors, U is the Hubbard-like onsite attraction, $n_{i\sigma} = a_{i\sigma}^\dagger a_{i\sigma}$ is electron number operator at a given site, $a_{i\sigma}$ ($a_{i\sigma}^\dagger$) is annihilation (creation) operator of an electron with spin σ, and local energies ϵ_i are assumed to be independent random variables at different lattice sites. For the validity of the standard "impurity" diagram technique [26, 27] we assume the Gaussian distribution for energy levels ϵ_i:

$$\mathcal{P}(\epsilon_i) = \frac{1}{\sqrt{2\pi}W}\exp\left(-\frac{\epsilon_i^2}{2W^2}\right). \qquad (2)$$

Distribution width W is the measure of disorder, while the Gaussian field of energy levels (independent on different sites—"white" noise correlation) induces the "impurity" scattering, which is described by the standard approach, based upon the calculation of the averaged Green's functions [27].

The generalized DMFT+Σ approach [14–17] extends the standard dynamical mean-field theory (DMFT) [11–13] introducing the additional "external" self-energy part (SEP) $\Sigma_{\mathbf{p}}(\varepsilon)$ (in general momentum dependent), which originates from any interaction outside the DMFT, and provides an effective procedure to calculate both single-particle and two-particle properties [18, 20]. The success of such a generalized approach is connected with the choice of single-particle Green's function in the following form:

$$G(\varepsilon, \mathbf{p}) = \frac{1}{\varepsilon + \mu - \varepsilon(\mathbf{p}) - \Sigma(\varepsilon) - \Sigma_{\mathbf{p}}(\varepsilon)}, \qquad (3)$$

where $\varepsilon(\mathbf{p})$ is the "bare" electronic dispersion, while the total SEP is an additive sum of Hubbard-like local SEP $\Sigma(\varepsilon)$ and "external" $\Sigma_{\mathbf{p}}(\varepsilon)$, neglecting the interference between Hubbard-like and "external" interactions. This allows us to conserve the system of self-consistent equations of the standard DMFT [11–13]. At the each step of DMFT iterations the "external" SEP $\Sigma_{\mathbf{p}}(\varepsilon)$ is recalculated with the use of some approximate scheme, corresponding to the form of additional interaction, while the local Green's function is also "dressed" by $\Sigma_{\mathbf{p}}(\varepsilon)$ at each step of the standard DMFT procedure.

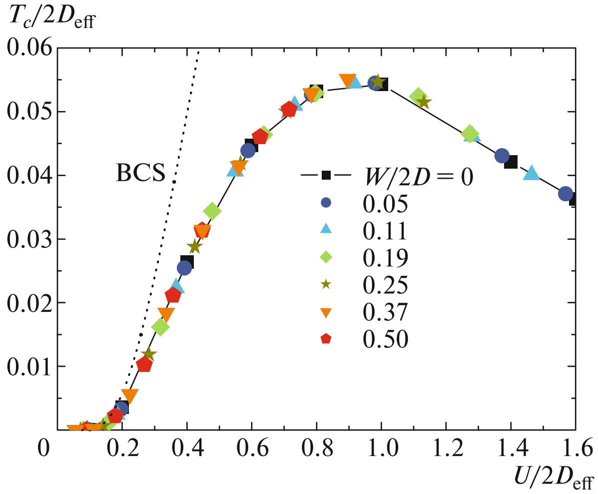

Fig. 1. (Color online) Universal dependence of the temperature of superconducting transition on the strength of Hubbard attraction for different levels of disorder.

The "external" SEP, entering DMFT+Σ cycle, in the problem of disorder scattering under consideration here [18, 19], is taken in the simplest (self-consistent Born) approximation, neglecting the "crossing" diagrams of impurity scattering, which gives:

$$\Sigma_{\mathbf{p}}(\varepsilon) \to \Sigma_{\rm imp}(\varepsilon) = W^2\sum_{\mathbf{p}} G(\varepsilon, \mathbf{p}). \qquad (4)$$

To solve the effective single Anderson impurity problem of DMFT we use here, as in our previous papers, the quite efficient impurity solver using the numerical renormalization group (NRG) [28].

In the following, we are using the "bare" band with semielliptic density of states (per unit cell with lattice parameter a and single spin projection), which is a rather good approximation in the three-dimensional case:

$$N_0(\varepsilon) = \frac{2}{\pi D^2}\sqrt{D^2 - \varepsilon^2}, \qquad (5)$$

where D defines the half–width of the conduction band.

In [23], we have shown that in the DMFT+Σ approach for the model with semi-elliptic density of states all effects of disorder upon single-particle properties reduce only to the band widening due to disorder, i.e., to the replacement $D \to D_{\rm eff}$, where $D_{\rm eff}$ is the effective half-width of the "bare" band in the absence of electronic correlations ($U = 0$), widened by disorder:

$$D_{\rm eff} = D\sqrt{1 + 4\frac{W^2}{D^2}}. \qquad (6)$$

The "bare" density of states (in the absence of U) "dressed" by disorder:

$$\tilde{N}_0(\xi) = \frac{2}{\pi D_{\rm eff}^2}\sqrt{D_{\rm eff}^2 - \varepsilon^2}, \qquad (7)$$

remains semielliptic also in the presence of disorder. It should be noted, that in other models of the "bare" band disorder effect is not reduced only to the widening of the band, changing also the form of the density of states, so that there is no complete universality of disorder influence on single-particle properties, reducing to a simple substitution $D \to D_{\rm eff}$. However, in the limit of strong enough disorder of interest to us, the "bare" band becomes practically semielliptic, restoring such universality [23].

All calculations below, as in our previous works, were performed for the rather typical case of the quarter–filled band (the number of electrons per lattice site is $n = 0.5$).

To consider superconductivity for the wide range of pairing interaction U, following [21, 23], we use the Nozieres–Schmitt-Rink approximation [10], which allows qualitatively correct (though approximate) description of the BCS–BEC crossover region. In this approach, we determine the critical temperature T_c using the usual BCS-type equation [23]:

$$1 = \frac{U}{2}\int_{-\infty}^{\infty} d\varepsilon \tilde{N}_0(\varepsilon)\frac{\tanh[(\varepsilon - \mu)/2T_c]}{\varepsilon - \mu}, \qquad (8)$$

with chemical potential μ determined via DMFT+Σ calculations for different values of U and W, i.e., from the standard equation for the number of electrons (band filling), determined by the Green's function given by Eq. (3), allowing us to find T_c for the wide range of the model parameters including the regions of BCS–BEC crossover and strong coupling, as well as for different levels of disorder. This reflects the physical meaning of the Nozieres–Schmitt-Rink approximation—in the weak coupling region, transition temperature is controlled by the equation for Cooper instability (8), while, in the strong coupling region, it is determined as BEC temperature controlled by chemical potential.

