SUPERCONDUCTIVITY AND LOCALIZATION

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Some misprints discovered in the original edition by World Scientific are corrected here by the author without additional comments.

Preface

In this book we present a review of theoretical and experimental works on the problem of mutual interplay of Anderson localization and superconductivity in strongly disordered systems. Superconductivity exists close to the metal–insulator transition in some disordered systems such as amorphous metals, superconducting compounds disordered by fast neutron irradiation etc. High-temperature superconductors are especially interesting from this point of view. Mainly bulk systems are considered, superconductor-insulator transition in purely two-dimensional disordered systems is not discussed in any detail, though nowadays it is a very active field of both experimental and theoretical research.

We start with brief discussion of modern aspects of localization theory including the basic concept of scaling, self–consistent theory and interaction effects. In fact we try to explain these concepts in some detail as this material is not so well known to the majority of people working in the field of superconductivity. After that we analyze disorder effects on Cooper pairing and superconducting transition temperature as well as Ginzburg—Landau equations for superconductors which are close to the Anderson transition. A necessary generalization of usual theory of "dirty" superconductors is formulated which allows to analyze anomalies of the main superconducting properties close to disorder-induced metal–insulator transition. Under very rigid conditions superconductivity may persist even in the localized phase (Anderson insulator).

Strong disordering leads to considerable reduction of superconducting transition temperature T_c and to important anomalies in the behavior of the upper critical field H_{c2}. Fluctuation effects are also discussed. In the vicinity of Anderson transition inhomogeneous superconductivity appears due to statistical fluctuations of the local density of states.

Although our interest is mainly in theoretical aspects of the problem, we briefly discuss a number of experiments demonstrating superconductivity close to the Anderson transition both in traditional and high–T_c superconductors. In traditional systems superconductivity is in most cases destroyed before metal–insulator transition. In case of high–T_c superconductors a number of anomalies shows that superconductivity is apparently conserved in the localized phase before it is suppressed by strong enough disorder.

Essentially these lecture notes are based on previous reviews [Sadovskii M.V. (1993); Sadovskii M.V. (1995); Sadovskii M.V. (1997)], and on a number of lectures and review talks by the author in different places and institutions. The material presented below in many respects reflects personal research interests and results of the author and therefore does not contain the thorough review of all works in the field.

The author is grateful to all his collaborators during the research work on superconductivity and localization, especially to Prof. L. N. Bulaevskii whose insights were so important at early stages of this work. In recent years I have benefited very much from the joint work with Dr. E. Z. Kuchinskii. I am also grateful to Dr. A.I.Posazhennikova for her help during the most recent work. Useful discussions of experimental situation with Prof. B. N. Goshchitskii and Prof. A. V. Mirmelstein are very much appreciated.

My research in this field in recent years was supported in part by the the grants of the Russian Foundation of Basic Research $N^o 93-02-2066$, $96-02-16065, 99-02-16285$ as well as by the grant the Russian Ministry of Science Program on High-Temperature Superconductivity $N^o 96-051$.

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Contents

Chapter 1

INTRODUCTION

The concept of electron localization [Anderson P.W. (1958)] is basic for the understanding of electron properties of disordered systems [Mott N.F. (1974); Mott N.F, Davis E.A. (1979)]. Besides these classic references a number of review papers exists, extensively discussing more modern aspects of this problem [Sadovskii M.V. (1981); Anderson Localization (1982); Lee P.A., Ramakrishnan T.V. (1985); Sadovskii M.V. (1986); Vollhardt D., Wölfle P. (1990); Suslov I.M. (1998)]. According to this concept introduction of sufficiently strong disorder into a metallic system leads to spatial localization of electronic states near the Fermi level and thus to a transition to dielectric state (Anderson transition). After this transition dc conductivity (at zero temperature, $T = 0$) vanishes, despite the finite value of electronic density of states at the Fermi level (at least in one-electron approximation).

At the same time it is well-known that even the smallest attraction of electrons close to the Fermi level leads to formation of Cooper pairs and the system becomes superconducting at sufficiently low temperatures [Bardeen J., Cooper L.N., Schrieffer J. (1957); Gennes de P.G. (1966)]. It is also known that the introduction of disorder which does not break the time–reversal invariance (normal, nonmagnetic impurities etc.) does not seriously influence superconducting transition temperature T_c and superconductivity in general (Anderson theorem) [Abrikosov A.A., Gorkov L.P. (1958); Abrikosov A.A., Gorkov L.P. (1959); Gorkov L.P. (1959); Anderson P.W. (1959)].

Thus a problem appears of the mutual interplay of these two possible electronic transitions in a disordered system which leads to quite different (even opposite) ground states (insulator or superconductor). This problem is very important both from theoretical and experimental points of view. Actually superconducting properties of many compounds depend strongly on structural disorder. In this respect we can mention amorphous systems (metallic glasses) and superconductors disordered by different forms of irradiation by high-energy particles (fast neutrons, electrons, heavy-ions etc.). It appears that in many of these systems superconductivity is realized when the system in normal state is quite close to the metal—insulator transition induced by disorder. In this case many anomalies of superconducting properties appear which cannot be satisfactorily explained within the standard theory of "dirty" superconductors [Gennes de P.G. (1966); Abrikosov A.A., Gorkov L.P. (1958); Abrikosov A.A., Gorkov L.P. (1959); Gorkov L.P. (1959); Anderson P.W. (1959)]. These include rather strong dependence of T_c on disorder in apparent contradiction with Anderson theorem, as well as some unusual behavior of the upper critical field H_{c2}.

The discovery of high-temperature superconductivity in metallic oxides [Bednorz J.G., Müller K.A. (1986); Bednorz J.G.,Müller K.A. (1988)] has lead to the entirely new opportunities in the studies of strong disorder effects in superconductors. Very soon it had been established that high—T_c superconductors are quite sensitive to structural disordering which leads to rather fast destruction of superconductivity and metal—insulator transition. However, the high values of initial T_c, as well as a small size of Cooper pairs and quasi-two dimensional nature of electronic states in these systems are very appropriate for the studies of the mutual interplay of localization and superconductivity [Sadovskii M.V. (1989)]. It may be stated with some confidence that in these systems superconductivity can be observed even in the region of localization (Anderson insulator).

This book is mainly concerned with theoretical aspects of localization and superconductivity close to Anderson transition. However, we shall pay some attention to experiments demonstrating the importance of localization phenomena for the correct analysis of superconductivity in strongly disordered systems. Special emphasis will be on the experiments with high–T_c superconductors. We shall limit ourselves with discussing only three-dimensional and quasi-two-dimensional (in case of HTSC) systems, practically excluding any discussion of purely two-dimensional systems, which are quite special both in respect to localization and superconductivity. In this case we refer a reader to a number of reviews [Ramakrishnan T.V. (1989);

Belevtsev B.I. (1990); Liu Y., Goldman A.M. (1994)] which are specifically concerned with superconductivity in purely two-dimensional case. Only sometime during our discussion we shall briefly refer to some aspects of this problem, which is becoming a field of very active research in recent years.

We must stress that the material presented in this book is concerned mainly with the personal interests of its author and we apologize to those people whose important contributions to this field will not be discussed in detail or even will be missed. At the same time, we shall try to present here some detailed discussion of rather wide number of problems, which hopefully may be useful for beginners. Of course, it is assumed that the reader has some basic knowledge of solid–state theory at the graduate level, as well as some standard theoretical approaches, such as Green's functions or modern theory of critical phenomena. In this Introduction we shall try to formulate the basic problems under study at some elementary level.

The usual theory of "dirty superconductors" [Gennes de P.G. (1966); Abrikosov A.A., Gorkov L.P. (1958); Abrikosov A.A., Gorkov L.P. (1959); Gorkov L.P. (1959); Anderson P.W. (1959)] is a cornerstone of our understanding of superconducting properties of disordered metals. It is based on the following main statements:

(1) As impurity concentration (disorder) grows a transition takes place from the "pure" limit, when the electron mean-free path l is much larger than the superconducting coherence length ξ_0: $l \gg \xi_0 = \hbar v_F/\pi\Delta_0$ to a "dirty" superconductor with $\xi_0 \gg l \gg \hbar/p_F$ (Here v_F, p_F—are Fermi velocity and momentum, Δ_0– is the zero – temperature energy gap). Transition temperature T_c changes only slightly, mainly due to small changes of Debye frequency ω_D and of pairing constant λ_p, which are due to relatively small changes in the electronic density of states under disordering. Transition from the free electron motion to diffusive one does not change T_c at all (Anderson's theorem). These statements practically ignore any disorder dependence of microscopic pairing interaction, which is assumed to be some constant as in the simplest BCS model.

(2) Superconducting coherence length ξ (at $T = 0$) determining the spatial scale of superconducting order-parameter (the size of a Cooper pair) diminishes with l so that $\xi \approx \sqrt{\xi_0 l}$ in the limit of $\hbar/p_F \ll l \ll \xi_0$.

(3) As ξ diminishes the critical region near T_c where thermodynamic fluctuations are important widens and is of the order of $\tau_G T_c$, where $\tau_G \sim [T_c N(E_F)\xi^3]^{-2}$ is the so called Ginzburg's parameter ($N(E_F)$ is electronic density of states at the Fermi level E_F). For "pure" superconductors $\tau_G \sim (T_c/E_F)^4 \ll 1$ and as l drops τ_G grows as ξ drops. However, in the limit of $l \gg \hbar/p_F$ the value of τ_G still remains very small.

Theory of "dirty" superconductors is the basis of our understanding of superconducting properties of many disordered alloys. However, the main results of this theory must be modified for the mean-free path values l of the order of inverse Fermi momentum $\hbar/p_F$ (i.e. of the order of interatomic distance). In three–dimensional systems the growth of disorder leads to destruction of diffusive motion of electrons and transition from extended to localized states at critical disorder determined by $l_c \approx \hbar/p_F$, i.e. to transition to Anderson insulator. This metal–insulator transition is reflected in a continuous drop to zero of the static metallic conductivity (at $T = 0$) as $l \to l_c$. For $l \gg l_c$ conductivity is determined by the usual Drude formula $\sigma_0 \sim l$, while for $l \to l_c$ it drops as $\sigma \sim (l - l_c)^\nu$, where ν is some critical exponent. Transition from diffusion to localization is realized at the conductivity scale of the order of the so-called "minimal metallic conductivity" $\sigma_c \approx (e^2 p_F/\pi^3\hbar^2) \approx (2-5)10^2 Ohm^{-1}cm^{-1}$. The usual theory of "dirty" superconductors does not consider localization effects and is valid for conductivities in the interval $(E_F/T_c)\sigma_c \gg \sigma \gg \sigma_c$.

Major changes in this picture occur for superconductors close to localization transition (i.e. for $\sigma \leq \sigma_c$) are:

(1) Assuming independence of the density of states at the Fermi level $N(E_F)$ and of the pairing constant λ_p from the value of the mean-free path l (disorder) we can show that T_c drops as disorder grows due to respective growth of Coulomb pseudopotential μ^*. This effect is due to the growth of retardation effects of Coulomb interaction within the Cooper pair as diffusion coefficient drops close to Anderson transition [Anderson P.W., Muttalib K.A., Ramakrishnan T.V (1983)]. T_c degradation starts even for $\sigma \gg \sigma_c$ and becomes fast for $\sigma < \sigma_c$ [Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)]. The changes in the density of states with disorder due to interaction effects (as well as the growth of spin fluctuations) also lead to the drop of T_c [Kuchinskii

E.Z., Sadovskii M.V., Erkabaev M.A. (1997)]. However, under very rigid conditions T_c can remain finite up to and after the Anderson transition.

(2) Close to the Anderson transition the usual expression for superconducting coherence length for a "dirty" limit $\xi = \sqrt{\xi_0 l}$ should be replaced by $\xi \approx (\xi_0 l^2)^{1/3}$ and it remains finite even below Anderson transition (i.e. in insulating phase) [Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985); Kapitulnik A., Kotliar G. (1985); Kotliar G., Kapitulnik A. (1986); Lee P.A., Ma M. (1985)], signalling the possibility of superconductivity in Anderson insulator. Obviously these results are valid only in case of finite T_c close to Anderson transition, which is possible only if very rigid conditions are satisfied.

(3) The growth of disorder as system moves to the transition leads to the growth of different kinds of fluctuations of superconducting order-parameter, both thermodynamic and also due to fluctuations of electronic characteristics of the system (e.g. local density of states) [Bulaevskii L.N., Sadovskii M.V. (1986); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989)].

In the following we shall present an extensive discussion of these and some of the other problems concerning the interplay of superconductivity and localization. However, first of all we shall briefly describe the main principles of modern theory of electron localization and physics of metal—insulator transition in disordered systems, which will be necessary for clear understanding of the main problem under discussion. After that we shall give rather detailed presentation of theoretical problem of superconductivity close to the Anderson transition. Finally, we shall describe the present experimental situation. We shall briefly describe some of the experiments with traditional superconductors, but our main emphasis will be on high–T_c oxides. We shall concentrate on the experiments with high–temperature–superconductors disordered by fast neutron irradiation which is one of the best methods to introduce disorder in a controlled fashion without any chemical (composition) changes. In this sense our review of experiments is also far from being complete, but we hope that it is full enough to claim that high–T_c systems are especially good for testing some of the main theoretical ideas, expressed throughout this book. Also we believe that better understanding of their properties under disordering may be important for the development of the general theory of high–temperature superconductivity.

Chapter 2

ANDERSON LOCALIZATION AND METAL-INSULATOR TRANSITION IN DISORDERED SYSTEMS

2.1 Basic Concepts of Localization

In recent years a number of review papers had appeared dealing with basic aspects of Anderson localization [Sadovskii M.V. (1981); Anderson Localization (1982); Lee P.A., Ramakrishnan T.V. (1985); Sadovskii M.V. (1986); Thouless D.J. (1974); Efros A.L. (1978); Vollhardt D., Wölfle P. (1990); Suslov I.M. (1998)]. Here we shall remind the main points of this theory and introduce the accepted terminology.

In 1958 Anderson [Anderson P.W. (1958)] has shown for the first time that the wave function of a quantum particle in a random potential can qualitatively change its nature if randomness becomes large enough. Usually, when disorder is small, the particle (e.g. electron) is scattered randomly and the wave function changes at the scale of the order of mean free path l. However, the wave function remains extended plane—wave—like (Bloch wave—like) through the system. In case of strong enough disorder, the wave function becomes localized, so that its amplitude (envelope) drops exponentially with distance from the center of localization $\mathbf{r}_0$:

$$|\psi(\mathbf{r})| \sim exp(|\mathbf{r} - \mathbf{r}_0|/R_{loc}) \tag{2.1}$$

where R_{loc} is localization length.

This situation is shown qualitatively in Fig. 2.1. The physical meaning of Anderson localization is relatively simple: coherent tunneling of electrons is possible only between energy levels with the same energy (e.g. between equivalent sites in crystalline lattice). However, in case of strong randomness the states with the same energy are too far apart in space for tunneling to

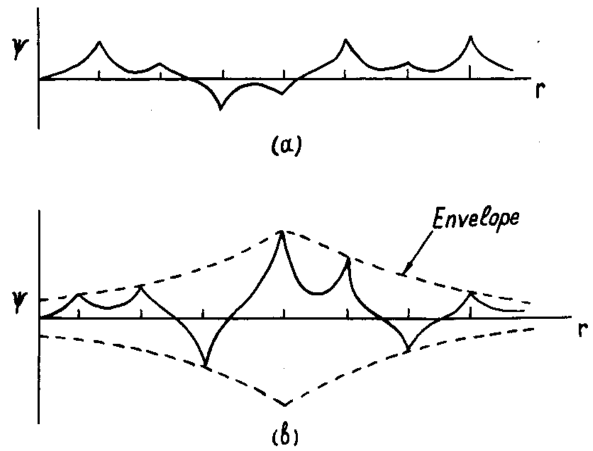

Fig. 2.1 Electron wave—function in a disordered system: (a) — extended state. (b) — localized state.

be effective.

At small disorder dc conductivity of a metal at $T = 0$ is determined by Drude expression:

$$\sigma_0 = \frac{ne^2}{m}\tau = \frac{ne^2}{p_F}l \tag{2.2}$$

where τ — is the mean free time, n — is electron density and e — its charge. Usual kinetic theory can be applied if

$$\frac{p_F l}{\hbar} \gg 1 \text{ or } \frac{E_F \tau}{\hbar} \gg 1 \tag{2.3}$$

which is a condition of weak scattering (disorder). From Eq. (2.2) and Eq. (2.3), taking into account $n = p_F^3/(3\pi^2\hbar^3)$, we can estimate the lower limit of conductivity for which Drude approximation is still valid:

$$\sigma_0 = \frac{e^2 p_F}{3\pi^2\hbar^2}(\frac{p_F l}{\hbar}) \gg \frac{e^2 p_F}{3\pi^2\hbar^2} \tag{2.4}$$

The conductivity value:

$$\sigma_c \approx \frac{e^2 p_F}{3\pi^2\hbar^2} \tag{2.5}$$

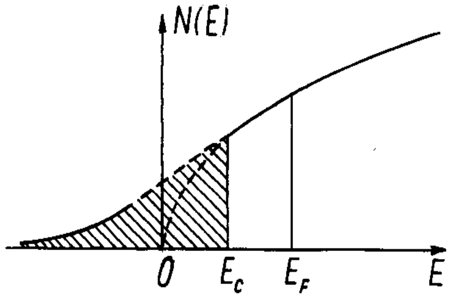

Fig. 2.2 Electron density of states near the band edge in a disordered system. Dashed is the region of localized states, E_c—is the mobility edge.

is usually called the "minimal metallic conductivity" [Mott N.F. (1974); Mott N.F, Davis E.A. (1979)]. As disorder grows the mean free path diminishes and becomes of the order of lattice spacing a, so that we reach $p_F l/\hbar \sim 1$, and the usual kinetic theory based upon Boltzmann equation becomes inapplicable. This was first noted by Ioffe and Regel [Ioffe A.F., Regel A.R. (1960)], who observed that at such disorder the qualitative form of wave function must change, transforming from extended to localized accompanied by metal—insulator transition. From Eq.(2.5) it is clear that this transition takes place at the conductivity scale of the order of $\sigma_c \sim (2-5)10^2 Ohm^{-1}cm^{-1}$ for typical $\hbar/p_F \sim a \sim (2-3)10^{-8}cm$.

Qualitative form of energy spectrum near the band—edge of a disordered system is shown in Fig. 2.2. When the Fermi level lies in the high—energy region electronic states close to it are slightly distorted plane waves. As Fermi energy moves towards the band—edge (or with the growth of disorder) the critical energy E_c (mobility edge) separating extended and localized states crosses the Fermi level. If E_F belongs to the region of localized states the system becomes insulating, conductivity is possible only for $T > 0$ or by exciting the carriers by alternating electric field. The appearance of these hopping mechanisms of conductivity signals Anderson transition [Mott N.F. (1974); Mott N.F, Davis E.A. (1979)].

One of the main problems is the qualitative behavior of conductivity when the Fermi level E_F crosses the mobility edge E_c (at $T = 0$). While Mott assumed the discontinuous drop of conductivity from σ_c to zero [Mott N.F. (1974); Mott N.F, Davis E.A. (1979)] modern approach [Sadovskii M.V. (1981); Anderson Localization (1982); Lee P.A., Ramakrishnan T.V.

(1985); Vollhardt D., Wölfle P. (1990)] based mainly on the scaling theory of localization [Abrahams E. et al. (1979)] demonstrates *continuous* transition. Experiments at low temperatures clearly confirm this type of behavior [Lee P.A., Ramakrishnan T.V. (1985)], and σ_c acts as a characteristic conductivity scale close to transition. Static conductivity of a metal at $T = 0$ close to Anderson transition within this approach is written as:

$$\sigma = A\frac{e^2}{\hbar\xi_{loc}} \approx \sigma_c \left|\frac{E_F - E_c}{E_c}\right|^{(d-2)\nu} \tag{2.6}$$

where A — is a numerical constant, d — is space dimension, and $\sigma_c \approx Ae^2/(\hbar a^{d-2})$. Here we introduced the correlation length of scaling theory diverging at the transition:

$$\xi_{loc} \approx \frac{\hbar}{p_F}\left|\frac{E_F - E_c}{E_c}\right|^{-\nu} \tag{2.7}$$

Critical exponent ν determines this divergence. In one—electron approximation and in the absence of magnetic scattering $\nu \sim 1$ [Lee P.A., Ramakrishnan T.V. (1985); Sadovskii M.V. (1986); Vollhardt D., Wölfle P. (1990); Wölfle P., Vollhardt D. (1982)]. In the region of localized states (i.e. for $E_F < E_c$) ξ_{loc} coincides with localization length of electrons R_{loc}. In metallic region ξ_{loc} determines the effective size of a sample at which "Ohmic" behavior appears, i.e. conductivity becomes independent of a sample size [Lee P.A., Ramakrishnan T.V. (1985); Imry Y. (1980)]. "Minimal metallic conductivity" σ_c determines, as we noted, the conductivity scale close to a transition.

In the vicinity of Anderson transition conductivity acquires an important frequency dependence [Wegner F.J. (1976); Shapiro B., Abrahams E. (1981a)]. For $E_F = E_c$ i.e. at the transition we have:

$$\sigma(\omega) \approx \sigma_c(i\omega\tau)^{\frac{d-2}{d}} \tag{2.8}$$

which is valid also close to the transition (from either side) for frequencies $\omega \gg \omega_c \sim [N(E_F)\xi_{loc}^d]^{-1}$. For $d = 3$ this is sometimes referred to as Götze's [Götze W. (1981)] $\omega^{1/3}$ – law, although this particular derivation was later acknowledged to be wrong [Belitz D., Gold A., Gotze W. (1981)].

The spatial dimension $d = 2$ is the so called "lower critical dimensionality" [Sadovskii M.V. (1981); Anderson Localization (1982); Lee P.A., Ramakrishnan T.V. (1985); Sadovskii M.V. (1986)]. For $d = 2$ all electronic states are localized for infinitesimal disorder [Abrahams E. et al. (1979)], and there is no Anderson transition. However recently a remarkable series of experiments on high-mobility Si-MOS structures, initiated by Pudalov [Kravchenko S.V. et al. (1995); Kravchenko S.V. et al. (1996); Popovic D., Fowler A.B., Washburn S. (1997)], demonstrated the existence of metal-insulator transition in two-dimensions, thus casting doubts on this general conclusion. Common belief at the moment is that these systems are somehow special, either due to unusual role of spin-orbit interaction [Pudalov V.M (1997)], or due to general effects or electron-electron interaction [Chakravarty S., Yin L., Abrahams E. (1998); Si Qimiao, Varma C.M. (1998)]. We shall not deal with this (quite interesting!) problem further, though some ideas relating it to superconductivity were already expressed [Phillips P., et al. (1998)].

Quasi—two—dimensional systems are especially interesting, mainly because most of high—T_c oxides demonstrate strongly anisotropic electronic properties. Here we shall make the simplest estimates for such systems on the line of Ioffe—Regel approach. Consider a system made of highly—conducting "planes" where the current carriers are "nearly—free", while the interplane tunneling is possible only due to some small transfer integral $w \ll E_F$ (E_F — is the Fermi energy of two—dimensional gas within the plane). Conductivity within the plane is determined for small disorder as:

$$\sigma_{\|} = e^2 D_{\|} N(E_F) \tag{2.9}$$

where $D_{\|} = v_F^2 \tau/2$, $N(E_F) = m/(\pi a_{\perp} \hbar^2)$, $a_{\perp}$ – is interplane spacing, which is noticeably larger than interatomic distance within the plane. Interplane conductivity is given by:

$$\sigma_{\perp} = e^2 D_{\perp} N(E_F) \tag{2.10}$$

where $D_{\perp} = (wa_{\perp})^2 \tau/\hbar^2$. The appropriate mean free paths are $l_{\|} = v_F \tau$, $l_{\perp} = wa_{\perp}\tau/\hbar$. Ioffe-Regel criterion for a quasi—two—dimensional system can be written as:

$$l_{\perp} = wa_{\perp}\tau/\hbar \sim a_{\perp} \tag{2.11}$$

which is equivalent to $w\tau/\hbar \sim 1$ — the condition of breaking of coherent tunneling between the planes. Elementary estimate shows that this corresponds to:

$$\sqrt{\sigma_{\|}\sigma_{\perp}} \sim \frac{e^2}{\sqrt{2}\pi\hbar a} \sim \sigma_c \tag{2.12}$$

where a — is interatomic distance *within the planes*. In isotropic case this reduces to Eq.(2.5). For strongly anisotropic system when $\sigma_{\|} \gg \sigma_{\perp}$ it is clear that Eq.(2.12) can be satisfied even for $\sigma_{\|} \gg \sigma_c$, because of small values of $\sigma_{\perp}$. Formally, for $\sigma_{\perp} \to 0$, critical value of $\sigma_{\|}$ diverges, that reflects on this elementary level the tendency towards complete localization for purely two—dimensional case.

The important property of energy spectrum in the region of localized states is its local discretness. As we noted above, the physical meaning of localization itself leads to a picture of close energy levels being far apart in space, despite the continuous nature of average density of states. Due to exponential decay of the localized wave functions it leads to the absence of tunneling [Anderson P.W. (1958)]. The energy spacing between levels of electrons localized within the sphere of the radius of the order of $R_{loc}(E)$ can be estimated [Mott N.F. (1974); Mott N.F, Davis E.A. (1979)] as:

$$\delta_{E_F} \approx [N(E_F)R^d_{loc}]^{-1} \tag{2.13}$$

As the metallic system moves toward Anderson transition, i.e. as the mean free path drops to interatomic distance and conductivity becomes less than $\sim 10^3 Ohm^{-1}cm^{-1}$ there appear the well known anomalies like the negative temperature coefficient of resistivity [Ioffe A.F., Regel A.R. (1960); Mooij J.H. (1973)]. These anomalies are apparently closely connected with localization phenomena [Lee P.A., Ramakrishnan T.V. (1985)].

Up to now we discussed Anderson transition, neglecting electron interactions. Its importance for the problem of metal–insulator transition in disordered systems has been known for a long time [Mott N.F. (1974)]. In recent years there was a serious progress in the general approach to a theory of "dirty" metals, based on the analysis of interference of impurity scattering and Coulomb interactions [Altshuler B.L., Aronov A.G. (1979); Altshuler B.L. et al. (1982); Altshuler B.L., Aronov A.G. (1985)]. Later we shall review its implications for the general picture of Anderson transition. Apparently the continuous nature of metal–insulator transition is not changed though interaction leads to a number of specific effects, e.g. in the behavior of the density of states at the Fermi level, as well as to the growth of magnetic (spin) fluctuations. However, recent experiments on purely two-dimensional systems [Kravchenko S.V. et al. (1995); Kravchenko S.V. et al. (1996); Popovic D., Fowler A.B., Washburn S. (1997)] have initiated some new ideas on the role of electron interactions [Si Qimiao, Varma C.M. (1998); Chakravarty S., Yin L., Abrahams E. (1998)].

Now we shall only briefly describe the concept of "soft" Coulomb gap appearing below the transition in the region of localized states [Efros A.L., Shklovskii B.L. (1975); Efros A.L. (1976); Shklovskii B.I., Efros A.L. (1979); Efros A.L., Shklovskii B.I. (1985)]. Coulomb interaction between localized electrons can be estimated as $e^2/\epsilon R_{loc}$, and it is obviously important if this energy is comparable with the local level spacing $[N(E_F)R^3_{loc}]^{-1}$ (for three—dimensional system). As a result a Coulomb pseudogap appears at the Fermi level with the width:

$$\Delta_c \approx (e^3/\epsilon^{3/2})[N(E_F)]^{1/2} \tag{2.14}$$

where ϵ is the dielectric constant. We shall see later that close to the Anderson transition $\epsilon \approx 4\pi e^2 N(E_F)R^2_{loc}$ and accordingly:

$$\Delta_c \approx [N(E_F)R^3_{loc}]^{-1} \approx \delta_{E_F} \tag{2.15}$$

so that in this case Coulomb effects are comparable with the effects of discretness of energy spectrum in localized phase. At the moment there is no complete theory connecting the localization region with metallic phase within the general approaches of interaction theory.

2.2 Elementary Scaling Theory of Localization

The behavior of electronic system close to the Anderson transition can be described by a scaling theory similar to that used in the theory of critical phenomena [Kadanoff L.P. et al. (1967); Wilson K.G., Kogut J. (1974); Patashinskii A.Z., Pokrovskii V.L. (1982)]. The main physical idea of this approach is based upon a series of scale transformations from smaller to larger "cells" in coordinate space with appropriate description of a system by transformed parameters of initial Hamiltonian. These transformations are usually called renormalization group. In the theory of critical phenomena this approach is usually motivated by the growth of correlation length of order—parameter fluctuations near the critical point [Kadanoff L.P. et al. (1967)]. This is analogous to the growth of localization length on the approach of mobility edge from Anderson insulator.

The accepted scaling approach to localization problem was proposed by Abrahams, Anderson, Licciardello and Ramakrishnan [Abrahams E. et al. (1979)]. In this theory localization is described in terms of *conductance* g as a function of a sample size L. For a small disorder ($p_F l/\hbar \gg 1$) the system is in a metallic state and conductivity σ is determined by Eq. (2.2) and is independent of a sample size if this size is much larger than the mean free path, $L \gg l$. Conductance is determined in this case just by Ohm law and for a d—dimensional hypercube we have:

$$g(L) = \sigma L^{d-2} \tag{2.16}$$

If electronic states near the Fermi level are localized, conductivity of an infinite system at $T = 0$ is zero and matrix elements for transitions between different electronic states drop exponentially on distances of the order of R_{loc}. Then it can be expected that for $L \gg R_{loc}$, the effective conductance becomes exponentially small:

$$g(L) \sim exp(-L/R_{loc}) \tag{2.17}$$

Elementary scaling theory of localization assumes that in general case the conductance of a hypercube of a size L satisfies the simplest differential equation of a renormalization group:

$$\frac{dlng(L)}{dlnL} = \beta_d(g(L)) \tag{2.18}$$

Most important assumption here is the dependence for $\beta_d(g)$ only on one variable g (one—parameter scaling). Then the qualitative behavior of β_g can be analyzed in a simplest possible way interpolating between limiting forms given by Eq. (2.16) and Eq. (2.17). For metallic phase (large g) from Eq. (2.16) and Eq.(2.18) we get:

$$\lim_{g\to\infty} \beta_d(g) \to d-2 \tag{2.19}$$

For insulator ($g \to 0$) from Eq. (2.18) and Eq. (2.17) it follows that:

$$\lim_{g\to 0} \beta_d(g) \to ln\frac{g}{g_c} \tag{2.20}$$

Assuming the existence of two perturbation expansions over the "charge" g in the limits of weak and strong "couplings" we can write correction to

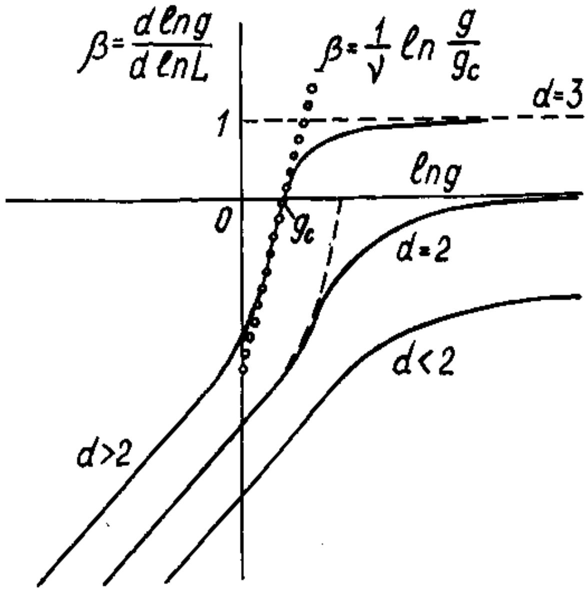

Fig. 2.3 Qualitative form of $\beta_d(g)$ for different d. Dashed line shows the behavior necessary to get discontinuous drop of conductivity at the mobility edge for $d = 2$.

Eq. (2.20) and Eq. (2.19) in the following form:

$$\beta_d(g \to 0) = ln\frac{g}{g_c}(1 + bg + \cdots) \tag{2.21}$$

$$\beta_d(g \to \infty) = d - 2 - \frac{\alpha}{g} + \cdots \qquad \alpha > 0 \tag{2.22}$$

Following these *assumptions* and supposing now monotonous and continuous form of $\beta_d(g)$ it is easy to plot it qualitatively for all g, as shown in Fig. 2.3.

All the previous equations are written for dimensionless conductance, which is measured in natural units of $e^2/\hbar \approx 2.510^{-4} Ohm^{-1} cm^{-1}$. We see that $\beta_d(g)$ definitely has no zeros for $d < 2$. If expansion Eq. (2.22) is valid there is no zero also for $d = 2$. For $d > 2$ β_d — function must have a zero: $\beta_d(g_c) = 0$. It is clear that $g_c \sim 1$ and no form of perturbation theory is valid near that zero. The existence of a zero of $\beta_d(g)$ corresponds to existence of an unstable fixed point of Eq. (2.18). The state of a system is supposedly determined by disorder at microscopic distances of the order of interatomic spacing a, i.e. by $g_0 = g(L = a)$. Using g_0 as an initial value and integrating Eq. (2.18) it is easy to find that for $g_0 > g_c$ conductivity $\sigma_L = g(L)L^{2-d}$ tends for $L \to \infty$ to a constant (metallic) value. For $g < g_c$ in the limit of $L \to \infty$ we get insulating behavior. Using for $g \sim g_c$ an approximation (shown with circles in Fig. 2.3):

$$\beta_d(g) \approx \frac{1}{\nu} ln \frac{g}{g_c} \approx \frac{1}{\nu} \frac{g - g_c}{g_c} \tag{2.23}$$

we obtain from Eq. (2.18) for $g_0 > g_c$ the following behavior of conductivity for $L \to \infty$:

$$\sigma \approx A \frac{e^2}{\hbar} \frac{g_c}{a^{d-2}} \left(ln \frac{g_0}{g_c} \right)^{(d-2)\nu} \approx A \frac{e^2}{\hbar} \frac{g_c}{a^{d-2}} \left(\frac{g_0 - g_c}{g_c} \right)^{(d-2)\nu} \tag{2.24}$$

where $A = const$ and we have explicitly introduced the conductivity scale of the order of σ_c. (Cf. Eq. (2.5)). We see that the existence of a fixed point leads to the existence of mobility edge, and behavior of $\beta_d(g)$ close to its zero determines the critical behavior at the Anderson transition. Under these assumptions conductivity continuously goes to zero for $g_0 \to g_c$, and the value of $\sigma_c \approx e^2/(\hbar a^{d-2})$ is characteristic scale of conductivity at the metal—insulator transition. To get a discontinuous drop of conductivity at the mobility edge $\beta_d(g)$ must be nonmonotonous as shown by dashed line for $d = 2$ in Fig. (2.3). This behavior seems more or less unphysical.

