Energy level fluctuations in finite systems near the metal-insulator transition

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(Submitted 22 February 1990)
Zh. Eksp. Teor. Fiz. **98**, 634–645 (August 1990)

The statistics of the one-electron energy levels in a finite, closed, disordered system near the Anderson transition is considered. The fluctuations $\langle[\delta N(E)]^2\rangle$ of the number of levels $N(E)$ are studied within an energy band of width E, as the realization of the random potential is varied. Within the framework of the self-consistent localization theory an interpolation scheme is constructed which describes the fluctuations of the number of energy levels over the entire range of existence of the system, from the weakly-ordered metal to the Anderson insulator. At the metal-insulator transition, the quantity $\langle[\delta N(E)]^2\rangle$ is proportional to the average number of levels in the interval $\langle N(E)\rangle$, similar to the case of a deep insulating state. The proportionality coefficient is of the order of 1/2. The results agree well with the qualitative arguments based on the scaling localization theory. The general description of the fluctuations in systems ranging from good metals to insulators agrees satisfactorily with the numerical model calculations.

1. INTRODUCTION

In recent years, great attention has been devoted to the study of statistical fluctuations of physical quantities in finite disordered systems (mesoscopy).[1] One of the important problems considered in the present work is the fluctuation of the number of energy levels within an energy band E as the realization of the random potential varies.[2,3] This problem is closely related to the statistical mesoscopic conductivity fluctuations, which have recently stimulated much interest.[2]

The fluctuation of the number of energy levels within a band of given width was first considered by Dyson.[4] In the work Al'tshuler and Shklovskiĭ,[2] this problem was solved in the case of a weakly disordered metal. Later on, Al'tshuler *et al.*[3] gave a qualitative analysis of the boundaries of the Anderson metal–insulator transition. This analysis was based on the simplest consideration of the ideas of the elementary scaling localization theory.[5] As a result, the ordering parameters were obtained, but a number of questions remained unanswered such as e.g., the role of the frequency (and possibly momentum) dependence of the diffusion coefficient near the mobility threshold. For an arbitrary disorder, Al'tshuler *et al.*[3] carried out direct numerical modeling which made possible a qualitative study of the statistics of the levels over the entire range from "good" metals to Anderson insulators. However, this analysis was possible only for very small systems, which in fact prevented a complete analysis of the transition to the ultimate insulator case.

In this work we demonstrate that, based on the self-consistent localization theory,[6,7] it is possible to develop a complete microscopic treatment of this problem and to build a consistent picture of the fluctuations in the entire region of metallic behavior of the system, up to the Anderson insulator (with a rather large localization radius). It is thus possible to analyze the role of the frequency dependence of the diffusion coefficient and to refine to a certain extent the results of Al'tshuler *et al.*[3] near the metal-insulator transition. These results are also in qualitative agreement with Al'tshuler's numerical analysis.[3]

2. THE MICROSCOPIC MODEL

The analysis of the level fluctuations can be performed by using the standard diagram technique for impurities δ. For this purpose, one should find the density-of-states correlator ν_ε for various energies ε_1 and ε_2:

$$K(\varepsilon_1,\varepsilon_2)=\langle\nu_{\varepsilon_1}\nu_{\varepsilon_2}\rangle-\langle\nu_{\varepsilon_1}\rangle\langle\nu_{\varepsilon_2}\rangle$$
$$=\frac{S^2}{\pi^2V^2}\int d\mathbf{r}_1\,d\mathbf{r}_2\{\langle\operatorname{Im}G_{\varepsilon_1}^R(\mathbf{r}_1\mathbf{r}_1)\operatorname{Im}G_{\varepsilon_2}^R(\mathbf{r}_2\mathbf{r}_2)\rangle$$
$$-\langle\operatorname{Im}G_{\varepsilon_1}^R(\mathbf{r}_1\mathbf{r}_1)\rangle\langle\operatorname{Im}G_{\varepsilon_2}^R(\mathbf{r}_2\mathbf{r}_2)\rangle\},\qquad(1)$$

where $G_\varepsilon^R(\mathbf{r}\mathbf{r}')$ is the retarded Green function of the electron with energy ε for the given realization of the impurity potential, S is the degree of degeneracy of the electronic states, V is the volume of the system and the angular brackets represent averaging over the impurity configurations.

The Green's correlator in equation (1) can easily be expressed in terms of the two-particle Green's function. The most significant part of it is determined by the diagrams shown in Fig. 1, which illustrates two equivalent ways of drawing the diagram that determines the correlator of the local densities of states. The shaded blocks and the wavy lines represent the diffusion propagator. A similar contribution is given by the diagram differing in the direction of the electron lines of one of the loops (the Cooperon contribution). In the final, the following expression is obtained for $K(\varepsilon_1,\varepsilon_2)^2$:

$$K(\varepsilon_1,\varepsilon_2)=-\frac{S^2}{\pi^2V^2}\operatorname{Re}\sum_{\mathbf{q}}\frac{1}{(\varepsilon_1-\varepsilon_2+iD_0q^2+i\gamma)^2}.\qquad(2)$$

where D_0 is the Drude diffusion coefficient of the electron, and γ is the broadening of the electronic levels due, e.g., to inelastic scattering processes. For an isolated sample in the shape of a parallellipiped of dimensions L_X, L_Y, L_Z the momentum quantization condition is[2]

$$q_\mu=\pi n_\mu/L_\mu,\quad n_\mu=0,\ \pm1,\ \pm2\ldots\ (\mu=x,\ y,\ z).\qquad(3)$$