In [23], it was shown that disorder's influence on the critical temperature T_c and single-particle characteristics (e.g., density of states) in the model with semielliptic "bare" density of states is universal and reduces only to the change of the effective bandwidth. In Fig. 1, just for illustrative purposes, we show the universal dependence of the critical temperature T_c on Hubbard attraction for different levels of disorder [23]. In the weak coupling region, the temperature of superconducting transition is described well by the BCS model (for comparison, in Fig. 1, the dashed line represents the dependence obtained for T_c from Eq. (8) with chemical potential independent of U and determined by quarter filling of the "bare" band), while for the strong coupling region the critical temperature is mainly determined by the condition of Bose conden-

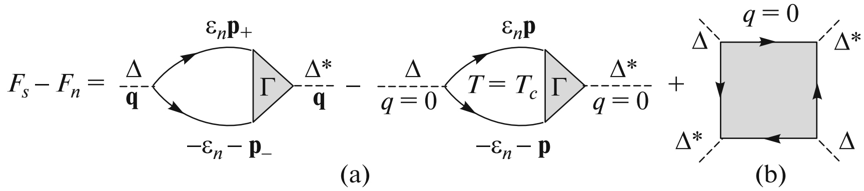

Fig. 2. Diagrammatic representation of Ginzburg–Landau expansion $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{q}/2$.

sation of Cooper pairs and drops with the growth of U as t^2/U, going through the maximum at $U/2D_{\text{eff}} \sim 1$.

The review of these and other results obtained for disordered Hubbard model in DMFT+Σ approximation can be found in [19].

3. GINZBURG–LANDAU EXPANSION

Ginzburg–Landau expansion for the difference of free-energy densities of superconducting and normal states is written in the standard form [27]:

$$F_s - F_n = A|\Delta_{\mathbf{q}}|^2 + q^2 C|\Delta_{\mathbf{q}}|^2 + \frac{B}{2}|\Delta_{\mathbf{q}}|^4, \qquad (9)$$

where $\Delta_{\mathbf{q}}$ is the Fourier component of the order parameter Δ.

This expansion (9) is determined by the loop–expansion diagrams for free-energy of an electron in the field of fluctuations of the order parameter (denoted by dashed lines) with small wavevector $\mathbf{q}$ [27], shown in Fig. 2 [27].

In the framework of the Nozieres–Schmitt-Rink approach [10], we use the weak coupling approximation to analyze Ginzburg–Landau coefficients, so that the "loops" with two and four Cooper vertices, shown in Fig. 2, do not contain contributions from Hubbard attraction and are "dressed" only by impurity scattering. However, like in the case of T_c calculation, the chemical potential, which is essentially dependent on the coupling strength and in the strong coupling limit actually controls the condition of Bose condensation of Cooper pairs, should be determined within full DMFT+Σ procedure.

In [24] it was shown that in this approach the coefficients A and B are determined by the following expressions:

$$A(T) = \frac{1}{U} - \int_{-\infty}^{\infty} d\varepsilon \tilde{N}_0(\varepsilon) \frac{\tanh[(\varepsilon - \mu)/2T]}{2(\varepsilon - \mu)}, \qquad (10)$$

$$B = \int_{-\infty}^{\infty} \frac{d\varepsilon}{2(\varepsilon - \mu)^3} \left(\tanh\frac{\varepsilon - \mu}{2T} - \frac{(\varepsilon - \mu)/2T}{\cosh^2[(\varepsilon - \mu)/2T]} \right) \tilde{N}_0(\varepsilon). \qquad (11)$$

For $T \to T_c$ the coefficient $A(T)$ takes the usual form:

$$A(T) \equiv \alpha(T - T_c). \qquad (12)$$

In BCS limit, where $T = T_c \to 0$, we obtain for coefficients α and B the standard result [27]:

$$\alpha_{BCS} = \frac{\tilde{N}_0(\mu)}{T_c}, \quad B_{BCS} = \frac{7\zeta(3)}{8\pi^2 T_c^2} \tilde{N}_0(\mu). \qquad (13)$$

In the general case, the coefficients A and B are determined only by the disorder widened density of states $\tilde{N}_0(\varepsilon)$ and chemical potential. Thus, in the case of semielliptic density of states the dependence of these coefficients on disorder is due only to the simple replacement $D \to D_{\text{eff}}$, leading to universal (independent of the level of disorder) curves for properly normalized dimensionless coefficients ($\alpha(2D_{\text{eff}})^2$ and $B(2D_{\text{eff}})^3$) on $U/2D_{\text{eff}}$) [24]. In fact, the coefficients α and B are rapidly suppressed with the growth of dimensionless coupling $U/2D_{\text{eff}}$.

It should be noted that Eqs. (10) and (11) for coefficients A and B were obtained in [24] using the exact Ward identities and remain valid also in the limit of arbitrarily large disorder (including the region of Anderson localization).

Universal dependence on disorder, related to widening of the band $D \to D_{\text{eff}}$, is observed, in particular, for specific heat discontinuity at the transition point, which is determined by coefficients α and B [24]:

$$C_s(T_c) - C_n(T_c) = T_c \frac{\alpha^2}{B}. \qquad (14)$$

From diagrammatic representation of Ginzburg–Landau expansion, shown in Fig. 2, it is clear that the coefficient C is determined by the coefficient before q^2 in a Cooper two-particle loop (first term in Fig. 2). Then we obtain the following expression:

$$C = -T \lim_{q \to 0} \sum_{n, \mathbf{p}, \mathbf{p}'} \frac{\Psi_{\mathbf{pp}'}(\varepsilon_n, \mathbf{q}) - \Psi_{\mathbf{pp}'}(\varepsilon_n, 0)}{q^2}, \qquad (15)$$

where $\Psi_{\mathbf{p},\mathbf{p}'}(\varepsilon_n, \mathbf{q})$ is a two-particle Green's function in a Cooper channel (see Fig. 3), "dressed" in the Nozieres–Schmitt-Rink approximation only by impurity scattering. In case of time-reversal invari-

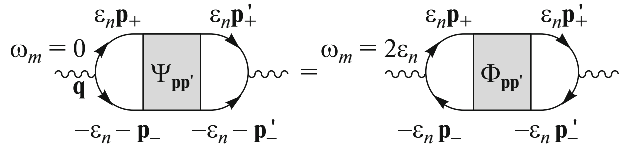

Fig. 3. The equality of loops in Cooper and diffusion channels under time-reversal invariance.

