Electron in a random field in a space of $d=4-\epsilon$ dimensions

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The problem of calculation of the averaged properties of an electron in a Gaussian random field is discussed. The equivalence of such problems with the theory of the zero-component field in a space of $d=4-\epsilon$ dimensions is used to show that the neighborhood of the mobility edge plays the role of the transition region from a weak to a strong coupling, which is analogous to the well-known situation in the Kondo problem. The scaling behavior in the transition region is not obtained.

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The description of electron states and the dynamics in the vicinity of the mobility edge represent an unsolved problem of the theory of disordered systems.[1] Attempts have been made[2–4] to describe the behavior of electrons in the vicinity of the mobility edge on the basis of the theory of critical phenomena in the vicinity of a phase transition of the second kind.[5] In particular, it was shown in Ref. 2 within the framework of the Anderson model[6] that the spatial behavior of the wave functions near the mobility edge can be described by scaling dependences with critical indices determined as in the problem of the phase transition with zero-component order parameter.[7,8] The analogy between the critical phenomena with zero-component order parameter and the problem of an electron in a random field is most transparent when the problem is formulated within the framework of the second traditional theory of disordered systems developed by Edwards (Ref. 9). However, it was noted in Ref. 2 that such an approach meets with serious difficulties, which reduces greatly the usefulness of such an analogy. It is our aim to study in detail the difficulties involved, which, unfortunately, have not been taken into account in the recent paper of Schuster.[10]

We shall consider an electron in the field of a random distribution of point scatterers and we shall calculate the one-particle Green's function $G(\mathbf{r}-\mathbf{r}'t)$ averaged over the random configurations of scatterers. In the limit $\rho\to\infty$, $V\to 0$, and $\rho V^2\to\text{const}$ (see Refs. 9 and 11), where ρ is the density and V is the potential of scatterers, the problem is equivalent to the motion of an electron in a Gaussian random field and the Green's function can be expressed in terms of the following functional integral[9,11]:

$$G(\mathbf{r}-\mathbf{r}'t)=\int\limits_{\mathbf{r}(0)=\mathbf{r}'}^{\mathbf{r}(t)=\mathbf{r}} D\mathbf{r}(\tau)\exp\left\{\frac{im}{2}\int\limits_0^t d\tau\,\dot{\mathbf{r}}^2(\tau)-\frac{\rho V^2}{2}\int\limits_0^t d\tau_1\int\limits_0^t d\tau_2\,\delta(\mathbf{r}(\tau_1)-\mathbf{r}(\tau_2))\right\}, \tag{1}$$

where we have used the model of the "white noise" impurity correlation function.[9,11,12] We shall perform the analytic continuation $t\to -i\beta$ in Eq. (1), which yields

$$G(\mathbf{r}-\mathbf{r}'\beta)=\int\limits_{\mathbf{r}(0)=\mathbf{r}'}^{\mathbf{r}(\beta)=\mathbf{r}} D\mathbf{r}(s)\exp\left\{-\frac{m}{2}\int\limits_0^\beta ds\,\dot{\mathbf{r}}^2(s)+\frac{\rho V^2}{2}\int\limits_0^\beta ds_1\int\limits_0^\beta ds_2\,\delta(\mathbf{r}(s_1)-\mathbf{r}(s_2))\right\}. \tag{2}$$

Such a functional integral describes the thermodynamics of a polymer chain with an attraction between links.[8,11] Following the method of de Gennes and des Cloizeaux applied to polymer chains with repulsion (the excluded volume problem),[7,8] we can show by a perturbation method that $G(\mathbf{r}-\mathbf{r}'\beta)$ or $G(\mathbf{p}\beta)$ in the momentum representation is determined by the inverse Laplace transform

$$G(p\beta)=\int\limits_{c-i\infty}^{c+i\infty}\frac{d\tau}{2\pi i}e^{\tau\beta}G(p\tau), \tag{3}$$

where $G(\mathbf{p}\tau)$ is the Green's function in a field theory with the Lagrangian

$$\mathscr{L}=\frac{1}{2}\sum_{j=1}^{n}\left\{\frac{1}{2m}(\nabla\Phi_j)^2+\tau\Phi_j^2\right\}-\frac{1}{8}\rho V^2\left(\sum_{j=1}^{n}\Phi_j^2\right)^2. \tag{4}$$