In order to evaluate the correlator of interest in the vi-

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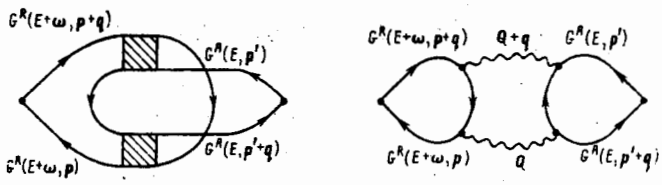

FIG. 1. The diagram that determines the local density-of-states correlator $K(\varepsilon_1,\varepsilon_2)$.

cinity of the mobility threshold, in the spirit of the self-consistent localization theory[6,7] one can replace the diffusion coefficient D_0 in Eq. (2) by the generalized diffusion coefficient $D_\varepsilon(\omega)$ determined from the following self-consistency equation

$$\frac{D_0}{D_\varepsilon(\omega)}=1+\frac{1}{\pi\nu(\varepsilon)}\sum_{|q|<k_0}\frac{1}{-i\omega+D_\varepsilon(\omega)q^2}, \tag{4}$$

where k_0 is the cutoff momentum described below.

The average number of levels in the band with width E centered at point ε on the energy scale can be expressed in terms of the density of states as follows:

$$N(E)=V\int_{\varepsilon-E/2}^{\varepsilon+E/2}d\varepsilon'\nu(\varepsilon'). \tag{5}$$

Thus, the mean square fluctuation of the number of levels is:

$$\langle[\delta N(E)]^2\rangle=V^2\int_{\varepsilon-E/2}^{\varepsilon+E/2}d\varepsilon_1\int_{\varepsilon-E/2}^{\varepsilon+E/2}d\varepsilon_2\,K(\varepsilon_1,\varepsilon_2). \tag{6}$$

In the self-consistency equation (2), one should take into account the inelastic damping γ. It enters this equation in the combination $\omega+i\gamma$. With this contribution accounted for, the generalized diffusion coefficient (u) is:

$$\frac{D_\varepsilon(\omega)}{D_0}=\begin{cases}1-3\lambda x_0=\alpha, & |\omega+i\gamma|\ll\omega_c, \quad \lambda<\lambda_c,\\ {}^3/_2\pi\lambda[-i(\omega+i\gamma)/\varepsilon]^{1/2}, & |\omega+i\gamma|\gg\omega_c,\\ ({}^3/_2\pi\lambda)^3\alpha^{-2}[-i(\omega+i\gamma)/\varepsilon], & |\omega+i\gamma|\ll\omega_c, \quad \lambda>\lambda_c.\end{cases} \tag{7}$$

Here, $\lambda=\gamma(\varepsilon)/\pi\varepsilon$, where $\gamma(\varepsilon)=\pi\rho U^2\nu(\varepsilon)$ is the "Born" collision frequency of electron with impurities (the impurity being pointlike),[8] ρ is the mixture concentration, U is its potential, and $x_0=k_0k_F$ is the cutoff parameter, where k_0 and k_F are the momentum cutoff and the Fermi momentum, respectively.

The diffusion approximation is valid for momenta of order l^{-1}, where l is the mean free path. In the metallic region, where $l^{-1}\gg k_F$ for the momentum cutoff one chooses $k_0=l^{-1}=\pi\lambda k_F$. In the insulator region, $l^{-1}\gg k_F$, and for the momentum cutoff one chooses k_0 on the order of k_F. The transition to the insulating state occurs for $\alpha=0$, i.e., for $\lambda=\lambda_c=(3\pi)^{-1/2}$. Therefore, from the condition of continuity of the momentum cutoff at the transition point, one obtains

$$x_0=\frac{k_0}{k_F}=\begin{cases}\pi\lambda, & \lambda<\lambda_c,\\ (\pi/3)^{1/2}, & \lambda>\lambda_c.\end{cases} \tag{8}$$

The characteristic frequency ω_c is chosen from the condition of continuity of Eqs. (7) for the absolute value of the diffusion coefficient. With these matching conditions we have

$$\omega_c=\varepsilon|\alpha|^3({}^3/_2\pi\lambda)^{-3}. \tag{9}$$

The density-of-states correlator $K(\varepsilon_1,\varepsilon_2)$ is symmetric with respect to the variables ε_1 and ε_2. Therefore, it is sufficient to consider only the case $\varepsilon_1\geqslant\varepsilon_2$. Then, instead of (2), one obtains:

$$K(\varepsilon_1,\varepsilon_2)=-\frac{S^2}{\pi^2V^2}\operatorname{Re}\sum_q[\varepsilon_1-\varepsilon_2+iD_{\varepsilon_2}(\varepsilon_1-\varepsilon_2)q^2+i\gamma]^{-2}$$

$$=K_{\varepsilon_2}(\varepsilon_1,\varepsilon_2). \tag{10}$$

After substituting the new variables $\omega=\varepsilon_1-\varepsilon_2$ and $w=\varepsilon+E/2-\varepsilon_2$ (Fig. 2) in Eq. (6), one obtains

$$\langle[\delta N(E)]^2\rangle=2V^2\int_0^E dw\int_0^w d\omega\,K_{\varepsilon+E/2-w}(\omega). \tag{11}$$