Integrating Eq. (2.18) with $\beta_d(g)$ from Eq. (2.23) with initial $g_0 < g_c$ gives:

$$g(L) \approx g_c exp \left\{ -A \left| ln \frac{g_0}{g_c} \right|^{\nu} \frac{L}{a} \right\} \tag{2.25}$$

From here it is clear (Cf. Eq. (2.7)) that:

$$R_{loc} \sim a \left| \frac{g_0 - g_c}{g_c} \right|^{-\nu} \tag{2.26}$$

and ν is the critical exponent of localization length. For $d = 2$ we have $\beta_d(g) < 0$ in the whole interval of g, so that $\sigma_{L\to\infty} \to 0$ for any initial value of g, so that there is no mobility edge and all states are localized.

For $d > 2$ limiting ourselves by those terms of perturbation expansion in g^{-1} shown in Eq. (2.22) we can solve $\beta_d(g_c) = 0$ to find:

$$g_c = \frac{\alpha}{d-2} \tag{2.27}$$

We can see that for $d \to 2$ the mobility edge goes to infinity which corresponds to complete localization in two—dimensional case. Now we have:

$$\beta_d(g \sim g_c) \approx (d-2)\left(\frac{g_0 - g_c}{g_c}\right) \tag{2.28}$$

and for the critical exponent of localization length we get (Cf. Eq. (2.23)):

$$\nu = \frac{1}{d-2} \tag{2.29}$$

which may be considered as the first term of ε—expansion near $d = 2$ (where $\varepsilon = d - 2$), i.e. near "lower critical dimension" for localization [Abrahams E. et al. (1979); Wegner F.J. (1979a); Shapiro B., Abrahams E. (1981b)]. Note that the expansion Eq. (2.22) can be reproduced in the framework of standard perturbation theory over impurity scattering [Gorkov L.P., Larkin A.I., Khmelnitskii D.E. (1979); Abrahams E., Ramakrishnan T.V. (1980)]. For $d = 3$ this gives $\alpha = \pi^{-3}$ (Cf.Ref.[Lee P.A., Ramakrishnan T.V. (1985)]).

Let us define now correlation length of localization transition as:

$$\xi_{loc} \sim a\left|\frac{g_0 - g_c}{g_c}\right|^{-\nu} \tag{2.30}$$

For $g_0 < g_c$ this length coincides with localization length R_{loc}. It is easy to see that Eq. (2.24) can be written as: [Wegner F.J. (1976)]

$$\sigma \approx A g_c \frac{e^2}{\hbar \xi_{loc}^{d-2}} \tag{2.31}$$

It follows that for $g > g_c$ correlation length ξ_{loc} determines behavior of conductivity close to the mobility edge, when this length becomes much larger than interatomic distance and mean free path.

Let us consider three—dimensional case in more details. Integrating Eq. (2.18) with $\beta_3(g) = 1 - g_c/g$ where $g_c = \alpha$ gives $g(L) = (\hbar/e^2)\sigma_L L = (\hbar/e^2)\sigma + g_c$ so that for a finite sample close to the mobility edge ($\xi_{loc} \gg l$) we obtain:

$$\sigma_L = \sigma + \frac{e^2 g_c}{\hbar L} \tag{2.32}$$

where in correspondence with Eq. (2.31)

$$\sigma \approx Ag_c \frac{e^2}{\hbar \xi_{loc}} \tag{2.33}$$

It follows that for $L \gg \xi_{loc} \gg l$ conductivity $\sigma_L \to \sigma$ while for $l \ll L \ll \xi_{loc}$ conductivity σ_L and the appropriate diffusion coefficient, determined by Einstein relation $\sigma = e^2 DN(E_F)$ are equal to:

$$\sigma_L \approx \frac{e^2 g_c}{\hbar L} \tag{2.34}$$

$$D_L \approx \frac{g_c}{N(E_F)} \frac{1}{\hbar L} \tag{2.35}$$

Thus in this latest case conductivity is not Ohmic, diffusion of electrons is "non—classical" [Anderson P.W., Muttalib K.A., Ramakrishnan T.V (1983); Lee P.A., Ramakrishnan T.V. (1985)]. From this discussion it is clear that the characteristic length ξ_{loc} in metallic region determines the scale on which conductivity becomes independent of sample size. Close to the mobility edge when $\xi_{loc} \to \infty$ only the samples with growing sizes $L \gg \xi_{loc}$ can be considered as macroscopic. These considerations allow to understand the physical meaning of diverging length ξ_{loc} of scaling theory in metallic region [Imry Y. (1980)]. Close to the mobility ξ_{loc} is considered as the only relevant length in the problem (with an exception of a sample size L) and the scaling hypothesis is equivalent to the assumption of:

$$g(L) = f\left(\frac{L}{\xi_{loc}}\right) \tag{2.36}$$

where $f(x)$—is some universal (for a given dimensionality d) function. In metallic region for $L \gg \xi_{loc} \gg l$ it is obvious that $f(x) \sim x^{d-2}$ which reproduces Eq. (2.31).

For finite frequencies ω of an external electric field a new length appears in the system [Shapiro B., Abrahams E. (1981a)]:

$$Ł_\omega = \left[\frac{D(\omega)}{\omega}\right]^{1/2} \tag{2.37}$$

where $D(\omega)$—is the frequency dependent diffusion coefficient. L_ω is a length of electron diffusion during one cycle of an external field. Close to the mobility edge ξ_{loc} is large and for $L_\omega < \xi_{loc}$, L and L_ω become the relevant length scale. In general, for finite ω localization transition is smeared, a sharp transition is realized only for $L^{-1} = L_\omega^{-1} = 0$. Thus for the finite frequency case the scaling hypothesis of Eq. (2.36) can be generalized as [Shapiro B., Abrahams E. (1981a)]:

$$g(L,\omega) = f\left(\frac{L}{\xi_{loc}}, \frac{L_\omega}{\xi_{loc}}\right) \tag{2.38}$$

where g denotes a real part of conductance. In metallic phase for $L \gg \xi_{loc}$ we have $g \sim L^{d-2}$ so that:

$$\begin{aligned}\sigma(\omega) = \frac{e^2}{\hbar} L^{2-d} f\left(\frac{L}{\xi_{loc}}, \frac{L_\omega}{\xi_{loc}}\right) \to \frac{e^2}{\hbar}\xi_{loc}^{2-d} f\left(\infty, \frac{L_\omega}{\xi_{loc}}\right) \\ \equiv \frac{e^2}{\hbar\xi_{loc}^{d-2}} F\left(\frac{\xi_{loc}}{L_\omega}\right)\end{aligned} \tag{2.39}$$

For small frequencies, when $L_\omega \gg \xi_{loc}$, we can write down the universal function $F(x)$ as $F(x) \approx Ag_c + Bx^{d-2}$ which reproduces Eq. (2.31) and the small frequency corrections found earlier in [Gorkov L.P., Larkin A.I., Khmelnitskii D.E. (1979)]. For $L_\omega \ll \xi_{loc}$ i.e. for high frequencies or close to mobility edge the relevant length is L_ω and frequency dependent part of conductivity is dominating. In particular at the mobility edge itself the length ξ_{loc} drops out and must cancel in Eq. (2.38) which leads to:

$$\sigma(\omega, E_F = E_c) \sim L_\omega^{2-d} \sim \left[\frac{\omega}{D(\omega)}\right]^{\frac{d-2}{2}} \tag{2.40}$$

On the other hand, according to Einstein relation we must have $\sigma(\omega) \sim D(\omega)$. Accordingly, from $[\omega/D(\omega)]^{(d-2)/2} \sim D(\omega)$ we get at the mobility edge:

$$\sigma(\omega, E_F = E_c) \sim D(\omega) \sim \omega^{\frac{d-2}{d}} \tag{2.41}$$

For $d = 3$ this leads [Wegner F.J. (1976); Götze W. (1981)] to $\sigma(\omega) \sim D(\omega) \sim \omega^{1/3}$. The crossover between different types of frequency dependence occurs for $L_\omega \sim \xi_{loc}$ which determines characteristic frequency [Shapiro B., Abrahams E. (1981a)]:

$$\omega_c \sim \frac{1}{\hbar\xi_{loc}^d N(E_F)} \tag{2.42}$$

The $\omega^{(d-2)/d}$—behavior is realized for $\omega \gg \omega_c$, while for $\omega \ll \omega_c$ we get small corrections of the order of $\sim \omega^{(d-2)/2}$ to Eq. (2.31).

Finally we must stress that for finite temperatures there appear *inelastic* scattering processes which destroy the phase correlations of wave functions at distances greater than a characteristic length of the order of $L_{\varphi} = \sqrt{D\tau_{\varphi}}$, where D is the diffusion coefficient due to *elastic* scattering processes considered above and τ_{φ} is the "dephasing" time due to inelastic processes [Altshuler B.L. et al. (1982)]. For $T > 0$ this length L_{φ} effectively replaces the sample size L in all expressions of scaling theory when $L \gg L_{\varphi}$, because on distances larger than L_{φ} all information on the nature of wave functions (e.g. whether they are localized or extended) is smeared out. Taking into account the usual low—temperature dependence like $\tau_{\varphi} \sim T^{-p}$ (where p is some integer, depending on the mechanism of inelastic scattering) this can lead to a non—trivial temperature dependence of conductivity, in particular to a possibility of a negative temperature coefficient of resistivity of "dirty" metals [Imry Y. (1980)] which are close to localization transition.* It is important to stress that similar expressions determine the temperature dependence of conductivity also for the localized phase until $L_{\varphi} < R_{loc}$. Only for $L_{\varphi} > R_{loc}$ the localized nature of wave functions starts to signal itself in temperature dependence of conductivity and the transition to exponentially activated hopping behavior takes place, which becomes complete for $T < [N(E_F)R^d_{loc}]^{-1}$.

2.3 Self—Consistent Theory of Localization

2.3.1 *Isotropic Systems*

It is obvious that qualitative scaling picture of Anderson transition described in the previous section requires microscopic justification. At the same time we need a practical method of explicit calculations for any physical characteristic of electronic system close to the mobility edge. Here we shall briefly describe the main principles of so called self—consistent theory of localization which while leaving aside some important points, leads to an

*Recently it was claimed [Golubev D.S., Zaikin A.D. (1998)] that the simple estimate of $\tau_{\varphi} \sim T^{-p}$ becomes invalid at low temperatures and τ_{φ} actually saturates as $T \to 0$ due to the effects of zero-point fluctuations. This claim was refuted in rather detailed discussion of [Aleiner I., Altshuler B.L., Gershenzon M. (1998)]. Probably more important is the experimental observation of τ_{φ} saturation at low temperatures [Mohanty P., Jariwala E.M., Webb R.A. (1997)], the reasons for which are not clear. Discussion still continues, we shall adopt a standard point of view, expressed above, which is definitely valid at least for not so low temperatures.

effective scheme for analysis of the relevant physical characteristics important for us. This approach, first formulated by Götze [Götze W. (1979); Götze W. (1981)] was later further developed by Vollhardt and Wölfle and other authors [Vollhardt D., Wölfle P. (1980); Vollhardt D., Wölfle P. (1982); Wölfle P., Vollhardt D. (1982); Myasnikov A.V., Sadovskii M.V. (1982); Kotov E.A., Sadovskii M.V. (1983); Sadovskii M.V. (1986); Vollhardt D., Wölfle P. (1990)].

Complete information concerning Anderson transition and transport in a disordered system is contained in the two—particle Green's function:

$$\Phi^{RA}_{\mathbf{pp}'}(E\omega\mathbf{q}) = -\frac{1}{2\pi i} < G^R(\mathbf{p}_+\mathbf{p}'_- E+\omega)G^A(\mathbf{p}'_-\mathbf{p}_- E) > \tag{2.43}$$

where $\mathbf{p}_{+-} = \mathbf{p} \pm (1/2)\mathbf{q}$, in most cases below E just coincides with the Fermi energy E_F. Angular brackets denote averaging over disorder. Graphically this Green's function is shown in Fig. 2.4. It is well known that this Green's function is determined by the Bethe—Salpeter equation of diagram technique for randomly spaced scatterers also shown graphically in Fig .2.4 [Edwards S.F. (1958); Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965); Vollhardt D., Wölfle P. (1980)]:

$$\Phi^{RA}_{\mathbf{pp}'}(E\mathbf{q}\omega) = G^R(E+\omega\mathbf{p}_+)G^A(E\mathbf{p}_-) \times \\ \left\{ -\frac{1}{2\pi i}\delta(\mathbf{p}-\mathbf{p}') + \sum_{\mathbf{p}''} U^E_{\mathbf{pp}''}(\mathbf{q}\omega)\Phi^{RA}_{\mathbf{p}''\mathbf{p}'}(E\mathbf{q}\omega) \right\} \tag{2.44}$$

where $G^{R,A}(E\mathbf{p})$ — is the averaged retarded (advanced) one—electron Green's function, while irreducible vertex part $U^E_{\mathbf{pp}'}(\mathbf{q}\omega)$ is determined by the sum of all diagrams which can not be cut over two electron lines (Cf. Fig. 2.4).

In fact, two—particle Green's function Eq. (2.43) contains even some abundant information and for the complete description of Anderson transition it is sufficient to know the two—particle Green's function summed over $\mathbf{pp}'$ [Vollhardt D., Wölfle P. (1980)]:

$$\Phi^{RA}_E(\mathbf{q}\omega) = -\frac{1}{2\pi i}\sum_{\mathbf{pp}'} < G^R(\mathbf{p}_+\mathbf{p}'_+ E+\omega)G^A(\mathbf{p}'_-\mathbf{p}_- E) > \tag{2.45}$$

Using Bethe—Salpeter equation Eq. (2.44) and exact Ward identities we can obtain a closed equation for $\Phi^{RA}_E(\mathbf{q}\omega)$ [Vollhardt D., Wölfle P. (1980);

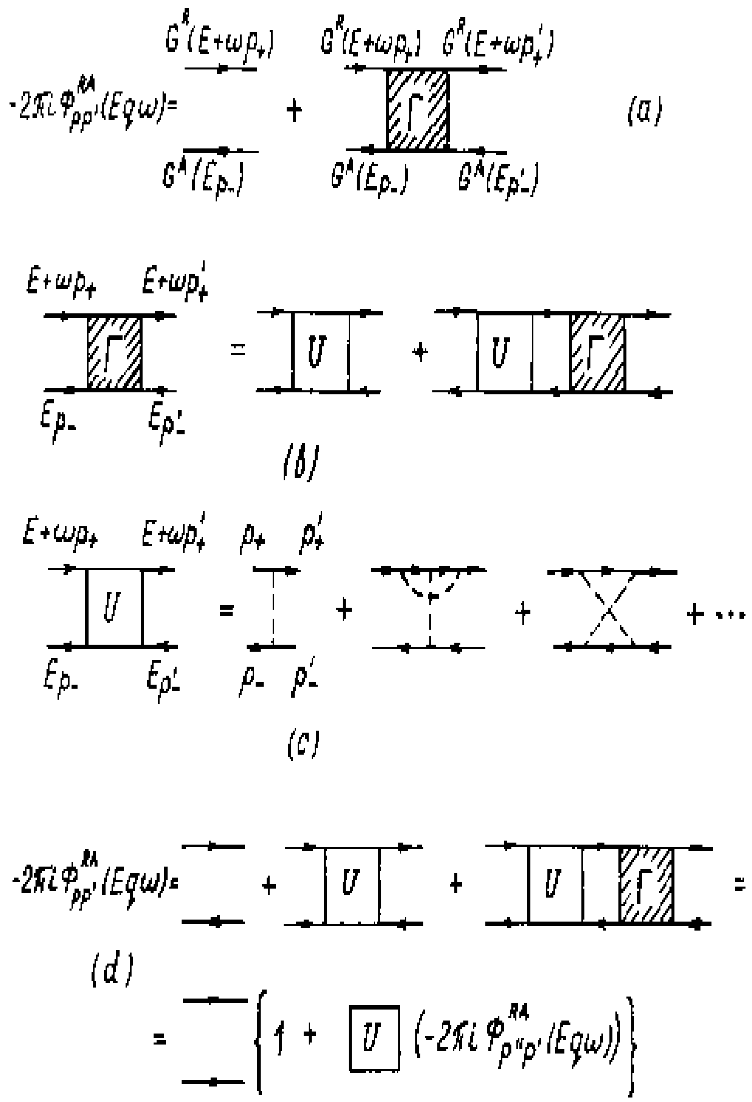

Fig. 2.4 Graphical representation of: (a) — two—electron Green's function $\Phi^{RA}_{\mathbf{pp}'}(E\mathbf{q}\omega)$; (b) — equation for full vertex part $\Gamma^{E}_{\mathbf{pp}'}(\mathbf{q}\omega)$; (c) — typical diagrams for irreducible vertex $U^{E}_{\mathbf{pp}'}(\mathbf{q};\omega)$; (d) — Bethe—Salpeter equation. Dashed line denotes "interaction" $U_0(\mathbf{p}-\mathbf{p}') = \rho|V(\mathbf{p}-\mathbf{p}')|^2$, where ρ — is density of scatterers, $V(\mathbf{p}-\mathbf{p}')$—is Fourier transform of a single scatterer potential.

Wölfle P., Vollhardt D. (1982); Sadovskii M.V. (1986)], and for small ω and $\mathbf{q}$ the solution of this equation has a typical diffusion–pole form:

$$\Phi_E^{RA}(\mathbf{q}\omega) = -N(E)\frac{1}{\omega + iD_E(\mathbf{q}\omega)q^2} \tag{2.46}$$

where $N(E)$—is electron density of states at energy E and the *generalized* diffusion coefficient $D_E(\mathbf{q}\omega)$ is expressed through the so called relaxation kernel $M_E(\mathbf{q}\omega)$:

$$D_E(\mathbf{q}\omega) = i\frac{2E}{dm}\frac{1}{M_E(\mathbf{q}\omega)} = \frac{v_F^2}{d}\frac{i}{M_E(\mathbf{q}\omega)} \tag{2.47}$$

where v_F is Fermi velocity of an electron. The retarded density—density response function at small ω and $\mathbf{q}$ is given by:

$$\chi^R(\mathbf{q}\omega) = \omega\Phi_E^{RA}(\mathbf{q}\omega) + N(E) + O(\omega, q^2) \tag{2.48}$$

or from Eq. (2.46):

$$\chi^R(\mathbf{q}\omega) = N(E)\frac{iD_E(\mathbf{q}\omega)q^2}{\omega + iD_E(\mathbf{q}\omega)q^2} \tag{2.49}$$

For relaxation kernel $M_E(\mathbf{q}\omega)$ (or for generalized diffusion coefficient) a self—consistency equation can be derived, which is actually the main equation of the theory [Vollhardt D., Wölfle P. (1980); Vollhardt D., Wölfle P. (1990); Wölfle P., Vollhardt D. (1982)].

For the one-electron Green's function $G^{R,A}(E\mathbf{p})$ we can use the simplest approximation:

$$G^{R,A}(E\mathbf{p}) = \frac{1}{E - \frac{\mathbf{p}^2}{2m} \pm i\gamma(E)} \tag{2.50}$$

where for point scatterers randomly distributed with spatial density ρ (V is scattering amplitude) we have:

$$\gamma = \frac{1}{2\tau} = \pi\rho V^2 N(E_F) \tag{2.51}$$

– the usual “Born” scattering rate. This result is obtained by summing the diagrams shown in Fig.2.5(a), which do not have any intesecting interaction lines, i.e. diagrams similar to that shown in Fig.2.5(b). Strictly speaking, this approximation can be used only if perturbation theory is applicable, i.e. $E \gg \gamma(E)$ [Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)].

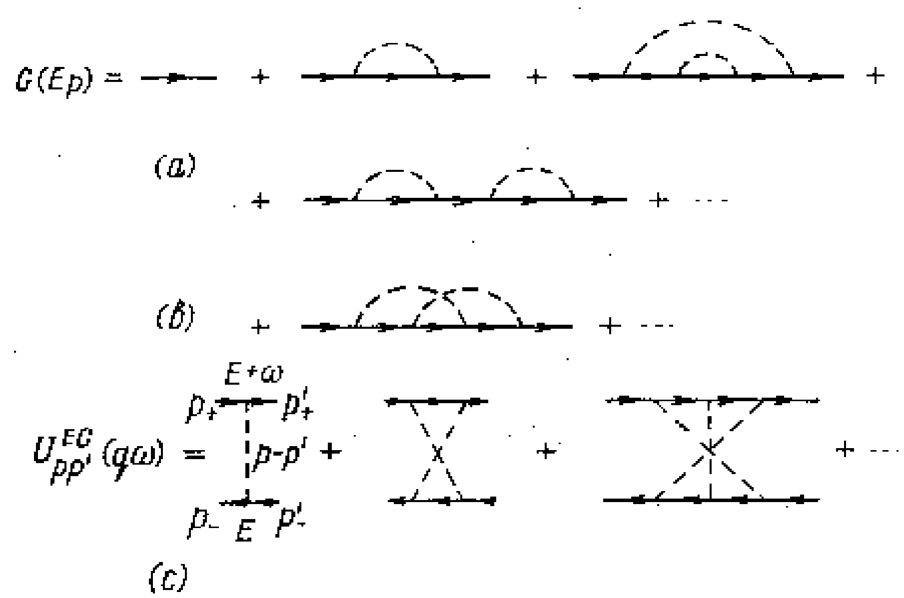

Fig. 2.5 (a) Diagrams determining the one-electron Green's function. (c) "Maximally—crossed" diagrams for irreducible vertex part of Bethe—Salpeter equation ("Cooperon").

However, it is assumed that we can use it up to the metal-insulator transition, which, in fact is justified by the general feeling that no singularities are associated with single-particle Green's function (density of states) at the transition. The next step of the derivation is actually central and it reduces to a major approximation for the irreducible vertex part $U^E_{\mathbf{pp}'}(\mathbf{q}\omega)$ in Bethe—Salpeter equation. The approximation of Vollhardt and Wölfle is based upon the use for $U^E_{\mathbf{pp}'}(\mathbf{q}\omega)$ of the sum of "maximally-crossed" graphs shown in Fig. 2.5(c). This series is easily summed and we get the co called "Cooperon" [Gorkov L.P., Larkin A.I., Khmelnitskii D.E. (1979); Vollhardt D., Wölfle P. (1980)]:

$$U^{EC}_{\mathbf{pp}'}(\mathbf{q}\omega) = \frac{2\gamma\rho V^2}{D_0(\mathbf{p}+\mathbf{p}')^2 + i\omega} \tag{2.52}$$

where

$$D_0 = \frac{E}{md\gamma} = \frac{1}{d}v_F^2\tau \tag{2.53}$$

is the classical (bare) diffusion coefficient determining Drude conductivity Eq. (2.2). These "maximally crossed" diagrams lead to the following quantum correction to diffusion coefficient:

$$\frac{\delta D(\omega)}{D_0} = -\frac{1}{\pi N(E)} \sum_{|\mathbf{k}|<k_0} \frac{1}{-i\omega + D_0 k^2} \tag{2.54}$$

Appropriate correction to relaxation kernel can be expressed via the correction to diffusion coefficient as:

$$\delta M_E(\omega) = -i\frac{2E_F}{dm}\frac{\delta D(\omega)}{D(\omega)^2} = -\frac{M_E(\omega)}{D(\omega)}\delta D(\omega) \tag{2.55}$$

Considering the usual Drude metal as the zeroth approximation we get:

$$\delta M_E(\omega) = -\frac{M_0}{D_0}\delta D(\omega) \tag{2.56}$$

The central point of the self-consistent theory of localization [Götze W. (1979)] reduces to the replacement of Drude diffusion coefficient D_0 in the diffusion pole of Eq.(2.54) by the generalized one $D(\omega)$. Using this relation in Eq.(2.56) we obtain the main equation of self-consistent theory of localization determining the relaxation kernel $M(0\omega)$ (for $\mathbf{q} = 0$) [Vollhardt D., Wölfle P. (1980); Wölfle P., Vollhardt D. (1982)]:

$$M_E(\omega) = 2i\gamma \left\{ 1 + \frac{1}{\pi N(E)} \sum_{|\mathbf{k}|<k_0} \frac{i}{\omega + \frac{2E}{dm}\frac{k^2}{M_E(\omega)}} \right\} \tag{2.57}$$

or the equivalent equation for the generalized diffusion coefficient itself:

$$\frac{D_0}{D_E(\omega)} = 1 + \frac{1}{\pi N(E)} \sum_{|\mathbf{k}|<k_0} \frac{1}{-i\omega + D_E(\omega)k^2} \tag{2.58}$$

Cut—off in momentum space in Eqs. (2.54), (2.57), (2.58) is determined by the limit of applicability of diffusion—pole approximation of Eq. (2.46) or Eq. (2.52) [Sadovskii M.V. (1986)]:

$$k_0 \approx Min\{p_F, l^{-1}\} \tag{2.59}$$

Close to the mobility edge $p_F \sim l^{-1}$. Note, that from here on we are generally using natural units with Planck constant $\hbar = 1$, however in some of the final expressions we shall write $\hbar$ explicitly.

Conductivity can be expressed as [Vollhardt D., Wölfle P. (1980); Wölfle P., Vollhardt D. (1982)]:

$$\sigma(\omega) = \frac{ne^2}{m}\frac{i}{\omega + M_E(\omega)} \to e^2 D_E(\omega) N(E) \text{ for } \omega \to 0 \tag{2.60}$$

where we have used $n/N(E) = 2E/d$. It is clear that for metallic phase $M_E(\omega \to 0) = i/\tau_E$, where τ_E is generalized mean free time. Far from Anderson transition (for weak disorder) $\tau_E \approx \tau$ from Eq. (2.51) and Eq. (2.60) reduces to standard Drude expression.

If the frequency behavior of relaxation kernel leads to the existence of a limit $lim_{\omega\to 0}\omega M_E(\mathbf{q}\omega)$ a singular contribution appears in Eq. (2.46) for $\omega \to 0$ [Götze W. (1981); Sadovskii M.V. (1986)]:

$$\Phi_E^{RA}(\mathbf{q}\omega) \approx -\frac{N(E)}{\omega}\frac{1}{1 - \frac{2E}{md}\frac{q^2}{\omega M_E(\mathbf{q}\omega)}} \approx -\frac{N(E)}{\omega}\frac{1}{1 + R_{loc}^2 q^2} \tag{2.61}$$

where we have defined:

$$R_{loc}^2(E) = -\frac{2E}{md} lim_{\omega \to 0} \frac{1}{\omega M_E(\omega)} \tag{2.62}$$

According to the general criterion of localization [Berezinskii V.L., Gorkov L.P. (1979); Sadovskii M.V. (1986)] (Cf. Appendix A) this behavior corresponds to the region of localized states. Using Eq. (A.16) we immediately obtain from Eq. (2.61) the singular contribution to Gorkov—Berezinskii spectral density (Cf. Eqs. (A.8), (A.9)):

$$\ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^F = \frac{1}{\pi N(E)} Im \Phi_E^{RA}(\mathbf{q}\omega) = A_E(\mathbf{q})\delta(\omega) \tag{2.63}$$

where

$$A_E(\mathbf{q}) = \frac{1}{1 + R_{loc}^2(E) q^2} \to 1 - R_{loc}^2(E) q^2 \text{ for } q \to 0 \tag{2.64}$$

From here and from Eq. (A.11) we can see that $R_{loc}(E)$ as defined in Eq. (2.62) is actually the localization length. It is useful to define a characteristic frequency [Vollhardt D., Wölfle P. (1980)]:

$$\omega_0^2(E) = -lim_{\omega \to 0} \omega M_E(\omega) > 0 \tag{2.65}$$

so that

$$R_{loc}(E) = \sqrt{\frac{2E}{md}}\frac{1}{\omega_0(E)} \tag{2.66}$$

Thus, the localization transition is signalled by the divergence of relaxation kernel for $\omega \to 0$ [Vollhardt D., Wölfle P. (1980)], so that two characteristic types of it behavior for $\mathbf{q} = 0$ and $\omega \to 0$ appear:

$$M_E(0\omega) \approx \begin{cases} \frac{i}{\tau_E} \text{ for } E \geq E_c \\ \frac{i}{\tau_E} - \frac{\omega_0^2(E)}{\omega} \text{ for } E \leq E_c \end{cases} \tag{2.67}$$

The frequency $\omega_0(E)$ is in some crude sense analogous to the order parameter in the usual theory of phase transitions. It appears in the localized phase signalling about Anderson transition.

From Eq. (A.16) neglecting nonsingular for $\omega \to 0$ and $\mathbf{q} = 0$ contribution from $Im\Phi_E^{RR}(\mathbf{q}\omega)$ we can get explicit expression for Berezinskii—Gorkov spectral density which is valid for small ω and $\mathbf{q}$ [Katsnelson M.I., Sadovskii M.V. (1984); Sadovskii M.V. (1986)]:

$$\ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^F = \begin{cases} \frac{1}{\pi}\frac{D_E q^2}{\omega^2 + (D_E q^2)^2} \text{ (Metal)} \\ A_E(\mathbf{q})\delta(\omega) + \frac{1}{\pi}\frac{D_E q^2}{\omega^2 + [\omega_0^2(E)\tau_E + D_E q^2]^2} \text{ (Insulator)} \end{cases} \tag{2.68}$$

where we have introduced renormalized diffusion coefficient, determined by relaxation time τ_E:

$$D_E = \frac{2E}{dm}\tau_E = \frac{1}{d}v_F^2\tau_E \tag{2.69}$$

Substituting Eq. (2.67) into self—consistency equation Eq. (2.57) we can obtain equations for τ_E and $\omega_0(E)$ [Vollhardt D., Wölfle P. (1982); Myasnikov A.V., Sadovskii M.V. (1982); Sadovskii M.V. (1986)] and thus determine all the relevant characteristics of the system. For $d > 2$ Eq. (2.57) and Eq. (2.58) do really describe metal—insulator transition [Vollhardt D., Wölfle P. (1982); Myasnikov A.V., Sadovskii M.V. (1982); Sadovskii M.V. (1986); Vollhardt D., Wölfle P. (1990)]. For $d = 2$ all electronic states are localized [Vollhardt D., Wölfle P. (1980)].

Below we present some of the results of this analysis which will be important for the following. For $2 < d < 4$ a correlation length similar to that of Eq. (2.7) and Eq. (2.30) appears:

$$\xi_{loc}(E) \sim \frac{1}{p_F}\left|\frac{E-E_c}{E_c}\right|^{-\nu} \quad \text{for } E \sim E_c \tag{2.70}$$

where $\nu = 1/(d-2)$. The position of the mobility edge is determined by a condition:

$$\left.\frac{E}{\gamma}\right|_{E=E_c} = \frac{d}{\pi(d-2)} \tag{2.71}$$

which follows if we assume the cut—off $k_0 = p_F$ in Eq. (2.57) and Eq. (2.58). Static conductivity in metallic phase ($E > E_c$) is given by (Cf. Eq. (2.31):

$$\sigma = \frac{\sigma_0}{[p_F\xi_{loc}(E)]^{d-2}} \tag{2.72}$$

where $\sigma_0 = (ne^2/m)\tau$ is usual Drude conductivity. In particular, for $d = 3$:

$$\left.\frac{E}{\gamma}\right|_{E=E_c} = p_F l|_{E=E_c} = \frac{3}{\pi} \tag{2.73}$$

in complete accordance with Ioffe—Regel criterion , and

$$\sigma = \frac{\sigma_0}{p_F\xi_{loc}(E)} \tag{2.74}$$

Critical exponent $\nu = 1$. Mean free path which follows from Eq. (2.73) corresponds to Drude conductivity :

$$\sigma_c = \left.\frac{ne^2}{m}\tau\right|_{E=E_c} = \frac{e^2 p_F}{3\pi^2\hbar^2}\left.\left(\frac{p_F l}{\hbar}\right)\right|_{E=E_c} = \frac{e^2 p_F}{\pi^3\hbar^2} \tag{2.75}$$

which is equivalent to elementary estimate of Eq. (2.5).

Eq. (2.74) can also be rewritten as [Bulaevskii L.N., Sadovskii M.V. (1985)] :

$$\sigma = \sigma_0\left\{1 - \frac{\sigma_c}{\sigma_0}\right\} = \sigma_0 - \sigma_c \tag{2.76}$$

where Drude conductivity σ_0 is now the measure of disorder. It is obvious that for small disorder (large mean free path) $\sigma_0 \gg \sigma_c$ and Eq. (2.76) reduces to $\sigma \approx \sigma_0$. As disorder grows (mean free path drops) conductivity $\sigma \to 0$ for $\sigma_0 \to \sigma_c$.

In dielectric phase ($E < E_c$) we have $\xi_{loc}(E) = R_{loc}(E)$ and finite $\omega_0^2(E)$ from Eq. (2.65) which tends to zero as $E \to E_c$ from below. This frequency determines dielectric function of insulating phase [Sadovskii M.V. (1986)] :

$$\epsilon(\omega \to 0) = 1 + \frac{\omega_p^2}{\omega_0^2(E)} = 1 + \kappa_D^2 R_{loc}^2(E) \sim \left|\frac{E - E_c}{E_c}\right|^{-2\nu} \tag{2.77}$$

where $\omega_p^2 = 4\pi ne^2/m$ is the square of plasma frequency, $\kappa_D^2 = 4\pi e^2 N(E)$ is the square of inverse screening length of a metal.

Thus the main results of self—consistent theory of localization coincide with the main predictions of elementary scaling theory of localization. Vollhardt and Wölfle had shown [Vollhardt D., Wölfle P. (1982); Wölfle P., Vollhardt D. (1982)] that equations of this theory and especially the main differential equation of renormalization group Eq. (2.18) for conductance may be explicitly derived from self—consistency equations Eq. (2.57) and Eq. (2.58) reformulated for a finite system by introduction of low—momentum cut—off at $k \sim 1/L$, where L is the system size.