Here, n is the number of components of the field Φ which should be set equal to zero at the end of the calculation to eliminate the "superfluous" diagrams with loops proportional to n which should not occur in the problem of an electron in a random field.[9] It should be noted that the interaction constant in Eq. (4) has "incorrect" sign, corresponding to an attraction of the "particles" of the field Φ. In fact, this incorrect sign makes the problem under study qualitatively different from the theory of phase transitions. This fact has not been noted in Ref. 10, which resulted in incorrect conclusions. It is well known that such a field-theoretic problem is unstable in the sense that the ground state does not exist.[13]

It can be shown that the spatial and time Fourier transform $G^R(\mathbf{p}E)$ of the retarded Green's function can be obtained from the Green's function $G(\mathbf{p}\tau)$ of the field theory defined by Eq. (4) by the analytic continuation $\tau\to -(E+i\delta)$.

It is well known that the problem defined by Eq. (4) can be solved in the parquet approximation[14] for a space of dimensionality $d=4$. The parquet diagrams represent a dominant sequence even for a space with $d=4-\varepsilon$ (see

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Ref. 15). The perturbation series for the Lagrangian defined by Eq. (4) containing only parquet diagrams represents an expansion in powers of the parameter us, where

$$s=\begin{cases}\ln\dfrac{a^{-1}}{\max\{\sqrt{2m\tau},\ p_i\}}; & d=4,\\ \dfrac{1}{\varepsilon}\left\{\dfrac{1}{\max a^{\varepsilon}[\sqrt{2m\tau},\ p_i]^{\varepsilon}}-1\right\}; & d=4-\varepsilon,\end{cases} \tag{5}$$

$$u=\frac{m^2a^{\varepsilon}}{2\pi^{d}}\rho V^2. \tag{6}$$

Here, u is a dimensionless coupling constant; p_i is the set of external momenta characterizing the diagram in question; a is a constant with dimensions of length related to a cut-off parameter which is introduced to remove the singularity of diverging integrals (this parameter represent the shortest length in the problem). We shall always go to the limit (when possible) $a\to 0$.

The vertex part $\Gamma(s)$ (four-line vertex) with all the external momenta of the same order of magnitude[14] plays the dominant role in the analysis of the perturbation theory. The parquet approximation yields[14]

$$\Gamma(s)=-\frac{u}{1-us}. \tag{7}$$

A pole at $s=u^{-1}$ indicates that the perturbation theory no longer holds for $s\gtrsim u^{-1}$.

We shall now discuss the physical consequences of such a pole. We shall introduce the quantity

$$Z_\beta=\int d^d r G(\mathbf{r}-\mathbf{r},\ \beta)=\int\limits_{c-i\infty}^{c+i\infty}\frac{d\tau}{2\pi i}e^{\tau\beta}\int\frac{d^dp}{(2\pi)^d}G(\mathbf{p}\tau) \tag{8}$$

playing the role of the partition sum of a polymer chain.[9] We can use the transformation from Z_β to $Z(t)$ which is achieved by the analytic continuation $\beta\to it$ to determine the electron density of states in a random field[9,11]:

$$N(E)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}dte^{iEt}Z(t). \tag{9}$$

We shall define a quantity

$$C=\int\frac{d^dp}{(2\pi)^d}\frac{\partial G(\mathbf{p}\tau)}{\partial\tau} \tag{10}$$

playing the role of specific heat in the theory of phase transitions.[14] The Ward identity

$$\frac{\partial G(\mathbf{p}\tau)}{\partial\tau}=\mathcal{T}(s)G^2(\mathbf{p}\tau) \tag{11}$$

can be used to calculate the specific heat[14]

$$C(s)=-\frac{m^2}{2\pi^2}\int\limits_0^s dt\mathcal{T}^2(t), \tag{12}$$

where

$$\mathcal{T}(s)=\exp\left\{2\int\limits_0^s dt\Gamma(t)\right\}. \tag{13}$$

In the case considered (n = 0), we obtain

$$C(s)=\frac{m^2}{\pi^2u}\{[1-us]^{1/2}-1\}. \tag{14}$$

Using the standard method of differentiation of the Laplace transform, we obtain from Eq. (9) the following result:

$$Z_\beta=-\frac{1}{\beta}\int\limits_{c-i\infty}^{c+i\infty}\frac{d\tau}{2\pi i}e^{\tau\beta}\frac{\partial}{\partial\tau}\int\frac{d^dp}{(2\pi)^d}G(\mathbf{p}\tau)$$

$$=-\frac{1}{\beta}\int\limits_{c-i\infty}^{c+i\infty}\frac{d\tau}{2\pi i}e^{\tau\beta}C(\tau). \tag{15}$$

Equations (14) and (15) then yield (for $p_i=0$)

$$Z_\beta=-\frac{m^2}{\pi^2u}\frac{1}{\beta}\int\limits_{c-i\infty}^{c+i\infty}\frac{d\tau}{2\pi i}e^{\tau\beta}\left\{\left[1-\left(\frac{E_{sc}}{\tau}\right)^{\varepsilon/2}\right]^{1/2}-1\right\}. \tag{16}$$

Here, $c>E_{sc}$, where

$$E_{sc}=\frac{1}{2ma^2}\left(\frac{u}{\varepsilon}\right)^{2/\varepsilon}. \tag{17}$$

Expanding the integrand in Eq. (16) in a series and applying the Laplace transform term by term, the analytic continuation $\beta\to it$, and the Fourier transformation (9), we obtain

$$N(E)=\frac{m^2}{\pi^2u}E\left\{\frac{1}{2}\frac{1}{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(2-\frac{\varepsilon}{2}\right)}\left(\frac{E}{E_{sc}}\right)^{-\varepsilon/2}\right.$$

$$\left.-\frac{1}{8}\frac{1}{\Gamma(\varepsilon)\Gamma(2-\varepsilon)}\left(\frac{E}{E_{sc}}\right)^{-\varepsilon}+\frac{1}{16}\frac{1}{\Gamma\left(\frac{3}{2}\varepsilon\right)\Gamma\left(2-\frac{3}{2}\varepsilon\right)}\left(\frac{E}{E_{sc}}\right)^{-\frac{3}{2}\varepsilon}+\dots\right\} \tag{18}$$

which holds for $E>E_{sc}$. In the limit $\varepsilon\to 0$, we obtain

$$N(E)=\frac{m^2}{\pi^2u}E\left\{\frac{1}{2}\frac{\varepsilon}{2}\left(\frac{E}{E_{sc}}\right)^{-\varepsilon/2}+\frac{1}{8}\varepsilon\left(\frac{E}{E_{sc}}\right)^{-\varepsilon}+\frac{1}{16}\frac{3}{2}\varepsilon\left(\frac{E}{E_{sc}}\right)^{-\frac{3}{2}\varepsilon}+\dots\right\}. \tag{19}$$

Equation (19) can be easily summed to yield

$$N(E)=N_0(E)\frac{1}{\sqrt{1-\left(\frac{E_{sc}}{E}\right)^{\varepsilon/2}}}, \tag{20}$$

where

$$N_0(E)=\frac{1}{(4\pi)^2}(2m)^{2-\varepsilon/2}E^{1-\varepsilon/2} \tag{21}$$

is the density of states of free electrons in a space of $d = 4 - \varepsilon$ dimensions.

Therefore, the parquet approximation to the interaction leads to a singularity of the density of states at $E = E_{sc}$, which reflects the presence of a nonphysical pole in the vertex defined by Eq. (7). The energy E_{sc} in Eq. (17) reduces exactly to the width of the "Ginzburg" critical region for this theory. The width of the critical region was estimated in Ref. 3 on the basis of simple physical arguments. The expansion (19) holds in the region $E \gg E_{sc}$, where the perturbation theory with respect to $u \ll 1$ is valid and it ceases to be applicable for $E \sim E_{sc}$.