Since we are considering a closed sample, the sum over the momentum in equation (10) contains a term with $q=0$ (see Ref. 3). This term should be distinguished since for $E,\gamma\ll D_\varepsilon/L^2$ the contribution of this term dominates. Here L is the dimension of the sample (in the following we will consider a cubic sample with side L). The contribution of this term in the fluctuation of the number of levels does not depend on the behavior of the diffusion coefficient and is equal to[2]

$$\langle[\delta N(E)]^2\rangle_0=(S^2/\pi^2)\ln(1+E^2/\gamma^2). \tag{12}$$

If the inelastic damping γ is much smaller than the average distance between the levels in the band $\delta=E/\langle N(E)\rangle\sim 1/\nu(\varepsilon)L^3\sim\varepsilon/(Lk_F)^3$, then the meaning of the width of the band is determined by δ. In this case

$$\langle[\delta N(E)]^2\rangle_0=2(S^2/\pi^2)\ln\langle N(E)\rangle. \tag{13}$$

Here $\langle N(E)\rangle$ is the average number of levels in the band. A description of the fluctuations of the number of levels (13) was given by Dyson.[4] In the following, the contribution $\langle[\delta N(E)]^2\rangle_0$ will be called Dyson contribution.

The remaining terms in the sum over momentum can be included by replacing the summation by integration. The upper limit must be limited by the momentum cutoff k_0, and the lower limit will be on the order L^{-1} However, if the contribution of the terms with $q\neq 0$ ($\langle[\delta N(E)]^2\rangle_q$) is much larger than that of the Dyson term, then the difference between zero and the lower limit of integration over q can be neglected. Then

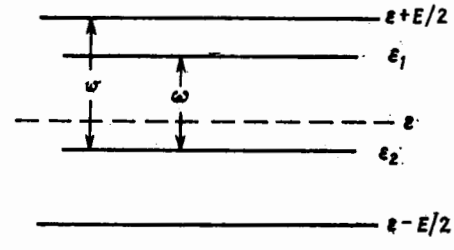

FIG. 2. The new integration variables.

$$\langle[\delta N(E)]^2\rangle_q = -\frac{S^2}{\pi^4} V \int_0^{k_0} q^2\, dq \int_0^{E} \times dw \int_0^{w} d\omega [\omega + i\gamma + iD_{\varepsilon+E/2-w}(\omega) q^2]^{-2}. \quad (14)$$

In the energy range where $\langle[\delta N(E)]^2\rangle_0 \gg \langle[\delta N(E)]^2\rangle_q$ the fluctuations will exhibit Dyson behavior.

3. MAIN RESULTS AND DISCUSSION

Let us consider a band centered at the Fermi level ($\varepsilon = \varepsilon_F$). Let us define $\Delta = |\varepsilon - \varepsilon_F|$, the energy distance between the Fermi level and the mobility threshold ε_c of the conductivity band and consider the vicinity of the metal-insulator transition, e.g., the case when $\Delta \ll \varepsilon$.

We are interested in bands with a large number of levels [$\langle N(E)\rangle \gg 1$] i.e. bands whose width is much larger than the average distance between the levels in the band:

$$E \gg \delta = E/\langle N(E)\rangle \sim \varepsilon/(Lk_F)^3.$$

If the band width satisfies $E \ll \Delta$, then the frequency dependence of the diffusion coefficient for various pairs of levels in the band varies only slightly, and the dependence on W in the diffusion coefficient in Eq. (14) can be neglected.

For $E \gg \Delta$ the frequency dependence of the diffusion coefficient in Eq. (14) depends critically on w and the energy dependence must be taken into account. In this case, the magnitude of the fluctuations is not much different from their magnitude at the transition point. For this reason, we consider directly the transition point. Using Eqs. (7) and (14), we obtain:

$$\langle[\delta N(E)]^2\rangle_q = \begin{cases} \dfrac{S^2}{4\pi^3} V k_F^3 \dfrac{E^2}{\varepsilon\gamma}, & E \ll \gamma, \\ \dfrac{S^2}{8\pi^2} V k_F^3 \dfrac{E}{\varepsilon}, & \gamma \ll E \ll (\gamma\varepsilon^2)^{1/3}, \\ S^2 \dfrac{\sqrt{2}+1}{8\pi^3} V k_F^3 \dfrac{E}{\varepsilon}, & (\gamma\varepsilon^2)^{1/3} \ll L^3. \end{cases} \quad (15)$$