ance (in the absence of magnetic field and magnetic impurities) and because of the static nature of impurity scattering "dressing" two-particle Green's function $\Psi_{\mathbf{p},\mathbf{p}'}(\varepsilon_n, \mathbf{q})$, we can reverse here the direction of all lower electron lines with simultaneous change of the sign of all momenta (see Fig. 3). As a result, we obtain:

$$\Psi_{\mathbf{p},\mathbf{p}'}(\varepsilon_n, \mathbf{q}) = \Phi_{\mathbf{p},\mathbf{p}'}(\omega_m = 2\varepsilon_n, \mathbf{q}), \qquad (16)$$

where ε_n are Fermionic Matsubara frequencies, $\mathbf{p}_{\pm} = \mathbf{p} \pm \frac{\mathbf{q}}{2}$, $\Phi_{\mathbf{p},\mathbf{p}'}(\omega_m = 2\varepsilon_n, \mathbf{q})$ is the two-particle Green's function in the diffusion channel, dressed by impurities. Then we obtain Cooper susceptibility as:

$$\chi(\mathbf{q}) = -T\sum_{n,\mathbf{p},\mathbf{p}'} \Psi_{\mathbf{p},\mathbf{p}'}(\varepsilon_n, \mathbf{q}) = -T\sum_{n,\mathbf{p},\mathbf{p}'} \Phi_{\mathbf{p},\mathbf{p}'}(\omega_m = 2\varepsilon_n, \mathbf{q}). \qquad (17)$$

Performing the standard summation over Fermionic Matsubara frequencies [26, 27], we obtain:

$$\chi(\mathbf{q}) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\varepsilon \,\mathrm{Im}\,\Phi^{RA}(\omega = 2\varepsilon, \mathbf{q}) \tanh\frac{\varepsilon}{2T}, \qquad (18)$$

where $\Phi^{RA}(\omega, \mathbf{q}) = \sum_{\mathbf{p},\mathbf{p}'} \Phi^{RA}_{\mathbf{p},\mathbf{p}'}(\omega, \mathbf{q})$. To find the loop $\Phi^{RA}(\omega, \mathbf{q})$ in strongly disordered case (e.g., in the region of Anderson localization) we can use the approximate self-consistent theory of localization [27, 29–33]. Then this loop contains the diffusion pole of the following form [19]:

$$\Phi^{RA}(\omega = 2\varepsilon, \mathbf{q}) = -\frac{\sum_{\mathbf{p}} \Delta G_{\mathbf{p}}(\varepsilon)}{\omega + iD(\omega)q^2}, \qquad (19)$$

where $\Delta G_{\mathbf{p}}(\varepsilon) = G^R(\varepsilon, \mathbf{p}) - G^A(-\varepsilon, \mathbf{p})$, G^R and G^A are the retarded and advanced Green's functions, and $D(\omega)$ is frequency dependent generalized diffusion coefficient. Then we obtain the coefficient C as:

$$C = \lim_{q\to 0} \frac{\chi(\mathbf{q}) - \chi(\mathbf{q} = 0)}{q^2}$$

$$= -\frac{1}{8\pi}\int_{-\infty}^{\infty} d\varepsilon \frac{\tanh\frac{\varepsilon}{2T}}{\varepsilon} \mathrm{Im}\left(\frac{iD(2\varepsilon)\sum_{\mathbf{p}} \Delta G_{\mathbf{p}}(\varepsilon)}{\varepsilon + i\delta}\right) \qquad (20)$$

$$= -\frac{1}{8\pi}\int_{-\infty}^{\infty} d\varepsilon \frac{\tanh\frac{\varepsilon}{2T}}{\varepsilon^2} \mathrm{Re}\left(D(2\varepsilon)\sum_{\mathbf{p}} \Delta G_{\mathbf{p}}(\varepsilon)\right) - \frac{1}{16T}\mathrm{Im}\left(D(0)\sum_{\mathbf{p}} \Delta G_{\mathbf{p}}(0)\right).$$

The generalized diffusion coefficient of the self-consistent theory of localization [27, 29–33] for our model can be found as the solution of the following self–consistency equation [18]:

$$D(\omega) = i\frac{\langle v\rangle^2}{d}\left(\omega - \Delta\Sigma^{RA}_{\mathrm{imp}}(\omega) + W^4\sum_{\mathbf{p}} (\Delta G_{\mathbf{p}}(\varepsilon))^2 \times \sum_{\mathbf{q}} \frac{1}{\omega + iD(\omega)q^2}\right)^{-1}, \qquad (21)$$

where $\omega = 2\varepsilon$, $\Delta\Sigma^{RA}_{\mathrm{imp}}(\omega) = \Sigma^R_{\mathrm{imp}}(\varepsilon) - \Sigma^A_{\mathrm{imp}}(-\varepsilon)$, d is space dimension, and velocity $\langle v\rangle$ is defined by the following expression:

$$\langle v\rangle = \frac{\sum_{\mathbf{p}} |\mathbf{v}_{\mathbf{p}}|\Delta G_{\mathbf{p}}(\varepsilon)}{\sum_{\mathbf{p}} \Delta G_{\mathbf{p}}(\varepsilon)}; \quad \mathbf{v}_{\mathbf{p}} = \frac{\partial\varepsilon(\mathbf{p})}{\partial\mathbf{p}}. \qquad (22)$$

Due to the limits of diffusion approximation summation over q in Eq. (21) should be limited by the following cut-off [27, 32]:

$$q < k_0 = Min\{l^{-1}, p_{\mathrm{F}}\}, \qquad (23)$$

where l is the mean free path due to elastic disorder scattering and p_{F} is Fermi momentum.

In the limit of weak disorder, when localization corrections are small, the Cooper susceptibility $\chi(\mathbf{q})$ and coefficient C related to it are determined by the "ladder" approximation. In this approximation coefficient C was studied by us in [25], where we obtained it in general analytic form. Let us now transform self-consistency Eq. (21) to make the obvious connection with exact "ladder" expression in the limit of weak disorder. In the "ladder" approximation, we just neglect the "maximally intersecting" diagrams entering the irreducible vertex. The second term in the r.h.s. of self-consistency Eq. (21) vanishes. Let us introduce the frequency dependent generalized diffusion coefficient in "ladder" approximation as:

$$D_0(\omega) = \frac{\langle v\rangle^2}{d}\frac{i}{\omega - \Delta\Sigma_{\rm imp}^{RA}(\omega)}. \quad (24)$$

Then $\frac{\langle v\rangle^2}{d}$ entering the self-consistency Eq. (21) can be rewritten via this diffusion coefficient D_0 in "ladder" approximation, so that Eq. (21) takes the following form:

$$D(\omega = 2\varepsilon) = \frac{D_0(\omega = 2\varepsilon)}{1 + \frac{W^4}{2\varepsilon - \Delta\Sigma_{\rm imp}^{RA}(\omega = 2\varepsilon)}\sum_{\mathbf{p}}(\Delta G_{\mathbf{p}}(\varepsilon))^2\sum_{\mathbf{q}}\frac{1}{2\varepsilon + iD(\omega = 2\varepsilon)q^2}}. \quad (25)$$