The results considered up to now are valid for $\omega \to 0$. Self—consistent theory of localization allows to study the frequency dependence of conductivity (generalized diffusion coefficient) [Wölfle P., Vollhardt D. (1982)]. At finite frequency the main Eq. (2.58) for the generalized diffusion coefficient for $d = 3$ can be rewritten as [Belitz D., Gold A., Gotze W. (1981); Wölfle P., Vollhardt D. (1982)]:

$$\frac{D_E(\omega)}{D_0} = 1 - \left(\frac{E_c}{E}\right)^{1/2} + \frac{\pi}{2}\left(\frac{E_c}{E}\right)^{1/2}\left\{-\frac{i\omega}{2\gamma}\frac{D_0}{D_E(\omega)}\right\}^{1/2} \tag{2.78}$$

which can be solved explicitly. With sufficient for our aims accuracy this solution may be written as:

$$D_E(\omega) \approx \begin{cases} D_E \qquad \omega \ll \omega_c \quad E \geq E_c \text{ (Metal)} \\ D_0\left(-\frac{i\omega}{2\gamma}\right)^{1/3} \qquad \omega \gg \omega_c \text{ (Metal and Insulator)} \\ D_E \frac{-i\omega}{-i\omega + \frac{3D_E}{v_F^2}\omega_0^2(E)} \qquad \omega \ll \omega_c \quad E < E_c \text{ (Insulator)} \end{cases} \tag{2.79}$$

where (Cf. Eq. (2.42)):

$$\omega_c \sim 2\gamma[p_F\xi_{loc}]^{-d} \sim \frac{1}{N(E)\xi_{loc}^d} \tag{2.80}$$

Here the renormalized diffusion coefficient:

$$D_E = \frac{D_0}{p_F \xi_{loc}(E)} \tag{2.81}$$

At the mobility edge itself $\xi_{loc}(E = E_c) = \infty$, so that $\omega_c = 0$ and we get the $\omega^{1/3}$—behavior (Cf. Eq. (2.41)):

$$D_E(\omega) = D_0 \left(-\frac{i\omega}{2\gamma} \right)^{1/3} \tag{2.82}$$

Note that ω_c is in fact determined by $D_E(\omega_c) \sim D_E \sim D_0(\omega_c/2\gamma)^{1/3}$. The meaning of the limit $\omega \to 0$ used above (Cf. e.g. Eq. (2.67)) is just that $\omega \ll \omega_c$. In particular, the expression Eq. (2.68) for Gorkov—Berezinskii spectral density is valid only for $\omega \ll \omega_c$. For $\omega_c \leq \omega \leq 2\gamma$, using Eq. (2.82) in Eq. (2.46) we get from Eq. (A.16):

$$\ll \rho_E \rho_{E+\omega} \gg^F_{\mathbf{q}} = \frac{\sqrt{3}}{2\pi} \frac{\alpha^{2/3}\omega^{1/3}q^2}{\omega^2 + \alpha^{2/3}\omega^{4/3}q^2 + \alpha^{4/3}\omega^{2/3}q^4} \tag{2.83}$$

where $\alpha = D_0 v_F/2\gamma = D_0 l \sim [N(E)]^{-1}$, where the last estimate is for $l \sim p_F^{-1}$. Eq. (2.83) is valid also at the mobility edge itself where $\omega_c = 0$. Obviously the correct estimate can be obtained from Eq. (2.68) by a simple replacement $D_E \to D_0(\omega/\gamma)^{1/3}$. It should be noted that the self-consistent theory approach to the frequency dependence of conductivity is clearly approximate. For example it is unable to reproduce the correct $Re\sigma(\omega) \sim \omega^2 ln^4\omega$ dependence for $\omega \to 0$ in the insulating state [Mott N.F, Davis E.A. (1979)]. This is apparently related to its inability to take the correct account of locally discrete nature of energy levels in Anderson insulators (Cf. below). However this is unimportant for our purposes while the general nature of frequency dependence at the mobility edge is apparently correctly reproduced.

In the following analysis we will also need a correlator of *local* densities of states defined in Eq. (A.3). This correlator can be expressed via two—particle Green's function as in Eq. (A.15). Neglecting nonsingular for small ω and q contribution from the second term of Eq. (A.15) and far from the Anderson transition (weak disorder) we can estimate the most important contribution to that correlator from the diagram shown in Fig. 2.6. [Bulaevskii L.N., Sadovskii M.V. (1986)]

The same contribution comes from the diagram which differs from that in Fig. 2.6 by direction of electron lines in one of the loops. Direct calcula-

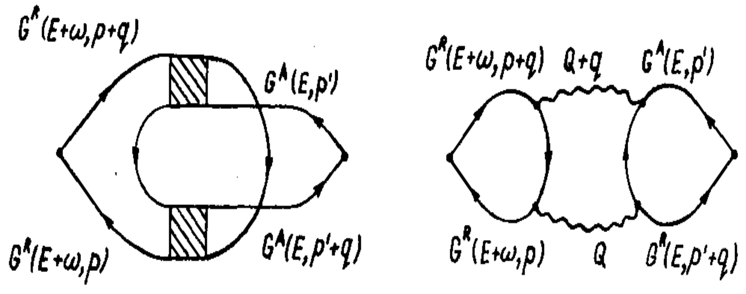

Fig. 2.6 Two equivalent forms of diagram for the correlator of local density of states. Wavy lines denote diffusion propagator, i.e. the sum of ladder diagrams.

tion gives:

$$\ll \rho_E \rho_{E+\omega} \gg^H_{\mathbf{q}} \sim \frac{N(E)}{\gamma^2}(\rho V^2)^2 Re \int d^d\mathbf{Q} \frac{1}{-i\omega + D_0\mathbf{Q}^2}\frac{1}{-i\omega + D_0(\mathbf{Q}+\mathbf{q})^2}$$
$$\sim \frac{1}{N(E)} Re \frac{1}{D_0^{d/2}} \frac{1}{(-i\omega + D_0 q^2)^{2-d/2}} \tag{2.84}$$

For the first time similar result for this correlator was found for some special model by Oppermann and Wegner [Oppermann R., Wegner F.J. (1979)]. For $d=3$ from Eq. (2.84) we find:

$$\ll \rho_E \rho_{E+\omega} \gg^H_{\mathbf{q}} \sim \frac{1}{N(E)D_0^{3/2}} \left\{ \frac{D_0 q^2}{\omega^2 + (D_0 q^2)^2} + [\omega^2 + (D_0 q^2)^2]^{-1/2} \right\}^{1/2} \tag{2.85}$$

It is obvious that for the estimates close to the mobility edge we can in the spirit of self—consistent theory of localization replace D_0 in Eq. (2.84) and Eq. (2.85) by the generalized diffusion coefficient $D(\omega)$. In particular, for system at the mobility edge ($\omega_c = 0$) $D_0 \rightarrow D_0(\omega/\gamma)^{1/3}$ in Eq. (2.85).

Surely, the self—consistent theory of localization is not free of some difficulties. Apparently the main problem is an uncontrollable nature of self—consistency procedure itself. In more details this is discussed in Refs.[Sadovskii M.V. (1986); Vollhardt D., Wölfle P. (1990)]. Here we shall concentrate only on some problems relevant for the future discussion. From the definition of generalized diffusion coefficient in Eq. (2.47) it is clear that it may be a function of both ω and $\mathbf{q}$, i.e. it can also possess spatial dispersion. Self—

consistent theory of localization deals only with the limit of $D_E(\mathbf{q} \to 0, \omega)$. At present it is not clear whether we can in any way introduce spatial dispersion into equations of self—consistent theory. Using scaling considerations the q—dependence of $D_E(\mathbf{q}\omega \to 0)$ can be estimated as follows [Lee P.A., Ramakrishnan T.V. (1985); Lee P.A. (1982)]. We have seen above that for the system of finite size of $L \ll \xi_{loc}$ elementary scaling theory of localization predicts the L—dependent diffusion coefficient $D_E \approx (g_c/N(E))/L^{d-2}$ (Cf. Eq. (2.35) for $d = 3$). From simple dimensional considerations we can try the replacement $L \to q^{-1}$ and get:

$$D_E(\omega \to 0\mathbf{q}) \approx \begin{cases} D_E \text{ for } q\xi_{loc} \ll 1 \\ \alpha q^{d-2} \text{ for } q\xi_{loc} \gg 1 \end{cases} \tag{2.86}$$

where $\alpha \sim g_c/N(E) \sim D_0 l$ and $E \sim E_c$, $l^{-1} \sim p_F$. Obviously an attempt to incorporate such q—dependence into equations of self–consistent theory of localization (like Eq. (2.57) and Eq. (2.58)) will radically change its structure. At the same time the L—dependence like $D_E \sim \alpha/L^{d-2}$ (for $L \ll \xi_{loc}$) can be directly derived from Eq. (2.58) as equations of elementary scaling theory are derived from it [Vollhardt D., Wölfle P. (1982); Wölfle P., Vollhardt D. (1982); Vollhardt D., Wölfle P. (1990)]. Thus the foundations for the simple replacement $L \to q^{-1}$ like in Eq. (2.86) are rather doubtful. More detailed analysis of wave number dependence of diffusion coefficient leading to Eq. (2.86) was given by Abrahams and Lee [Abrahams E., Lee P.A. (1986)] within the scaling approach. However, the complete solution of this problem is apparently still absent. In an important paper [Suslov I.M. (1995)] it was shown, using some general relations obtained in [Sadovskii M.V. (1986)] (Cf. also Appendix A), that Eq.(2.86) actually contradicts the general localization criterion of Berezinskii and Gorkov, from which it follows directly that at the localization transition the static diffusion coefficient $D(\omega = 0, \mathbf{q})$ vanishes for all q simultaneously. The detailed analysis performed in [Suslov I.M. (1995)] demonstrates the absence of any significant spatial dispersion of diffusion coefficient on the scale of $q \sim \xi^{-1}$, while its presence on the scale of $q \sim p_F$ is irrelevant for the critical behavior of the system close to the Anderson transition. Direct analysis, leading to the same conclusions within the formalism of self–consistent theory of localization, was performed in [Novokshonov S.G., Groshev A.G. (1998)]. More importantly, on the basis of symmetry approach formulated in [Suslov I.M. (1995)] it was even argued that the *exact* critical behavior at the mobility edge coincides with that predicted by the self-consistent theory of localization.

Finally we would like to stress that self–consistent theory of localization can not be applied "deep" inside localization region. Its derivation is based on a kind of extrapolation of "metallic" expressions and it does not take into account local discreteness of energy spectrum in the region of localized states as discussed in previous section. This is reflected in the form of one–particle Green's function used in self–consistent theory [Vollhardt D., Wölfle P. (1980); Wölfle P., Vollhardt D. (1982); Vollhardt D., Wölfle P. (1990); Sadovskii M.V. (1986)]. It does not describe the effects of local level repulsion, though it does not contradict it [Ohkawa F.J. (1982)]. Thus self–consistent theory of localization can be applied within localized region only until local energy spacing given by Eq. (2.13) is much smaller than other relevant energies of the problem under consideration. In fact this always leads to a condition of sufficiently large localization length R_{loc}, i.e. the system must be in some sense close to the mobility edge.

2.3.2 *Quasi-Two-Dimensional Systems*

Self–consistent theory of localization for quasi–two–dimensional systems was first analyzed by Prigodin and Firsov [Prigodin V.N., Firsov Yu.A. (1984)]. The electronic spectrum of a quasi–two–dimensional system can be modelled by nearly–free electrons within highly conducting planes and tight–binding approximation for interplane electron transfer:

$$E(\mathbf{p}) - E_F = v_F(|\mathbf{p}_{\|}| - p_F) - w\varphi(p_{\perp}) \tag{2.87}$$

Here w is the interplane transfer integral and $\varphi(p_{\perp}) = cosp_{\perp}a_{\perp}$, where $-\pi/a\perp \leq p_{\perp} \leq \pi/a_{\perp}$. Then the equations of self—consistent theory of localization for anisotropic generalized diffusion coefficient take the following form [Prigodin V.N., Firsov Yu.A. (1984)]:

$$D_j(\omega) = D_j^0 - \frac{1}{\pi N(E_F)}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{D_j(\omega)}{-i\omega + D_{\|}(\omega)q_{\|}^2 + D_{\perp}(\omega)(1-\varphi(q_{\perp}))} \tag{2.88}$$

where $j = \|, \perp$, and $D_{\|}^0 = v_F^2\tau/2$, $D_{\perp}^0 = (wa_{\perp})^2\tau$ are inplane and interplane bare Drude diffusion coefficients, τ is the mean free time due to elastic scattering (disorder). This approach is in complete correspondence with the analysis of Wölfle and Bhatt [Wölfle P., Bhatt R.N. (1984)] who has shown that the effects of anisotropy can be completely absorbed into anisotropic diffusion coefficient. It can be seen that the initial anisotropy of diffusion coefficient does not change as disorder grows up to the Anderson transition and in fact we have only to find one unknown ratio:

$$d(\omega) = \frac{D_j(\omega)}{D_j^0} = \frac{\sigma_j(\omega)}{\sigma_j^0} \tag{2.89}$$

which is determined by algebraic equation following from Eq. (2.88):

$$d(\omega) = 1 - \frac{1}{2\pi E_F \tau} \times$$

$$ln \frac{2}{[-i\omega\tau/d(\omega)] + (w\tau)^2 + [(-i\omega\tau/d(\omega))(-i\omega/d(\omega) + 2w^2\tau^2)]^{1/2}} \tag{2.90}$$

Due to a quasi–two–dimensional nature of the system there is no complete localization for any degree of disorder which is typical for purely two–dimensional system. However the tendency for a system to become localized at lower disorder than in isotropic case is clearly seen. All states at the Fermi level become localized only for $w < w_c$, where

$$w_c = \sqrt{2}\tau^{-1} exp(-\pi E_F \tau) \tag{2.91}$$

Thus the condition of localization is actually more stringent than given by the simplest Ioffe—Regel type estimate as in Eq. (2.11). For fixed w the mobility edge appears at:

$$E_F = E_c = \frac{1}{\pi\tau} ln \left(\frac{\sqrt{2}}{w\tau} \right) \tag{2.92}$$

Thus in case of strong anisotropy when $w\tau \ll 1$ localization can in principle take place even in case of $E_F \gg \tau^{-1}$, i.e. at relatively weak disorder. These estimates are in qualitative accordance with Eq.(2.11), which is valid in case of relatively strong disorder $E_F \tau \sim 1$.

In the metallic phase close to the Anderson transition:

$$\sigma_j = \sigma_j^0 \frac{E_F - E_c}{E_c} \tag{2.93}$$

For $w \to 0$ we have $E_c \to \infty$ which reflects complete localization in two dimensions. We can also define inplane Drude conductivity at $E_F = E_c$ as a kind of a "minimal metallic conductivity" in this case as a characteristic conductivity scale at the transition:

$$\sigma_{\parallel}^{c}=e^{2}N(E_{F})D_{\parallel}^{0}(E_{F}=E_{c})=\frac{1}{\pi^{2}}\frac{e^{2}}{\hbar a_{\perp}}ln\left(\frac{\sqrt{2}\hbar}{w\tau}\right)\approx\frac{1}{\pi^{2}}\frac{e^{2}}{\hbar a_{\perp}}ln\left(\frac{E_{F}}{w}\right) \tag{2.94}$$

where we have used $N(E_F)=m/(\pi a_{\perp}\hbar^2)$, m is inplane effective mass, and the last equality is valid for $E_F\tau/\hbar\sim 1$, i.e. for a case of sufficiently strong disorder. For the time being we again use $\hbar$ explicitly. From these estimates it is clear that inplane "minimal conductivity" is logarithmically enhanced in comparison with usual estimates (Cf. Eq. (2.5)). This logarithmic enhancement grows as the interplane overlap of electronic wave functions diminishes. Accordingly in case of small overlap ($w\tau/\hbar\ll 1$) this conductivity scale may be significantly larger than $(3-5)10^2 Ohm^{-1}cm^{-1}$ which is characteristic for isotropic systems. Thus in quasi–two–dimensional case Anderson transition may take place at relatively high values of inplane conductivity. For a typical estimate in a high—T_c system we can take something like $E_F/w>10$ so that the value of $\sigma_{\parallel}^{c}$ may exceed $10^3 Ohm^{-1}cm^{-1}$. Obviously these estimates are in qualitative accordance with elementary estimates based upon Ioffe—Regel criterion of Eq. (2.11) and Eq. (2.12). Similar conclusions can be deduced from the analysis presented in Ref.[Li Qiming, Soukoulis C.M, Economou E.N. (1989)] where it was shown by a different method that in case of anisotropic Anderson model the growth of anisotropy leads to a significant drop of a critical disorder necessary to localize all states in a conduction band.

Now let us quote some results for the frequency dependence of generalized diffusion coefficient in quasi—two—dimensional case which follow from the solution of Eq. (2.90) [Prigodin V.N., Firsov Yu.A. (1984)]. We shall limit ourselves only to the results valid close to the mobility edge in metallic phase:

$$d(\omega)\approx\begin{cases}\frac{E_F-E_c}{E_c} & \omega\ll\omega_c\\ (2\pi E_F w\tau^2)^{-2/3}(-i\omega\tau)^{1/3} & \omega_c\ll\omega\ll w^2\tau\\ 1-\frac{1}{2\pi E_F\tau}ln\left(\frac{1}{-i\omega\tau}\right) & w^2\tau\ll\omega\ll\tau^{-1}\end{cases} \tag{2.95}$$

where

$$\omega_c\approx[2\pi E_F w\tau^2]^2\frac{1}{\tau}\left|\frac{E_F-E_c}{E_c}\right|^3 \tag{2.96}$$

From these expressions we can see the crossover from $\omega^{1/3}$—behavior typical for isotropic three–dimensional systems to logarithmic dependence on frequency which is characteristic for two—dimensional systems.

2.3.3 *Self-Consistent Theory of Localization in Magnetic Field*

Early version of self-consistent theory of localization as proposed by Vollhardt and Wölfle was essentially based upon time-reversal invariance [Vollhardt D., Wölfle P. (1980); Wölfle P., Vollhardt D. (1982)]. This property is obviously absent in the presence of an external magnetic field. In this case in addition to Eq. (2.45) we have to consider two—particle Green's function in particle—particle (Cooper) channel:

$$\Psi_E^{RA}(\mathbf{q},\omega) = -\frac{1}{2\pi i}\sum_{\mathbf{p}_+\mathbf{p}'_-} < G^R(\mathbf{p}_+,\mathbf{p}'_+,E+\omega)G^A(-\mathbf{p}'_-,-\mathbf{p}_-,E > \tag{2.97}$$

which for small ω and $\mathbf{q}$ again has diffusion pole form like that of Eq. (2.46), but with *different* diffusion coefficient. Appropriate generalization of self–consistent theory of localization was proposed by Yoshioka, Ono and Fukuyama [Yoshioka D., Ono Y., Fukuyama H. (1981)]. This theory is based on the following system of coupled equations for relaxation kernels $M_j(\mathbf{q},\omega)$, corresponding to diffusion coefficients in particle—hole and particle—particle channels:

$$M_1 = 2i\gamma\left\{1 - \frac{1}{\pi N(E)}\sum_{n=0}^{N_0}\frac{2}{\pi L_H}\int_0^{\sqrt{q_0^2-4m\omega_H(n+1/2)}}\frac{dq_z}{2\pi}\right.$$
$$\left.\frac{1}{\omega - \frac{D_0}{\tau M_2}[q_z^2+4m\omega_H(n+1/2)]}\right\} \tag{2.98}$$

$$M_2 = 2i\gamma\left\{1 - \frac{1}{\pi N(E)}\sum_{|\mathbf{q}|<q_0}\frac{1}{\omega - D_0q^2/(\tau M_1)}\right\} \tag{2.99}$$

Here $\omega_H = eH/mc$ is cyclotron frequency, $L_H = (c/eH)^{1/2}$ is magnetic length and $N_0 = q_0^2/4m\omega_H$. These equations form the basis of self-consistent theory of localization in the absence of time-reversal invariance and were extensively studied in Refs.[Yoshioka D., Ono Y., Fukuyama H. (1981); Ono Y., Yoshioka D., Fukuyama H. (1981); Kotov E.A. (1988); Kotov E.A. (1989); Kuchinskii E.Z., Sadovskii M.V. (1991a)]. Alternative formulations of self–consistent theory in magnetic field were given in Refs. [Ting C.S. (1982); Cai W., Ting C.S. (1985a); Cai W., Ting C.S. (1985b); Belitz D. (1984); Theumann A., Pires Idiart M.A. (1991)]. All these approaches lead to qualitatively similar results. Here we shall concentrate on formulations given in Ref. [Kuchinskii E.Z., Sadovskii M.V. (1991a)].

Let us introduce the dimensionless parameter $\lambda = \gamma/\pi E$ as a measure of disorder and generalized diffusion coefficients in diffusion and Cooper channels D_1 and D_2 defined as in Eq. (2.47) with M replaced by M_1 and M_2 respectively. We shall use dimensionless $d_j = D_j/D_0$ $(j = 1, 2)$ in the following.

We are mainly interested in diffusion coefficient in the Cooper channel, which as we shall see defines the upper critical field of a superconductor. Both this coefficient as well as the usual one are determined by the following equations which follow from Eq. (2.98) and Eq. (2.99) after the use of Poisson summation over Landau levels in the first equation which allows one to separate the usual diffusion coefficient independent of magnetic field and the field—dependent part:

$$\begin{cases} d_1 = (1 + \frac{3\lambda - \delta_2 - \Delta_2}{d_2})^{-1} \\ d_2 = (1 + \frac{3\lambda - \delta_1}{d_1})^{-1} \end{cases} \tag{2.100}$$

where

$$\delta_j = (3/2\pi\lambda)^{3/2}(-i\omega/E)^{1/2} d_j^{-1/2} \tag{2.101}$$

and

$$\Delta_2 = -3\lambda \sum_{p=1}^{\infty} (-1)^p \int_0^1 dx 2 \int_0^{\sqrt{1-x}} dy \frac{\cos(2\pi p x_0^2/c^2)}{y^2 + x + 3/2\pi\lambda(-i\omega/E)/(d_2 x_0^2)} \tag{2.102}$$

where $c = (2\omega_H/E)^{1/2}$.In the following we have to solve Eqs. (2.100) for the case of small δ_j and Δ_2. Limiting ourselves to terms linear in δ_1, δ_2 and Δ_2 we obtain:

$$\frac{d_1}{d_2} = 1 + \frac{\Delta_2}{1 + 3\lambda} \tag{2.103}$$

Using Eq. (2.103) in Eqs. (2.100) we get an equation for diffusion coefficient in Cooper channel:

$$d_2 = 1 - 3\lambda + \delta_2 + \frac{3\lambda}{1+3\lambda}\Delta_2 \tag{2.104}$$

Introducing Δ_1 which differs from Δ_2 by the replacement of d_2 by d_1 we can write down also the approximate equation for the usual diffusion coefficient:

$$d_1 = 1 - 3\lambda + \delta_1 + \frac{1}{1+3\lambda}\Delta_1 \tag{2.105}$$

In the absence of magnetic field ($\Delta_1 = \Delta_2 = 0$) Eq. (2.104) and Eq. (2.105) are the same and lead to standard results of self—consistent theory quoted above. Eq. (2.104) can be written as:

$$2mD_2 = \pm(\frac{\omega_c}{E})^{1/3} + (-\frac{i\omega}{E})^{1/2}(2mD_2)^{-1/2} + \frac{3\lambda}{1+3\lambda}\Delta_2 \tag{2.106}$$

where + corresponds to metallic, and − to insulating phases, while characteristic frequency

$$\omega_c = \left(\frac{|1-3\lambda|}{(3/2)\pi\lambda}\right)^3 E \tag{2.107}$$

can be considered as a measure of disorder and separate regions with different frequency dependencies of diffusion coefficient.

Neglecting in Eq. (2.102) terms oscillating with magnetic field (these oscillations are connected with sharp cut—off in momentum space used above and disappear for smooth cut—off) we get:

$$\Delta_2 = -(2\omega_H/E)^{1/2}\sum_{p=1}^{\infty}\frac{(-1)^p}{p^{1/2}}f(2\pi\rho\kappa) \tag{2.108}$$

where

$$f(y) = \sqrt{2/\pi}\int_0^\infty \frac{cos(t)dt}{\sqrt{t+y}}; \qquad \kappa = \frac{-i\omega/E}{2\omega_H/E}\frac{1}{2mD_2} \tag{2.109}$$

This gives:

$$\Delta_2 = \begin{cases} W(2\omega_H/E)^{1/2} & |\kappa| \ll 1 \\ \frac{1}{48}\left((-i\omega/E)\frac{1}{2mD_2}\right)^{-3/2}(2\omega_H/E)^2 & |\kappa| \gg 1 \end{cases} \tag{2.110}$$

where $W = -\sum_{p=1}^{\infty}(-1)^p/p^{1/2} \approx 0.603$.

Solutions of Eq. (2.106) for different limiting cases can be found in Ref. [Kuchinskii E.Z., Sadovskii M.V. (1991a)]. Comparison of Eq. (2.105) and Eq. (2.104) shows that the usual diffusion coefficient D_1 is given by the same expressions as D_2 with the replacement of the coefficient $3\lambda/(1+3\lambda)$ before the field—dependent correction by $1/(1+3\lambda)$. Here we only quote the results for D_2 in case of $\omega_c/E \ll (\omega_H/E)^{3/2}$, valid close to the transition in the absense of magnetic field:

$$D_2 = \frac{1}{2m}\left\{\pm(\omega_c/E)^{1/3} + \left[\frac{3\lambda}{1+3\lambda}\right] W(2\omega_H/E)^{1/2}\right\} \approx \frac{1}{4m}W(2\omega_H/E)^{1/2} \qquad \omega \ll \omega_c^{\star} \tag{2.111}$$

$$D_2 = \frac{1}{2m}\left\{(-i\omega/E)^{1/3} + \frac{2}{3}\left[\frac{3\lambda}{1+3\lambda}\right]\frac{1}{48}\frac{(2\omega_H/E)^2}{(-i\omega/E)}\right\} \qquad \omega \gg \omega_c^{\star} \tag{2.112}$$

where $\omega_c^{\star} = (W/2)^3(2\omega_H/E)^{3/2}E$.

Note that for high frequencies larger than $\omega_c^{\star}$ the correction term becomes quadratic in field which differs from usual square root behavior at low frequencies.

It is easy to see that in the absence of the external magnetic field these equations reduce to the usual self-consistency equation as derived by Vollhardt and Wölfle with a single relaxation kernel.

Let us finally quote some results for the purely two-dimensional case [Kuchinskii E.Z., Sadovskii M.V. (1993a)]. Self-consistent equations for the diffusion coefficients take now the following form:

$$\begin{aligned}\frac{D_0}{D_2} &= 1 + \frac{1}{\pi N(E)}\sum_{|\mathbf{q}|<q_0}\frac{1}{\omega + D_1 q^2}\\ \frac{D_0}{D_1} &= 1 + \frac{1}{\pi N(E)}\sum_{|\mathbf{k}|<q_0}\frac{1}{\omega + D_2 k^2}\end{aligned} \tag{2.113}$$

where $k^2 = 4m\omega_H(n+\frac{1}{2})$, and we assume that ω here is imaginary (Matsubara) frequency, which simplifies the analysis. Actually only the dependence on the Matsubara's frequencies are important for further applications to superconductivity.

Introduce again the dimensionless diffusion coefficients $d_1 = \frac{D_1}{D_0}$, $d_2 =$

$\frac{D_2}{D_0}$, so that Eqs.(2.113) are rewritten as:

$$\begin{aligned} \frac{1}{d_2} &= 1 + \frac{\lambda}{d_1} ln(1 + d_1 \frac{1}{2\omega\tau}) \\ \frac{1}{d_1} &= 1 + \frac{\lambda}{d_2} \sum_{n=0}^{N_0} \frac{1}{n + \frac{1}{2} + \frac{\omega}{4m\omega_H D_0} \frac{1}{d_2}} \end{aligned} \tag{2.114}$$

where $N_0 = \frac{1}{8m\omega_H D_0 \tau}$—is the number of Landau levels below the cutoff. We assume that the magnetic field is low enough, so that $N_0 \gg 1$, i.e.

$$H \ll \frac{\Phi_0}{D_0 \tau} \tag{2.115}$$

With sufficient for further use accuracy we can write down the following solution for the diffusion coefficient in Cooper channel:

For weak magnetic field $\omega_H \ll \frac{\lambda e^{-1/\lambda}}{\tau}$

$$d_2 = \begin{cases} 1 & \text{for } \omega \gg \frac{e^{-1/\lambda}}{2\tau} \\ 2\omega\tau e^{1/\lambda} & \text{for } \omega \ll \frac{e^{-1/\lambda}}{2\tau} \end{cases} \tag{2.116}$$

and we can neglect the magnetic field influence upon diffusion.

For larger fields $\omega_H \gg \frac{\lambda e^{-1/\lambda}}{\tau}$

$$d_2 = \begin{cases} 1 & \text{for } \omega \gg \frac{e^{-1/\lambda}}{2\tau} \\ \frac{1}{\lambda ln(1/2\omega\tau)} & \text{for } \frac{e^{-1/\lambda^2 lnQ}}{2\tau} \ll \omega \ll \frac{e^{-1/\lambda}}{2\tau} \\ 2\omega\tau\lambda ln Q e^{1/\lambda^2 lnQ} & \text{for } \omega \ll \frac{e^{-1/\lambda^2 lnQ}}{2\tau} \end{cases} \tag{2.117}$$

where $Q = \frac{\pi\gamma\lambda}{\tau\omega_H}$, $\gamma \approx 1.781$. Here we neglect the magnetic field corrections small in comparison to the d_2 value in the absence of magnetic field given by Eq.(2.116).

2.3.4 *NMR Relaxation and Localization*

Measuring NMR relaxation time T_1 is one of the experiments in which we can apparently get clear evidence about electron localization in disordered systems. Actually, this fact was noted long ago by Warren [Warren W.W. (1971)], who used this method to study metal-insulator transition in semiconductor melts. Warren's qualitative treatment was later confirmed by Götze and Ketterlee [Götze W., Ketterlee W. (1983)]. The main conclusion drawn in these works was a growth of the reciprocal relaxation time

T_1^{-1} in the metallic phase as the system approaches the Anderson transition (as compared with the ordinary Korringa relaxation rate in metals [Narat A. (1967)]). Later, in addition to the results of [Warren W.W. (1971); Narat A. (1967)], we have shown that on the insulator side of the Anderson transition T_1^{-1} decreases with electron localization length (with increasing disorder) [Zhdanov Yu.I. et al. (1992)]. Thus, the Anderson transition can be accompanied by a peak in T_1^{-1}, which can serve as an independent experimental proof that localization transition is realized. Here we shall follow the treatment of [Zhdanov Yu.I. et al. (1992)].

In system with free electrons the main mechanism responsible for the coupling between the electrons and nuclear spins is the Fermi contact interaction, which leads to the following expression for T_1^{-1} [Narat A. (1967)]:

$$\frac{1}{T_1} = 2A^2 \frac{T}{g_e^2 \mu_B^2 \omega_N} \int \frac{d^3\mathbf{q}}{(2\pi)^3} Im\chi(\mathbf{q}, \omega_N) \tag{2.118}$$

where

$$\chi(\mathbf{q}, \omega_N) = g_e^2 \mu_B^2 < \sum_{\mathbf{pp}'} \sum_{\nu\nu'} \phi_{\nu'\uparrow}(\mathbf{p}_+) \phi^\star_{\nu'\uparrow}(\mathbf{p}'_+) \phi_{\nu\downarrow}(\mathbf{p}_-) \phi^\star_{\nu\downarrow}(\mathbf{p}'_-)$$
$$\frac{f_{\nu'\uparrow} - f_{\nu\downarrow}}{\varepsilon_{\nu\downarrow} - \varepsilon_{\nu'\uparrow} + \omega_N + i\delta} > \tag{2.119}$$

is the electron susceptibility at the wave vector $\mathbf{q}$ and the NMR frequency ω_N. Here A is the usual hyperfine coupling constant, g_e the electron gyromagnetic ratio, μ_B the Bohr magneton, f_ν the Fermi function, T is the temperature. Here we use the representation of the exact wave functions $\phi_\nu(\mathbf{r})$ of an electron in the random field of a disordered system, and ε_ν are the respective exact energy eigenvalues (Cf. Appendix A). The angle brackets stand for averaging over random configurations, and the arrows denote the direction of electron spin.

Using the inequality $\omega_N \ll \omega_e \ll T$, where ω_e is the electron Zeeman frequency, we get by direct calculation:

$$\frac{1}{T_1} = 2A^2 T N(E_F)\pi \int \frac{d^3\mathbf{q}}{(2\pi)^3} \ll \rho_{E_F}\rho_{E_F+\omega_e} \gg^F_{\mathbf{q}}$$
$$= 2A^2 T \int \frac{d^3\mathbf{q}}{(2\pi)^3} Im\Phi^{RA}_{E_F}(\mathbf{q}, \omega_e) \tag{2.120}$$

where we have introduced the Fourier transform of Berezinskii-Gorkov spectral density (A.2) and used its relation to the averaged two-particle Green's function (A.15) (Cf. Appendix A).

Now we can use the results of self-consistent theory of localization, e.g. Eqs.(2.68),(2.83) and calculate T_1^{-1} explicitly. We have only to introduce the upper cut-off in the integral over $\mathbf{q}$ in (2.120) similar to (2.59). Direct calculations for the metallic region ($E_F > E_c$) yield [Zhdanov Yu.I. et al. (1992)]:

$$\frac{1}{T_1} = \frac{1}{T_1^K} \begin{cases} 1 + \frac{\pi}{3}\frac{\sigma_c^2}{\sigma(\sigma+\sigma_c)} \text{ if } \sigma > \sigma_0 \left(\frac{\omega_e}{E_F}\right)^{1/3} \\ \left(\frac{E_F}{\omega_e}\right)^{1/3} \text{ if } \sigma \leq \sigma_c \left(\frac{\omega_e}{E_F}\right)^{1/3} \end{cases} \tag{2.121}$$

where T_1^K is the Korringa relaxation time [Narat A. (1967)], which determines the relaxation of nuclear spins in a "good" metal:

$$\frac{1}{T_1^K} = 2A^2 T\pi[N(E_F)]^2. \tag{2.122}$$

The characteristic conductivity $\sigma_c = e^2 p_F/\pi^3\hbar^2$ coincides with Mott's minimal metallic conductivity. The unity in the right-hand side of (2.121) is written explicitly, so that this expression agrees with the usual result in the "pure" limit of $p_F l \gg 1$, when $\sigma \gg \sigma_c$. The first expression in (2.121) practically coincides with the respective expression in Ref.[Götze W., Ketterlee W. (1983)], but here we have reduced it to a form more convenient for comparison with the experimental data.

Equation (2.121) shows that in a system approaching an Anderson transition NMR relaxation rate T_1^{-1} grows considerably in comparison with Korringa relaxation rate. Moreover, in the narrow range near the transition point T_1 begins to depend on magnetic field (via ω_e). Unfortunately, this dependence emerges only for $\sigma \leq \sigma_c(E_F/\omega_e)^{-1/3} \ll \sigma_c$, that is, in an extremely narrow neighborhood of the transition, which means that observing the $\omega_e^{-1/3}$ – dependence of NMR relaxation rate is extremely difficult experimentally, even for systems with low values of Fermi energy, of the doped semiconductor type. In this range the temperature dependence of conductivity, ignored above, begins to play an essential role. On the other hand, the increase in the relaxation rate T_1^{-1} as conductivity decreases (disorder grows) should be observable in a fairly broad region, starting from the Ioffe-Regel limit of $\sigma \sim 10^3 Ohm^{-1}cm^{-1}$. This behavior was actually observed in

Warren's experiments [Warren W.W. (1971)] with liquid semiconductors.