By comparing with the term $\langle[\delta N(E)]^2\rangle_0$, we see that the region of Dyson behavior is absent. Finally, for the transition point we obtain :

$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle} = \begin{cases} \dfrac{S}{\pi}\dfrac{E}{\gamma}, & E \ll \gamma, \\ S/2, & \gamma \ll E \ll (\gamma\varepsilon^2)^{1/3}, \\ \dfrac{\sqrt{2}+1}{2\pi} S \approx 0.38S, & (\gamma\varepsilon^2)^{1/3} \ll E. \end{cases} \quad (16)$$

The fluctuation behavior for $\gamma \ll E \ll (\gamma\varepsilon^2)^{1/3}$ is determined by the Götze frequency dependence[9] for the generalized diffusion coefficient [$D_\varepsilon(\omega) \propto \omega^{1/3}$]. This result is correct within the framework of the self-consistent localization theory.[9]

For $E \gg (\gamma\varepsilon^2)^{1/3}$ the magnitude of the fluctuations is determined by the total contribution of pairs of levels with metallic [$D\hat{e}(\omega) = \text{const}$] and insulating [$D_\varepsilon(\omega) \propto -i\omega$] behavior of the diffusion coefficient, i.e., by the frequencies smaller than the characteristic frequency. This result is to a great extent determined by the method chosen to assure the continuity of expression (7) for the diffusion coefficient from various frequency regions, and by the choice of the characteristic frequency (9). Therefore, particular care should be exercised in the numerical evaluation of expression $\langle[\delta N(E)]^2\rangle/\langle N(E)\rangle \approx 0.38S$ for $E \gg (\gamma\varepsilon^2)^{1/3}$.

In the case $\gamma \ll \delta$ the smearing of the band width is determined by the average distance between the levels in the band:

$$\delta = E/\langle N(E)\rangle \sim \varepsilon/(Lk_F)^3.$$

Thus, for the fluctuations in the transition point, we obtain

$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle} \approx \begin{cases} S/2, & E \ll \varepsilon/Lk_F, \\ 0.38S, & E \gg \varepsilon/Lk_F. \end{cases} \quad (17)$$

Let us consider now the vicinity of the transition point. We introduce the correlation length $\xi \sim (\varepsilon/\Delta) k_F^{-1}$, which in the insulator region has the meaning of localization radius. In the metallic region ξ determines the scale on which Ohm's law for the conductivity is valid[10] and hence the validity of the Dyson description of energy level repulsion.[4]

The characteristic frequency ω_c for the bands with $E \ll \Delta$ changes little for different pairs of levels, and is determined by the average dispersion of the levels in a region of the sample with dimension $\sim\xi$ [$\omega_c \sim \Delta^3/\varepsilon^2 \sim 1/\nu(\varepsilon)\xi^3$]. Therefore, for $\gamma \gg \Delta^3/\varepsilon^2$ any further increase of ξ influences neither the behavior of the diffusion coefficient [$D_\varepsilon(\omega) \propto (-i\omega)^{1/3}$ depends little on ξ] nor that of the fluctuations. The fluctuations have the same behavior as at the transition point. For $\gamma \ll \delta \sim \varepsilon/(Lk_F)^3$, the average distance between the levels δ determines the spreading of the band, i.e., it plays the role of γ, and the fluctuations [for $\Delta \ll \varepsilon/Lk_F$, i.e., for $\xi \gg L$] behave in the same way as at the transition point.

Substituting the expressions for the diffusion coefficient from Eq. (7) into Eq. (14), and comparing with the Dyson term, we find the fluctuation behavior in the vicinity of the transition point.

A. Fluctuation behavior for $\gamma \ll \varepsilon/(Lk_F)^2 \sim \delta$

In this case, the role of γ is played by the distances between the levels in the band. This case corresponds to the low-temperature and small-sample limit. In fact, at low temperatures, the main contribution to damping is due to electron–electron scattering processes. Therefore, we have $\gamma \sim T^2/\varepsilon$, and in order to fulfill the condition $\gamma \ll \varepsilon/(Lk_F)^2$ even at temperatures of the order of 10^{-2} K, samples of dimensions smaller than 10^{-4} cm are needed. However, consideration of this case is necessary for the description of level fluctuations in the limit $T = 0$ and for comparison with the numerical modeling results given in Reference 3.

1. Fluctuations in the metallic region, $L \gg \xi$ (Fig. 3, curve 2). For $E \ll \Delta(\xi)/(Lk_F)^2 \sim D(\xi)/L^2 \equiv E_c$ the Dyson behavior of fluctuations is valid: $\langle[\delta N(E)]^2\rangle = 2(S/\pi)^2 \ln\langle N(E)\rangle$. Here, $E_c = D(\xi)/L^2$ is the energy necessary for the electron to diffuse through the entire sample, and $D(\xi) = \Delta(\xi)(D_0/\varepsilon)$ is the metallic behavior of the diffusion coefficient for small frequencies ($\omega < \omega_c$).