Using the approach of [25], the diffusion coefficient $D_0(\omega = 2\varepsilon)$ in the "ladder" approximation can be derived analytically. In fact, in the "ladder" approximation the two-particle Green's function (19) takes the following form:

$$\Phi_0^{RA}(\omega = 2\varepsilon, \mathbf{q}) = -\frac{\sum_{\mathbf{p}}\Delta G_{\mathbf{p}}(\varepsilon)}{\omega + iD_0(\omega = 2\varepsilon)q^2}. \quad (26)$$

Then we obtain:

$$\varphi(\varepsilon, \mathbf{q} = 0) \equiv \lim_{q\to 0}\frac{\Phi_0^{RA}(\omega = 2\varepsilon, \mathbf{q}) - \Phi_0^{RA}(\omega = 2\varepsilon, \mathbf{q} = 0)}{q^2} = \frac{i\sum_{\mathbf{p}}\Delta G_{\mathbf{p}}(\varepsilon)}{\omega^2}D_0(\omega = 2\varepsilon). \quad (27)$$

Then the diffusion coefficient D_0 can be written as:

$$D_0 = \frac{\varphi(\varepsilon, \mathbf{q} = 0)(2\varepsilon)^2}{i\sum_{\mathbf{p}}\Delta G_{\mathbf{p}}(\varepsilon)}. \quad (28)$$

In [25] using the exact Ward identity we have shown, that in the "ladder" approximation $\varphi(\varepsilon, \mathbf{q} = 0)$ can be represented as:

$$\varphi(\varepsilon, \mathbf{q} = 0)(2\varepsilon)^2 = \sum_{\mathbf{p}}v_x^2 G^R(\varepsilon, \mathbf{p})G^A(-\varepsilon, \mathbf{p}) + \frac{1}{2}\sum_{\mathbf{p}}\frac{\partial^2\varepsilon(\mathbf{p})}{\partial p_x^2}(G^R(\varepsilon, \mathbf{p}) + G^A(-\varepsilon, \mathbf{p})), \quad (29)$$

where $v_x = \frac{\partial\varepsilon(\mathbf{p})}{\partial p_x}$.

Finally, using Eqs. (28), (29) we find the diffusion coefficient D_0 in the "ladder" approximation. Using self-consistency Eq. (25) we determine the generalized diffusion coefficient, and then using Eq. (20) we find the coefficient C. In the limit of weak disorder, when the "ladder" approximation works well and generalized diffusion coefficient just coincides with the diffusion coefficient in the "ladder" approximation, we obtain for coefficient C the result obtained in [25]:

$$C_0 = -\frac{1}{8\pi}\int_{-\infty}^{\infty}d\varepsilon\frac{\tanh\frac{\varepsilon}{2T}}{\varepsilon^2} \times\sum_{\mathbf{p}}\left(v_x^2\,{\rm Im}(G^R(\varepsilon, \mathbf{p})G^A(-\varepsilon, \mathbf{p})) + \frac{\partial^2\varepsilon_{\mathbf{p}}}{\partial p_x^2}{\rm Im}\,G^R(\varepsilon, \mathbf{p})\right) + \frac{1}{16T}\sum_{\mathbf{p}}\left(v_x^2\,{\rm Re}(G^R(0, \mathbf{p})G^A(0, \mathbf{p})) + \frac{\partial^2\varepsilon_{\mathbf{p}}}{\partial p_x^2}{\rm Re}\,G^R(0, \mathbf{p})\right). \quad (30)$$

Now we can use the iteration scheme to find the coefficient C, which in the limit of weak disorder reproduces the results of the "ladder" approximation, while in the limit of strong disorder takes into account the effects of Anderson localization (in the framework of the self-consistent theory of localization).

In numerical calculations using Eqs. (28) and (29) we first find the "ladder" diffusion coefficient D_0 for the given value of $\omega = 2\varepsilon$. Then, solving by iterations the transcendental self-consistency Eq. (25), we determine the generalized diffusion coefficient at this frequency. After that, using Eq. (20) we calculate the Ginzburg–Landau coefficient C.

In [18] it was shown, that in DMFT+Σ approximation for the Anderson–Hubbard model the critical disorder for Anderson metal–insulator transition $W/2D = 0.37$ and is independent of the value of the Hubbard interaction U. The approach developed here allows determination of the C coefficient also in the region of Anderson insulator at disorder levels $W/2D > 0.37$.

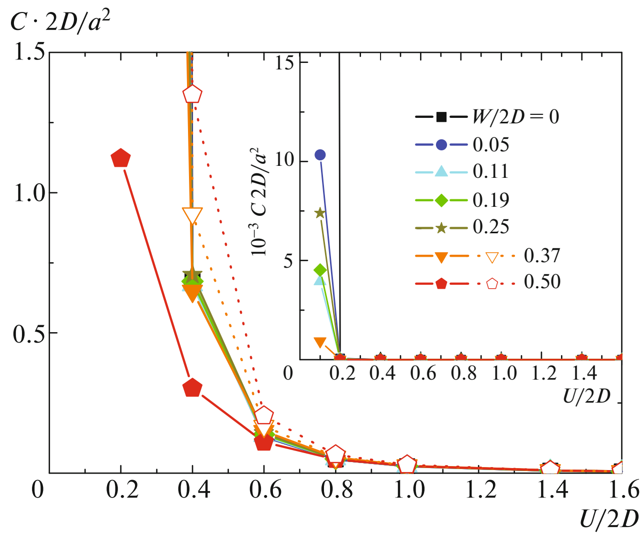

Fig. 4. (Color online) Dependence of C coefficient on the strength of Hubbard attraction for different levels of disorder (a is lattice parameter). Filled symbols and continuous lines correspond to calculations taking into account localization corrections. Unfilled symbols and dashed lines correspond to the "ladder" approximation.