In the localization region ($E_F < E_c$), calculations similar to those that lead to (2.121) yield:

$$\frac{1}{T_1} \approx \frac{1}{T_1^K} \begin{cases} \frac{1}{(p_F l)^2}\left(\frac{\omega_e}{\gamma}\right)^{-1/3} \approx \left(\frac{E_F}{\omega_e}\right)^{-1/3} \text{ if } R_{loc}(E_F) \geq (N(E_F)\omega_e)^{-1/3} \\ p_F R_{loc}(E_F) \text{ if } R_{loc}(E_F) \leq (N(E_F)\omega_e)^{-1/3}. \end{cases} \tag{2.123}$$

Obviously, here too the $\omega_e^{-1/3}$ – dependence operates only within a very narrow region near the transition, where the localization length $R_{loc}(E_F)$ is extremely large. As Fermi level E_F moves deeper into the localization region (disorder increases), $R_{loc}(E_F)$ decreases, so that the relaxation rate T_1^{-1} diminishes together with $p_F R_{loc}$. The second expression in (2.123) is valid up to $p_F R_{loc} \sim 1$, when T_1^{-1} returns to values of the order of the Korringa rate. For smaller values of R_{loc} approximations based on self-consistent theory of localization can not be used, since in this theory R_{loc} cannot be smaller than roughly p_F^{-1}. Thus, T_1^K/T_1 passes through a maximum when the Fermi level crosses the mobility edge. Experimental observation of such a maximum might serve as an independent indication of Anderson transition.

The physical meaning of these results is simple. NMR relaxation occurs via nuclear spin being flipped by some conduction electron flying past the nucleus and flipping its own spin. As the system approaches the Anderson transition, conduction electrons in the vicinity of the given nuclear spin diffuse more slowly and, hence, interact with this spin during increased time interval, leading to the growth of relaxation rate T_1^{-1} [Warren W.W. (1971)]. This effect is most pronounced at the transition point, but subsequently, as electrons remain localized in a region with dimensions of the order of R_{loc} surrounding the given nucleus, localization length decreases as the degree of disorder grows and an electron "carries away" less and less information about the nuclear spin, leading to the decrease in T_1^{-1}.

In the above analysis we ignored the temperature dependence of conductivity. The zero value of σ corresponds to the Anderson transition at $T = 0$, while in experiments (at finite T) conductivity is always finite. For this reason and also because of the effects of a negative temperature coefficient of resistivity in metallic region it is usually very difficult to determine the transition point from conductivity measurements. Measurements of NMR relaxation offer some advantages. We also ignored, up to now, electron-electron interactions,which complicate the entire picture of NMR relaxation, especially due to formation of localized magnetic moments (Cf. below). Some aspect of this problem were investigated in Refs.[Gan Zi-Zhao, Lee P.A. (1986); Shastry B.S., Abrahams E. (1994)]. It is worth noting that we ignored the effect of the external magnetic field, in which the NMR experiment is conducted. This effect on localization is not small (Cf.above) and shifts the Anderson transition to the region of larger disorder. For this reason, in NMR experiments we can only find the transition point of the metal-insulator transition in an external magnetic field.

2.4 Phase Transition Analogy and Scaling for Correlators

Scaling description of a system close to Anderson transition can be developed also on the basis of some analogies with usual phase transitions [Sadovskii M.V. (1981); Sadovskii M.V. (1986); Lee P.A., Ramakrishnan T.V. (1985)]. Most successful in this respect is an approach initially proposed by Wegner [Wegner F.J. (1979a); Wegner F.J. (1979b); Wegner F.J. (1982)].

Let us consider Eq. (2.68) and Eq. (2.84) which define basic electronic correlators (spectral densities) in a disordered system. For the metallic region we can write:

$$K_F(\mathbf{q}\omega) \equiv N(E) \ll \rho_E \rho_{E+\omega} \gg^F_{\mathbf{q}} \sim Re\frac{N(E)}{-i\omega + D_E q^2} \tag{2.124}$$

$$K_H(\mathbf{q}\omega) \equiv N(E) \ll \rho_E \rho_{E+\omega} \gg^H_{\mathbf{q}} \sim Re\frac{D_E^{d/2}}{(-i\omega + D_E q^2)^{2-d/2}} \tag{2.125}$$

Wegner has noted [Wegner F.J. (1979b); Oppermann R., Wegner F.J. (1979)] that these expressions are in some sense similar to analogous expressions for transversal and longitudinal susceptibilities of a ferromagnet [Patashinskii A.Z., Pokrovskii V.L. (1982)]:

$$\chi_\perp(\mathbf{q}) = \frac{M}{H + \rho_s q^2} \tag{2.126}$$

$$\chi_\parallel(\mathbf{q}) \sim \frac{1}{(H + \rho_s q^2)^{2-d/2}} \tag{2.127}$$

Table 2.1 Anderson transition and ferromagnet close to Curie point T_c.

Localization	Ferromagnet
$E - E_c$	$T - T_c$
$N(E)$	M
K_F	$\chi_\perp$
K_H	$\chi_\parallel$
$-i\omega$	H
$D(E)$	ρ_s
ξ_{loc}	ξ

where M is magnetization, H is external magnetic field and ρ_s is the spin—stiffness coefficient. Comparing Eqs. (2.124) with Eq. (2.126) and Eq. (2.125) with Eq. (2.127) we can write down a correspondence between electron diffusion in a random system and a ferromagnet as given in Table. I.

Now we can use the main ideas of scaling approach in the theory of critical phenomena [Kadanoff L.P. et al. (1967); Wilson K.G., Kogut J. (1974); Patashinskii A.Z., Pokrovskii V.L. (1982); Ma Shang-keng (1976)] and formulate similar expressions for electronic system close to the Anderson transition. As was noted above scaling theory is based upon an assumption that a singular behavior of physical parameters of a system close to a phase transition appears due to large scale (long wave—length) fluctuations of order—parameter (e.g. magnetization) close to critical temperature T_c. Scaling hypothesis claims that singular dependencies on $T - T_c$ reflect the divergence of correlation length of these fluctuations ξ and this length is the only relevant length—scale in the critical region. Scaling approach is based upon an idea of scale transformations and dimensional analysis. Under the scale transformation the spatial interval Δx changes to $\Delta x'$, according to:

$$\Delta x \to \Delta x' = s^{-1}\Delta x \tag{2.128}$$

Accordingly for the wave—vector:

$$\mathbf{q} \to \mathbf{q}' = s\mathbf{q} \tag{2.129}$$

Scaling dimension [Ma Shang-keng (1976)] of a physical quantity A is equal to λ if under scale transformations defined by Eq. (2.128) and Eq. (2.129)

Table 2.2 Scaling dimensions in the theory of critical phenomena.

ξ	q	M	H
-1	$+1$	$1/2(d-2+\eta)$	$1/2(d+2-\eta)$

we get:

$$A \to A' = As^{\lambda} \tag{2.130}$$

Scaling dimensions for the main characteristics of a ferromagnet are given in terms of standard critical exponents [Ma Shang-keng (1976)] in Table II.

Correlation length of the theory of critical phenomena behaves like:

$$\xi \sim |T - T_c|^{-\nu} \tag{2.131}$$

The knowledge of scaling dimension of a given physical quantity allows to determine its dependence on ξ, i.e. on $T - T_c$. For example magnetization M behaves according to Table II as:

$$M \sim \xi^{-1/2(d-2+\eta)} \sim |T - T_c|^{\beta} \tag{2.132}$$

where the critical exponent of magnetization equals

$$\beta = \frac{1}{2}\nu(d - 2 + \eta) \tag{2.133}$$

Magnetic susceptibility is given by:

$$\chi(\mathbf{q}, T - T_c) = \xi^{2-\eta} g(q\xi) \tag{2.134}$$

where $g(x)$ is some universal function, such that $g(0) \sim const$, $g(x \to \infty) \sim x^{-(2-\eta)}$. From Eq. (2.134) we get standard results:

$$\chi(0, T - T_c) \approx \xi^{2-\eta} g(0) \sim |T - T_c|^{-\gamma} \tag{2.135}$$

where $\gamma = (2 - \eta)\nu$ is the susceptibility exponent. Analogously:

$$\chi(\mathbf{q}, T = T_c) \sim q^{-2+\eta} \tag{2.136}$$

Here η is sometimes called Fisher's exponent.

It is easy to see that Eq. (2.134) is equivalent to scaling relation (H—dependence is taken from Table II):

$$\chi(s\mathbf{q}, s^{-1}\xi, s^{1/2(d+2-\eta)}H) = s^{-(2-\eta)}\chi(\mathbf{q}, \xi, H) \tag{2.137}$$

It is convenient to make transformation $|T - T_c| \rightarrow b|T - T_c|$ so that $\xi \rightarrow b^{-\nu}\xi$ which is equivalent to the choice of $s = b^{\nu}$. Then Eq. (2.137) transforms to:

$$\chi(b^{\nu}\mathbf{q}, b^{-\nu}\xi, b^{\nu(d+2-\eta)/2}H) = b^{-\gamma}\chi(\mathbf{q}, \xi, H) \tag{2.138}$$

Finally note that close to Curie point the spin—stiffness coefficient ρ_s satisfies the so—called Josephson relation [Patashinskii A.Z., Pokrovskii V.L. (1982)]:

$$\rho_s \sim |T - T_c|^{(d-2)\nu} \tag{2.139}$$

and tends to zero as $T \rightarrow T_c$ from within the condensed phase.

Consider now the analogy formulated in Table I. Density of states $N(E)$ is nonsingular at the mobility edge [Thouless D.J. (1974); Sadovskii M.V. (1986)]. Then considering $N(E)$ as an analog of magnetization M we have to assume $\beta = 0$, i.e. at the localization transition:

$$\eta = 2 - d \tag{2.140}$$

and the "order—parameter" $N(E)$ is nonsingular at the transition $E = E_c$. Accordingly we have $\gamma = d\nu$. Josephson relation Eq. (2.139) now takes the form:

$$D_E \sim |E - E_c|^{(d-2)\nu} \tag{2.141}$$

i.e. in fact is equivalent to Wegner's relation for conductivity given by Eq. (2.31). Correlation length exponent ν remains unknown.

For electronic correlators of Eq. (2.124) and Eq. (2.125) from Eq. (2.138) we obtain scaling relations [Wegner F.J. (1979b); Wegner F.J. (1982)]:

$$K_{F,H}(b^{\nu}\mathbf{q}, b^{d\nu}\omega, b(E - E_c)) = b^{-d\nu}K_{F,H}(\mathbf{q}, \omega, E - E_c) \tag{2.142}$$

Taking $\nu = 1/(d-2)$ form Eq. (2.29) for $d = 3$ and $E = E_c$ (i.e. at the mobility edge itself) we get from Eq. (2.142):

$$K_{F,H}(b\mathbf{q}, b^3\omega) = b^{-3}K_{F,H}(\mathbf{q}\omega) \tag{2.143}$$

which is equivalent to:

$$K_{F,H}(\mathbf{q}\omega) = L_{\omega}^3 F_{F,H}(qL_{\omega}) \tag{2.144}$$

where $F_{F,H}(x)$ is some universal function and we introduced characteristic length:

$$L_\omega = [\omega N(E)]^{-1/3} \tag{2.145}$$

Note that the same scaling dependence follows e.g. for $K_H(\mathbf{q}\omega)$ from Eq. (2.84) or Eq. (2.85) after a simple replacement of D_0 by a diffusion coefficient given by:

$$D_{E=E_c}(\mathbf{q}\omega) = L_\omega^{-1} f(qL_\omega) \tag{2.146}$$

where $f(x \to 0) \to 1$ and $f(x \to \infty) \to x$. In particular in the limit of $qL_\omega \to 0$ we get $F(x) = (1+x^4)^{-1/4}$ and the replacement $D_0 \to D_0(\omega/\gamma)^{1/3}$ mentioned in connection with Eq. (2.85) is valid. On the other hand from Eq. (2.140) it follows that at $\omega = 0$ we get from Eq. (2.136):

$$K(\mathbf{q}, \omega = 0, E = E_c) \sim q^{-d} \tag{2.147}$$

which is equivalent to Eq. (2.68) if we take $D_{E=E_c}(\omega = 0, \mathbf{q}) = \alpha q^{d-2}$ (Cf. Eq. (2.86)) in contradiction with self-consistent theory.

Microscopic justification of this scaling hypothesis can be done with one or another variant of field—theory approach based upon nonlinear σ—model [Wegner F.J. (1979a); Wegner F.J. (1979b); Wegner F.J. (1982)]. There exist several alternative schemes of "mapping" of the problem of an electron in a random field onto field—theoretic formalism of nonlinear σ—models [Wegner F.J. (1979b); Wegner F.J. (1979b); Efetov K.B., Larkin A.I., Khmelnitskii D.E. (1980); Schäfer L., Wegner F.J. (1980); McKane A.J., Stone M. (1981); Schäfer L., Pruisken A.M.M. (1982); Efetov K.B. (1983)]. The main physical justification of this approach is to represent an effective Hamiltonian of an electronic system in a form similar to analogous Hamiltonian of Heisenberg ferromagnet below Curie point:

$$\mathcal{H} = \frac{1}{2}\left(\frac{\partial \mathbf{M}}{\partial x_\alpha}\right)^2 - \mathbf{HM}; \qquad \mathbf{M^2} = const \tag{2.148}$$

As a result an effective Hamiltonian for an electron in a random field in terms of interacting modes responsible for the critical behavior close to mobility edge appears. Following Ref.[Efetov K.B., Larkin A.I., Khmelnitskii D.E. (1980)] we can introduce an "order—parameter" as a $2n \times 2n$ matrix

$\hat{Q}$ (n—integer). Every matrix element of $\hat{Q}$ can be represented as:

$$Q_{ij} = \begin{pmatrix} D_{ij} & \Delta_{ij} \\ -\Delta^{\star}_{ij} & D^{\star}_{ij} \end{pmatrix} \tag{2.149}$$

where $D_{ij} = D^{\star}_{ji}$ and $\Delta_{ij} = -\Delta^{\star}_{ji}$, i.e. are elements of Hermitian and anti-symmetric matrices respectively. Analogously $\mathbf{M}^2 = const$ in a ferromagnet $\hat{Q}$—matrix must satisfy the condition:

$$\hat{Q}^2 = 1; \qquad Tr\hat{Q} = 0 \tag{2.150}$$

Effective Hamiltonian for diffusion modes takes the following form [Wegner F.J. (1979a); Wegner F.J. (1979b)]:

$$\mathcal{H} = D_0 Tr(-i\nabla\hat{Q})^2 - i\omega Tr\hat{\Lambda}\hat{Q} \tag{2.151}$$

Here $\hat{\Lambda}$ is the diagonal matrix with first n elements equal to 1 and the remaining n are -1. Correlation function of D—elements corresponds to diffuson, while that of Δ—elements to Cooperon. Parameter n should be put equal to zero at the end of calculations in the spirit of famous "replica trick" in the theory of disordered systems [Ma Shang-keng (1976); Sadovskii M.V. (1981)].

This formalism is useful also for the analysis of different kinds of external perturbations, such as external magnetic field, magnetic impurities, spin—orbital scattering etc. [Efetov K.B., Larkin A.I., Khmelnitskii D.E. (1980)]. Standard methods of renormalization group using perturbation theory over $(p_F l)^{-1} \ll 1$ reproduce all the main results of elementary scaling theory of localization, including the qualitative form of β—function as in Fig. 2.3. However the formalism of σ—model approach is quite complicated and practically does not allow to get explicit expressions for physical characteristics of the system, especially in localized phase.

Many problems of fundamental nature still remain unresolved. Most important are questions concerning the role of nonperturbative contributions close to the mobility edge [Sadovskii M.V. (1981); Sadovskii M.V. (1986); Kravtsov V.E., Lerner I.V. (1985); Efetov K.B. (1983); Altshuler B.L., Kravtsov V.E., Lerner I.V. (1986)]. Note especially strong criticism of one—parameter scaling in Refs.[Kravtsov V.E., Lerner I.V. (1985); Altshuler B.L., Kravtsov V.E., Lerner I.V. (1986)]. Among a lot of results obtained within σ—model approach we wish to mention an important paper by Lerner [Lerner I.V. (1988)] where a distribution function for local density of states in a system close to Anderson transition was determined and shown to be essentially non—Gaussian. Very important results within field-theory approach to localization were obtained in recent years by Suslov [Suslov I.M. (1998)], apparently solving some of the major problems connected with "ghost" poles, appearing in this theory similarly to basic models of quantum field theory [Sadovskii M.V. (1981); Sadovskii M.V. (1986)]. He also formulated quite different analogy (from that discussed above) between ferromagnet and localization problem – that of isotropic ferromagnet with *infinite* number of magnetization components, which allowed him to lift contradictions with self-consistent theory of localization and argue that critical exponents, obtained within this theory, are in fact exact.

For our future analysis it is important to stress that in most cases the results of σ—model approach practically coincide with predictions of self—consistent theory of localization which also neglects all nonperturbative effects, except those determined by some infinite resummation of diagrams. It must be stressed that self—consistent theory is based upon some uncontrollable ad hoc assumptions and in this respect it is not as well justified as σ—model approach. However this simple theory as we have seen above allows practical calculation of any interesting characteristic of an electronic system close to mobility edge including the localized phase.

2.5 Interaction Effects and Anderson Transition

The main unsolved problem of the theory of metal—insulator transition in disordered systems is the role of electron—electron interactions. The importance of interactions for this problem has been known for a long time [Mott N.F. (1974)]. In recent years the decisive importance of interactions was revealed in the theory of "dirty metals" [Altshuler B.L., Aronov A.G. (1979); Altshuler B.L. et al. (1982); Altshuler B.L., Aronov A.G. (1985)], as well as in the concept of Coulomb gap at the Fermi level of strongly localized electrons [Efros A.L., Shklovskii B.L. (1975); Efros A.L. (1976); Shklovskii B.I., Efros A.L. (1979); Efros A.L., Shklovskii B.I. (1985)]. We have already briefly discussed Coulomb gap. It appears for strongly localized states. In case of "dirty metals" diffusive nature of electronic transport leads to special interference effects between Coulomb interaction and disorder scattering [Altshuler B.L., Aronov A.G. (1979); Altshuler B.L.,

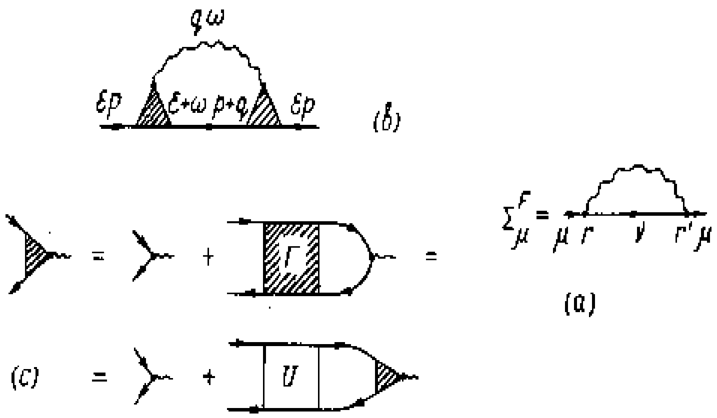

Fig. 2.7 Lowest order interaction corrections: (a) Simplest Fock correction for self— energy in exact eigenstate representation. (b) Equivalent diagram in momentum representation. (c) "Triangular" vertex defining diffusion renormalization. U—irreducible impurity scattering vertex, Γ—full impurity scattering vertex. Wavy line denotes inter-electron interaction.

Aronov A.G. (1985)]. Most important is an appearance of some kind of a precursor to Coulomb gap already in metallic state. It is connected with simple exchange correction to electron self–energy, shown by diagrams of Fig.2.7. This correction leads to the following cusp–like correction to one–particle density of states in case of the screened Coulomb interaction in three–dimensional system [Altshuler B.L., Aronov A.G. (1979)]:

$$\delta N(E) = \frac{|E - E_F|^{1/2}}{2\sqrt{2}\pi^2 D_0^{3/2}} \tag{2.152}$$

where D_0 is the usual Drude diffusion coefficient. In two—dimensional case this correction is logarithmic [Altshuler B.L., Aronov A.G., Lee P.A. (1980); Altshuler B.L., Aronov A.G. (1985)]. General belief is that this cusp somehow transforms into Coulomb gap as system moves from metal to insulator. However, up to now there is no complete solution for this problem, though we shall discuss some attempts below.

Early attempt to describe electron—electron interactions in Anderson insulators in a Fermi—liquid like scheme was undertaken in Ref. [Fleishman L., Anderson P.W. (1980)]. Simple generalization of the theory of "dirty metals" [Altshuler B.L., Aronov A.G. (1979); Altshuler B.L. et al. (1982);

Altshuler B.L., Aronov A.G. (1985)] along the lines of self—consistent theory of localization was proposed in Refs.[Katsnelson M.I., Sadovskii M.V. (1983); Katsnelson M.I., Sadovskii M.V. (1984); Sadovskii M.V. (1986)] and later developed in [Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A. (1997)]. We shall discuss this approach in some detail in the next section. The most general approach to this problem was introduced by McMillan [McMillan W.L. (1981)] who proposed to describe the metal—insulator transition in a disordered system by a scaling scheme similar in spirit to elementary scaling theory of localization of noninteracting electrons discussed above. He formulated a simple system of coupled differential equations of renormalization group for two effective "charges": dimensionless conductance g and single–particle density of states $N(E)$. Later it was realized that this simple scheme can not be correct because it assumed for conductivity the relation like Eq. (2.60) with density of states while the correct Einstein relation for interacting system contains electron compressibility $dn/d\zeta$ (ζ is chemical potential) [Kubo R. (1957); Lee P.A. (1982); Finkelstein A.M. (1983)], which is not renormalized close to the metal—insulator transition as opposed to the density of states. The most comprehensive approach to a scaling description of metal—insulator transition in disordered systems was formulated by Finkelstein [Finkelstein A.M. (1983); Finkelstein A.M. (1984a); Finkelstein A.M. (1984b); Finkelstein A.M. (1990)]. Unfortunately more or less explicit solutions were only obtained neglecting the scattering and interaction processes in Cooper channel which are mainly responsible, as we have seen above, for localization itself. Some attempts in this direction were undertaken only in Ref.[Finkelstein A.M. (1984b)]. This approach is still under very active discussion [Castellani C. et al. (1984); Castellani C., Kotliar G., Lee P.A. (1987); Kotliar G. (1987); Kirkpatrick T.R., Belitz D. (1989); Belitz D., Kirkpatrick T.R. (1989); Kirkpatrick T.R., Belitz D. (1990); Belitz D., Kirkpatrick T.R. (1990); Belitz D., Kirkpatrick T.R. (1991); Belitz D., Kirkpatrick T.R. (1994)] and demonstrate fundamental importance of interactions. However the problem is still unresolved and most of these works consider only the metallic side of transition with no serious attempts to analyze the insulating state. Very interesting direction of current research is that connected with studies of Hubbard model in infinite dimensions ($d = \infty$), where a kind of exact approach to strong correlations is possible [Vollhardt D. (1993); Georges A., et al. (1996)]. Within this approach many aspects of disordered Hubbard model were also studied. However, localization effects are, unfortunately, unaccessible within $d = \infty$ – approximation, while the regular procedure to calculate $1/d$ corrections is still under development.

Below we consider only some qualitative results of this approach, following mainly Refs.[Castellani C., Kotliar G., Lee P.A. (1987); Kotliar G. (1987)]. Fermi liquid theory survives the introduction of disorder [Betbeder-Matibet O., Nozieres P. (1966)], although with some important corrections [Altshuler B.L., Aronov A.G. (1979); Altshuler B.L., Aronov A.G. (1985)], and is actually valid up to metal—insulator transition [Finkelstein A.M. (1983); Finkelstein A.M. (1984a); Castellani C., Kotliar G., Lee P.A. (1987); Kotliar G. (1987)]. In the absence of translation invariance there is no momentum conservation and we have to use some unknown exact eigenstate $\phi_\nu(\mathbf{r})$ representation for electrons in random field to characterize quasi-particles with energies ε_ν (Cf. Ref.[Migdal A.B. (1983)]). The free energy as a functional of quasi—particle distribution function $n_s(\varepsilon_\nu, \mathbf{r})$ (s—spin variable) is written as in usual Fermi liquid theory:

$$F\{n_s(\varepsilon_\nu, \mathbf{r})\} = \sum_{s,\nu} \int d\mathbf{r} n_s(\varepsilon_\nu \mathbf{r})(\varepsilon_\nu - \zeta) + \frac{1}{2} \sum_{ss'} \int d^d\mathbf{r} \delta N_s(\mathbf{r}) \delta N_{s'}(\mathbf{r}) f_{ss'} \tag{2.153}$$

where $N_s = \sum_\nu n_s(\varepsilon_\nu \mathbf{r})$ is the total density per spin and $f_{ss'} = f^s + ss' f^a$ is the quasi—particle interaction function. The angular dependence of f—function in dirty case can be neglected, because $n_s(\varepsilon_\nu \mathbf{r})$ is assumed to describe electrons on distances larger than mean free path there only s—wave scattering is important and Fermi—liquid interaction becomes point—like. In an external spin dependent field V_s the quasi—particle distribution function obeys a kinetic equation:

$$\frac{\partial}{\partial t} n_s - D\nabla^2 n_s + \left(\frac{\partial n_s}{\partial \varepsilon}\right)(-D\nabla^2)[V_s + \sum_{s'} f_{ss'} N_{s'}] = 0 \tag{2.154}$$

where D is quasi—particle diffusion coefficient. Eq. (2.154) is obtained from usual Fermi—liquid kinetic equation [Migdal A.B. (1983)] by replacing $v_F \partial/\partial\mathbf{r}$ by $-D\nabla^2$ which reflects a crossover from ballistic to diffusive transport in disordered system. Solving Eq. (2.154) for density—density and spin—spin response functions one gets [Finkelstein A.M. (1983); Finkelstein A.M. (1984a); Castellani C. et al. (1984)]:

$$\chi_\rho(\mathbf{q}\omega) = \frac{dn}{d\zeta}\frac{D_\rho q^2}{D_\rho q^2 - i\omega} \tag{2.155}$$

$$\chi_s(\mathbf{q}\omega) = \frac{\chi D_s q^2}{D_s q^2 - i\omega} \tag{2.156}$$

where $dn/d\zeta = N(E_F)/(1+F_0^s)$, $\chi = N(E_F)\mu_B^2/(1+F_0^a)$ (μ_B is Bohr's magneton) and

$$D_\rho = D(1+F_0^s) \tag{2.157}$$

$$D_s = D(1+F_0^a) \tag{2.158}$$

Landau parameters $F_0^{s,a}$ are defined by

$$N(E_F)f^s = F_0^s \qquad N(E_F)f^a = F_0^a \tag{2.159}$$

Here $N(E_F)$ is quasi-particle density of states at the Fermi level (for both spin directions). If we neglect Fermi—liquid renormalization effects Eq. (2.155) reduces to Eq. (2.49). Conductivity is given now by $\sigma = e^2D(dn/d\zeta)$.

As system moves towards metal—insulator transition Hubbard—like interaction of electrons close to a given impurity site becomes more and more important. It has been known for a long time [Mott N.F. (1974); Sadovskii M.V. (1986)] that this interaction leads to the appearance of a band of single—occupied states just below the Fermi level of a system on the dielectric side of Anderson transition. These states actually simulate paramagnetic centers and lead to Curie—like contribution (diverging as temperature $T \to 0$) [Mott N.F. (1974); Sadovskii M.V. (1986)]. Thus on the metallic side of transition static magnetic susceptibility χ is expected to diverge since it is infinite (at $T = 0$) on the insulating side. At the same time $dn/d\zeta$ remains finite. Therefore $D_s/D_\rho = (dn/d\zeta)/\chi$ goes to zero, i.e. *spin diffusion is much slower than charge diffusion* close to metal—insulator transition. This fact was first noted in Ref. [Finkelstein A.M. (1984b)] where it was assumed that it leads to a possibility of local magnetic appearing in metallic phase before a transition. It is interesting to note that the slowing down of spin diffusion due to interactions was actually discovered long before [Fulde P., Luther A. (1968)] it appeared in the context of interaction picture of metal—insulator transition. This idea was further elaborated in Refs. [Kirkpatrick T.R., Belitz D. (1990); Belitz D., Kirkpatrick T.R. (1990); Belitz D., Kirkpatrick T.R. (1991)], where extensive discussion of this magnetic transition was given. There is an interesting problem why these localized moments are not quenched by the Kondo effect. This can apparently be explained by the local fluctuations of Kondo temperature due to fluctuations of local density of states induced by disorder [Dobrosavljevič V., Kirkpatrick T.R., Kotliar G.] . The resulting distribution of Kondo temperatures is shown to be singular enough to induce diverging magnetic susceptibility as $T \to 0$.

The idea of paramagnetic moments appearing already in metallic phase apparently can much simplify the analysis of metal—insulator transition and allow its description by equations of elementary scaling theory of localization [Altshuler B.L., Aronov A.G. (1983a); Altshuler B.L., Aronov A.G. (1983b); Altshuler B.L., Aronov A.G. (1985)]. In general case electron interactions in diffusion channel can be classified by total spin of an electron and hole j [Altshuler B.L., Aronov A.G. (1985)]. It can be shown that all interaction corrections with $j = 0$ do not depend on electron—electron coupling constant (charge) and are universal [Altshuler B.L., Aronov A.G. (1985)]. If paramagnetic scattering is operating in the system it dumps scattering processes in Cooper (localization) channel [Lee P.A. (1980)] as well as interaction processes in diffusion channel with $j = 1$ [Altshuler B.L., Aronov A.G. (1985)]. In this case only interaction processes with $j = 0$ determine corrections to classical (Drude) conductivity. Due to universal nature of these corrections (independence of electronic charge) their structure is actually coincide with that of localization corrections (Cooperon) [Altshuler B.L., Aronov A.G. (1983a); Altshuler B.L., Aronov A.G. (1983b)]. This means that renormalization group has only one effective "charge" — dimensionless conductance g. In this case differential equation for the conductance of a finite system is again given by Eq. (2.18) with the same asymptotic forms of $\beta_d(g)$. This approach is valid for systems with linear size $L < L_T = \sqrt{\hbar D/T}$. This length L_T replaces in the theory of interacting electrons characteristic length of phase coherence L_φ of non-interacting theory. The appearance of this new length is due to the fact that characteristic time of interaction processes [Altshuler B.L., Aronov A.G. (1985)] is $\sim \hbar/T$. We must stress that these arguments are probably oversimplified as Refs.[Finkelstein A.M. (1983); Finkelstein A.M. (1984a); Finkelstein A.M. (1984b); Castellani C. et al. (1984)] had demonstrated the relevance of interaction in the sense of appearance of additional coupling constants ("charges"). Also it is in no way clear that local moments appearing within this approach are acting just as usual paramagnetic scatterers. However, the simple scheme following from Refs.[Altshuler B.L., Aronov A.G. (1983a); Altshuler B.L., Aronov A.G. (1983b)] seems to be too attractive on physical grounds just to be neglected. As in noninteracting case for $d = 3$ Eq. (2.18) again possesses unstable fixed point responsible for the existence of mobility edge and absence of minimal metallic conductivity at the metal—insulator transition. However, in this case there are no special reasons to believe that the critical exponent ν of localization correlation length ξ_{loc} will coincide with its value for noninteracting theory. At finite temperatures as in usual scaling picture conductivity for $d = 3$ is given by [Altshuler B.L., Aronov A.G. (1983a); Altshuler B.L., Aronov A.G. (1983b); Altshuler B.L., Aronov A.G. (1985)]:

$$\sigma \approx \frac{e^2}{\hbar\xi_{loc}} f\left(\frac{\xi_{loc}}{L_T}\right) \tag{2.160}$$

As system approaches insulating phase $\xi_{loc} \to \infty$. For $\xi_{loc} \ll L_T$ we have $f(\xi_{loc}/L_T) = A + B(\xi_{loc}/L_T)$, where A and B are some numerical constants. Thus in this region conductivity corrections are proportional to $\sqrt{T}$ [Altshuler B.L., Aronov A.G. (1979)]. In case of $\xi_{loc} \gg L_T$, i.e. very close to transition:

$$\sigma \approx C\frac{e^2}{\hbar L_T} = C\frac{e^2}{\hbar}\sqrt{T/D\hbar} \tag{2.161}$$

where again $C \sim 1$. Using Einstein relation [Kubo R. (1957)] $\sigma = e^2 D(dn/d\zeta)$ we immediately obtain:

$$D = \frac{C^{2/3}}{\hbar} T^{1/3} \left(\frac{dn}{d\zeta}\right)^{-2/3} \tag{2.162}$$

and

$$\sigma = C^{2/3}\frac{e^2}{\hbar}\left(T\frac{dn}{d\zeta}\right)^{1/3} \tag{2.163}$$

which is valid for $L_T < \xi_{loc}$, where $L_T = [C/(Tdn/d\zeta)]^{1/3}$.

In case of a system in alternating electric field with frequency $\omega \gg T/\hbar$ the relevant length becomes $L_\omega = [D/\omega]^{1/2}$ as in Eq. (2.37). Accordingly for $L_\omega \ll \xi_{loc}$ instead of Eq. (2.163) we get:

$$\sigma(\omega) \approx \frac{e^2}{\hbar}\left(\omega\frac{dn}{d\zeta}\right)^{1/3} \tag{2.164}$$

which is analogous to Eq. (2.41) and Eq. (2.82). However we must note that this result can not be considered very reliable since the dynamical critical exponent in general case is an independent one [Finkelstein A.M. (1984a); Finkelstein A.M. (1984b)].

The metal—insulator transition can be viewed as a gradual breakdown of the Fermi—liquid state [Kotliar G. (1987)]. As we approach the transition different Fermi—liquid parameters, such as D, $N(E_F)$, χ etc. change continuously and at a critical point some of these may either diverge or go to zero. This behavior is related to the divergence of correlation length ξ_{loc} characterized by a critical exponent ν. On the insulating side of the transition this length can be also interpreted as the scale inside which a Fermi liquid description of the system still holds.

2.5.1 *Self-consistent theory with interactions*

At present we are in a need of some kind of new approach to the theory of interacting electrons in disordered systems which probably may be formulated along the lines of the self-consistent theory of localization. The hope is to provide an effective formalism to calculate the basic physical properties of the system in an interpolating scheme from metallic to insulating state. Below we briefly describe an attempt to construct such self-consistent approach [Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A. (1997)]. The basic idea is in equal footing (additive) treatment of both localization and interaction corrections to the current relaxation kernel defining the generalized diffusion coefficient in Eq.(2.47). As a zeroth approximation we take the Drude metal and consider the simplest localization and interaction corrections, so that the relaxation kernel takes the following form:

$$M(\omega) = M_0 + \delta M(\omega) \tag{2.165}$$

where $\delta M(\omega) = \delta M_l(\omega) + \delta M_c(\omega) = -\frac{M_0}{D_0}(\delta D_l(\omega) + \delta D_c(\omega))$. Here the localization correction to diffusion coefficient $D_l(\omega)$ is defined by the usual sum of "maximally crossed" diagrams which yields:

$$\frac{\delta D(\omega)}{D_0} = -\frac{1}{\pi N_0(E_F)} \sum_{|\mathbf{q}|<k_0} \frac{1}{-i\omega + D_0 q^2} \tag{2.166}$$

while the Coulomb correction $D_c(\omega)$ is given by

$$\frac{\delta D_c(\omega)}{D_0} = \frac{\delta\sigma(\omega)}{2e^2 N_0(E_F) D_0} = \tag{2.167}$$

$$= \frac{8i}{\pi d}\mu D_0 \frac{1}{\pi N_0(E_F)} \int_{\omega}^{\infty} d\Omega \sum_{|\mathbf{q}|<k_0} \frac{q^2}{(-i(\Omega+\omega) + D(\Omega+\omega)q^2)(-i\Omega + D(\Omega)q^2)^2}$$

where $\mu = N_0(E_F)v_0$ is the dimensionless point-like interaction with $N_0(E_F)$ now denoting the single-spin density of states at the Fermi level for noninteracting case. Conductivity correction $\delta\sigma$ due to interactions was defined by the lowest-order diagrams shown in Fig.2.8 which were for the first time analyzed in Ref.[Altshuler B.L. et al. (1980)], neglecting localization corrections. Physically this means, that we assume rather weak interaction between electrons and disorder remains crucial parameter, determining the metal-insulator transition. Strongly correlated systems (e.g. undergoing Mott transition) are definitely outside our scheme.