For $E_c \sim \Delta/(Lk_F)^2 \ll E \ll \Delta^3/\varepsilon^2 \sim 1/\nu(\varepsilon)\xi^3$ the fluctuations increase and become equal to

$$\langle[\delta N(E)]^2\rangle = S^2\frac{(3\pi)^{1/4}}{12\pi^3}Vk_F^3\left(\frac{E}{\Delta}\right)^{1/2} = S^2\frac{2^{1/2}}{6\pi^3}\left(\frac{E}{E_c}\right)^{1/2},$$
$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle} = \frac{S}{(3\pi)^{1/4}}\frac{E^{1/2}\varepsilon}{\Delta^{3/2}}. \quad (18)$$

The fluctuations increase[2,3] because for $E \gg E_c$ in the time $\sim E^{-1}$, the electron manages to traverse only a small portion of the sample of the order of $\sim [D(\xi)/E]^{1/2}$. Separate cubes of this size have practically independent systems of levels and their fluctuations add together.

For $1/\nu(\varepsilon)\xi^3 \sim \Delta^3/\varepsilon^2 \ll E \ll \Delta$ we have:

$$\langle[\delta N]^2\rangle = S^2\frac{2^{1/2}}{4\pi^3}Vk_F^3\frac{E}{\varepsilon}, \quad \frac{\langle[\delta N]^2\rangle}{\langle N\rangle} = S\frac{2^{1/2}}{\pi} \approx 0.45S. \quad (19)$$

The change in magnitude of the fluctuations is due to the fact that if for $E \ll \Delta^3/\varepsilon^2 \sim \omega_c$ the diffusion coefficient has metallic behavior [$D(\omega) = \text{const}$] over the entire band, then for $E \gg \Delta^3/\varepsilon^2 \sim \omega_c$ the frequency region where the diffusion coefficient has metallic behavior is bounded above by the characteristic frequency ω_c. This frequency region determines the result (19). Numerically, expression (19) depends to a great extent on how the solutions for the diffusion coefficient from various frequency regions (7) are joined and on the choice of the characteristic frequency (9). However, equations (18) and (19) coincide completely with those obtained in Ref. 3 based on the scaling theory of localization. This confirms the successful choice of the method for joining the solutions for the diffusion coefficient from various frequency regions in the metallic case.

For $E \gg \Delta$, we are practically at the transition point (17), so we have

$$\frac{\langle[\delta N]^2\rangle}{\langle N\rangle} = \frac{\sqrt{2}+1}{2\pi}S \approx 0.38S.$$

This result, as already mentioned, also results from the region of frequencies lower than the characteristic frequency. However, unlike the previous case (19), here the characteristic frequencies ω_c differ strongly for different positions of the lower level, i.e., for different ω, and the main contribution to fluctuations comes from both the region with metallic behavior [$D_\varepsilon(\omega) = \text{const}$] and the region with insulator behavior [$D_\varepsilon(\omega) \propto -i\omega$].

2. Fluctuations in the metallic region, $L \sim \xi$ (Fig. 3, curve 3). With increasing disorder, the correlation length ξ in the metallic region increases, and $\Delta(\xi)$ decreases. For $L \sim \xi$, we obtain $\Delta \sim \varepsilon/Lk_F$ and $\Delta/(Lk_F)^2 \sim \Delta^3/\varepsilon^2 \sim \varepsilon/(Lk_T)^3$. Thus, the region of validity of the Dyson behavior of the form (18) vanishes. We obtain

$$\frac{\langle[\delta N]^2\rangle}{\langle N\rangle}\gamma^2 \approx \begin{cases} 0.45S, & E \ll \varepsilon/Lk_F, \\ 0.38S, & E \gg \varepsilon/Lk_F. \end{cases} \quad (20)$$

3. Fluctuations in the metallic region, $L \gg \xi$ (Fig. 3, curve 6). For $E \ll \Delta^3/\varepsilon^2 \sim 1/\nu(\varepsilon)\xi^3$, the self-consistent localization theory does not provide good values for the magnitude of fluctuations (Appendix 1). This is due to the need to consider the discrete character of the levels in the insulating region. In fact, the average energy distance between the levels in a sample of dimensions of the order of ξ is $\sim 1/\nu(\varepsilon)\xi^3 \sim \Delta^3/\varepsilon^2$. Therefore, the levels in a band with width much smaller than Δ^3/ε^2 will be spread out over distances larger than ξ. Therefore, the levels overlap weakly and are independent, and the fluctuations will be the same as in "deep" insulators:

$$\langle[\delta N]^2\rangle/\langle N\rangle = S. \quad (21)$$

For $\Delta^3/\varepsilon^2 \ll E \ll \Delta$ we have:

$$\langle[\delta N]^2\rangle = \frac{S^2}{4\pi^3}Vk_F^3\frac{E}{\varepsilon}, \quad \frac{\langle[\delta N]^2\rangle}{\langle N\rangle} = \frac{S}{\pi} \approx 0.32S. \quad (22)$$

This fluctuation magnitude is determined by the range of frequencies smaller than the characteristic frequency ω_c, with a diffusion coefficient exhibiting insulating behavior. As in the case of Eq. (19), the result expressed by Eq. (22) depends qualitatively on the method used to join the solutions for the diffusion coefficient from different frequency regions, and on the choice of ω_c.