4. MAIN RESULTS

The coherence length at given temperature $\xi(T)$ gives a characteristic scale of inhomogeneities of the order parameter Δ:

$$\xi^2(T) = -\frac{C}{A}. \tag{31}$$

Coefficient A changes its sign and becomes zero at a critical temperature: $A = \alpha(T - T_c)$, so that

$$\xi(T) = \frac{\xi}{\sqrt{1 - T/T_c}}, \tag{32}$$

where we have introduced the coherence length of a superconductor:

$$\xi = \sqrt{\frac{C}{\alpha T_c}}, \tag{33}$$

which reduces to a standard expression in the weak coupling region and in the absence of disorder [27]:

$$\xi_{BCS} = \sqrt{\frac{C_{BCS}}{\alpha_{BCS} T_c}} = \sqrt{\frac{7\zeta(3)}{16\pi^2 d}}\frac{v_F}{T_c}. \tag{34}$$

Penetration depth of magnetic field into superconductor is defined by:

$$\lambda^2(T) = -\frac{c^2}{32\pi e^2}\frac{B}{AC}. \tag{35}$$

Then:

$$\lambda(T) = \frac{\lambda}{\sqrt{1 - T/T_c}}, \tag{36}$$

where we have introduced:

$$\lambda^2 = \frac{c^2}{32\pi e^2}\frac{B}{\alpha C T_c}, \tag{37}$$

which in the absence of disorder has the form:

$$\lambda^2_{BCS} = \frac{c^2}{32\pi e^2}\frac{B_{BCS}}{\alpha_{BCS} C_{BCS} T_c} = \frac{c^2}{16\pi e^2}\frac{d}{N_0(\mu) v_F^2}. \tag{38}$$

As λ_{BCS} is independent of T_c, i.e., of coupling strength, it is convenient to use for normalization of penetration depth λ (37) at arbitrary U and W.

Close to T_c the upper critical magnetic field H_{c2} is determined by the Ginzburg–Landau coefficients as:

$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2(T)} = -\frac{\Phi_0}{2\pi}\frac{A}{C}, \tag{39}$$

where $\Phi_0 = c\pi/e$ is a magnetic flux quantum. Then the slope of the upper critical filed close to T_c is given by:

$$\frac{dH_{c2}}{dT} = \frac{\Phi_0}{2\pi}\frac{\alpha}{C}. \tag{40}$$

In Fig. 4 we show the dependence of coefficient C on the strength of Hubbard attraction for different disorder levels. In this figure and in the following we use filled symbols and continuous lines corresponding to the results of calculations taking into account localization corrections, while unfilled symbols and dashed lines correspond to calculations in the "ladder" approximation. Coefficient C is essentially a two-particle characteristic and it does not follow universal behavior on disorder, as in case of coefficients A and B, and disorder dependence here is not reduced only to widening of effective bandwidth by disorder. Correspondingly, in the dependence of C on coupling strength, where all energies are normalized by effective bandwidth $2D_{\rm eff}$, we do not observe a universal curve for different levels of disorder [25], in contrast to similar dependencies for coefficients α and B. In fact, coefficient C is rapidly suppressed with the growth of coupling strength. Especially strong suppression is observed in the weak coupling region (cf. insert in Fig. 4). Localization corrections become relevant in the limit of strong enough disorder ($W/2D > 0.25$). Under such strong disordering localization corrections significantly suppress coefficient C in weak coupling region (cf. dashed lines ("ladder" approximation) and continuous curves (with localization corrections) for $W/2D = 0.37$ and 0.5). In strong coupling region for $U/2D > 1$ localization corrections, in fact, do not change the value of coefficient C, as compared to the results of "ladder" approximation, even in the limit of strong disorder for $W/2D > 0.37$, where the system becomes an Anderson insulator.

In Fig. 5, we show the dependencies of coefficient C on disorder level for different values of coupling strength $U/2D$. In the limit of weak coupling ($U/2D = 0.1$), we

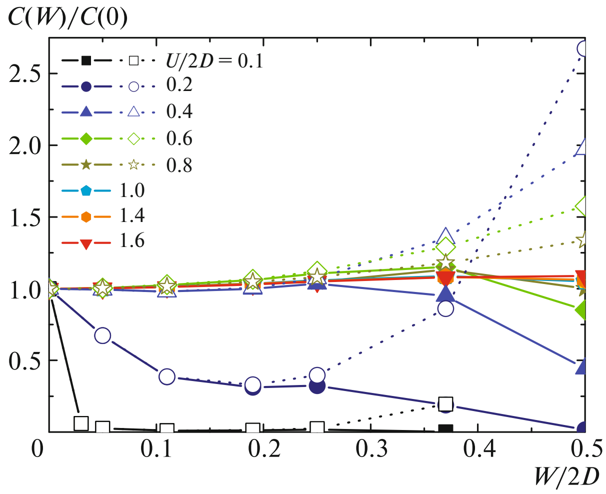

Fig. 5. (Color online) Dependence of coefficient C normalized by its value in the absence of disorder for different values of Hubbard attraction U. Dashed lines—“ladder” approximation, continuous curves—calculations with the account of localization corrections.

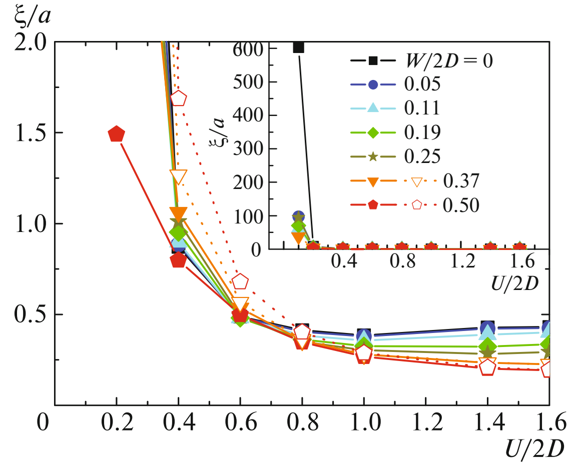

Fig. 6. (Color online) Dependence of coherence length on the strength of Hubbard attraction U for different disorder levels. Inset: the rapid growth of coherence length with diminishing coupling in BCS limit.

observe rather rapid suppression of coefficient C with the growth of disorder in case of weak enough impurity scattering. In the region of strong enough disorder in the “ladder” approximation, we can observe some growth of coefficient C with the increase of disorder, which is related mainly with significant widening of the band by such strong disorder and corresponding drop of the effective coupling $U/2D_{\mathrm{eff}}$. However, localization corrections, which are significant at large disorder $W/2D > 0.25$, actually lead to suppression of coefficient C with the growth of disorder in the limit of strong impurity scattering. In the intermediate coupling region ($U/2D = 0.4$–0.6) coefficient C in the “ladder” approximation is only growing slightly with increasing disorder. In the BEC limit ($U/2D > 1$) coefficient C is practically independent of impurity scattering both in the “ladder” approximation and with the account of localization corrections. In the BEC limit the account of localization corrections in fact do not change the value of C in comparison with the “ladder” approximation.

As the Ginzburg–Landau expansion coefficient α and B demonstrate the universal dependence on disorder, Anderson localization in fact does not influence them at all, while coefficient C in the weak coupling region is strongly affected by localization corrections, being almost independent of them in the BEC limit, the physical properties depending on C will be also significantly changed by localization corrections in the weak coupling region, becoming practically independent of localization in the BEC limit.