It was shown in Ref.[Altshuler B.L. et al. (1980)] that the total contribution of diagrams (a), (b) and (c) is actually zero and conductivity correction reduces to that determined by diagrams (d) and (e). Here we neglect also the so called Hartree corrections to conductivity [Altshuler B.L., Aronov A.G. (1985); Altshuler B.L. et al. (1980)], which is valid in the limit of $2k_F/\kappa_D \gg 1$, where κ_D - is the inverse screening length. This inequality, strictly speaking, is valid for systems with low electronic density, which are most interesting for experimental studies of disorder induced metal-insulator transitions. Also, if we remember the divergence of screening length at the metal-insulator transition, we can guess that this approximation becomes better as we approach the transition. The point-like interaction model used above has to be understood only in this sense.

Self-consistency procedure is reduced to the replacement of D_0 by the generalized diffusion coefficient in all diffusion denominators. As a result we obtain the following *integral* equation for the generalized diffusion equation:

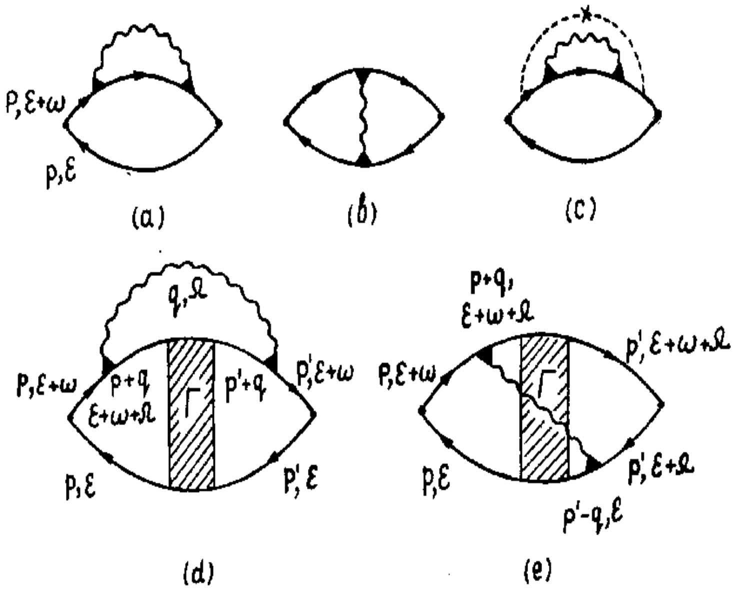

Fig. 2.8 Lowest order interaction corrections to conductivity.

$$\frac{D_0}{D(\omega)} = 1 + \frac{1}{\pi N_0(E_F)} \int_{|\mathbf{q}|<k_0} \frac{d^d q}{(2\pi)^d} \frac{1}{-i\omega + D(\omega)q^2} -$$

$$-\frac{8i}{\pi d}\mu D_0 \frac{1}{\pi N_0(E_F)} \int_{\omega}^{\infty} d\Omega \int_{|\mathbf{q}|<\mathbf{k_0}} \frac{d^d q}{(2\pi)^d}$$

$$\frac{q^2}{(-i(\Omega+\omega) + D(\Omega+\omega)q^2)(-i\Omega + D(\Omega)q^2)^2} \tag{2.168}$$

This equation forms the basis of the proposed self-consistent approach. In the absence of interactions ($\mu = 0$) it obviously reduces to the usual self-consistent theory of localization. Cut-off in momentum space is assumed here to be $k_0 = x_0 Min[p_F, l^{-1}]$ with $x_0 \sim 1$. Let us transform it to dimensionless imaginary Matsubara frequencies which is simpler for further analysis as it makes our integral equation real: $\frac{-i\omega}{D_0 k_0^2} \rightarrow \omega$, $\frac{-i\Omega}{D_0 k_0^2} \rightarrow \Omega$, and also introduce the dimensionless diffusion coefficient $d(\omega) = \frac{D(\omega)}{D_0}$. In these notations integral equation (2.168) takes the following form:

$$\frac{1}{d(\omega)} = 1 + \frac{1}{d(\omega)} d\lambda x_0^{d-2} \int_0^1 \frac{dy y^{d-1}}{y^2 + \frac{\omega}{d(\omega)}} +$$

$$+\frac{8}{\pi}\mu\lambda x_0^{d-2} \int_\omega^\infty \frac{d\Omega}{d(\omega+\Omega)d^2(\Omega)} \int_0^1 \frac{dy y^{d+1}}{\left(y^2 + \frac{\omega+\Omega}{d(\omega+\Omega)}\right)\left(y^2 + \frac{\Omega}{d(\Omega)}\right)^2} \quad (2.169)$$

where $\lambda = \gamma/\pi E_F = 1/2\pi E_F \tau$ is the usual disorder parameter. In the following we shall limit ourselves only to the case of spatial dimension d=3. Diffusion coefficient of the self-consistent theory of localization (2.79) in these notations reduces to:

$$d(\omega) = \begin{cases} \alpha = 1 - 3\lambda x_0 \approx \frac{E_F - E_c}{E_c} & \omega \ll \omega_c, \quad \alpha > 0 \quad \text{Metal} \\ \left(\frac{\pi}{2} 3\lambda x_0\right)^{\frac{2}{3}} \omega^{\frac{1}{3}} & \omega \gg \omega_c \quad \text{Metal \& Insulator} \\ \frac{\left(\frac{\pi}{2} 3\lambda x_0\right)^2}{\alpha^2} \omega = (\xi_{loc} k_0)^2 \omega & \omega \ll \omega_c, \quad \alpha < 0 \quad \text{Insulator} \end{cases} \quad (2.170)$$

where $\omega_c = \frac{|\alpha|^3}{\left(\frac{\pi}{2} 3\lambda x_0\right)^2}$ and ξ_{loc} - is the localization length, x_0 - the dimensionless cutoff. Let us introduce $K(\omega) = \frac{\omega}{d(\omega)}$ and analyze Eq.(2.169) assuming that $K(\omega)$, $K(\Omega)$ and $K(\omega + \Omega) \ll 1$. Expanding the right-hand side of Eq.(2.169) over these small parameters we obtain:

$$\frac{\alpha}{d(\omega)} = 1 - \frac{\pi}{2} \frac{3\lambda x_0}{d(\omega)} K^{1/2}(\omega) +$$

$$+2\mu\lambda x_0 \int_\omega^\infty \frac{d\Omega}{d(\omega+\Omega)d^2(\Omega)} \frac{K^{1/2}(\Omega) + 2K^{1/2}(\omega+\Omega)}{\left(K^{1/2}(\Omega) + K^{1/2}(\omega+\Omega)\right)^2} \quad (2.171)$$

Consider the metallic phase and look for diffusion coefficient $d(\omega)$ solution in the following form:

$$d(\omega) = \begin{cases} d & \omega \ll \omega_c \\ d\left(\frac{\omega}{\omega_c}\right)^{\frac{1}{3}} & \omega \gg \omega_c \end{cases} \quad (2.172)$$

Substituting (2.172) into Eq.(2.171) we find d and ω_c and for the diffusion coefficient we obtain:

$$d(\omega) = \begin{cases} \alpha - \alpha^* & \omega \ll \omega_c \\ \left(\frac{\pi}{2} 3\lambda x_0\right)^{\frac{2}{3}} \omega^{\frac{1}{3}} & \omega \gg \omega_c \end{cases} \tag{2.173}$$

where $\omega_c = \frac{|\alpha - \alpha^*|^3}{\left(\frac{\pi}{2} 3\lambda x_0\right)^2}$, $\alpha^* = c\mu$, $c \approx 0.89$.

Thus for the metallic phase we come to a very simple qualitative conclusion — Anderson transition persists and the conductivity exponent remains $\nu = 1$. The transition itself has shifted to the region of weaker disorder $\alpha = \alpha^* = c\mu$–interaction facilitates transition to insulating state. The frequency behavior of diffusion coefficient in metallic phase is qualitatively similar to that in the usual self-consistent theory of localization (2.170). In the region of high frequencies $\omega \gg \omega_c$ the behavior of diffusion coefficient remains unchanged after the introduction of interelectron interactions.

Consider now the insulating phase. In the region of high frequencies $\omega \gg \omega_c$ the diffusion coefficient obviously possesses the frequency dependence like $d(\omega) \sim \omega^{1/3}$. Assume that for small frequencies it is also some power of the frequency:

$$d(\omega) = \begin{cases} d\left(\frac{\omega}{\omega_c}\right)^{\delta} & \omega \ll \omega_c \\ d\left(\frac{\omega}{\omega_c}\right)^{\frac{1}{3}} & \omega \gg \omega_c \end{cases} \tag{2.174}$$

where δ is some exponent to be determined.

Substituting (2.174) into (2.171) and considering the case of $\alpha < 0$ (insulating phase of the usual self-consistent theory of localization) and $|\alpha| \gg \alpha^*$, we get:

$$d(\omega) = \begin{cases} \frac{\left(\frac{\pi}{2} 3\lambda x_0\right)^2}{\alpha^2}\omega = (\xi_{loc} k_0)^2 \omega & \omega^* \ll \omega \ll \omega_c \\ \left(\frac{\pi}{2} 3\lambda x_0\right)^{\frac{2}{3}} \omega^{\frac{1}{3}} & \omega \gg \omega_c \end{cases} \tag{2.175}$$

where $\omega_c = \frac{|\alpha|^3}{\left(\frac{\pi}{2} 3\lambda x_0\right)^2}$, while $\omega^* \approx 0.1\mu \frac{\alpha^2}{\left(\frac{\pi}{2} 3\lambda x_0\right)^2} = 0.1 \frac{\mu}{(\xi_{loc} k_0)^2}$ —is some new characteristic frequency defined by the interactions. Note that $\omega^* \to 0$ as we approach the transition point when $\xi_{loc} \to \infty$.

Thus, sufficiently deep inside the insulating phase when $\alpha < 0$ $|\alpha| \gg \alpha^*$ and for the frequencies $\omega \gg \omega^*$, the diffusion coefficient remains the same as in the self-consistent theory of localization, i.e. at small frequencies it is linear over frequency, while for the higher frequencies it is $\sim \omega^{1/3}$.

The analysis of Eq.(2.171) shows that for the frequencies $\omega \ll \omega^*$ it is impossible to find the power-like dependence for $d(\omega)$, i.e. the diffusion coefficient in the insulating phase is apparently can not be represented in the form of $d(\omega) = d\frac{\omega^*}{\omega_c}\left(\frac{\omega}{\omega^*}\right)^\delta$, where δ - is some unknown exponent. Because of this we were unable to find any analytical treatment of Eq.(2.171) in the region of $\omega \ll \omega^*$ within the insulating phase.

Consider now the system behavior not very deep inside the insulating phase when $\alpha - \alpha^* < 0$ while $\alpha > 0$, that is when the system without interaction would be within the metallic phase. Let us assume that the frequency behavior of the diffusion coefficient for $\omega \ll \omega_c$ possesses the power-like form, i.e. the diffusion coefficient is defined by the expression (2.174). Substituting (2.174) into (2.171) we get $\delta = \frac{1}{3}$. As a result for the diffusion coefficient we get:

$$d(\omega) = \begin{cases} \left(4.2\frac{\mu\lambda x_0}{\alpha}\right)^{\frac{2}{3}} \omega^{\frac{1}{3}} & \omega \ll \omega_c \\ \left(\frac{\pi}{2}3\lambda x_0\right)^{\frac{2}{3}} \omega^{\frac{1}{3}} & \omega \gg \omega_c \end{cases} \tag{2.176}$$

where $\omega_c = \frac{|\alpha-\alpha^*|^3}{\left(\frac{\pi}{2}3\lambda x_0\right)^2}$. Naturally, the exact solution for the diffusion coefficient should show the continuous change of frequency behavior around $\omega \sim \omega_c$.

Thus, within the insulating phase close to transition point, where the system without interactions should have been metallic, the diffusion coefficient behaves as $\sim \omega^{1/3}$, everywhere, though for the low frequency region the coefficient of $\omega^{1/3}$ differs from that of the usual self-consistent theory of localization and explicitly depends upon the interaction constant.

We have also performed the numerical analysis of the integral equation (2.169) for the wide region of frequencies, both for metallic (Fig.2.9) and insulating phases (Fig.2.10).

Solution was achieved by a simple iteration procedure using the results of the usual self-consistent theory of localization as an initial approximation. Numerical data are in good correspondence with our analytical estimates. In the region of high frequencies, both for metallic and insulating phases, the

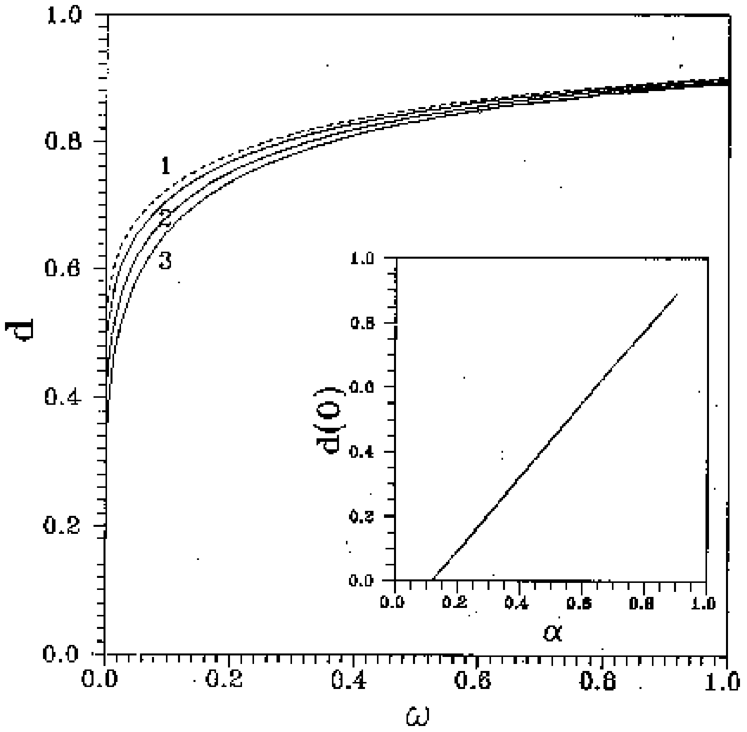

Fig. 2.9 Dependence of dimensionless generalized diffusion coefficient on dimensionless Matsubara frequency in metallic phase ($\alpha = 0.5$), obtained by numerical solution for different values of μ: **1.** 0.24; **2.** 0.6; **3.** 0.95; Dashed line — the usual self-consistent theory of localization, $\mu = 0$. At the insert: Dependence of static diffusion coefficient ($d = \frac{D(0)}{D_0}$) on disorder for $\mu = 0.24$.

frequency behavior of diffusion coefficient is very close to that defined by the usual self-consistent theory of localization. In the region of small frequencies within the metallic phase diffusion coefficient $d(\omega)$ diminishes as interaction grows. Dependence of static generalized diffusion coefficient on disorder for $\mu = 0.24$ is shown at the insert of Fig.2.9, and is practically linear. Metal-insulator transition in this case is observed at $\alpha = \alpha^* = \mu$, where $c \approx 0,5$, which is also in good correspondence with our qualitative analysis. Within the insulating phase for the region of small frequencies ($\omega \ll \omega^*$) we observe significant deviations from predictions of the usual self-consistent theory of localization. Diffusion coefficient is apparently nonanalytic in frequency

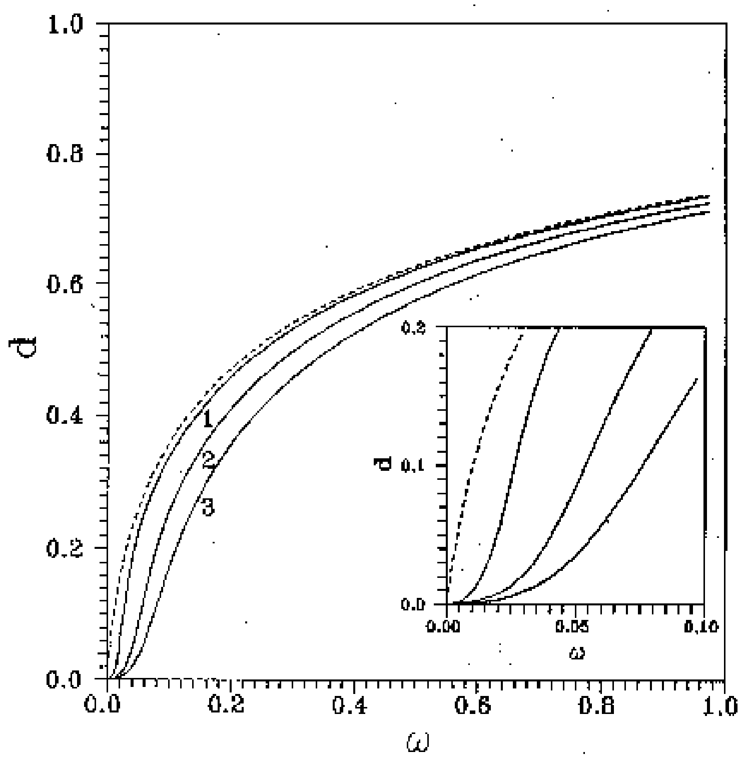

Fig. 2.10 Dependence of dimensionless generalized diffusion coefficient on dimensionless Matsubara frequency in dielectric phase ($\alpha = -0.5$), obtained by numerical solution for different values of μ: **1.** 0.12; **2.** 0.6; **3.** 1.2; Dashed line — the usual self-consistent theory of localization, $\mu = 0$.

here and we clearly see the tendency to formation of some kind of effective gap for the frequencies $\omega \ll \omega^*$, with this "gap" closing as interactions are turned off.

The above numerical analysis was performed in Matsubara frequency region, which was used in writing down the Eq.(2.169). For real frequencies similar calculations were done in [Kuchinskii E.Z., Erkabaev M.A. (1997)]. Of course, it requires more heavy numerical work to solve coupled system of integral equations for real and imaginary parts of generalized diffusion coefficient, which is necessary in this case. Appropriate results are shown in Figs. 2.11,2.12,2.13.

In Refs.[Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A.

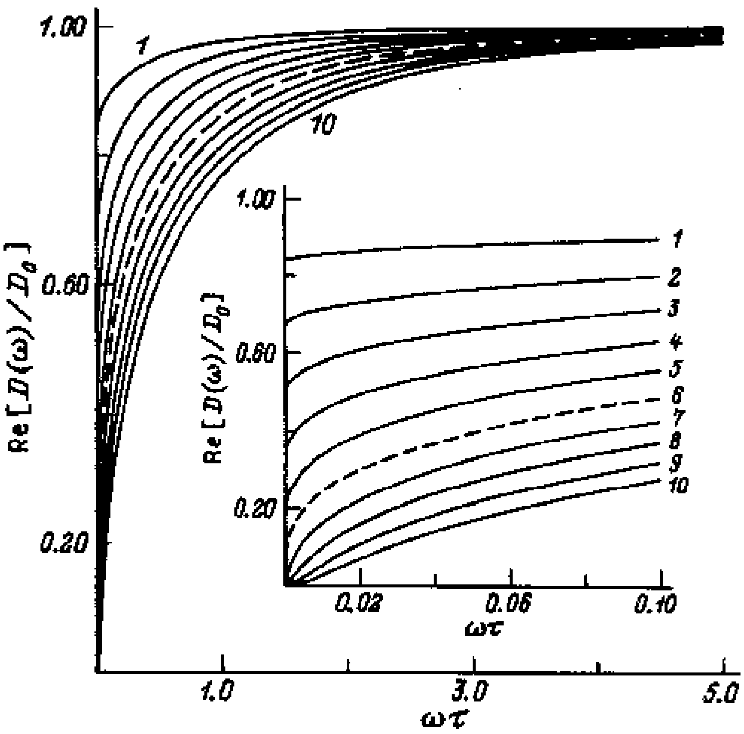

Fig. 2.11 Frequency dependence of the real part of generalized diffusion coefficient for dimensionless interaction parameter $\frac{8}{3\pi}\mu = 1$ and different values of disorder parameter $3\lambda x_0$: 1 – 0.1,...,5 – 0.5 – in metallic phase; 7 – 0.7,...,10 – 1.0 – in insulating phase. Dashed curve 6 corresponds to transition point: $3\lambda x_0 \approx 0.6$.

(1997)] a study was made of the gradual evolution of the tunneling density of states from metallic to insulating region, demonstrating the continuous transformation of a cusp singularity of Eq.(2.152) in a metal into a kind of interaction induced pseudogap at the Fermi level in an insulator, which is in some respects similar to the Coulomb gap of Refs.[Efros A.L., Shklovskii B.L. (1975); Efros A.L. (1976); Shklovskii B.I., Efros A.L. (1979); Efros A.L., Shklovskii B.I. (1985)]. Consider the effects of interelectron interactions upon the single-particle ("tunneling") density of states which is

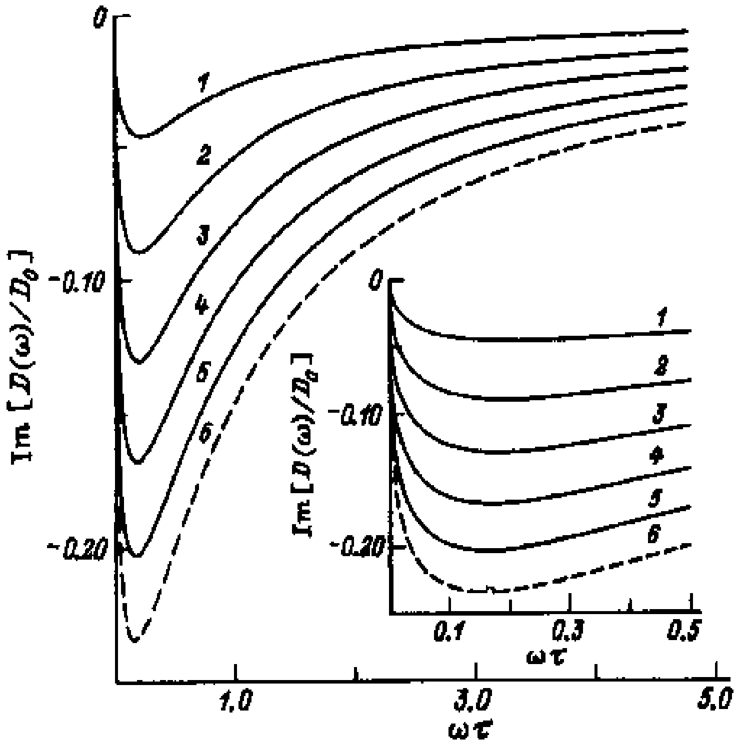

Fig. 2.12 Frequency dependence of the imaginary part of generalized diffusion coefficient for dimensionless interaction parameter $\frac{8}{3\pi}\mu = 1$ and different values of disorder parameter $3\lambda x_0$: 1 – 0.1,...,6 – 0.6 – in metallic phase up to metal-insulator transition at $3\lambda x_0 \approx 0.6$.

defined by the well-known relation:

$$N(\varepsilon) = -\frac{1}{\pi}\int \frac{d^3\mathbf{p}}{(2\pi)^3} ImG^R(\mathbf{p},\varepsilon) \tag{2.177}$$

where $\varepsilon = E - E_F$ - is electronic energy with respect to the Fermi level and $G^R(\mathbf{p},\varepsilon)$ - is the retarded Green's function defined by:

$$G^R(\mathbf{p},\varepsilon) = \frac{1}{\varepsilon - \xi_{\mathbf{p}} + i\gamma - \Sigma^R_{ee}(\varepsilon,\mathbf{p})} \tag{2.178}$$

where $\xi_{\mathbf{p}} = v_F(|\mathbf{p}| - p_F)$. Consider the so called "Fock" contribution to the self-energy part $\Sigma^R_{ee}(\varepsilon,\mathbf{p})$ given by diagrams shown in Fig.2.7(b,c):

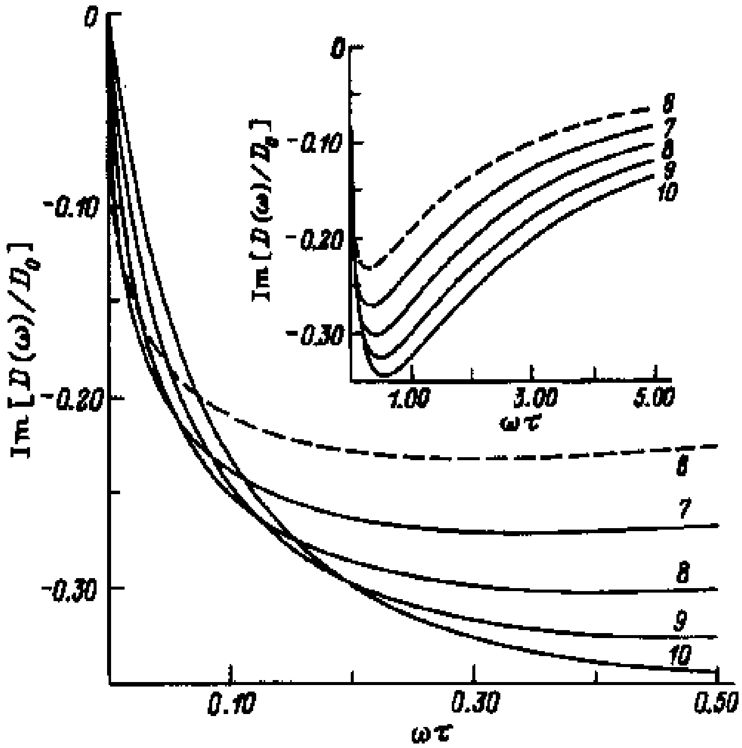

Fig. 2.13 Frequency dependence of the imaginary part of generalized diffusion coefficient for dimensionless interaction parameter $\frac{8}{3\pi}\mu = 1$ and different values of disorder parameter $3\lambda x_0$: 6 – 0.6,...,10 – 1.0 – in insulating phase. Dashed curve 6 corresponds to transition point: $3\lambda x_0 \approx 0.6$.

$$\Sigma^R_{ee}(\varepsilon,\mathbf{p}) = i\int \frac{d^3\mathbf{q}}{(2\pi)^3}\int_\varepsilon^{1/\tau}\frac{d\omega}{2\pi}G_0^A(\mathbf{p}-\mathbf{q},\varepsilon-\omega)v(\mathbf{q})\gamma^2(\mathbf{q},\omega) \approx$$

$$\approx i\gamma^2\mu G_0^A(\mathbf{p},\varepsilon)\left(f(\varepsilon)+ig(\varepsilon)\right), \tag{2.179}$$

where $f(\varepsilon) = ReS(\varepsilon)$ and $g(\varepsilon) = ImS(\varepsilon)$ with

$$S(\varepsilon) = 4N_0^{-1}(0)\int_{|\mathbf{q}|<\mathbf{k_0}}\frac{d^3\mathbf{q}}{(2\pi)^3}\int_\varepsilon^{1/\tau}\frac{d\omega}{2\pi}\frac{1}{\left(-i\omega+D_E(\omega)q^2\right)^2} \tag{2.180}$$

and $G_0^A(\mathbf{p},\varepsilon)$ – the advanced Green's function without interaction contributions. Here and in the following we denote as $N_0(0)$ the density of states at the Fermi level in the absence of interactions.

Substituting the expression for $\Sigma^R_{ee}(\varepsilon,\mathbf{p})$ from (2.179) into (2.178) we obtain as $\varepsilon \to 0$:

$$\frac{N(\varepsilon)}{N_0(0)} \approx -\frac{1}{\pi}\int_{-\infty}^{\infty} d\xi_{\mathbf{p}} Im G^R(\mathbf{p},\varepsilon) = \tag{2.181}$$

$$= \frac{\gamma}{\pi}\int_{-B}^{B} d\xi_{\mathbf{p}} \frac{\xi_{\mathbf{p}}^2+\gamma^2+\mu\gamma^2 g(\varepsilon)}{\left(\xi_{\mathbf{p}}^2+\gamma^2+\mu\gamma^2 g(\varepsilon)\right)^2+\left(\mu\gamma^2 f(\varepsilon)\right)^2}$$

where $2B$ – is the effective band width. Qualitative analysis of these expressions for the density of states was performed in [Kuchinskii E.Z. et al. (1995)]. Here we shall present only some of the results, obtained after extensive numerical calculations in [Kuchinskii E.Z., Erkabaev M.A. (1997)], solving the nonlinear integral equation (2.168) both in metallic and insulating phases and using these results to calculate density of states. The results of these calculations for the band of the finite width with $B = E_F$ are shown in Fig.2.14. These results demonstrate the formation of some kind of "Coulomb" pseudogap at the Fermi level as system moves through metal-insulator transition with the growth of disorder. Square-root cusp at the Fermi level in metallic phase transforms to rather narrow (logarithmic) singularity at the transition point and then to power-like behavior in insulating phase †. Of course, this last behavior is similar, but not that of "Coulomb gap" of Refs.[Efros A.L., Shklovskii B.L. (1975); Efros A.L. (1976); Shklovskii B.I., Efros A.L. (1979); Efros A.L., Shklovskii B.I. (1985)], which is due to long-range Coulomb forces, totally neglected in our calculations. The effective width of our "pseudogap" grows with the growth of disorder.

Similar behavior of the density of states was observed experimentally in a number of disordered systems close to the metal-insulator transition [Lee P.A., Ramakrishnan T.V. (1985); Belitz D., Kirkpatrick T.R. (1994)], from amorphous alloys [McMillan W.L., Mochel J. (1981); Imry Y., Ovadyahu Z.

†In the case of the band of infinite width square-root behavior persists in insulating phase [Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A. (1997)]

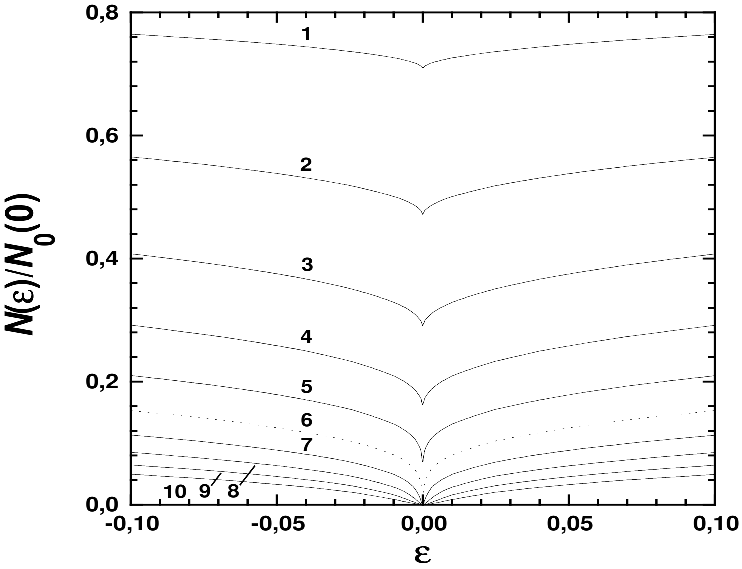

Fig. 2.14 Density of states in the case of the band with finite width $2E_F$ for dimensionless interaction parameter $\frac{8}{3\pi}\mu = 1$ and different values of disorder parameter $3\lambda x_0$: $1 - 0.1$,...,$5 - 0.5$ – in metallic phase; $7 - 0.7$,...,$10 - 1.0$ – in insulating phase. Dashed curve 6 corresponds to transition point at $3\lambda x_0 \approx 0.6$. Energies are given in units of $D_0 k_0^2$.

(1982); Hertel G. et al. (1983)] to disordered single-crystals of copper-oxides [Srikanth H. et al. (1992)]. Note, however, that though our results are very similar to these experiments qualitatively, they fail quantitatively as in the experiments the effective width of the "pseudogap" at the metal-insulator transition itself is rather large and is definitely not logarithmic. At the same time, our results provide rather attractive interpolation scheme qualitatively valid in large interval of system parameters – from slightly disordered metal up to disorder–induced metal-insulator transition and through to insulating phase. The general drop of the density of states in rather wide energy interval with the growth of disorder is also similar to that observed in the experiments.

2.5.2 *Disorder in "Marginal" Fermi–liquid*

For high—T_c superconductors problems of interplay of localization and interactions become especially important because of unusual nature of normal state of these systems. In the absence of accepted theory of this normal state we shall limit ourselves only to few remarks on one specific model. The so called "marginal" Fermi—liquid theory [Varma C.M. et al. (1989); Kuroda Y., Varma C.M. (1990)] is a promising semi–phenomenological description of both normal and superconducting properties of these systems. We shall see that localization effects are apparently greatly enhanced in this case [Kotliar G., Varma C.M. (1990); Varma C.M. (1990)].

Basically the idea of "marginal" Fermi–liquid is expressed by the following form of one–particle Green's function [Varma C.M. et al. (1989)]:

$$G(E\mathbf{p}) = \frac{Z_{\mathbf{p}}}{\varepsilon - \xi_{\mathbf{p}} - i\gamma_{\mathbf{p}}} + G_{incoh} \tag{2.182}$$

where $\xi_{\mathbf{p}}$ is renormalized quasi—particle energy, $\gamma_{\mathbf{p}} \sim Max[\varepsilon, T]$ is anomalous (linear) decay—rate for these quasiparticles which is quite different from quadratic in ε or T decay—rate of the usual Fermi—liquid theory [Migdal A.B. (1983)]. The concept of "marginality" arises due to peculiar behavior of quasiparticle residue:

$$Z_{\mathbf{p}}^{-1} \approx ln\frac{\tilde{\omega}_c}{|\xi_{\mathbf{p}}|} \approx ln\frac{\tilde{\omega}_c}{|\varepsilon|} \tag{2.183}$$

where $\tilde{\omega}_c$ is characteristic frequency scale of some kind of electronic excitations, which is the phenomenological parameter of the theory. From Eq. (2.183) it is clear that quasiparticle contribution to Green's function Eq. (2.182) vanishes precisely at the Fermi level, while exists close to it though with logarithmically reduced weight. Note that in the case of usual Fermi—liquid $Z_{\mathbf{p}} \approx 1$ [Migdal A.B. (1983)].