For $E \gg \Delta$, we are practically at the transition point i.e.,

$$\frac{\langle[\delta N]^2\rangle}{\langle N\rangle} = \frac{\sqrt{2}+1}{2\pi}S \approx 0.38S.$$

4. Fluctuations in the insulating region, $L \sim \xi$ (Fig. 3, curve 5). As we approach the transition from the insulating side, ξ increases, Δ decreases, and for $L \sim \xi$ we obtain $\Delta^3/\varepsilon^2 \sim \varepsilon/(Lk_F)^3 \sim \delta$ and $\Delta \sim \varepsilon/Lk_F$, i.e., the region in which Eq. (21) holds vanishes:

$$\frac{\langle[\delta N]^2\rangle}{\langle N\rangle} \approx \begin{cases} 0.32S, & E \ll \varepsilon/Lk_F, \\ 0.38S, & E \gg \varepsilon/Lk_F. \end{cases} \quad (23)$$

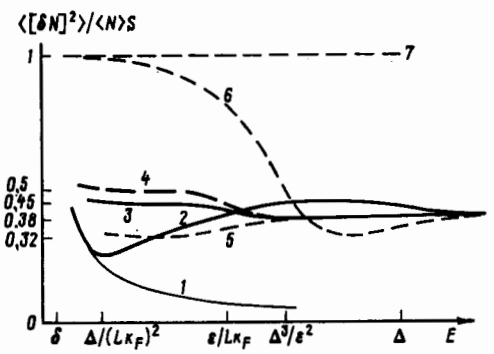

FIG. 3. Qualitative behavior of the fluctuations of the number of energy levels for $\gamma \ll \delta \sim \varepsilon/(Lk_F)^3$: 1—Dyson behavior of the fluctuations; 2—in the metallic region ($\xi \ll L$); 3—in the metallic region ($\xi \sim L$); 4—at the metal-to-insulator transition point; 5—in the insulating region ($\xi \sim L$); 6—in the insulating region ($\xi \ll L$); 7—in the "deep" insulator.

B. The behavior of fluctuations for $\gamma \gg \varepsilon/(Lk_F)^3 \sim \delta$

The behavior of fluctuations for $\gamma \gg \delta$ is shown in Fig. 4, and the corresponding analytic expressions are given in Appendix 2.

1. Fluctuations in the metallic region, $\gamma \ll \Delta/(Lk_F)^2$. For $E \ll \Delta/(Lk_F)^2 \sim E_c$, the fluctuations exhibit Dyson behavior which is $\gamma \gg \delta$ has the form

$$\langle[\delta N(E)]^2\rangle = \frac{S^2}{\pi^2}\ln\left(1+\frac{E^2}{\gamma^2}\right).$$

For the rest, the fluctuation behavior coincides with that described in case *A* for the metallic region with $L \gg \xi$.

2. Fluctuations in the metallic region $\Delta/(Lk_F)^2 \ll \gamma \ll \Delta^3/\varepsilon^2$. The region where Dyson behavior holds vanish.

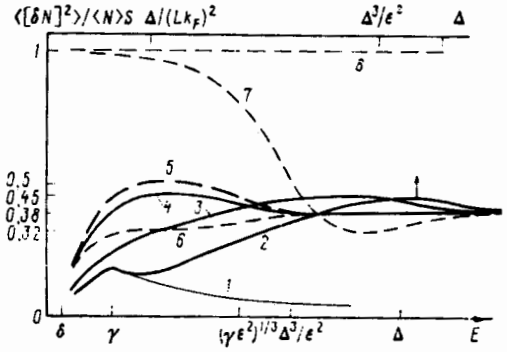

FIG. 4. Qualitative behavior of the fluctuations of the number of energy levels for $\gamma \gg \delta \sim \varepsilon/(Lk_F)^3$: 1—Dyson behavior of the fluctuations; 2—in the metallic region, $\gamma \ll \Delta/(Lk_F)^2$; 3—in the metallic region, $\Delta/(Lk_F)^2 \ll \gamma \ll \Delta^3/\varepsilon^2$; 4—in the metallic region, $\gamma \sim \Delta^3/\varepsilon^2$; 5—in the point of metal-to-insulator transition; 6—in the insulating region, $\gamma \sim \Delta^3/\varepsilon^2$; 7—in the insulating region, $\gamma \ll \Delta^3/\varepsilon^2$; 8— in the "deep" insulator.

For $E \ll \gamma$, the fluctuations are larger than Dyson fluctuations and have the form

$$\langle [\delta N]^2 \rangle = S^2 \frac{(3\pi)^{1/2}}{2^{1/2}\pi^3} V k_F^3 \frac{E^2}{\Delta^{1/2}\gamma^{1/2}}.$$

For $E \gg \gamma$, the fluctuations have the same form as in case 1 for $E \gg \Delta/(Lk_F)^2$.