Let us now discuss the behavior of physical properties. Dependence of coherence length on Hubbard attraction strength is shown in Fig. 6. We can see that in the weak coupling region (cf. insert at Fig. 6) coherence length rapidly drops with the growth of U for any disorder, reaching the value of the order of lattice parameter a in the intermediate coupling region of $U/2D \approx 0.4$–0.6. Further growth of coupling strength changes the coherence length only slightly. The account of localization corrections for coherence length is significant only at large disorder ($W/2D > 0.25$). We see that localization corrections lead to significant suppression of coherence length in the BCS limit of weak coupling and practically do not change the coherence length in the BEC limit.

In Fig. 7, we show the dependence of penetration depth, normalized by its BCS value in the absence of disorder (38), on the strength of Hubbard attraction U for different levels of disorder. In the absence of impurity scattering, penetration depth grows with the increase of the coupling strength. In BCS weak coupling limit disorder leads to a fast growth of penetration depth (for “dirty” BCS superconductors $\lambda \sim l^{-1/2}$, where l is the mean free path). In BEC strong coupling limit disorder only slightly diminish the penetration depth (cf. Fig. 10a). This leads to suppression of penetration depth with disorder with the growth of Hubbard attraction strength in the region of weak enough coupling and to the growth of λ with U in BEC strong coupling region. The account of localization corrections is significant only in the limit of strong disorder ($W/2D > 0.25$) and leads to noticeable growth of penetration depth as compared to the “ladder” approximation in the weak coupling region. In the BEC limit the influence of localization on penetration depth is just insignificant.

Dependence of the slope of the upper critical magnetic field on the strength of Hubbard attraction for different disorder levels is shown in Fig. 8. In the limit of weak enough impurity scattering, until Anderson

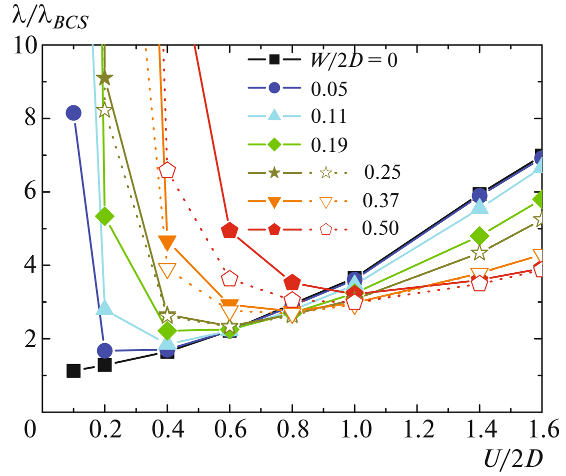

Fig. 7. (Color online) Dependence of penetration depth, normalized by its BCS value in the limit of weak coupling, on the strength of Hubbard attraction U for different levels of disorder.

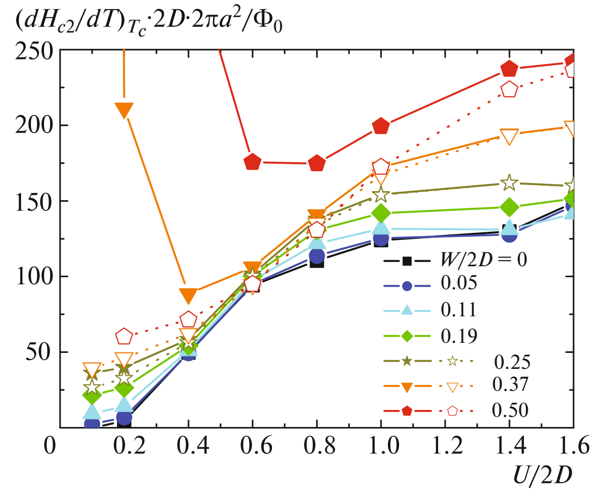

Fig. 8. (Color online) Dependence of the slope of the upper critical field on the strength of Hubbard attraction U for different level of disorder.

localization corrections remain unimportant, the slope of the upper critical field grows with the growth of the coupling strength. The fast growth of the slope is observed with the growth of U in the region of weak enough coupling, while in the limit of strong coupling the slope is rather weakly dependent on $U/2D$. In the region of strong enough disorder ($W/2D > 0.25$) the account of localization corrections becomes quite important—it qualitatively changes the behavior of the upper critical held. While the "ladder" approximation (dashed curves) conserves the behavior of the slope of the upper critical held typical for the region of weak disorder, where the slope grows with the growth of the coupling strength, the account of Anderson localization ($W/2D \geq 0.37$) leads to a strong increase of the slope of the upper critical held in the weak coupling limit. As a result, in Anderson insulator the slope of the upper critical hied rapidly drops with the growth of U in the weak coupling limit and just insignificantly grows with the growth of U in BEC limit. Note that the account of localization corrections is also unimportant for for the slope of the upper critical held in the strong coupling limit.

Let us consider now dependencies of physical properties on disorder. In Fig. 9 we show dependence of coherence length ξ on disorder for different values of coupling. In the BCS limit for weak coupling and for weak enough impurity scattering we observe the standard "dirty" superconductor dependence $\xi \propto l^{1/2}$, i.e., the coherence length rapidly drops with the growth of disorder (cf. insert in Fig. 9a). However, at strong enough disorder in "ladder" approximation (dashed lines) coherence length starts to grow with disorder (cf. Fig. 9b and insert in Fig. 9a), which is mainly related to the widening of the band by disorder and corresponding suppression of $U/2D_{\text{eff}}$. Taking into account localization corrections leads to noticeable suppression of coherence length in comparison with the "ladder" approximation in the limit of strong disorder, which leads to restoration of general suppression of ξ with the growth of disorder in this limit. In the standard BCS model with a bare band of infinite width coherence length drops with the growth of disorder $\xi \propto l^{1/2}$ and close to Anderson transition this suppression of ξ even accelerates, so that $\xi \propto l^{2/3}$ [7–9], which differs from the present model here, where close to Anderson coherence length is rather weakly dependent on disorder, which is related to significant widening of the band by disorder. With growth of coupling, for $U/2D > 0.4$–0.6 coherence length ξ becomes of the order of lattice parameter and is almost disorder independent, while in BEC limit of very strong coupling $U/2D = 1.4$, 1.6 the growth of disorder up to very strong values ($W/2D = 0.5$) leads to suppression of coherence length approximately by the factor of two (cf. Fig. 9b). Again we see, that in the limit of strong coupling the account of localization corrections is rather insignificant.