For disordered system we can estimate the impurity contribution to the scattering rate of quasi—particles as (it is given by diagram similar to that in Fig.2.7(b), but with wavy dynamic interaction line replaced by dashed static impurity line) [Kotliar G., Varma C.M. (1990)]:

$$\gamma = 2\rho V^2 Z Im \sum_{\mathbf{p}} \Lambda^2(\mathbf{p}+\mathbf{q},\mathbf{p}) G(\mathbf{p}+\mathbf{q}\varepsilon) \approx$$
$$2\pi\rho V^2 Z^2 \Lambda^2(\mathbf{q}\to 0) N(E_F) \approx Z\Lambda^2\gamma_0 \qquad (2.184)$$

where Λ is the appropriate vertex—part renormalized by Fermi—liquid effects, ρ again is impurity concentration, V is impurity potential and $N(E_F) = Z^{-1}N_0(E_F)$ is the renormalized density of states in the Fermi—liquid. Here $N_0(E_F)$ is density of states for noninteracting electrons at the Fermi level, γ_0 is scattering rate for noninteracting case. To get the last relation in Eq. (2.184) a weak dependence of vertices and self—energy on momentum was assumed. Now we can use the Ward identity for $\Lambda(\mathbf{q}\to 0\omega = 0)$ vertex of disordered Fermi—liquid theory [Betbeder-Matibet O., Nozieres P. (1966); Castellani C., Kotliar G., Lee P.A. (1987); Kotliar G. (1987)]:

$$\Lambda(\mathbf{q}\to 0\omega = 0) = (1+F_0^s)^{-1}Z^{-1} \qquad (2.185)$$

where F_0^s is Landau parameter introduced above. As a result we can easily get a simple relation between the mean free paths of interacting and noninteracting quasiparticles [Kotliar G., Varma C.M. (1990); Varma C.M. (1990)]:

$$l = (p_F/m^\star)\gamma^{-1} = (p_F/m)\gamma_0^{-1}/\Lambda^2(\mathbf{q}\to 0) = l_0(1+F_0^s)^2 Z^2 \qquad (2.186)$$

Here $m^\star = Z^{-1}m$ is the effective mass of quasiparticle. Assuming $F_0^s \approx const < 1$ and using Eq. (2.183) we get at $T = 0$:

$$l = \frac{l_0}{\left[ln\frac{\tilde{\omega}_c}{|\varepsilon|}\right]^2} \qquad (2.187)$$

Then from usual Ioffe—Regel criterion for localization $p_F l \approx 1$ we obtain that all quasiparticle state within the region of the order of

$$|\varepsilon_c| \approx \tilde{\omega}_c exp(-\sqrt{p_F l}) \qquad (2.188)$$

around the Fermi—level in high—T_c oxides are localized even for the case of weak impurity scattering $p_F l \gg 1$. For realistic estimates of $\tilde{\omega}_c \approx 0.1 - 0.2eV$ [Varma C.M. et al. (1989)] and $p_F l < 5$ the width of this localized band may easily be of the order of hundreds of degrees K, while for $p_F l \approx 10$ and $\tilde{\omega}_c \approx 1000K$ we get $|\varepsilon_c| \approx 40K$. Obviously this band grows with disorder as the mean free path l_0 drops. We can safely neglect this localization for $T \gg |\varepsilon_c|$, but for low enough temperatures localization effects become important and all states are localized in the ground state.

Of course, the formal divergence of the mean free path denominator in Eq.(2.187) is unphysical. Single-impurity scattering cannot overcome the so called unitarity limit [Kotliar G., Varma C.M. (1990)], so that we must always have:

$$l \geq \frac{p_F^2}{4\pi\rho} \tag{2.189}$$

In a typical metal with $p_F \sim a^{-1}$ this leads to $l \geq 1/4\pi\rho a^2$ and Ioffe-Regel criterion $l \leq a$ can be easily satisfied for large impurity concentrations $\rho \sim a^{-3}$. Thus the singularity in Eq.(2.187) does not mean that localization can appear for arbitrarily low concentration of impurities. We can safely speak only about the significant enhancement of localization effects in marginal Fermi liquids.

These ideas are still at rather elementary level and we may quote only few papers attempting to put them on more sound basis of scaling theory of metal–insulator transition of interacting electrons [Ng T.K. (1991)] or general picture of disorder effects in non Fermi–liquids [Varma C.M. (1997)].

Chapter 3

SUPERCONDUCTIVITY AND LOCALIZATION: STATISTICAL MEAN-FIELD APPROACH

3.1 BCS Model and Anderson Theorem

We shall start our analysis of superconductivity in strongly disordered systems within the framework of simple BCS—model [Bardeen J., Cooper L.N., Schrieffer J. (1957); Gennes de P.G. (1966)] which assumes the existence of some kind of effective electron—electron attraction within energy region of the order of $2 < \omega >$ around the Fermi level. In usual superconductors $< \omega >\sim \omega_D$, where ω_D is Debye frequency, because pairing is determined by electron—phonon mechanism, however we shall use some effective $< \omega >$ as an average frequency of some kind of Bose—like excitations responsible for pairing, e.g. in high—T_c superconductors. At the moment we shall not discuss microscopic nature of this attraction which in general case is determined by the balance of attraction due to Boson—exchange and Coulomb repulsion. Here we just assume (as always is done in simple BCS—approach) that this effective attraction is described by some interaction constant g, which is considered just as a parameter. More detailed microscopic approach will be given in later sections.

Nontrivial results concerning superconductivity in disordered systems were obtained very soon since the discovery of BCS—theory [Abrikosov A.A., Gorkov L.P. (1958); Abrikosov A.A., Gorkov L.P. (1959); Gorkov L.P. (1959); Anderson P.W. (1959)]. The concept of "dirty" superconductor described the experimentally very important case of the mean free path l short in comparison with superconducting coherence length $\xi_0 \sim \hbar v_F/T_c$, i.e. the case when:

$$\xi_0 \gg l \gg \hbar/p_F \tag{3.1}$$

Already in this case of not so strongly disordered (in the sense of closeness to metal—insulator transition) system Cooper pairing takes place not between electrons with opposite momenta and spins as in regular case, but between time—reversed exact eigenstates of electrons in disordered system [Anderson P.W. (1959); Gennes de P.G. (1966)]:

$$(\mathbf{p}_\uparrow, -\mathbf{p}_\downarrow) \Longrightarrow (\phi_\nu(\mathbf{r})_\uparrow, \phi^\star_\nu(\mathbf{r})_\downarrow) \tag{3.2}$$

In the following we consider mainly singlet isotropic (s-wave) pairing. Some aspects of anisotropic pairing will be analyzed later. The underlying physics is simple: in disordered systems such as e.g. an alloy the electron momentum becomes badly determined due to the lack of translational invariance. However, in random potential field we can always define exact eigenstates $\phi_\nu(\mathbf{r})$, which are just solutions of Schroedinger equation in this random field (for a given configuration of this field). We don't need to know the explicit form of these eigenstates at all, the pairing partner of $\phi_\nu(\mathbf{r})$ is being given by time—reversed $\phi^\star_\nu(\mathbf{r})$. This leads to a relative stability of a superconducting state with respect to disordering in the absence of scattering mechanisms which break the time—reversal invariance such as e.g. magnetic impurities.

Within standard Green's function approach superconducting system is described by Gorkov equations [Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965); Gorkov L.P. (1958)] which in coordinate representation take the form:

$$\mathcal{G}_\uparrow(\mathbf{r}\mathbf{r}'\varepsilon_n) = G_\uparrow(\mathbf{r}\mathbf{r}'\varepsilon_n) - \int d\mathbf{r}'' G_\uparrow(\mathbf{r}\mathbf{r}''\varepsilon_n)\Delta(\mathbf{r}'')\mathcal{F}(\mathbf{r}''\mathbf{r}'\varepsilon_n) \tag{3.3}$$

$$\mathcal{F}(\mathbf{r}\mathbf{r}'\varepsilon_n) = \int d\mathbf{r}'' G^\star_\downarrow(\mathbf{r}\mathbf{r}''\varepsilon_n)\Delta^\star(\mathbf{r}'')\mathcal{G}_\uparrow(\mathbf{r}''\mathbf{r}'\varepsilon_n) \tag{3.4}$$

where $G(\mathbf{r}\mathbf{r}'\varepsilon_n)$ is an exact one—electron Matsubara Green's function of the normal state and superconducting order—parameter (gap) $\Delta(\mathbf{r})$ is determined by self—consistent gap equation:

$$\Delta(\mathbf{r}) = gT\sum_{\varepsilon_n} \mathcal{F}^\star(\mathbf{r}\mathbf{r}\varepsilon_n) \tag{3.5}$$

where $\mathcal{F}(\mathbf{r}\mathbf{r}'\varepsilon_n)$ is (antisymmetric over spin variables) anomalous Gorkov Green's function, $\varepsilon_n = (2n+1)\pi T$.

If we consider temperatures close to superconducting transition temperature T_c, when $\Delta(\mathbf{r})$ is small, $\mathcal{F}(\mathbf{rr}'\varepsilon_n)$ can be obtained from the linearized equation:

$$\mathcal{F}(\mathbf{rr}'\varepsilon_n) = \int d\mathbf{r}'' G^{\star}_{\downarrow}(\mathbf{rr}''\varepsilon_n)\Delta^{\star}(\mathbf{r}'')G_{\uparrow}(\mathbf{r}''\mathbf{r}'\varepsilon_n) \tag{3.6}$$

Then the linearized gap equation determining T_c takes the form:

$$\Delta(\mathbf{r}) = gT \int d\mathbf{r}' \sum_{\varepsilon_n} K(\mathbf{rr}'\varepsilon_n)\Delta(\mathbf{r}') \tag{3.7}$$

where the kernel:

$$K(\mathbf{rr}'\varepsilon_n) = G_{\uparrow}(\mathbf{rr}'\varepsilon_n)G^{\star}_{\downarrow}(\mathbf{r}'\mathbf{r}\varepsilon_n) \tag{3.8}$$

is formed by exact one—electron Green's functions of a normal metal. Now we can use an exact eigenstate representation for an electron in a random field of a disordered system to write (Cf. Eq. (A.13)):

$$G_{\uparrow}(\mathbf{rr}'\varepsilon_n) = \sum_{\nu} \frac{\phi_{\nu\uparrow}(\mathbf{r})\phi^{\star}_{\nu\uparrow}(\mathbf{r}')}{i\varepsilon_n - \varepsilon_\nu} \tag{3.9}$$

where ε_ν are exact energy levels of an electron in disordered system. Then

$$K(\mathbf{rr}'\varepsilon_n) = Tg \sum_{\mu\nu} \frac{\phi_{\nu\uparrow}(\mathbf{r})\phi^{\star}_{\nu\uparrow}(\mathbf{r}')\phi^{\star}_{\mu\downarrow}(\mathbf{r}')\phi_{\mu\downarrow}(\mathbf{r})}{(i\varepsilon_n - \epsilon_\nu)(-i\varepsilon_n + \epsilon_\mu)} \tag{3.10}$$

In the following for brevity we shall drop spin variables always assuming singlet pairing. In case of a system with time—reversal invariance (i.e. in the absence of an external magnetic field, magnetic impurities etc.) Eq. (3.10) can be rewritten as:

$$K(\mathbf{rr}'\varepsilon_n) = G(\mathbf{rr}'\varepsilon_n)G(\mathbf{r}'\mathbf{r} - \varepsilon_n) = \sum_{\mu\nu} \frac{\phi_{\nu}(\mathbf{r})\phi^{\star}_{\nu}(\mathbf{r}')\phi_{\mu}(\mathbf{r}')\phi^{\star}_{\mu}(\mathbf{r})}{(i\varepsilon_n - \varepsilon_\nu)(-i\varepsilon_n - \varepsilon_\mu)} \tag{3.11}$$

Averaging over disorder we get:

$$< \Delta(\mathbf{r}) >= gT \int d\mathbf{r}' \sum_{\varepsilon_n} < K(\mathbf{rr}'\varepsilon_n)\Delta(\mathbf{r}') > \tag{3.12}$$

Practically in all papers on the superconductivity in disordered systems it is assumed that we can make simplest decoupling in Eq. (3.12) to get the following linearized equation for the average order—parameter:

$$< \Delta(\mathbf{r}) >= gT \int d\mathbf{r}' \sum_{\varepsilon_n} K(\mathbf{r}-\mathbf{r}'\varepsilon_n) < \Delta(\mathbf{r}') > \tag{3.13}$$

where the averaged kernel in case of time—invariance is given by:

$$\begin{aligned} K(\mathbf{r}-\mathbf{r}'\varepsilon_n) = K^\star(\mathbf{r}-\mathbf{r}'\varepsilon_n) =< K(\mathbf{r}\mathbf{r}'\varepsilon_n) >= \\ =< \sum_{\mu\nu} \frac{\phi_\nu(\mathbf{r})\phi^\star_\mu(\mathbf{r})\phi_\mu(\mathbf{r}')\phi^\star_\nu(\mathbf{r}')}{(i\varepsilon_n-\varepsilon_\nu)(-i\varepsilon_n-\varepsilon_\mu)} >= \\ = \int_{-\infty}^{\infty} dEN(E) \int_{-\infty}^{\infty} d\omega \frac{\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F}{(i\varepsilon_n+E)(E+\omega-i\varepsilon_n)} \end{aligned} \tag{3.14}$$

where we have introduced Gorkov—Berezinskii spectral density [Berezinskii V.L., Gorkov L.P. (1979)] (Cf. Eq. (A2)):

$$\begin{aligned} \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F = \\ \frac{1}{N(E)} < \sum_{\mu\nu} \phi^\star_\nu(\mathbf{r})\phi_\mu(\mathbf{r})\phi^\star_\mu(\mathbf{r}')\phi_\nu(\mathbf{r}')\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'}) > \end{aligned} \tag{3.15}$$

Here $N(E)$ is an exact electron density of states per *one spin direction* as it always appears in superconductivity theory (above, while discussing localization we almost always used density of states for both spin directions).

Usually the decoupling procedure used in Eq. (3.12) to reduce it to Eq. (3.13) is justified by the assumption that the averaging of $\Delta(\mathbf{r})$ and of Green's functions in Eq. (3.12) forming the kernel can be performed independently because of essentially different spatial scales [Gorkov L.P. (1959)]: $\Delta(\mathbf{r})$ changes at a scale of the order of coherence length (Cooper pair size) ξ, while $G(\mathbf{r}\mathbf{r}'\varepsilon_n)$ are oscillating on the scale of interatomic distance $a \sim \hbar/p_F$, and we always have $\xi \gg a$. Actually it is clear that this decoupling is valid only if the order—parameter is *self—averaging* (i.e. in fact nonrandom) quantity: $\Delta(\mathbf{r}) =< \Delta(\mathbf{r}) >, \quad < \Delta^2(\mathbf{r}) >=< \Delta(\mathbf{r}) >^2$. Below we shall see that for a system close to mobility edge the property of self—averageness of $\Delta(\mathbf{r})$ is absent and situation is actually highly nontrivial. In this case the so called *statistical* fluctuations [Bulaevskii L.N., Sadovskii M.V. (1986)] leading to inequality of $< \Delta^2(\mathbf{r}) >$ and $< \Delta(\mathbf{r}) >^2$ become quite important. However, we shall start with what we call statistical mean—field approach which completely neglects these fluctuations and allows the simple analysis using Eq. (3.13), as a necessary first step to understand superconductivity in strongly disordered systems, which will allow to find most of the important deviations from the usual theory of "dirty" superconductors. The role of statistical fluctuations will be analyzed later.

If we look for the solution of Eq. (3.13) $\Delta(\mathbf{r}) = const$ (homogeneous gap), we immediately obtain the following equation for superconducting transition temperature T_c:

$$1 = gT_c \int d\mathbf{r} \sum_{\varepsilon_n} K(\mathbf{r} - \mathbf{r}'\varepsilon_n) =$$

$$= gT_c \int d\mathbf{r} \sum_{\varepsilon_n} \int_{-\infty}^{\infty} dEN(E) \int_{-\infty}^{\infty} d\omega \frac{\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F}{(E + i\varepsilon_n)(E + \omega - i\varepsilon_n)} \quad (3.16)$$

Using the general sum—rule given in Eq. (A.5) [Berezinskii V.L., Gorkov L.P. (1979)]:

$$\int d\mathbf{r} \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F = \delta(\omega) \quad (3.17)$$

we immediately reduce Eq. (3.16) to a standard BCS form:

$$1 = gT_c \int_{-\infty}^{\infty} dEN(E) \sum_{\varepsilon_n} \frac{1}{E^2 + \varepsilon_n^2} = g \int_0^{<\omega>} dEN(E) \frac{1}{E} th \frac{E}{2T_c} \quad (3.18)$$

where we introduced the usual cut—off at $E \sim 2 < \omega >$. Note that $N(E)$ here is an exact one-particle density of states (per one spin direction) in a normal state of a disordered system. From Eq. (3.18) we get the usual result:

$$T_c = \frac{2\gamma}{\pi} < \omega > exp\left(-\frac{1}{\lambda_p}\right) \quad (3.19)$$

where $\lambda_p = gN(E_F)$ is dimensionless pairing constant, $ln\gamma = C = 0.577...$ is Euler constant. This is the notorious Anderson theorem: in the absence of scattering processes breaking time—reversal invariance disorder influence T_c only through the possible changes of the density of states $N(E_F)$ under disordering (which are usually relatively small).

Due to the sum—rule of Eq. (3.17) all singularities of Berezinskii—Gorkov spectral density reflecting possible localization transition do not appear in equation determining T_c: there is no explicit contribution from $\delta(\omega)$ term of Eq. (A.8) and Eq. (3.18) has the same form both in metallic and localized phases (Cf. Ref.[Takagi H., Kuroda Y. (1982)]).

The only limitation here which appears on the physical grounds is connected with the local discreteness of electronic spectrum in localized phase discussed above. It is clear that Cooper pairing is possible in localized phase only between electrons with centers of localization within the distance of the order of $\sim R_{loc}(E)$, because only in that case their wave functions overlap [Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)]. However, these states are splitted in energy by δ_E defined in Eq. (2.13). Obviously, we have to demand that superconducting gap Δ (at $T = 0$, $\Delta \sim T_c$) be much larger than this δ_E:

$$\Delta \sim T_c \gg \delta_E \sim \frac{1}{N(E)R_{loc}^3(E)} \tag{3.20}$$

i.e. on the energy interval of the order of $\Delta \sim T_c$ there must be many discrete levels, with centers of localization within distance $\sim R_{loc}(E)$ from each other. In this case the problem of Cooper pairs formation within $\sim R_{loc}(E)$ is qualitatively the same as in metallic state, e.g. we can replace summation over discrete levels ε_ν by integration. Analogous problem was considered previously in case of Cooper pairing of nucleons in finite nuclei [Migdal A.B. (1983)] and also of Cooper pairing of electrons in small metallic particles (granular metals) [Imry Y., Strongin M. (1981); Mühlschlegel B., Scalapino D.J., Denton R. (1972)]. For strongly anisotropic high—T_c systems we must similarly have [Sadovskii M.V. (1989)]:

$$\Delta \sim T_c \gg [N(E)R_{loc}^a R_{loc}^b R_{loc}^c]^{-1} \tag{3.21}$$

where we have introduced the appropriate values of localization lengths along the axes of an orthorhombic lattice.

Obviously Eq. (3.20) is equivalent to a condition of large enough localization length:

$$R_{loc}(E) \gg [N(E)\Delta]^{-1/3} \sim (\xi_0/p_F^2)^{1/3} \sim (\xi_0 l^2)^{1/3} \tag{3.22}$$

i.e. the system must be close enough to mobility edge or just slightly localized. Here we used the usual estimate of mean free path close to Anderson transition $l \sim p_F^{-1}$. Below we shall see that Eq. (3.22) is just a condition that Cooper pairs must be much smaller than localization length, only in this case Cooper pairing is possible in localized phase [Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)].

3.2 T_c Degradation

In usual BCS model discussed above pairing interaction g is assumed to be a given constant in the vicinity of the Fermi level. In more realistic approach this interaction is determined by the balance of interelectron attraction, due e.g. to electron—phonon coupling (as in traditional superconductors) or some other Boson—exchange mechanism (as is apparently the case in high—T_c superconductors), and Coulomb repulsion. It is clear that in strongly disordered system all these interactions can, in principle, be strongly renormalized in comparison with "pure" case. The aim of this section is to discuss these effects on the approach to metal—insulator transition induced by disorder.

Usually the Coulomb repulsion within Cooper pair is strongly reduced in comparison with electron—phonon attraction due to a retarded nature of electron—phonon coupling [Gennes de P.G. (1966)]. Characteristic time of electron—phonon interaction is of the order of ω_D^{-1}, while for Coulomb interaction in "pure" metal it is determined by $\sim \hbar/E_F$—the time during which electrons "pass" each other in the pair. Due to metallic screening both interactions are more or less point—like. However, in a disordered metal ballistic transport changes to diffusion and as disorder grows electron motion becomes slower effectively leading to the growth of Coulomb repulsion within Cooper pair and the appropriate drop of T_c as was first claimed by Anderson, Muttalib and Ramakrishnan [Anderson P.W., Muttalib K.A., Ramakrishnan T.V (1983)]. Actually electron—phonon interaction can also change under disordering but a common belief is that these changes are less significant than in case of Coulomb interaction [Keck B., Schmid A. (1976); Fleurov V.N., Kondratenko P.S., Kozlov A.N. (1980)]. This problem is still under active discussion and some alternative points of view were expressed [Belitz D. (1987a); Belitz D. (1987b); Belitz D. (1989)]. However, the general agreement is that some kind of diffusion renormalization of effective interaction of electrons within Cooper pair provides effective mechanism of T_c degradation under disordering. Later in this section we shall also consider the possible mechanisms of T_c degradation under disordering due to magnetic fluctuations (or local moments) which appear close to metal—insulator transition. Possible relation of these mechanisms to enhanced Coulomb effects will also be discussed. Also we have seen above, that important renormalization of the density of states close to the Fermi level takes place due to electron-electron interaction as system approaches metal-insulator, leading to “Coulomb” pseudogap. Obviously this leads to additional mechanism of T_c degradation [Kuchinskii E.Z., Sadovskii M.V., Erkabaev M.A. (1997)], which also will be analyzed here. At present it is not very clear which (if any) of these mechanisms is dominant in real systems. Most probably all are effective simultaneously, while there is no theory taking all of these into account in a united manner.

The general problem of T_c degradation under disordering becomes much more complicated in case of high—temperature superconductors because of unknown nature of pairing in these systems. However, we believe that the mechanisms discussed here are also operational in these systems, while of course it is difficult to say much about disorder effects upon attractive interactions leading to Cooper pair formation in these systems.

If we assume some kind of spin-independent Boson—exchange (phonons, excitons etc.) model of pairing interaction, the T_c can be obtained from the generalized Eliashberg equations and thus be given by the famous Allen—Dynes expression [Allen P.B., Dynes R.C. (1975)]:

$$T_c = \frac{f_1 f_2}{1.20}\omega_{log} exp\left\{-\frac{1.04(1+\lambda)}{\lambda-\mu^\star(1+0.62\lambda)}\right\} \tag{3.23}$$

where

$$f_1 = [1+(\lambda/\lambda_1)^{3/2}]^{1/3}; \qquad f_2 = 1+\frac{[<\omega^2>^{1/2}/\omega_{log}-1]\lambda^2}{\lambda^2+\lambda_2^2}$$
$$\lambda_1 = 2.46(1+3.8\mu^\star); \qquad \lambda_2 = 1.82(1+6.3\mu^\star)\frac{<\omega^2>^{1/2}}{\omega_{log}} \tag{3.24}$$

Here ω_{log} is the mean logarithmic frequency and $<\omega>^2$ is the mean square frequency of Bosons responsible for pairing (the averaging is over the spectrum of these Bosons), $\mu^\star$ is the Coulomb pseudopotential, λ is the dimensionless pairing constant due to Boson—exchange. Strictly speaking, Allen-Dynes formula has been derived for the electron—phonon model, with certain assumptions about the phonon spectrum. Its use for general Boson—exchange model here serves only for illustrative purposes. At relatively weak coupling $\lambda \leq 1.5$ Allen—Dynes expression effectively reduces to McMillan formula [McMillan W.L. (1968)]:

$$T_c = \frac{\omega_{log}}{1.20} exp\left\{-\frac{1.04(1+\lambda)}{\lambda-\mu^\star(1+0.62\lambda)}\right\} \tag{3.25}$$

which for the weak coupling gives the usual BCS result $T_c \sim <\omega> exp(\frac{-1}{\lambda-\mu^\star})$.

For very strong pairing interaction Eq. (3.23) gives the asymptotic behavior $T_c \approx 0.18\sqrt{\lambda < \omega^2 >}$. Below we shall mostly limit ourselves to weak coupling approximation. Coulomb pseudopotential $\mu^\star$ in the "pure" system is given by:

$$\mu^\star = \frac{\mu}{1 + \mu ln \frac{E_F}{<\omega>}} \tag{3.26}$$

where μ is the dimensionless Coulomb constant. The mechanism of T_c degradation under disordering due to the growth of Coulomb repulsion is reflected in the appropriate growth of $\mu^\star$ [Anderson P.W., Muttalib K.A., Ramakrishnan T.V (1983); Bulaevskii L.N., Sadovskii M.V. (1985)].

The singlet gap function with a simple s-wave symmetry which we have discussed above has a non-zero amplitude at zero separation of the two electrons in the pair. Thus it must pay the energy price of short-range repulsion due to a finite μ. In recent years a number of new mechanisms of superconducting pairing were proposed which try to eliminate the effect of repulsion assuming a pair wavefunction which vanishes at zero separation. Qualitatively this is equivalent to the requirement that the sum over all momentum of the BCS gap function Δ must vanish [Abrahams E. (1992)]:

$$\Delta(\mathbf{r}) =< \psi \uparrow (\mathbf{r}) \psi \downarrow (\mathbf{r}) >= \sum_{\mathbf{k}} \Delta(\mathbf{k}) = 0 \tag{3.27}$$

A number of rather exotic schemes for this were proposed [Abrahams E. (1992)], but probably the simplest way of satisfying this requirement is by means of higher angular momentum pairing, e.g. d-wave which became very popular as a possible explanation of high-T_c superconductivity within the spin-fluctuation exchange mechanism [Bickers N., Scalapino D., White S.R. (1989); Monthoux P., Balatsky A.V., Pines D. (1991); Monthoux P., Balatsky A.V., Pines D. (1992); Monthoux P., Pines D. (1993); Monthoux P., Pines D. (1994)]. The sum in Eq.(3.27) is then zero because the gap changes sign as $\mathbf{k}$ goes around the Fermi surface. This leads to a large extent to cancellation of Coulomb pseudopotential effects which just do not enter the formula for T_c. As an example of such formula we quote the "BCS-like" expression proposed within antiferromagnetic spin-fluctuation mechanism in [Monthoux P., Balatsky A.V., Pines D. (1991); Monthoux P., Pines D. (1993)] (actually as a good fit to their numerical data, obtained solving the appropriate Eliashberg equations):

$$T_c = \alpha\omega_{SF}(T_c)\frac{\xi^2_{SF}(T_c)}{a^2}exp\left(-\frac{1}{\lambda(T_c)}\right)$$
$$\equiv \alpha\frac{\Gamma(T_c)}{\pi^2}exp\left(-\frac{1}{\lambda(T_c)}\right) \qquad (3.28)$$

where ω_{SF} – effective paramagnon energy of the order of $10-100K$ (depending on compound), ξ_{SF} – correlation length of these antiferromagnetic fluctuations, which may be of the order of several interatomic spacings a, so that effective "Boson" energy Γ is typically of the order of $0.4-0.5eV$, while the dimensionless coupling constant $\lambda(T_c) = \eta g^2_{eff}(T_c)\chi_0(T_c)N(E_F)$ varies from 0.48 to 0.33 also depending on compound. Here g_{eff} – temperature dependent interaction constant, χ_0 – experimentally measurable spin susceptibility, while α and η – material constants of order of unity. Eq.(3.28) can really explain the values of T_c in a number of high-T_c compounds within intermediate to weak coupling approach, using semi-phenomenological values (experimental) of all essential parameters, entering this expression [Monthoux P., Balatsky A.V., Pines D. (1991); Monthoux P., Pines D. (1993)]. For us it is interesting to note that almost all of these parameters may significantly change under disordering. At present it is rather difficult to provide any kind of serious microscopic analysis of disorder effect within this approach.

However, we shall see below that irrespective to microscopic mechanisms of pairing, anisotropic d–wave pairing is extremely sensitive to any kind of disordering and superconductivity is destroyed in fact long before localization transition, at least within the standard BCS-like approach which assumes disorder independent pairing interaction. The problems which appear here with respect to the experiments on disordering high-T_c systems (which demonstrate unusual stability to disordering in this sense) and the possible ways out will be analyzed later in this section, as well as during our discussion of experimental situation.

Among mechanisms discussed for high-T_c superconductors we also mention different types of so-called van-Hove scenarios [Friedel J. (1989); Newns D.M., Pattnaik P.C., Tsuei C.C. (1991); Abrikosov A.A., Campuzano J.C., Gofron K. (1993); Dagotto E., Nazarenko A., Moreo A. (1995)], which are based upon the idea of T_c-enhancement due to some kind of the density of states singularity close to the Fermi level. For all such mechanisms rather strong T_c suppression may be due to the potential scattering smoothing out these singularities. We shall not discuss these mechanisms here as having almost nothing to do with localization effects.

3.2.1 *Coulomb Kernel*

Let us use again the exact eigenstate $\phi_\nu(\mathbf{r})$ representation for an electron in random system, with exact energy levels ε_ν. These functions and energies may correspond either to extended or to localized states. Consider the one—electron Green's function in this representation and take its diagonal element $G_{\nu\nu}(\varepsilon)$. The influence of interaction is described by the appropriate irreducible self—energy $\Sigma_\nu(\varepsilon)$ [Migdal A.B. (1983); Abrahams E., Anderson P.W., Lee P.A. (1981)]:

$$G_{\nu\nu}(\varepsilon) = \frac{1}{\varepsilon - \varepsilon_\nu - \Sigma_\nu(\varepsilon)} \tag{3.29}$$

Here energy zero is at the Fermi level. Let us introduce a "self—energy" $\Sigma_E(\varepsilon)$ averaged over some surface of constant energy $E = \varepsilon_\nu$ and over random field configurations [Abrahams E., Anderson P.W., Lee P.A. (1981)]:

$$\Sigma_E(\varepsilon) = \frac{1}{N(E)} < \sum_\nu \delta(E - \varepsilon_\nu)\Sigma_\nu(\varepsilon) > \tag{3.30}$$

Consider model with short—range static interelectron interaction $v(\mathbf{r} - \mathbf{r}')$. Then for the simplest Fock diagram shown in Fig. 2.7 we find:

$$\Sigma_\mu^F = -\int d\mathbf{r} \int d\mathbf{r}' v(\mathbf{r} - \mathbf{r}') \sum_\nu f_\nu \phi_\mu^\star(\mathbf{r}')\phi_\nu^\star(\mathbf{r})\phi_\mu(\mathbf{r})\phi_\nu(\mathbf{r}') \tag{3.31}$$

where $f_\nu = f(\varepsilon_\nu)$ is Fermi distribution function. Accordingly from Eq. (3.30) we get [Katsnelson M.I., Sadovskii M.V. (1984)]:

$$\Sigma_E^F = -\int_{-\infty}^{\infty} d\omega f(E + \omega) \int d\mathbf{r} \int d\mathbf{r}' v(\mathbf{r} - \mathbf{r}') \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F \tag{3.32}$$

where we again introduced Berezinskii—Gorkov spectral density defined in Eq. (A.2) or Eq. (3.15).

Let us define the Coulomb kernel by the following functional derivative:

$$K_c(E - E') = -\frac{\delta \Sigma_E^F}{\delta f(E')} \tag{3.33}$$

which characterize the change of electron energy due to a variation of its distribution function. It is easy to see that:

$$K_c(\omega) = \frac{1}{N(E)} < \sum_{\mu\nu} < \mu\nu|v(\mathbf{r}-\mathbf{r}')|\nu\mu > \delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_\mu) >=$$
$$= \int d\mathbf{r} \int d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F \tag{3.34}$$

is actually Fock—type matrix element of interaction averaged over two surfaces of constant energy E and $E' = E+\omega$ and over disorder. We can use $K_c(\omega)$ as a kernel in the linearized gap equation (Cf. Appendix B) determining T_c which is a reasonable generalization of a Coulomb kernel used in the theory of ordered superconductors [Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)]. In momentum representation:

$$K_c(\omega) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} v(\mathbf{q}) \ll \rho_E\rho_{E+\omega} \gg^F_{\mathbf{q}} \tag{3.35}$$

In the weak coupling approximation over pairing interaction it is the only relevant Coulomb contribution in the gap equation (Cf. Appendix B), in case of strong coupling there are additional contributions, e.g. connected with diffusional renormalization of the density of states Eq. (2.152) [Belitz D. (1987a); Belitz D. (1987b); Belitz D. (1989); Belitz D. (1985); Maekawa S., Ebisawa H., Fukuyama H. (1985)]. We refer to these papers for the detailed analysis of the effects of small corrections to the density of states upon T_c. Much stronger effects due to "Coulomb" pseudogap formation are discussed later in this section.