3. Fluctuations in the metallic and insulating region, $\gamma \sim \Delta^3/\varepsilon^2$. For $E \ll \gamma$, the fluctuations have the form

$$\frac{\langle [\delta N]^2 \rangle}{\langle N \rangle} = \frac{S}{\pi} \frac{E}{\gamma}.$$

For $E \gg \gamma$ they are as in case *A* for $L \sim \xi$:

$$\frac{\langle [\delta N]^2 \rangle}{\langle N \rangle} = \text{const},$$

and the constant values are the same as in case *A* for the metallic and the insulating regions with $L \sim \xi$ respectively. However, the small variation of the constant quantities which in case *A* occurs for $E \sim \varepsilon/Lk_F$, happens here for $E \sim (\gamma\varepsilon^2)^{1/3}$.

4. Fluctuations in the insulating region, $\gamma \ll \Delta^3/\varepsilon^2$. The fluctuations coincide with those described in case *A* for the insulating region with $L \gg \xi$.

4. CONCLUSIONS

Use of the self-consistent localization theory allows one to obtain a picture of the number of energy level fluctuations in systems from weakly disordered metals to Anderson insulators (with rather large localization radius). Qualitatively, the picture of the fluctuation behavior agrees with the development based on the scaling localization theory.[3] Taking into account the frequency dependence of the diffusion coefficient leads to an insignificant variation of the ratio $\langle [\delta N]^2 \rangle / \langle N \rangle$ near the transition point, as compared to that in Ref. 3. The qualitative conclusion that, in the vicinity of the transition, this ratio depends weakly on the width of the band is still valid. From this work, it follows that at the transition point we have $\langle [\delta N]^2 \rangle / \langle N \rangle \approx 1/2$. However, one should note the fact that the numerical value of the ratio $\langle [\delta N]^2 \rangle / \langle N \rangle$ near the transition point may be slightly different from that obtained in this work. The fact is that these quantities are to a great extent determined by the way the solutions for $D_\varepsilon(\omega)$ from different frequency regions are joined and by the choice of the characteristic frequency ω_c. One should therefore expect that, e.g., the magnitude of the fluctuations, and therefore also $\langle [\delta N]^2 \rangle / \langle N \rangle$, in the vicinity of the transition on the insulating side will be somewhat larger than on the metal side, because of the decrease of level repulsion when the disorder increases. In our calculations, this is not the case (see Figs. 3 and 4). At the present time, it is not clear whether this nonmonotonicity has any physical significance. It is also possible that, in the insulating region, it is necessary to take into account the momentum dependence of the diffusion coefficient. This appears to be indicated by the inapplicability of approximations based on the self-consistent localization theory, in the insulating region where $E \ll 1/\nu(\varepsilon)\xi^2$. The picture we have obtained for the behavior of the fluctuations also agrees well with the numerical results obtained in Ref. 3. For example, near the transition point, the numerical value of $\langle [\delta N]^2 \rangle / \langle N \rangle$ is indeed close to 1/2. One should also note that the numerical model developed in Ref. 3 included only the region with $E < Lk_F$, i.e., did not allow one to consider in the insulating region the transition "knee" from the behavior of $\langle [\delta N]^2 \rangle / \langle N \rangle = 1$ for $E \ll 1/\nu(\varepsilon)\xi^3$ to $\langle [\delta N]^2 \rangle / \langle N \rangle \approx 1/2$ for $E \gg 1/\nu(\varepsilon)\xi^3 > \varepsilon/Lk_F$ (see Fig. 3).

The idea on which this work is based originated during useful discussions with B. L. Al'tshuler, to whom the author expresses his warm gratitude.

APPENDIX I: THE FLUCTUATIONS OF THE NUMBER OF LEVELS IN THE INSULATING REGION [$\gamma, E \ll \Delta^3/\varepsilon^2 \sim 1/\nu(\varepsilon)\xi^3$]

In the insulating region, for E, $\gamma \ll \Delta^3/\varepsilon^2$, the diffusion coefficient has the form

$$D(\omega) = -i(\omega + i\gamma)\xi^2, \tag{A1.1}$$

where $\xi = (3\pi)^{1/2}(\varepsilon/\Delta)k_F^{-1}$ is the localization radius. Then, using Eq. (14), one obtains the following expression for the fluctuations:

$$\langle [\delta N(E)]^2 \rangle = \frac{S^2}{8\pi^3} \frac{V}{\xi^3} \ln\left(1 + \frac{E^2}{\gamma^2}\right) = \frac{1}{8\pi} \frac{V}{\xi^3} \langle [\delta N(E)]^2 \rangle_0. \tag{A1.2}$$

The result expressed by (A1.2) can be interpreted as the sum of independent Dyson fluctuations in V/ξ^3 cubes with dimensions of the order of ξ. The fluctuations in the cubes are independent since for $E \ll 1/\nu(\varepsilon)\xi^3$ the levels are spread out over distances greater than ξ, and overlap each other only weakly. However, inside the same cube of dimension of order ξ, the average number of levels in the band with $E \ll 1/\nu(\varepsilon)\xi^2$ will be much smaller than unity, and our description cannot be applied. In fact, the bands with these widths require consideration of the discrete character of the levels. The levels in the band overlap each other weakly and the fluctuations will be the same as those in a deep insulator, i.e., $\langle [\Delta N]^2 \rangle / \langle N \rangle = 1$.