Dependence of penetration depth on disorder for different values of Hubbard attraction is shown in Fig. 10a. In weak coupling limit disorder in accordance with the theory of "dirty" superconductors leads to the growth of penetration depth ($\lambda \propto l^{-1/2}$). With increase of the coupling strength the growth of penetration depth slow down and in the limit of very strong coupling, for $U/2D = 1.4$, 1.6, penetration depth is even slightly suppressed by disorder. The account of localization corrections leads to some quantitative growth of penetration depth in comparison with the results of the "ladder" approximation in the weak coupling region. Qualitatively the dependence of penetration depth on disorder does not change. In BEC limit of strong coupling the account of localization corrections is rather irrelevant. In Fig. 10b we show the disorder dependence of dimensionless Ginzburg–Landau $\kappa = \lambda/\xi$.

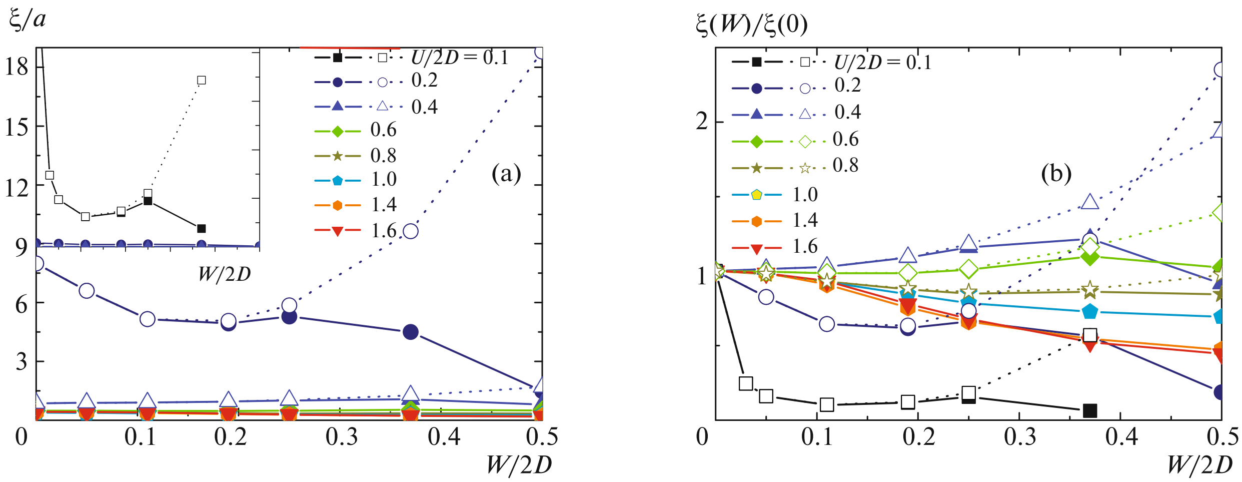

Fig. 9. (Color online) Dependence of coherence length on disorder for different values of Hubbard attraction, (a) coherence length normalized by lattice parameter *a*. Inset: dependence of coherence length on disorder in weak coupling limit, (b) coherence length normalized by its value in the absence of disorder.

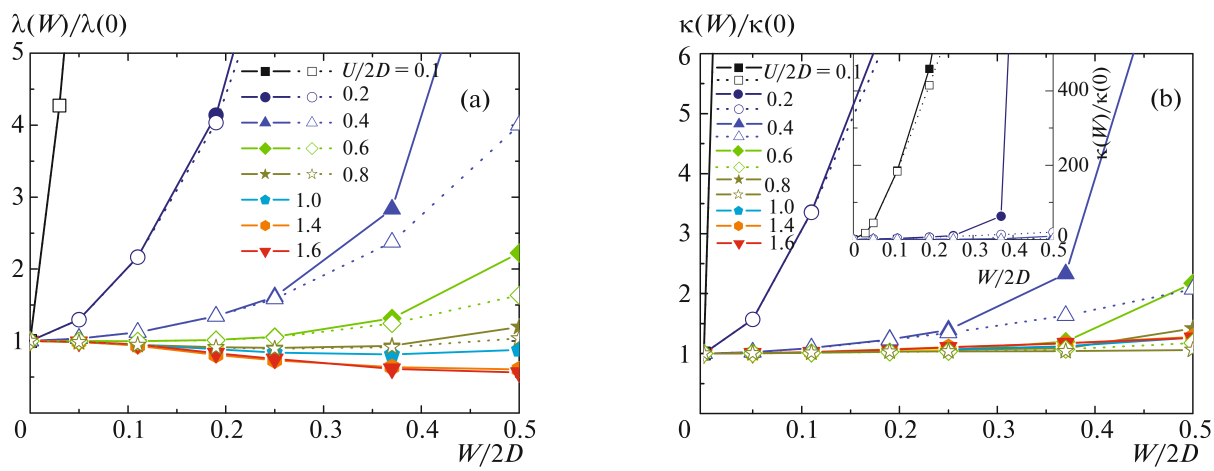

Fig. 10. (Color online) Dependence of penetration depth (a) and Ginzburg–Landau parameter (b) on disorder level for different values of Hubbard attraction. Inset shows the growth of the Ginzburg–Landau parameter with disorder in weak coupling limit.

We can see, that in the weak coupling limit Ginzburg–Landau parameter is rapidly growing with disorder (cf. insert in Fig. 10b) in accordance with the theory of "dirty" superconductors, where $\kappa \propto l^{-1}$. With the increase of coupling strength the growth of the Ginzburg–Landau parameter with disorder slows down and in the limit of strong coupling $U/2D > 1$ parameter κ is practically disorder independent. The account of localization corrections quantitatively increases Ginzburg–Landau parameter in Anderson insulator phase ($W/2D \geq 0.37$) in the strong coupling region. In the strong coupling region localization corrections are again irrelevant.