In the following we assume point—like interaction: $v(\mathbf{q}) = v_0$. During our discussion of localization we have discovered that for small $\omega \ll \gamma$ and $q \ll l^{-1}$ Gorkov—Berezinskii spectral density acquires a diffusional contribution:

$$\ll \rho_E\rho_{E+\omega} \gg^{Fdiff}_{\mathbf{q}} = \frac{1}{\pi N(E)} Im\Phi^{RA}_E(\mathbf{q}\omega) \tag{3.36}$$

where

$$\Phi^{RA}_E(\mathbf{q}\omega) = -\frac{N(E)}{\omega + iD_E(\omega)q^2} \tag{3.37}$$

and the generalized diffusion coefficient in metallic phase is given by:

$$D_E(\omega) \approx \begin{cases} D_E & |\omega| \ll \omega_c \approx 2\gamma(\sigma/\sigma_c)^3 \\ D_0 \left(-\frac{i\omega}{2\gamma}\right)^{1/3} & |\omega| \gg \omega_c \end{cases} \tag{3.38}$$

In the absence of disorder this diffusional contribution disappears and the kernel $K_c(\omega)$ for $|\omega| < E_F$ reduces to usual Coulomb potential $\mu = N(E)v_o$ [Gennes de P.G. (1966); Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)]. Accordingly we can use the following approximation [Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$K_c(\omega) \approx \mu\theta(E_F - |\omega|) + K_c^{diff}(\omega) \tag{3.39}$$

where

$$K_c^{diff}(\omega) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} v_0 \ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^{F diff} \tag{3.40}$$

This form of the Coulomb kernel gives correct interpolation between the strong disorder limit and "pure" case. Note, that in case of disordered system besides diffusional contribution which contains singularities associated with Anderson transition there also appear "regular" contributions to $K_c(\omega)$ which may be modelled by μ, making it different from its value in "pure" system. Diffusional term in $K_c(\omega)$ is connected with diffusion renormalization of electron—electron interaction vertex [Altshuler B.L., Aronov A.G. (1979); Altshuler B.L. et al. (1982); Abrahams E., Anderson P.W., Lee P.A. (1981); Lee P.A. (1982); Katsnelson M.I., Sadovskii M.V. (1983); Katsnelson M.I., Sadovskii M.V. (1984)]. Fig. 2.7 shows diagrams of standard perturbation theory responsible for this renormalization. In case of the approach based upon self—consistent theory of localization "triangular" vertex defined by Fig. 2.7 (c) is given by [Katsnelson M.I., Sadovskii M.V. (1983); Kotov E.A., Sadovskii M.V. (1985)]:

$$\gamma^{RA}(\mathbf{q}\omega) \approx \frac{2\gamma}{-i\omega + D_E(\omega)q^2} \qquad \omega \ll \gamma \quad q \ll l^{-1} \tag{3.41}$$

Singularity of Eq. (3.41) for small ω and q leads to significant growth of interaction in disordered system. Actually this expression is the same as in a "dirty" metal [Altshuler B.L., Aronov A.G. (1979)] but with the replacement of Drude diffusion coefficient by the generalized one.

3.2.2 Electron—Phonon Interaction

The case of electron—phonon interaction is different. Diffusion renormalization of electron—phonon vertex is less important because the relevant corrections compensate each other if we take into account impurity vibrations [Pippard A.B. (1955); Keck B., Schmid A. (1976); Fleurov V.N., Kondratenko P.S., Kozlov A.N. (1980)]. Surely the value of electron—phonon contribution to pairing interaction does change in a disordered system in comparison with "pure" case [Keck B., Schmid A. (1976)]. However, these changes are relatively insignificant in the sense of absence of drastic changes at the Anderson transition. We shall demonstrate the absence of diffusion renormalization of electron—phonon vertex using the lowest order diagrams of perturbation theory following the approach of Ref. [Fleurov V.N., Kondratenko P.S., Kozlov A.N. (1980)].

Let us limit our analysis to homogeneous continuous medium. The appearance of deformation $\mathbf{u}$ leads to the variation of density of the medium given by $\delta\rho = -\rho div\mathbf{u}$. Accordingly, taking into account the electroneutrality condition we get the variation of electron density as $\delta n = -ndiv\mathbf{u}$. This leads to the following change of the free electron Green's function:

$$\begin{aligned}\delta G^{-1}(E\mathbf{p}) = -ndiv\mathbf{u}\frac{d}{dn}[E - v_F(|\mathbf{p}| - p_F)] = \\ = -nv_F div\mathbf{u}\frac{dp_F}{dn} = -\frac{1}{3}v_F p_F div\mathbf{u}\end{aligned} \tag{3.42}$$

where we have used $n = p_F^3/(3\pi^2)$. Let us define electron—phonon vertex $\mathbf{\Lambda}$ by:

$$\frac{\delta G}{\delta\mathbf{u}} = G\mathbf{\Lambda}G = -G\frac{\delta G^{-1}}{\delta\mathbf{u}}G; \qquad \mathbf{\Lambda} = -\frac{\delta G^{-1}}{\delta\mathbf{u}} \tag{3.43}$$

For $\mathbf{u}(\mathbf{r},t) = \mathbf{u}exp(i\mathbf{qr} - i\omega t)$ we get from Eq. (3.42):

$$\delta G^{-1}(E\mathbf{p}) = -i\mathbf{qu}\frac{v_F p_F}{3} \tag{3.44}$$

so that the "bare" electron—phonon vertex (i is vector index):

$$\Lambda_{1i}^{(0)} = iq_i\frac{v_F p_F}{3} \tag{3.45}$$

Consider the system with impurities randomly placed at points $\mathbf{R}_n$ which

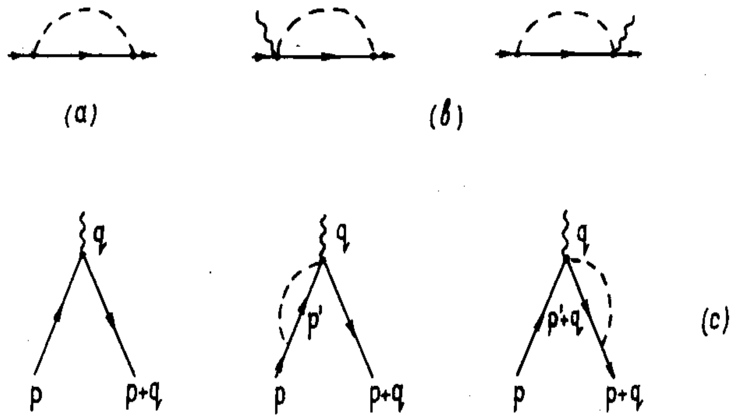

Fig. 3.1 Electron—phonon interaction and impurity scattering: (a) Self—energy due to impurity scattering. (b) Diagrams representing changes of (a) due to impurity vibrations. (c) Diagrams for "bare" electron—phonon vertex in case of vibrating impurities.

create the potential:

$$U(\mathbf{r}) = \sum_n V(\mathbf{r} - \mathbf{R}_n) \tag{3.46}$$

Vibrations of the medium lead to vibrations of impurity atoms, so that $\mathbf{R}_n \to \mathbf{R}_{0n} + \mathbf{u}_n(t)$ with $\mathbf{u}_n(t) = \mathbf{u}exp(i\mathbf{q}\mathbf{R}_{0n} - i\omega t)$. Random field of static impurities leads to a simplest self—energy correction given by Fig. 3.1 (a) [Edwards S.F. (1958); Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)].

Impurity vibrations can be accounted for by the additional interaction term:

$$\delta V(\mathbf{r} - \mathbf{R}_n) = \frac{\partial V(\mathbf{r} - \mathbf{R}_{n0})}{\partial \mathbf{R}_{n0}} \mathbf{u} exp(i\mathbf{q}\mathbf{R}_{n0} - i\omega t)$$

so that

$$\Lambda_{2i} u_i = \frac{\delta \Sigma}{\delta u_i} u_i =$$

$$= < \sum_n \left\{ \frac{\partial V(\mathbf{r} - \mathbf{R}_{n0})}{\partial R^i_{n0}} G(\mathbf{r}t, \mathbf{r}'t') V(\mathbf{r}' - \mathbf{R}_{n0}) u_{in} + \right.$$

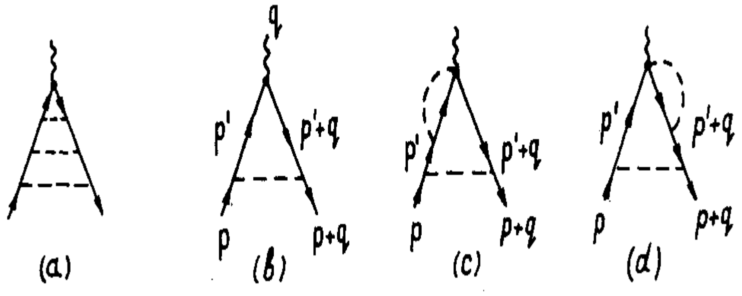

Fig. 3.2 Electron—phonon vertex renormalization: (a) Impurity "ladder" (diffusion) renormalization. (b), (c), (d) Simplest corrections due to impurity vibrations.

$$+V(\mathbf{r}-\mathbf{R}_{n0})G(\mathbf{r}t,\mathbf{r}'t')\frac{\partial V(\mathbf{r}'-\mathbf{R}_{n0})}{\partial R^i_{n0}}u_{in}\Big\} > \tag{3.47}$$

where the angular brackets define as usual the averaging over random impurity positions. In momentum representation and for point—like impurities we get in lowest order over ω/E_F and q/p_F:

$$\begin{aligned}\Lambda_{2i}(\mathbf{p},\mathbf{q}) = \rho V^2 \int \frac{d^3\mathbf{p}'}{(2\pi)^3}[-i(p_i-p'_i)G(E\mathbf{p}') + i(p'_i-p_i)G(E\mathbf{p}')] = \\ = 2\rho V^2 \int \frac{d^3\mathbf{p}'}{(2\pi)^3}[-i(p_i-p'_i)G(E\mathbf{p}')] = \\ = 2\pi\rho V^2 N(E)p_i = 2\gamma p_i\end{aligned} \tag{3.48}$$

The relevant diagrams are shown in Fig. 3.1 (b) [Nielsen P.E., Taylor P.L. (1974)].

"Bare" electron—phonon vertex is thus given by the sum of three diagrams shown in Fig. 3.1 (b) and reduces to:

$$\Lambda^{(0)}_i = \Lambda^{(0)}_{1i} + \Lambda^{(0)}_{2i} = iq_i\frac{v_Fp_F}{3} + 2\gamma p_i \tag{3.49}$$

Diffusion renormalization of electron—phonon vertex can appear due to impurity scattering ladder corrections as shown in Fig. 3.2 (a). Similar diagrams shown in Fig. 2.7 (c) lead to diffusion renormalization of Coulomb vertex. However, in case of electron—phonon interaction we have to make the same renormalization of three diagrams of Fig. 2.7 (c). Let us consider simplest corrections shown in Fig. 3.2 (b,c,d). For the contribution of graph of Fig. 3.2 (b) we have:

$$\Lambda^{(1)}_{1i} = \rho V^2 i q_i \frac{v_F p_F}{3}\int \frac{d^3\mathbf{p}'}{(2\pi)^3} G(E\mathbf{p}')G(E+\omega\mathbf{p}'+\mathbf{q}) \approx$$
$$\approx iq_i \frac{v_F p_F}{3}[1 + i\omega/2\gamma - D_0 q^2/2\gamma] \approx iq_i \frac{v_F p_F}{3} \qquad \omega, q \to 0 \tag{3.50}$$

and for the sum of graphs of Fig. 3.2 (c,d):

$$\Lambda^{(1)}_{2i} = 2\rho V^2 \gamma \int \frac{d^3\mathbf{p}'}{(2\pi)^3} G(E\mathbf{p}')G(E+\omega\mathbf{p}'+\mathbf{q})p'_i \approx$$
$$\approx 2\rho V^2 \gamma q_i \int \frac{d^3\mathbf{p}'}{(2\pi)^3} p'_i G(E\mathbf{p}') \frac{\partial}{\partial p_i} G(E+\omega\mathbf{p}') \approx$$
$$\approx 2\gamma\rho V^2 q_i p_F \int \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{v_F}{3} G(E\mathbf{p}')G^2(E\mathbf{p}') = -iq_i \frac{v_F p_F}{3} \tag{3.51}$$

Thus for $\omega \to 0$, $q \to 0$ we obtain:

$$\Lambda^{(1)}_{1i} + \Lambda^{(1)}_{2i} = 0 \tag{3.52}$$

and we have total cancellation of initial diagrams contributing to diffusion ladder. Apparently there is no diffusion renormalization of electron—phonon vertex (for $\omega, q \to 0$): this cancellation is valid for any graph obtained from the simplest corrections by adding further impurity lines to the ladder. Similar cancellation apparently takes place in case of adding to diagrams of Fig.3.2 (b,c,d) corrections due to maximally crossed impurity lines (Cooper channel). Thus there is no significant change of electron—phonon vertex due to Cooperon and the only relevant contribution to electron—phonon vertex in impure system is defined by the sum of diagrams of Fig. 3.1 (b) leading to Eq. (3.49) which does not contain diffusion type renormalization. Localization appears via generalized diffusion coefficient which replaces the Drude one. Thus localization singularities does not appear in electron—phonon vertex, though surely this interaction is really changed by disorder scattering in comparison with "pure" case. However, the question of whether localization effects contribute to renormalization of electron—phonon coupling is still under discussion [Belitz D. (1987a)]. Probably more important aspect of this problem is reflected by the fact that superconductivity is actually determined not by electron—phonon vertex itself, but by the famous integral expression over the phonon spectrum of Eliashberg function $\alpha^2(\omega)F(\omega)$ which defines the pairing constant λ [Allen P.B., Dynes

R.C. (1975); Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)]. This integration will apparently smooth out all possible singularities.

In the following we shall model pairing interaction due to phonon exchange by some constant λ as in BCS model. Of course we must stress that this constant is different from that in regular metal. It is constant in a sense that it does not contain singularities due to metal—insulator transition. Electron—phonon kernel in the linearized gap equation (Cf. Appendix B) can be taken in the simplest form:

$$K_{ph}(E,E') = \begin{cases} -\lambda & |E|,|E'| < \omega_D \\ 0 & |E|,|E'| > \omega_D \end{cases} \tag{3.53}$$

and consider λ as relatively weakly dependent on disordering. More detailed discussion of electron—phonon pairing in disordered systems can be found in Refs. [Keck B., Schmid A. (1976); Belitz D. (1987a); Belitz D. (1987b)].

As we mentioned above it is quite difficult to speculate on disorder dependence of pairing interaction in high—temperature superconductors. In case of the "marginal" Fermi—liquid approach [Varma C.M. et al. (1989); Kuroda Y., Varma C.M. (1990)] pairing interaction can be modelled as in Eq. (3.53) with the replacement of Debye frequency ω_D by some phenomenological *electronic* frequency $\tilde{\omega}_c$ which we briefly mentioned above while discussing localization in "marginal" Fermi—liquid. In the following we shall just assume that this pairing interaction is weakly dependent on disorder as in the case of phonon mechanism of pairing.

3.2.3 *Metallic Region*

In metallic region we can use Eqs. (3.35—3.37) and Eq. (3.39) and find the diffusional contribution to Coulomb kernel:

$$\begin{aligned} K_c^{diff}(\omega) &= -\int \frac{d^3\mathbf{q}}{(2\pi)^3} v_0 Im \frac{1}{\omega + iD_E q^2} \approx \\ &\approx \frac{v_0}{2\pi^3}\left[\frac{1}{|D_E(\omega)l|} - \frac{|\omega|^{1/2}}{|D_E^{3/2}(\omega)|}\right] \approx \\ &\approx \frac{v_0}{2\pi^3}\begin{cases} \frac{1}{D_E l} - \frac{|\omega|^{1/2}}{D_E^{3/2}} & |\omega| \ll \omega_c \\ \frac{1}{D_0 l}\left(\frac{\omega}{2\gamma}\right)^{-1/3} & |\omega| \gg \omega_c \end{cases} \end{aligned} \tag{3.54}$$

Accordingly for the Coulomb kernel defined by Eq. (3.39) we get [Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$K_c(\omega) = \mu\theta(E_F - |\omega|) + \frac{\mu}{p_F l}\begin{cases} \frac{\sigma_c}{\sigma} & |\omega| < \omega_c \\ \frac{1}{p_F l}\left(\frac{\omega}{2\gamma}\right)^{-1/3} & \omega_c < \omega < \gamma \sim E_F \end{cases} \tag{3.55}$$

Upper limit cut—off in the integral in Eq. (3.54) was taken $\sim l^{-1}$. Rough estimate of contribution of higher momenta can be achieved introducing cut—off $\sim p_F$ (Cf. Ref.[Belitz D. (1985)]). This will cancel $(p_F l)^{-1}$ in Eq. (3.55). Close to Anderson transition $l^{-1} \sim p_F$ and this correction is irrelevant. We shall assume that far from transition these higher momenta corrections can be included in the definition of μ. From Eq. (3.55) we can see that diffusion renormalization of Coulomb kernel leads to substantial growth of Coulomb repulsion close to Anderson transition (i.e. when conductivity drops below σ_c—"minimal metallic conductivity").

Superconducting transition temperature T_c is determined by the linearized gap equation [Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)] which in the weak coupling approximation can be written as (Cf. Appendix B) [Dolgov O.V. (1977); Dolgov O.V., Sadovskii M.V. (1984)]:

$$\begin{aligned}\Delta(\omega) = \lambda\theta(<\omega> - \omega)\int_0^{<\omega>}\frac{d\omega'}{\omega'}\Delta(\omega')th\frac{\omega'}{2T_c} - \\ -\theta(E_F - \omega)\int_0^{E_F}\frac{d\omega'}{\omega'}K_c(\omega - \omega')\Delta(\omega')th\frac{\omega'}{2T_c}\end{aligned} \tag{3.56}$$

Consider metallic region and take $\omega_c \gg <\omega>$ which in accordance with ω_c estimate given in Eq. (3.38) roughly corresponds to $\sigma \geq \sigma_c$ for typical $E_F/ <\omega> \sim 10^2$, so that the system is not very close to Anderson transition. The change of T_c due to diffusion contribution in Coulomb kernel Eq. (3.55) can be determined by perturbation theory over $K_c^{diff}(\omega)$ in gap equation. First iteration of Eq. (3.56) gives:

$$\frac{\delta T_c}{T_{co}} \approx \frac{\int\limits_0^\infty \frac{d\omega}{\omega}\int\limits_0^\infty \frac{d\omega'}{\omega'}\Delta_0(\omega)th\frac{\omega}{2T_{co}}K_c^{diff}(\omega - \omega')\Delta_0(\omega')th\frac{\omega'}{2T_{co}}}{\frac{1}{2T_{co}}\int\limits_0^\infty d\omega[\Delta_0(\omega)]^2[ch\frac{\omega}{2T_{co}}]^{-2}} \tag{3.57}$$

where $\Delta_0(\omega)$ is the usual "two—step" solution of Eg. (3.56) [Gennes de P.G. (1966); Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)] which is valid for standard form of Coulomb kernel $K_c(\omega) = \mu\theta(E_F - |\omega|)$, and

$$T_{co} = 1.13 < \omega > exp\left(-\frac{1}{\lambda - \mu_0^\star}\right) \tag{3.58}$$

is a critical temperature in regular superconductor when the Coulomb pseudopotential is given by :

$$\mu_0^\star = \frac{\mu}{1 + \mu ln\frac{E_F}{<\omega>}} \tag{3.59}$$

Using the first relation in Eq. (3.55) we get from Eq. (3.57):

$$\frac{\delta T_c}{T_{c0}} \approx -\frac{\mu}{(\lambda - \mu_0^\star)^2}\frac{1}{p_F l}\frac{\sigma_c}{\sigma} \tag{3.60}$$

This change of T_c is equivalent to the following change of Coulomb pseudopotential [Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$\delta\mu^\star \approx \mu\frac{\sigma_c^2}{\sigma(\sigma + \sigma_c)} \tag{3.61}$$

where we have used Eq. (2.76) and $p_F l \approx \sigma_0/\sigma_c = (\sigma + \sigma_c)/\sigma_c$ to replace $p_F l$ in Eq. (3.60). As we noted above this later factor disappears from Eq. (3.60) if we use cut—off at $q \sim p_F$ in Eq. (3.54). According to Eq. (3.61) Coulomb pseudopotential $\mu^\star$ grows as σ drops and this dependence is more strong than a similar one obtained in Ref.[Anderson P.W., Muttalib K.A., Ramakrishnan T.V (1983)], which is connected with our use of the results of self—consistent theory of localization. Method of Ref.[Anderson P.W., Muttalib K.A., Ramakrishnan T.V (1983)] is based upon the use of q—dependence of diffusion coefficient as given by Eq. (2.86) . Our expression for $\delta\mu^\star$ leads to a significant growth of $\mu^\star$ for conductivities $\sigma \leq 10^3 Ohm^{-1}cm^{-1}$. This growth can easily explain the typical T_c degradation in "very dirty" superconductors as their conductivity in normal state drops approaching the Ioffe—Regel limit [Ioffe A.F., Regel A.R. (1960)]. At the same time expressions for $\mu^\star$ proposed in Ref.[Anderson P.W., Muttalib K.A., Ramakrishnan T.V (1983)] can explain experimental data only under the assumption that a characteristic conductivity scale determining $\mu^\star$ is an order of magnitude larger than Ioffe—Regel limit, for which we see no serious grounds. Additional extensive discussion can be found in Ref.[Belitz D. (1987a)].

Let us consider now the situation at the mobility edge itself, when $\sigma = 0$ and $\omega_c = 0$ so that $K_c(\omega)$ is determined by the second expression in Eq. (3.55) for all frequencies below $\gamma \sim E_F$. In this case we can show [Bulaevskii L.N., Sadovskii M.V. (1985)] that the influence of Coulomb repulsion on T_c is again described by effective pseudopotential $\mu^\star$ which can be estimated as:

$$\mu^\star \sim \alpha\mu \left(\frac{<\omega>}{2\gamma}\right)^{-1/3} \qquad \alpha \sim 1 \tag{3.62}$$

In this case T_c may remain finite at the mobility edge only under very strict conditions: both $E_F \sim \gamma$ and μ must be very small, while λ must be at least close to unity. As a crude estimate we can demand something like $\lambda \sim 1, \mu \leq 0.2$ and $E_F \leq 10^3 T_{c0}$. Obviously only some narrow band superconductors like Chevrel phases can satisfy these conditions among traditional systems. High—T_c superconductors are especially promising. Experimental situation will be discussed later.

Using Eq. (3.61) and Eq. (3.62) we can write down a simple interpolation formula for the conductivity dependence of $\mu^\star$ [Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$\mu^\star \approx \mu_0^\star + \frac{\alpha\mu(<\omega>/2\gamma)^{-1/3} - \mu_0^\star}{1 + (<\omega>/2\gamma)^{-1/3}\sigma(\sigma+\sigma_c)/\sigma_c^2} \tag{3.63}$$

To get an expression via observable parameters we may take into account the simple relation $<\omega>/\gamma \approx (<\omega>/E_F)(1+\sigma/\sigma_c)$. These expressions describe continuous crossover from the region of weak localization corrections to the vicinity of Anderson transition where its influence upon T_c becomes very strong. This crossover takes place at $\omega_c \sim <\omega>$.

3.2.4 *Localization Region*

Let us now consider Anderson insulator. According to Eq. (3.35) and Eq. (A.9) Coulomb kernel acquires in this case $\delta(\omega)$—contribution:

$$K_c^{loc}(\omega) = v_0 A_E \delta(\omega) = v_0 \frac{1}{N(E)} < \sum_\nu \delta(E-\varepsilon_\nu)|\phi_\nu(\mathbf{r})|^2|\phi_\nu(\mathbf{r})|^2 > \tag{3.64}$$

$$A_E = A_E(\mathbf{r}-\mathbf{r}')|_{\mathbf{r}=\mathbf{r}'} \sim R_{loc}^{-3} \tag{3.65}$$

which is actually connected with "Hubbard—like" repulsion of electrons in a single quantum state becoming nonzero in localization region [Mott N.F.

(1971); Katsnelson M.I., Sadovskii M.V. (1983); Sadovskii M.V. (1986)]. This mechanism acts in addition to diffusion contributions in Coulomb pseudopotential $\mu^{\star}$ considered above, which are due to "regular" part of Gorkov—Berezinskii spectral density. Using Eq. (3.65) as a full Coulomb vertex in linearized gap equation (3.56) we can solve it exactly [Bulaevskii L.N., Sadovskii M.V. (1985)] and find:

$$\Delta(\omega) = \frac{\theta(<\omega> - |\omega|)\Delta_1}{1 + \frac{\mu A_E}{2N(E)}\frac{1}{\omega}th\frac{\omega}{2T_c}} \tag{3.66}$$

where

$$\Delta_1 = \lambda \int_0^{<\omega>} d\omega \Delta(\omega)\frac{1}{\omega}th\frac{\omega}{2T_c} \tag{3.67}$$

and equation for T_c takes the form:

$$1 = \lambda \int_0^{<\omega>} d\omega \frac{th\frac{\omega}{2T_c}}{\omega + \frac{\mu A_E}{2N(E)}th\frac{\omega}{2T_c}} \tag{3.68}$$

To account for "regular" diffusion contributions to $\mu^{\star}$ we can just replace here $\lambda \to \lambda^{\star} = \lambda - \mu^{\star}$, where $\mu^{\star}$ is given by Eq. (3.62). Then our equation for T_c can be approximately represented by[Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$ln\frac{T^{\star}}{T_c} \approx \Psi\left(\frac{1}{2} + \frac{\mu A_E}{4T_c N(E)}\right) - \Psi\left(\frac{1}{2}\right) \tag{3.69}$$

where $\Psi(x)$ is digamma function, and $T^{\star}$ is taken to be equal to T_c of the system at the mobility edge which is given by Eq. (3.58) with $\mu_0^{\star}$ replaced by $\mu^{\star}$ from Eq. (3.62). Here we slightly overestimate the role of Coulomb repulsion in localization region. We can see that this additional "Hubbard—like" repulsion acts upon T_c as magnetic impurities [Gennes de P.G. (1966); Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)] with effective spin—flip scattering rate:

$$\frac{1}{\tau_{sf}} = \pi\frac{\mu A_E}{N(E)} \sim \frac{\mu}{N(E)R_{loc}^3(E)} \tag{3.70}$$

Obviously this result is connected with the appearance below the mobility edge of the "band" of singly occupied electron states of the width [Mott N.F. (1971); Katsnelson M.I., Sadovskii M.V. (1983); Sadovskii M.V. (1986);

Kamimura H. (1980)] $v_0 R_{loc}^{-3}$. Superconductivity persists until $\tau_{sf}^{-1} < 0.57 T_c^{\star}$, i.e. until

$$R_{loc}(E) > \left[\frac{\mu}{N(E)T_c^{\star}}\right]^{1/3} \sim (\xi_0 p_F^{-2})^{1/3} \sim (\xi_0 l^2)^{1/3} \tag{3.71}$$

where the last estimates are valid for typical values of parameters and correspond to the simple estimate of Eq. (3.20). Thus the Coulomb repulsion in a single (localized) quantum state leads to a sharp reduction of T_c below the mobility edge even if superconductivity survived up to the Anderson transition. Another interpretation of this effect is the influence of "free" spins of Mott's band of singly occupied states below the Fermi level of Anderson insulator.

Coulomb gap [Efros A.L., Shklovskii B.L. (1975); Efros A.L. (1976); Shklovskii B.I., Efros A.L. (1979); Shklovskii B.I., Efros A.L. (1979); Efros A.L., Shklovskii B.I. (1985)] effects in rough approximation can be neglected here [Bulaevskii L.N., Sadovskii M.V. (1985)] because according to the estimates given in Eq.(2.14) and Eq. (2.15) the Coulomb gap width:

$$\Delta_c \sim [N(E)R_{loc}^3(E)]^{-1} \ll T_c \sim \Delta \tag{3.72}$$

i.e. is small in comparison to superconducting gap Δ (or T_c) under conditions given by Eq. (3.20) which is necessary for the observation of superconductivity in localization region. However, we shall see below, that renormalization of the density of states due to interactions becomes important even in metallic phase and acts as an independent mechanism of T_c degradation.

3.2.5 *Spin Fluctuations*

As we mentioned during our discussion of interaction effects upon Anderson transition the role of magnetic fluctuations (spin effects) in general becomes stronger as we approach metal—insulator transition. The band of single—occupied states is being formed below the Fermi level of Anderson insulator, which is equivalent to the appearance of localized moments [Mott N.F. (1971); Sadovskii M.V. (1986); Kamimura H. (1980)]. These effects actually may become important already before metal—insulator transition [Finkelstein A.M. (1984b); Kirkpatrick T.R., Belitz D. (1990); Belitz D., Kirkpatrick T.R. (1990); Belitz D., Kirkpatrick T.R. (1991); Altshuler B.L., Aronov A.G. (1983a); Altshuler B.L., Aronov A.G. (1983b);

Altshuler B.L., Aronov A.G. (1985)], and lead to additional mechanism of T_c degradation. Unfortunately there is no complete theoretical understanding of these effects and accordingly only few estimates can be done concerning superconductivity. Here we shall mention only some of these crude estimates following Refs. [Fukuyama H. (1985a); Fukuyama H. (1985b); Ebisawa H., Fukuyama H., Maekawa S. (1985)].

In the framework of Hubbard model with weak disorder it can be shown [Fukuyama H. (1985a)] that the spin susceptibility is represented by:

$$\chi_s = \frac{\chi_0}{1 - UN(E) + \gamma_0 - \gamma'} = \frac{\chi_0}{\eta_0 - \gamma'} \equiv \frac{\chi_0}{\eta} \tag{3.73}$$

where χ_0 is spin susceptibility of free electrons, $\eta_0 = 1 - UN(E) + \gamma_0$ is enhancement factor for the ordered case (U is Hubbard interaction, γ_0 is correlation correction to RPA approximation), γ' is the correction due to the interference of Hubbard interaction and disorder scattering:

$$\gamma' = B\lambda^2 \qquad B = 6\sqrt{3}\pi^2[N(E)U]^2\left\{1 - \frac{1}{2}UN(E)\right\} \tag{3.74}$$

Here $\lambda = 1/(2\pi E\tau) = 1/(p_F l)$ is the usual perturbation theory parameter for disorder scattering. As $\gamma' > 0$ we can see from Eq. (3.73) that disordering leads to diminishing denominator $\eta = \eta_0 - \gamma'$. If we reach a critical disorder defined by:

$$\lambda_c = \sqrt{\frac{\eta_0}{B}} \qquad p_F l|_c = 2\sqrt{\frac{B}{\eta_0}} \tag{3.75}$$

we get $\chi_s \to \infty$. It should be stressed that this divergence of χ_s in a disordered system must not be identified with any kind of ferromagnetic instability but may signify something like the appearance of a spin—glass state or just of localized moments. In any case it means the growth of spin dependent effects under disordering.

If the initial enhancement of spin susceptibility is strong enough (e.g. due to a large U), i.e. $\eta_0 \ll 1$, the critical disorder defined by Eq. (3.75) may be lower than the critical disorder for Anderson localization, appearing at $p_F l \sim 1$. Then these spin dependent effects may become important well before Anderson transition. In the opposite case these effects will appear only very close to metal—insulator transition. In general case the relation between these two transitions depends on parameters.

If spin fluctuations are strong enough ($\eta \ll 1$) a strong mechanism for T_c degradation in superconducting state appears [Fukuyama H. (1985b)] analogous to similar effect due to magnetic impurities [Gennes de P.G. (1966); Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)]:

$$ln\frac{T_{c0}}{T_c} = \Psi\left(\frac{1}{2}+\rho\right) - \Psi\left(\frac{1}{2}\right) \tag{3.76}$$

where [Fukuyama H. (1985b)]:

$$\rho = \frac{9\sqrt{3}\pi}{2}\lambda^2\frac{UN(E)}{\eta} = \frac{9\sqrt{3}}{2}\left[\frac{UN(E)}{B}\right]\frac{\lambda^2}{\lambda_c^2-\lambda^2} \tag{3.77}$$

As ρ from Eq. (3.77) diverges as $(\lambda_c - \lambda)^{-1}$ for $\lambda \to \lambda_c$ superconducting transition temperature T_c drops to zero.

If $\lambda_c \ll 1$, which is possible for $\eta_0 \ll 1$, superconductivity will be destroyed long before metal—insulator transition. In the opposite case this mechanism may lead to its destruction on the either side of metal—insulator transition depending on the parameters of the system, such as U. In general we need a more accurate analysis which must include the mutual interplay of magnetic fluctuations and disorder scattering leading to metal—insulator transition. In any case magnetic mechanisms of T_c degradation close to metal—insulator transition may be as important as Coulomb effects considered above.

3.2.6 *Density of States Effects*

In Chapter 2 we formulated a kind of a theory [Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A. (1997)] of metal-insulator transitions which generalizes the self-consistent theory of localization [Wölfle P., Vollhardt D. (1982); Sadovskii M.V. (1986); Vollhardt D., Wölfle P. (1990)] taking into account the effects of electron-electron interaction. This approach has allowed us to study the behavior of the generalized diffusion coefficient for the wide interval of disorder parameter both for metallic and insulating regions. These results were used in calculations of one-particle density of states with the account of interelectron interactions. Calculations demonstrate the formation and the growth of the "Coulomb pseudogap" in the density of states close to the Fermi level. In metallic region this behavior of the density of states corresponds to the usual square-root Altshuler-Aronov correction [Altshuler B.L., Aronov A.G. (1979); Altshuler B.L., Aronov A.G. (1985)].

As disorder parameter grows and system moves towards the metal-insulator transition this "pseudogap" deepens, while the effective region of square-root behavior diminishes, and at the point of the metal-insulator transition the density of states at the Fermi level becomes equal to zero — we obtain a kind of a "Coulomb gap". In the insulating region, for the band of the finite width, we obtain the typical quadratic behavior of the density of states close to the Fermi level, reminiscent of the Coulomb gap of Efros and Shklovskii [Efros A.L., Shklovskii B.L. (1975); Shklovskii B.I., Efros A.L. (1979); Efros A.L., Shklovskii B.I. (1985)], widening with the further growth of disorder. Such behavior of the density of states is in qualitative agreement with experiments on the number of disordered systems close to the metal-insulator transition [Belitz D., Kirkpatrick T.R. (1994)], from amorphous alloys [McMillan W.L., Mochel J. (1981); Imry Y., Ovadyahu Z. (1982); Hertel G. et al. (1983); Bishop D.J., Spencer E.G., Dynes R.C. (1985)] to disordered single-crystals of metallic oxides, including high-temperature superconductors [Srikanth H. et al. (1992)]. Here we use the results of these calculations of the density of states for the numerical study of "Coulomb gap" effects on the T_c suppression for superconductors which are close to the metal-insulator transition [Kuchinskii E.Z., Sadovskii M.V., Erkabaev M.A. (1997)].

We shall analyze superconductivity within the framework of the simplest BCS-model. In the weak coupling approximation the linearized gap-equation takes the following form (Cf. Appendix B):

$$\Delta(\xi) = -\int_{-\infty}^{\infty} d\xi' V(\xi,\xi') N(\xi') \frac{1}{2\xi'} th(\frac{\xi'}{2T_c}) \Delta(\xi'), \tag{3.78}$$

where $N(\xi)$ - is the averaged on disorder density of states which includes the effects of electron-electron interaction, $V(\xi,\xi')$ - is the effective pairing interaction. The only difference with the standard approach is in the account of non-trivial dependence of $N(\xi)$ on the electron energy ξ, close to the Fermi level E_F.