APPENDIX 2: THE FLUCTUATIONS OF THE NUMBER OF ENERGY LEVELS [$\gamma \gg \delta \sim \varepsilon/(Lk_F)^3$]

1. The fluctuations in the metallic region, $\gamma \ll /(Lk_F)^2$ (Fig. 4, curve 2). For $E \ll \Delta/(Lk_F)^2$ the fluctuations exhibit Dyson behavior

$$\langle[\delta N(E)]^2\rangle=\frac{S^2}{\pi^2}\ln\left(1+\frac{E^2}{\gamma^2}\right).$$

In the other cases, one obtains

$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle}=\begin{cases}\dfrac{S}{(3\pi)^{1/4}}\dfrac{eE^{1/2}}{\Delta^{3/4}}, & \dfrac{\Delta}{(Lk_F)^2}\ll E\ll\dfrac{\Delta^3}{e^2},\\ \dfrac{2^{1/2}}{\pi}S\approx 0.45S, & \dfrac{\Delta^3}{e^2}\ll E\ll\Delta,\\ \dfrac{2^{1/2}+1}{2\pi}S\approx 0.38S, & \Delta\ll E.\end{cases}\tag{A2.1}$$

2. The fluctuations in the metallic region, $\Delta/(Lk_F)^2\ll\gamma\ll\Delta^3/\varepsilon^2$ (Fig. 4, curve 3). The region of validity of the Dyson behavior of the fluctuations vanishes, and the fluctuations have the form:

$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle}=\begin{cases}S\dfrac{(3\pi)^{1/4}}{2^{1/2}\pi}\dfrac{eE}{\Delta^{3/4}\gamma^{1/4}}, & E\ll\gamma,\\ \dfrac{S}{(3\pi)^{1/4}}\dfrac{eE^{1/2}}{\Delta^{3/4}}, & \gamma\ll E\ll\dfrac{\Delta^3}{e^2},\\ \dfrac{2^{1/2}}{\pi}S\approx 0.45S, & \dfrac{\Delta^3}{e^2}\ll E\ll\Delta,\\ \dfrac{2^{1/2}+1}{2\pi}S\approx 0,38S, & \Delta\ll E.\end{cases}\tag{A2.2}$$

3. The fluctuations in the metallic region, $\gamma\sim\Delta^3/\varepsilon^2$ (Fig. 4, curve 4):

$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle}=\begin{cases}\dfrac{S}{\pi}\dfrac{E}{\gamma}, & E\ll\gamma,\\ \dfrac{2^{1/2}}{\pi}S\approx 0,45S, & \gamma\ll E\ll(\gamma\varepsilon^2)^{1/3},\\ \dfrac{2^{1/2}+1}{2\pi}S\approx 0,38S, & (\gamma\varepsilon^2)^{1/3}\ll E.\end{cases}\tag{A2.3}$$

4. The fluctuations at the transition point (Fig. 4, curve 5):

$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle}=\begin{cases}\dfrac{S}{\pi}\dfrac{E}{\gamma}, & E\ll\gamma,\\ S/2, & \gamma\ll E\ll(\gamma\varepsilon^2)^{1/3},\\ \dfrac{2^{1/2}+1}{2\pi}S\approx 0,38S, & (\gamma\varepsilon^2)^{1/3}\ll E.\end{cases}\tag{A2.4}$$

5. The fluctuations in the insulating region, $\gamma\sim\Delta^3/\varepsilon^2$ (Fig. 4, curve 6):

$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle}=\begin{cases}\dfrac{S}{\pi}\dfrac{E}{\gamma}, & E\ll\gamma,\\ S/\pi\approx 0,32S, & \gamma\ll E\ll(\gamma\varepsilon^2)^{1/3},\\ \dfrac{2^{1/2}+1}{\pi}S\approx 0.38S, & (\gamma\varepsilon^2)^{1/3}\ll E.\end{cases}\tag{A2.5}$$

6. The fluctuations in the insulating region, $\gamma\ll\Delta^3/\varepsilon^2$ (Fig. 4, curve 7):

$$\frac{\langle[\delta N(E)]^2\rangle}{\langle N(E)\rangle}=\begin{cases}S, & E\ll\Delta^3/\varepsilon^2,\\ S/\pi\approx 0.32S, & \Delta^3/\varepsilon^2\ll E\ll\Delta,\\ \dfrac{2^{1/2}+1}{\pi}S\approx 0.38S, & \Delta\ll E.\end{cases}\tag{A2.6}$$

[1)] Expression (19) yields a result, in the vicinity of the transition point, which is twice as large as that obtained in Reference 3. This is due to the fact that in the work reported in Reference 3, the conductance in the transition point was chosen with the spin degeneration $S=2$ taken into account, and the expression for the fluctuations of the number of levels was written in the absence of degeneration.

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Translated by I. Nelson