The approximation of the "self-consistent field" of Zittartz–Langer–Edwards[9,12] (or the method of the optimum Lifshits fluctuation[16]) is widely used in the theory of disordered systems. Such an approximation can be used to estimate the asymptotic behavior of the tail in the density of states corresponding to localized electron states. To compare this result with the perturbation theory, we shall generalize the corresponding expressions to a space of $d = 4 - \varepsilon$ dimensions.

We shall assume[9,12,16] that an electron in the region of localized states moves in an effective potential well of width R and the asymptotic behavior of the tail of the density of states is governed by the lowest states of the electron in the well. In a space of dimensionality d, we obtain[9]

$$N(E)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\,\exp\left\{-it\left[\frac{d\pi^2}{2mR^2}-E\right]-\frac{\rho V^2t^2}{2R^d}\right\}$$
$$=\left\{\frac{R^d}{2\pi\rho V^2}\right\}^{1/2}\exp\left\{-\frac{\left[\frac{d\pi^2}{2mR^2}-E^2\right]R^d}{2\rho V^2}\right\}. \quad (22)$$

The width of the well (radius of localization) is determined by

$$\frac{d}{dR}R^d\left\{\frac{d\pi^2}{2mR^2}-E\right\}^2=0, \quad (23)$$

which yields

$$R_0=\left\{\frac{2m(-E)}{(4-d)\pi^2}\right\}^{-1/2}, \quad (24)$$

$$N(E)\propto\exp\left\{-\frac{\pi^2}{\varepsilon}\left(\frac{|E|}{E_{sc}}\right)^{\varepsilon/2}\right\};\ E<0;\ \varepsilon\to 0. \quad (25)$$

We have omitted the exponential factor. The validity of such an approximation is given by $|E| \gg E_{sc}$, where the quantity E_{sc} is again defined by Eq. (17). The existence of an infinite tail in the density of states is due to the Gaussian nature of the random field, which contains arbitrarily deep fluctuations. This only reflects the aforementioned absence of the ground state in the field theory defined by Eq. (4). It is quite clear that this is not a fundamental problem.

We shall now try to explain the behavior of the vertex part in the field theory defined by Eq. (4) corresponding to the density of states defined by Eq. (25). It follows (if the exponential factor is omitted) that [see Eqs. (10)–(13)] Eq. (25) corresponds to

$$C(\tau)\sim\exp\left\{-\frac{\pi^2}{\varepsilon}\left(\frac{\tau}{E_{sc}}\right)^{\varepsilon/2}\right\};\ \mathcal{T}(s)\sim\exp\left\{-\frac{\pi^2}{2\varepsilon}us\right\}. \quad (26)$$

Therefore, Eq. (13) ($us \gg 1$) yields

$$\Gamma(s)=\frac{1}{2}\frac{d}{ds}\ln\mathcal{T}(s)\approx-\frac{u}{\varepsilon}\frac{\pi^2}{4}+0\left(\frac{1}{us}\right). \quad (27)$$

Therefore, we can define two different types of condition for the effective interaction in our theory. Regarded as a function of the parameter us, the effective interaction can be either of a strong or weak type:

$$\Gamma(s)=\begin{cases}-\frac{u}{1-us},\ us\ll 1,\\ -\frac{\pi^2}{4}\frac{u}{\varepsilon}+0\left(\frac{1}{us}\right),\ us\gg 1.\end{cases} \quad (28)$$

The region $us \sim 1$ (intermediate coupling) lies outside the limits of validity of our approximations and, therefore, can be studied only numerically.

It is quite clear that the problem under study is equivalent to the Kondo problem and is characterized by a transition from the weak-coupling conditions defined by $u \ll 1$ to strong-coupling conditions when the effective interaction constant is of the order of $u/\varepsilon \gg 1$. Following the analogy with the behavior of the effective interaction in the Kondo problem, we may assume that the transition between the two conditions is continuous.[17] In contrast to the results of Ref. 10, such a behavior has nothing in common with the scaling theory of critical phenomena. In that sense, the theory of Edwards is complementary to the description of localization due to Anderson, where, as shown in Ref. 2, the behavior near the mobility edge is governed by the scaling theory of critical phenomena with $n = 0$. Furthermore, there is a similarity between the two theories which manifests itself in the universal role of the space of $d = 4$ dimensions.

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