In Fig. 11 we show the disorder dependence of the slope of the upper critical field. In the weak coupling limit we again observe the behavior typical for "dirty" superconductors—the slope of the upper critical field grows with the growth of disorder (cf. Fig. 11a and the insert in Fig. 11b). The account of localization corrections in weak coupling limit sharply increases the slope of the upper critical field in comparison with the result of the "ladder" approximation in the region of Anderson insulator ($W/2D \geq 0.37$). As a result, in an Anderson insulator the slope of the upper critical field grows with the increase of impurity scattering much faster than in the "ladder" approximation. In intermediate coupling region ($U/2D$ = 0.4–0.8) the slope of the upper critical field is practically independent of impurity scattering in the region of weak disorder. In the "ladder" approximation such behavior is conserved also in the region of strong disorder. However, the account of localization corrections leads to significant growth of the slope with disorder in Anderson insulator phase. In the limit of very strong coupling and weak disorder the slope of the upper critical field can even slightly diminish with disorder, but in the limit of

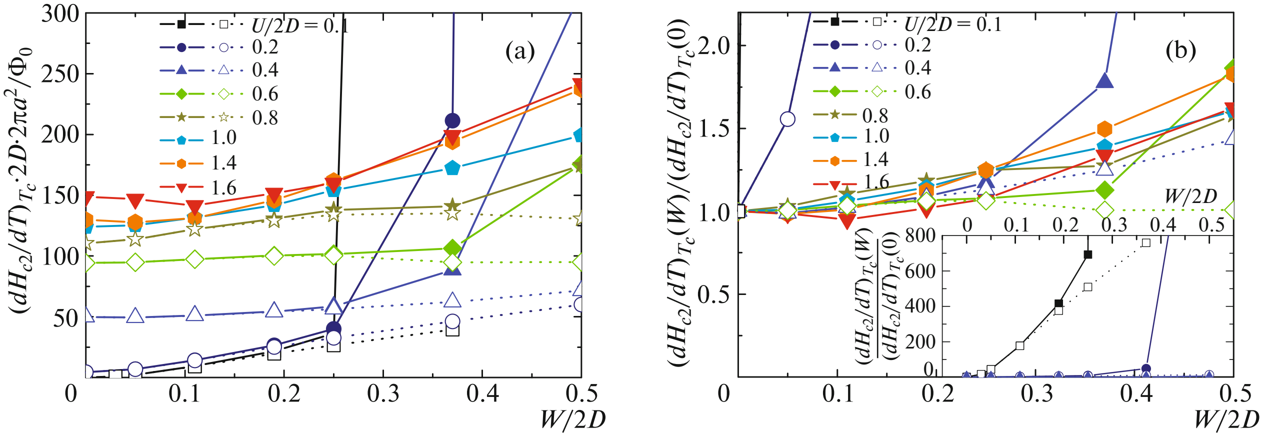

Fig. 11. (Color online) Dependence of the slope of the upper critical field (a) and this slope, normalized by its value in the absence of disorder (b), on disorder for different values of Hubbard attraction strength. In the inset we show the growth of the slope with disorder in weak coupling region.

strong disorder the slope grows with the growth of impurity scattering. In the BEC limit the account of localization corrections is irrelevant and only slightly changes the slope of the upper critical field as compared with the results of the "ladder" approximation.

5. CONCLUSIONS

In this paper in the framework of the Nozieres–Schmitt-Rink approximation and DMFT+Σ generalization of dynamical mean field theory we have studied the effects of disorder (including the strong disorder region of Anderson localization) on the Ginzburg–Landau coefficients and related physical properties close to T_c in disordered Anderson–Hubbard model with attraction. Calculations were done for the wide range of attractive potentials U, from weak coupling region $U/2D_{\rm eff} \ll 1$, where instability of normal phase and superconductivity is well described by the BCS model, up to the strong coupling limit $U/2D_{\rm eff} \gg 1$, where the transition into the superconducting state is due to Bose condensation of compact Cooper pairs, forming at a temperature much higher than the temperature of superconducting transition.

The growth of the coupling strength U leads to rapid suppression of all Ginzburg–Landau coefficients. The coherence length ξ rapidly drops with the growth of coupling and for $U/2D \approx 0.4$ becomes on the order of lattice spacing and only slightly changes with further increase of coupling. Penetration depth in "clean" superconductors grows with U, while in "dirty" superconductors it drops in the weak coupling and grows in BEC limit, passing through the minimum in the intermediate coupling region $U/2D \approx 0.4$–0.8. In the region of weak enough disorder ($W/2D < 0.37$), when Anderson localization effect are not very important, the slope of the upper critical field grows with the growth of U. However, in the limit of weak coupling in Anderson insulator phase localization effects sharply increase the slope of the upper critical field, while in BEC limit of strong coupling localization effects become unimportant. As a result, the slope of the upper critical field drops with the growth of U in BCS limit, passing through the minimum at $U/2D \approx 0.4$–0.8. The specific heat discontinuity grows with Hubbard attraction U in the weak coupling region and drops in the strong coupling limit, passing through the maximum at $U/2D_{\rm eff} \approx 0.55$ [24].

Disorder influence (including the strong disorder in the region of Anderson localization) upon the critical temperature T_c and Ginzburg–Landau coefficients A and B and the related discontinuity of specific heat is universal and is completely determined only by disorder widening of the bare band, i.e., by the replacement $D \to D_{\rm eff}$. Thus, even in the strong coupling region, the critical temperature and Ginzburg–Landau coefficients A and B satisfy the generalized Anderson theorem—all influence of disorder is related only to the change of the density of states. Disorder influence on coefficient C is not universal and is related not only to the bare band widening.

Coefficient C is sensitive to the effects of Anderson localization. We have studied this effect for a wide range of disorder, including the region of Anderson insulator. To compare and extract explicitly effects of Anderson localization we also studied coefficient C in the "ladder" approximation for disorder scattering. In the weak coupling limit $U/2D_{\rm eff} \ll 1$ and weak disorder $W/2D < 0.37$ the behavior of coefficient C and related physical properties is well described by the theory of "dirty" superconductors—coefficient C and coherence length rapidly drop with the growth of disorder, while penetration depth and the slope of the upper critical field grow. In the region of strong disorder (in an Anderson insulator) in BCS limit the behavior of coefficient C is strongly affected by localization effects. In the "ladder" approximation the band widening effect leads to the growth of coefficient C with the growth of W [25], however localization effects restore suppression of coefficient C by disorder and in Anderson insulator phase. Correspondingly, localization effects significantly change physical properties, related to coefficient C, so that for these properties qualitatively follow the dependencies characteristic for "dirty" superconductors—the coherence length is suppressed by disorder, while the penetration depth and the slope of the upper critical field grow with the growth of disorder. In the BCS–BEC crossover region and in the BEC limit coefficient C and all related physical properties are rather weakly dependent on disorder. In particular, in BEC limit both coherence length and penetration depth are slightly suppressed by disorder, so that their ratio (Ginzburg–Landau parameter) is practically disorder independent. In the BEC limit the effects of Anderson localization rather weakly affect the coefficient C and the related physical characteristics.

It should be noted that all results were derived here under implicit assumption of the self-averaging nature of superconducting order parameter entering the Ginzburg–Landau expansion, which is connected with our use of the standard "impurity" diagram technique [26, 27]. It is well known [9], that this assumption becomes, in the general case, inapplicable close to Anderson metal–insulator transition, due to strong fluctuations of the local density of states developing here [34] and inhomogeneous picture of superconducting transition [35]. This problem is very interesting in the context of the superconductivity in the BCS–BEC crossover region and in the region of strong coupling and deserves further studies.

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