In BCS theory we usually assume the existence of some effective electron-electron attraction, which is determined by the balance of pairing attraction due to electron-phonon interaction and Coulomb repulsion. Thus we consider the effective pairing interaction in the following simple form:

$$V(\xi,\xi') = V_c(\xi,\xi') + V_{ph}(\xi,\xi'), \tag{3.79}$$

where $V_c(\xi,\xi') = V_c\theta(E_F - |\xi|)\theta(E_F - |\xi'|)$ and $V_{ph}(\xi,\xi') = -V_{ph}\theta(\omega_D - |\xi|)\theta(\omega_D - |\xi'|)$ - are the respectively the electron-electron and electron-phonon interactions, ω_D - is the Debye frequency, or the characteristic frequency of any other Boson excitation responsible for pairing. The constants $V_c > 0$ and $V_{ph} > 0$ correspond to repulsion and attraction which effectively operate on rather different intervals of energy: $E_F \gg \omega_D$. After using this expression in Eq.(3.78) and some transformations using the even-odd properties of the gap function $\Delta(\xi)$ we obtain:

$$\begin{aligned}
\Delta(\xi) &= [V_{ph}\theta(\omega_D - \xi) - V_c\theta(E_F - \xi)] \int\limits_0^{\omega_D} d\xi' N(\xi')\frac{1}{\xi'}th(\frac{\xi'}{2T_c})\Delta(\xi') - \\
&- V_c\theta(E_F - \xi)\int\limits_{\omega_D}^{E_F} d\xi' N(\xi')\frac{1}{\xi'}th(\frac{\xi'}{2T_c})\Delta(\xi').
\end{aligned} \tag{3.80}$$

We look for the solution of this equation in the usual two-step form [Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)]:

$$\Delta(\xi) = \begin{cases} \Delta_{ph}, & |\xi| < \omega_D, \\ \Delta_c, & \omega_D < |\xi| < E_F, \end{cases} \tag{3.81}$$

where Δ_{ph}, Δ_c - are some constants which are determined from the following system of homogeneous linear equations, which is obtained after the substitution of Eq.(3.81) into Eq.(3.80):

$$\begin{aligned}
\{1 - (V_{ph} - V_c)N_0(0)K(\frac{\omega_D}{2T_c})\}\Delta_{ph} + V_cN_0(0)[K(\frac{E_F}{2T_c}) - K(\frac{\omega_D}{2T_c})]\Delta_c = 0,\\
V_cN_0(0)K(\frac{\omega_D}{2T_c})\Delta_{ph} + \{1 + V_cN_0(0)[K(\frac{E_F}{2T_c}) - K(\frac{\omega_D}{2T_c})]\Delta_c = 0,
\end{aligned} \tag{3.82}$$

where $N_0(0)$ - is the one-electron density of states of noninteracting electrons at the Fermi level and we introduce

$$K(\xi) = \int\limits_0^{\xi} d\xi' \frac{1}{\xi'} th(\xi') \left[\frac{N(2T_c\xi')}{N_0(0)} \right]. \tag{3.83}$$

Equation for T_c follows from the usual zero-determinant condition for this homogeneous system:

$$\begin{aligned} (\lambda \quad - \quad \mu^*)K(\frac{\omega_D}{2T_c}) = 1, \\ \mu^* = \mu\{1 \quad + \quad \mu[K(\frac{E_F}{2T_c}) - K(\frac{\omega_D}{2T_c})]\}^{-1}, \end{aligned} \tag{3.84}$$

where μ^* - is the Coulomb pseudopotential, $\mu = V_c N_0(0)$ - is the Coulomb constant, $\lambda = V_{ph} N_0(0)$ - is the pairing constant due to electron-phonon interaction. In the clean limit, when the density of states at the Fermi level is constant, this reduces to the usual BCS-expression for T_c.

Equation (3.84) for T_c was solved numerically for disorder parameter changing in wide interval both for metallic and insulating regions. The density of states was calculated taking into account lowest order corrections over electron-electron interaction as in Chapter 2 (Cf.(2.177),(2.179)) and in [Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A. (1997)]:

$$N(\xi) = -\frac{1}{\pi} Im \int \frac{d^3\mathbf{p}}{(2\pi)^3} G^R(\mathbf{p}, \xi), \tag{3.85}$$

where

$$G^{R(A)}(\mathbf{p}, \xi) = \frac{1}{\xi - \xi_{\mathbf{p}} \pm i\gamma - \Sigma_{ee}^{R(A)}(\mathbf{p}, \xi)} \tag{3.86}$$

- is the retarded (advanced) electron Green's function, $\Sigma_{ee}^{R(A)}(\mathbf{p}, \xi)$ - is "Fock" contribution to electron self-energy [Altshuler B.L., Aronov A.G. (1979); Altshuler B.L., Aronov A.G. (1985); Kuchinskii E.Z. et al. (1995)]:

$$\Sigma_{ee}^{R(A)}(\mathbf{p}, \xi) \approx 4i\gamma^2 \mu N_0^{-1}(0) G_0^{A(R)}(\mathbf{p}, \xi) \int\limits_{\xi}^{\infty} \frac{d\omega}{2\pi} \int\limits_{|\mathbf{q}|<k_0} \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{[-i\omega + D(\omega)q^2]^2}. \tag{3.87}$$

Here $D(\omega)$ - is the generalized diffusion coefficient, which is determined from the self-consistent nonlinear integral equation (2.168) of Chapter 2 [Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A. (1997)]:

$$\frac{D(\omega)}{D_0} = 1 - \frac{1}{\pi N_0(0)} \frac{D(\omega)}{D_0} \int\limits_{|\mathbf{q}|<k_0} \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{-i\omega + D(\omega)q^2} +$$

$$\frac{8i}{3\pi} \frac{\mu D_0}{\pi N_0(0)} \int\limits_{\omega}^{\infty} d\Omega \int\limits_{|\mathbf{q}|<k_0} \frac{q^2}{(-i(\Omega+\omega) + D(\Omega+\omega)q^2)(-i\Omega + D(\Omega)q^2)^2} \tag{3.88}$$

where $D_0 = E_F/\ 3m\gamma$ - is the usual Drude diffusion coefficient, $\gamma = 1/2\tau$ - Born scattering rate, τ - mean free time, $k_0 = Min\{p_F, l^{-1}\}$ - cut-off in the momentum space, p_F - Fermi momentum, l - mean free path. The data on static conductivity used below were also obtained by numerical solution of Eq.(3.88) [Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A. (1997)].

In Fig.2.14 we have shown the behavior of the density of states close to the Fermi level which demonstrated the evolution of the "Coulomb pseudogap" as disorder grows. This behavior obviously leads to superconducting T_c suppression as the system moves towards the metal-insulator transition.

Fig.3.3 demonstrates T_c suppression as disorder parameter $(p_F l)^{-1}$ grows for different values of the Coulomb constant μ and fixed value of pairing constant λ. For large μ and growing disorder $(p_F l)^{-1}$ the value of T_c drops rather fast and becomes zero in metallic region far enough from metal-insulator transition. For smaller values of μ this drop of T_c with growing disorder $(p_F l)^{-1}$ becomes slower and for small μ and large enough λ (dashed curves) we get the possibility of superconductivity persisting even in the insulating region. This possibility is clearly seen at the insert in Fig.3.3, where we show the T_c dependence on the static conductivity of the system σ for the appropriate values of λ and μ. For large values of μ as conductivity σ drops T_c also drops and superconductivity is completely suppressed rather far from the metal-insulator transition. For small values of μ this drop of T_c with σ becomes slower and for sufficiently large values of λ (dashed curves) T_c remains finite even in the case of $\sigma \to 0$.

Fig.3.4 demonstrates T_c degradation with the growth of disorder parameter $(p_F l)^{-1}$ for different values of the pairing constant λ for the fixed value

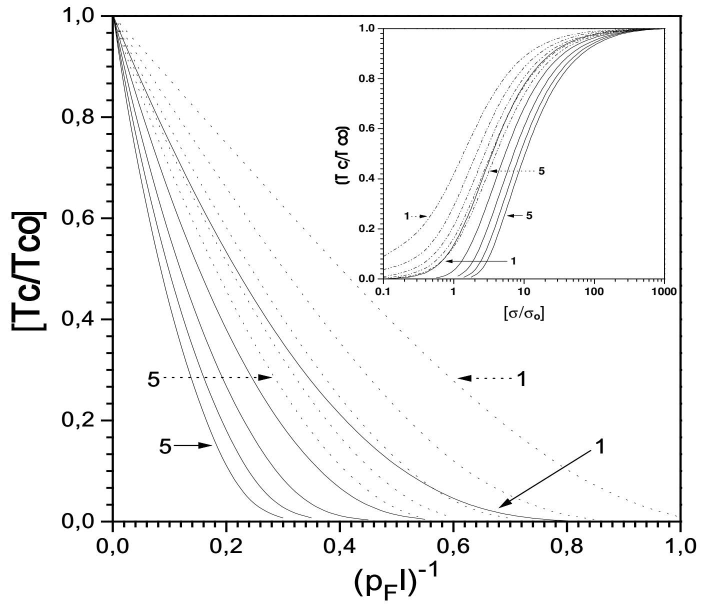

Fig. 3.3 T_c degradation as a function of disorder parameter $(p_F l)^{-1}$ for fixed pairing constant λ ($\lambda = 0.5$–full curves, $\lambda = 1.0$–dashed curves) and for different values of dimensionless Coulomb constant $\frac{8}{3\pi}\mu$: $1 - 0.2,...,5 - 1.0$. At the insert we show T_c dependence on static conductivity σ for the appropriate values of pairing constant λ and Coulomb repulsion μ.

of the Coulomb constant μ. For small λ and disorder parameter $(p_F l)^{-1}$ growing the value of T_c drops rather fast and becomes zero in the metallic state far from the metal-insulator transition. As λ grows this drop of T_c becomes slower and for large enough values of λ superconductivity is completely suppressed only somewhere in the insulating region. At the insert in

Fig.3.4 we show the dependence of the Coulomb pseudopotential μ^* on the disorder parameter $(p_F l)^{-1}$ for the appropriate values of λ and μ demonstrating rather insignificant growth of μ^* with disorder $(p_F l)^{-1}$ close to the point where superconductivity is completely suppressed. Apparently this behavior is natural enough because we neglect all the processes renormalizing the matrix element of Coulomb interaction in Eq.(3.79) due Anderson localization which can actually lead to rather important growth of Coulomb pseudopotential close to the metal-insulator transition as was shown in previous sections. Also we neglect here all possible effects of spin-fluctuations. Later in the book we shall try to compare these predictions with experiments on real systems [Kuchinskii E.Z., Sadovskii M.V., Erkabaev M.A. (1997)].

3.2.7 *Localization and d-wave Pairing*

There is a growing body of experimental evidence in high-T_c superconductors that indicate that the pairing state is of $d_{x^2-y^2}$ symmetry [Wollman D.A. et al. (1994); Tsuei C.C. et al. (1994); Annett J., Goldenfeld N., Leggett A.J. (1996)]. In superconductors with an anisotropic order parameter, both magnetic and non-magnetic impurities are pair breaking.For d–wave symmetry, the effect of non-magnetic impurities is equivalent to magnetic impurities in s-wave superconductors [Gorkov L.P., Kalugin P.A. (1985); Radke R.J. et al. (1993)]. Effectively this means that superconductivity in such systems in most cases cannot persist until disorder becomes high enough to transform the system into Anderson insulator. The situation is different for the so called extended s–wave symmetry. This corresponds to an order parameter with uniform sign which could, in particular, vanish at certain directions in momentum space [Fehrenbacher R., Norman M.R. (1994)]. Point impurities are not pair–breaking in this case, but they are "pair–weakening": for small impurity concentration T_c decreases linearly with disorder, but the critical impurity concentration is formally infinite, i.e. Anderson's theorem works after essential isotropisation of the gap[Markowitz D., Kadanoff L.P. (1963)].

We shall present now some of the relevant equations along the lines of our discussion of the Anderson theorem. More traditional derivation of the same results will be given in the next Chapter. We shall consider d-wave pairing on two-dimensional lattice induced by the following interaction

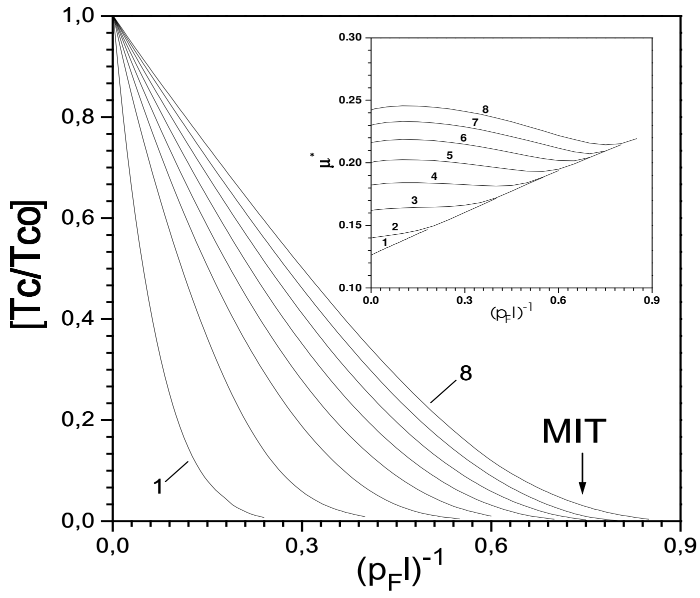

Fig. 3.4 T_c degradation as a function of disorder parameter $(p_F l)^{-1}$ for fixed value of Coulomb constant $\frac{8}{3\pi}\mu = 0.4$ and for different values of pairing constant λ: $1 - 0.3, 2 - 0.4, \ldots, 8 - 1.0$. At the insert we show the dependence of effective Coulomb pseudopotential μ^* on disorder parameter $(p_F l)^{-1}$) for appropriate values of pairing constant λ and Coulomb repulsion μ. An arrow shows the point of metal-insulator transition.

Hamiltonian:

$$H_{int} = -g \sum_{\mathbf{r}} \hat{\Delta}^{\dagger}_{\mathbf{r}} \hat{\Delta}_{\mathbf{r}} \tag{3.89}$$

where $\mathbf{r}$ denotes lattice sites. This Hamiltonian corresponds to an instantaneous anisotropic attractive interaction with an implicit cutoff at a characteristic energy $< \omega >$. In order to model $d_{x^2-y^2}$ symmetry we choose $\hat{\Delta}^{\dagger}_{\mathbf{r}}$, expressed via electron creation operators $c^{\dagger}_{\mathbf{r}}$ in the following form:

$$\hat{\Delta}^{\dagger}_{\mathbf{r}} = \frac{1}{\sqrt{2}} \sum_{\delta} \epsilon_{\delta} \left(c^{\dagger}_{\mathbf{r}\uparrow} c^{\dagger}_{\mathbf{r}+\delta\downarrow} - c^{\dagger}_{\mathbf{r}\downarrow} c^{\dagger}_{\mathbf{r}+\delta\uparrow} \right) \tag{3.90}$$

with $\delta = \pm \mathbf{e}_1, \pm \mathbf{e}_2$ being the lattice vectors, and $\epsilon_{\pm \mathbf{e}_1} = -\epsilon_{\pm \mathbf{e}_2} = 1$.

Next we can perform the analysis similar to that used in deriving Eqs. (3.7)—(3.16) and find that now we again have Eq.(3.16) determining the critical temperature T_c with the kernel $K(\mathbf{r}\mathbf{r}'\varepsilon_n)$ in the exact eigenstates representation taking the following form [Rojo A.G., Balseiro C.A. (1998)]:

$$K(\mathbf{r}\mathbf{r}'\varepsilon_n) = gT < \sum_{\mu\nu\delta\delta'} \epsilon_{\delta}\epsilon_{\delta'} \frac{\phi^*_{\mu}(\mathbf{r})\phi^*_{\nu}(\mathbf{r}+\delta)\phi_{\nu}(\mathbf{r}')\phi_{\mu}(\mathbf{r}'+\delta')}{(\varepsilon_{\nu} - i\varepsilon_n)(\varepsilon_{\mu} + i\varepsilon_n)} >= \tag{3.91}$$

$$= \int_{-\infty}^{\infty} dEN(E) \int_{-\infty}^{\infty} d\omega \frac{\ll d_E(\mathbf{r}) d_{E+\omega}(\mathbf{r}') \gg}{(i\varepsilon_n + E)(E + \omega - i\varepsilon_n)}$$

where we have introduced the spectral density:

$$\ll d_E(\mathbf{r}) d_{E+\omega}(\mathbf{r}') \gg = \frac{1}{N(E)} \times$$
$$< \sum_{\mu\nu} \sum_{\delta\delta'} \epsilon_{\delta}\epsilon_{\delta'} \phi^{\star}_{\nu}(\mathbf{r}+\delta)\phi_{\mu}(\mathbf{r})\phi^{\star}_{\mu}(\mathbf{r}')\phi_{\nu}(\mathbf{r}'+\delta')\delta(E-\varepsilon_{\nu})\delta(E+\omega-\varepsilon_{\nu'}) > \tag{3.92}$$

Now we can rewrite Eq.(3.16) for T_c as:

$$1 = gT_c \int_{-\infty}^{\infty} dEN(E) \int_{-\infty}^{\infty} d\omega \sum_{\varepsilon_n} \frac{g(\omega)}{(E + i\varepsilon_n)(E + \omega - i\varepsilon_n)} \tag{3.93}$$

where

$$g(\omega) = \int \mathbf{dr}' \ll d_E(\mathbf{r}) d_{E+\omega}(\mathbf{r}') \gg = \ll d_E d_{E+\omega} \gg_{\mathbf{q}=\mathbf{0}} \tag{3.94}$$

No sum rules similar to that given by Eqs.(A.5)-(A.6) exist for the spectral density of Eq.(3.92). However, it can be easily expressed via the Green's functions and we obtain the following relations similar to those obtained in Appendix A:

$$\ll d_E d_{E+\omega} \gg_{\mathbf{q}} = \frac{1}{\pi N(E)} Im \left\{ \Phi^{RA}_{Ed}(\omega\mathbf{q}) - \Phi^{RR}_{Ed}(\omega\mathbf{q}) \right\} \tag{3.95}$$

where

$$\Phi^{RA(R)}_{Ed}(\mathbf{q}\omega) = -\frac{1}{2\pi i} \sum_{\mathbf{p}\mathbf{p}'} \gamma^d_{\mathbf{p}} < G^R(\mathbf{p}_+\mathbf{p}'_+ E + \omega) G^{A(R)}(\mathbf{p}'_- \mathbf{p}_- E) > \gamma^d_{\mathbf{p}'} \tag{3.96}$$

with the vertices $\gamma^d_{\mathbf{p}} = \cos p_x a - \cos p_y a$ for d–wave. If from now on we ignore the lattice effects then $\gamma^d_{\mathbf{p}} = \cos 2\theta_{\mathbf{p}}$, which corresponds to a gap function $\Delta(\mathbf{p}) = \Delta(T)\cos 2\theta_{\mathbf{p}}$, where $\theta_{\mathbf{p}}$ is the polar angle in the plane [Fehrenbacher R., Norman M.R. (1994)]. Similar expressions will determine T_c for the case of anisotropic s-wave pairing with the vertices $\gamma^d_{\mathbf{p}}$ replaced by appropriate angle-dependent expressions [Fehrenbacher R., Norman M.R. (1994)].

Now we can write as usual:

$$g(\omega) = \frac{1}{\pi N(E)} Im \left\{ \Phi^{RA}_{Ed}(\omega\mathbf{q} = 0) \right\} = \tag{3.97}$$

$$= \frac{1}{\pi N(E)} Im \left\{ -\frac{1}{2\pi i} \sum_{\mathbf{p}\mathbf{p}'} \cos 2\theta_{\mathbf{p}} \Phi^{RA}_{\mathbf{p}\mathbf{p}'}(E\omega\mathbf{q} = 0) \cos 2\theta_{\mathbf{p}'} \right\}$$

Here $\Phi^{RA}_{\mathbf{p}\mathbf{p}'}(E\omega\mathbf{q} = 0)$ obeys the $\mathbf{q} = 0$ limit of the Bethe-Salpeter equation Eq.(2.44) which is easily transformed to the following kinetic equation [Vollhardt D., Wölfle P. (1980)]:

$$(\omega - \frac{i}{\tau})\Phi^{RA}_{\mathbf{p}\mathbf{p}'}(E\omega) = -\Delta G_{\mathbf{p}} \left[\delta(\mathbf{p} - \mathbf{p}') + \sum_{\mathbf{p}''} U^E_{\mathbf{p}\mathbf{p}''}(\omega) \Phi^{RA}_{\mathbf{p}''\mathbf{p}'}(E\omega) \right] \tag{3.98}$$

with $\Delta G_{\mathbf{p}} \equiv G^R(\mathbf{p}E + \omega) - G^A(\mathbf{p}E)$. If we replace in (3.98) the irreducible vertex by the bare vertex $U_0 = \rho V^2$, we obtain finally:

$$g(\omega) = \frac{1}{4\pi} \frac{\tau}{1 + (\omega\tau)^2} \ . \tag{3.99}$$

with the usual scattering rate $1/\tau = 2\pi\rho V^2 N(E)$. Inserting (3.99) in (3.93) and following the standard analysis [Gennes de P.G. (1966)] we obtain the well known expression for the critical temperature:

$$\ln(T_{c0}/T_c) = \Psi(1/2 + 1/4\pi\tau T_c) - \Psi(1/2) \tag{3.100}$$

where $\Psi(x)$–is the di-Gamma function, and which is identical to Abrikosov–Gorkov dependence in the case of magnetic impurity scattering in superconductors [Abrikosov A.A., Gorkov L.P. (1960); Gennes de P.G. (1966)]. However here the normal potential scattering rate is operational leading to very fast degradation of T_c — superconducting state is completely destroyed for critical scattering rate $1/\tau > 2\gamma_c = 1.76T_{c0}$. This scattering rate actually defines the appropriate critical concentration of impurities as well as the critical value of residual resistivity of the normal phase:

$$\rho_{AG} = \frac{2m\gamma_c}{ne^2} = \frac{8\pi\gamma_c}{\omega_p^2} \tag{3.101}$$

where n and m are electron concentration and mass, ω_p is plasma frequency of electrons [Radke R.J. et al. (1993)]. Actually this result does not depend on spatial dimensionality of the system. It is very convenient to use the relation similar to (3.101) to express $1/\tau = 2\gamma$ in (3.100) through experimentally measurable resistivity ρ to compare the observed dependences of T_c on ρ with predictions of (3.100) [Radke R.J. et al. (1993)].

It seems that effectively this makes practically impossible to reach the Anderson transition before superconductivity is destroyed: critical disorder for metal-insulator transition is determined by $1/\tau \sim E_F \gg T_c$. Some initial hope is to analyze the quasi-two-dimensional case, where this critical disorder can be reduced due to a small enough interplane transfer integral w as in Eqs.(2.91)-(2.92). Localization appears for $w < w_c = \frac{\sqrt{2}}{\tau}exp(-\pi E_F\tau)$ and taking as an estimate $1/\tau \approx T_{c0}$, so that superconductivity is still possible, we can arrive at the following criterion of coexistence of localization and superconductivity:

$$w < T_{c0}exp(-\pi E_F/T_{c0}) \tag{3.102}$$

In typical situation even for high-temperature superconductors we have $T_{c0} < 0.1E_F$ and inequality in Eq.(3.102) can be satisfied only for extremely anisotropic systems with $w \ll T_{c0}$. Most known superconductors apparently fail in this respect.

However, there is enough experimental data on the closeness of e.g. radiationally disordered high-T_c systems to the disorder induced metal-insulator transition, which will be discussed in the experimental part of our book. The obvious question arises – how to reconcile these data and all the evidence for d-wave pairing in these systems? Of course, the above reasoning does not apply to the case of anisotropic s-wave pairing, where Anderson theorem effectively works for large degrees of disorder [Borkovski L.S., Hirschfeld P.J. (1994); Fehrenbacher R., Norman M.R. (1994)]. In this respect the experiments on disordering in high-T_c systems can become crucial in solving the problem of the nature of pairing (and thus of its microscopic mechanisms) in these systems. But still, the evidence for d–wave pairing in high-T_c superconductors is compelling enough [Annett J., Goldenfeld N., Leggett A.J. (1996)]. One of the possible explanations of greater stability of d–wave pairing may be due to the effectiveness of *anisotropic* impurity scattering in these systems, which really makes d–wave pairing more stable to disordering, than in the standard case, described by Eq.(3.100) [Haran G., Nagi A.D.S. (1996); Posazhennikova A.I., Sadovskii M.V. (1997a)]. We shall discuss this model in the next Chapter. However, this explanation seems to be rather non-universal and limited. There is an obvious need of some more general reason for greater stability of d–wave pairing in copper oxides. Such a general idea was proposed on qualitative level in [Posazhennikova A.I., Sadovskii M.V. (1997b)].

Consider the (opposite to the usual BSC-picture) limit of extremely strong pairing interaction, leading to compact Boson formation [Nozieres P., Schmitt-Rink S. (1985)]. In this case T_c is determined by the temperature of Bose condensation of free Bosons. In case of an impure system Bose-condensation point can be determined using the general method valid for any interacting Bose-system [Patashinskii A.Z., Pokrovskii V.L. (1982)]. The number of particles (Bosons) with momentum $\mathbf{p}$ is given by:

$$n_{\mathbf{p}} = -T \sum_n G(\mathbf{p}, \omega_n), \tag{3.103}$$

where

$$G(\mathbf{p}, \omega_n) = \frac{1}{i\omega_n + \mu - \frac{p^2}{2m} - \Sigma(\mathbf{p}, \omega_n)}. \tag{3.104}$$

where $\omega_n = 2n\pi T$ is the even Matsubara frequency, μ is the chemical potential, $\Sigma(\mathbf{p}, \omega_n)$ – self–energy of Bosons, interacting with random impurities. In the simplest approximation used below Σ is independent of momentum:

$\Sigma(\mathbf{p}, \omega_n) \equiv \Sigma(\omega_n)$.

Bose-condensation leads to the divergence of the number of particles n_0 with momentum $p = 0$. The divergence of n_0 according to (3.103) may be due to two reasons: either the series diverges, or one or few terms in it diverge. Terms of (3.103) with large ω_n behave as in an ideal Bose-gas [Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)]. In an ideal Bose-gas ($\Sigma = 0$) n_0 diverges due to the divergence of the term with $\omega_n = 0$, while the sum of remaining terms is finite[Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)]. Thus at the transition point at least one of the terms in (3.103) becomes infinite for $p = 0$, so that:

$$G^{-1}(0, \omega_n) = 0. \tag{3.105}$$

If this equation is valid for $n \neq 0$, then it is equivalent to two real equations (for real and imaginary parts of G^{-1}), so that it determines an isolated singular point in the plane of $\mu - T$. In an ideal Bose-gas the transition line is determined by $\mu = 0$. From continuity arguments it can be expected that in an interacting case the phase transition will take place along some line in this plane. This situation is described by Eq.(3.105) with $\omega_n = 0$ (if $\Sigma(\mathbf{p}, 0)$ is real) [Patashinskii A.Z., Pokrovskii V.L. (1982)]. Thus in case of impure system condensation point can be determined by the following equation:

$$\mu_p - \Sigma(0) = 0 \tag{3.106}$$

where μ_p is the chemical potential of pairs and $\Sigma(0)$ is the zero-frequency limit of Boson self-energy due to impurity scattering, which in case of impurities with point-like potential V, randomly distributed in space with concentration n_{imp} (we introduce this notation herenot to mix impurity concentration with resistivity ρ), reduces to the one-loop expression, corresponding to diagram shown in Fig.3.5:

$$\Sigma(\omega_n) = n_{imp} V^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{i\omega_n - \frac{\mathbf{p}^2}{2m^\star} + \mu_p} \tag{3.107}$$

where $m^\star = 2m$ is the mass of the pair, and we assume temperatures $T > T_c$. In the following we consider only three-dimensional systems.

Let us show that in this approximation $\Sigma(0)$ is real. From (3.107) we can obtain the following expressions for real and imaginary parts of Σ:

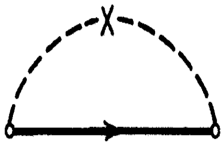

Fig. 3.5 One-loop self-energy part of Boson in an impure system.

$$Re\Sigma(\omega_n) = -\frac{n_{imp}V^2}{2\pi^2}\int_0^\infty dp\frac{p^2(\frac{p^2}{2m^\star}-\mu_p)}{\omega_n^2+(\frac{p^2}{2m^\star}-\mu_p)^2}, \tag{3.108}$$

$$Im\Sigma(\omega_n) = -\frac{n_{imp}V^2}{2\pi^2}\omega_n\int_0^\infty dp\frac{p^2}{\omega_n^2+(\frac{p^2}{2m^\star}-\mu_p)^2}. \tag{3.109}$$

Direct calculations yield:

$$Re\Sigma(\omega_n) = E_{0c} + \frac{n_{imp}V^2m^{\star 3/2}}{2\pi}\sqrt{\sqrt{\mu_p^2+\omega_n^2}-\mu_p^2}, \tag{3.110}$$

$$Im\Sigma(\omega_n) = -\frac{n_{imp}V^2m^{\star 3/2}}{2\pi}\frac{\omega_n}{\sqrt{\sqrt{\mu_p^2+\omega_n^2}-\mu_p^2}}, \tag{3.111}$$

where $E_{0c} = -\frac{m^\star}{\pi^2}n_{imp}V^2p_0$ – is the band-edge shift due to impurity scattering [Sadovskii M.V. (1986)], p_0 – is some cut-off in momentum space of the order of inverse lattice spacing a^{-1}. Then we see that really $Im\Sigma(0) = 0$, and $\Sigma(0)$ can be written as:

$$\Sigma(0) = Re\tilde{\Sigma}(0) + E_{0c} \tag{3.112}$$

and

$$Re\tilde{\Sigma}(0) = \frac{1}{\sqrt{2}\pi}n_{imp}V^2m^{\star 3/2}\sqrt{|\mu_p|} \tag{3.113}$$

Actually, E_{0c} leads just to renormalization of the chemical potential: $\tilde{\mu} = \mu_p - E_{0c}$, so that in renormalized form Eq.(3.106) reduces to:

$$\tilde{\mu}\left(1-\frac{1}{\sqrt{2|\tilde{\mu}|}\pi}n_{imp}V^2m^{\star 3/2}sign\tilde{\mu}\right) = 0 \tag{3.114}$$

with the only relevant ($\tilde{\mu} < 0$ for Bosons at $T > T_c$) solution of $\tilde{\mu} = 0$, i.e. $\mu_p - E_{0c} = 0$, determining the Bose condensation temperature of the impure system by the standard equation:

$$\frac{n}{2} = g \int_{-\infty}^{\infty} d\varepsilon N(\varepsilon) \frac{1}{e^{\frac{\varepsilon}{T_c}} - 1} \tag{3.115}$$

where $g = 2s + 1$ (for Bosons of spin s), $N(\varepsilon)$ is the impurity averaged density of states, which in case of the simplest approximation of Eq.(3.107) just reduces to $N(E - E_{0c})$ - the usual free particle expression with energy ε calculated with respect to the shifted band-edge. Obviously we obtain the standard expression for T_c [Landau L.D., Lifshits E.M. (1976)]:

$$T_c = \frac{3.31}{g^{2/3}} \frac{(n/2)^{2/3}}{m^\star} \tag{3.116}$$

which is *independent of disorder*. The only possible disorder effect may be connected with exponentially small "Lifshits tail" in the density of states in Eq.(3.115) due to localization [Lifshits I.M., Gredeskul S.A., Pastur L.A. (1982)], which is neglected in our simplest approximation of Eq.(3.107). Thus, our conclusion is that in case of extremely strong pairing interaction (compact Boson picture of superconductivity) T_c is practically disorder independent for *any* value of the spin of Cooper pair, e.g. s-wave, d-wave etc.

It was shown rather long ago by Nozieres and Schmitt-Rink [Nozieres P., Schmitt-Rink S. (1985)] for non impure superconductor that as the strength of the pairing interaction grows, there is a smooth crossover of T_c from the weak-coupling BCS-picture to that of compact Bosons. In the impure case similar analysis for T_c can apparently be performed solving the following coupled system of equations generalizing similar equations of Ref.[Nozieres P., Schmitt-Rink S. (1985)] — the usual equation for BCS instability:

$$1 - \chi(0,0) = 0 \tag{3.117}$$

and the equation for Fermion density (chemical potential of electrons μ):

$$\frac{1}{2}(n - n_f) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d\omega}{\pi} \frac{1}{exp(\frac{\omega}{T_c}) - 1} \frac{\partial}{\partial \mu} \delta(\mathbf{q}\omega) \tag{3.118}$$

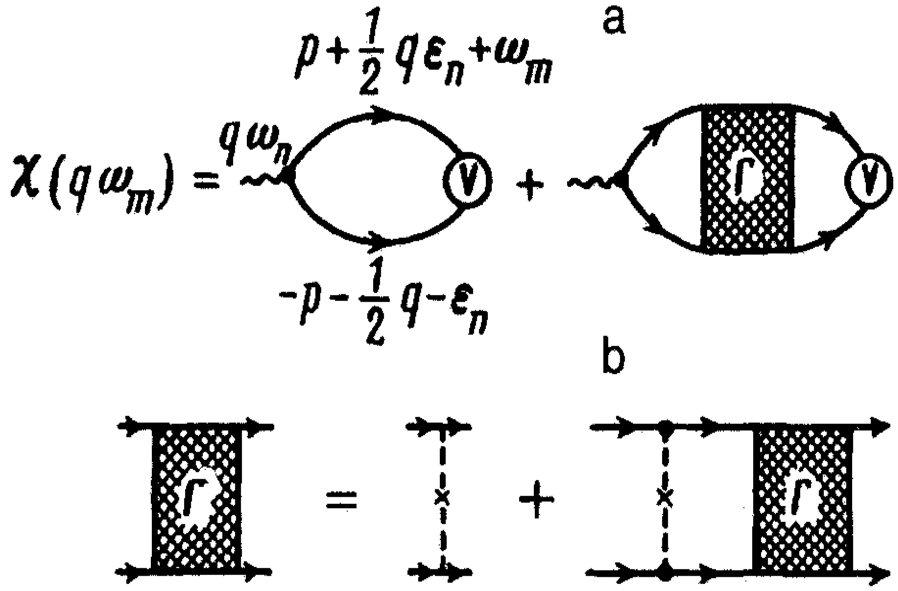

Fig. 3.6 Diagrammatical representation of Cooper susceptibility $\chi(\mathbf{q}\omega)$ - (a). Γ – impurity scattering vertex-part in Cooper channel in "ladder" approximation - (b).

where $n_f(\mu, T_c)$ is the free Fermion part of density,

$$\delta(\mathbf{q}\omega) = arctg\frac{Im\chi(\mathbf{q}\omega)}{1 - Re\chi(\mathbf{q}\omega)}, \tag{3.119}$$

and Cooper susceptibility $\chi(\mathbf{q}\omega)$ is determined by diagrams shown in Fig.3.6. In this figure the vertices contain the symmetry factors for different types of pairing, e.g. in case of cubic lattice [Scalapino D.J., Loh E., Hirsch J. (1987)]:

$$\begin{aligned}
\psi_s(\mathbf{p}) &= 1 \qquad \text{(isotropic } s\text{-wave)}\\
\psi_{s'}(\mathbf{p}) &= \cos p_x a + \cos p_y a + \cos p_z a \qquad \text{(anisotropic } s\text{-wave)}\\
\psi_{d_{x^2-y^2}}(\mathbf{p}) &= \cos p_x a - \cos p_y a \qquad (d\text{-wave})\\
\psi_{d_{3z^2-r^2}}(\mathbf{p}) &= 2\cos p_z a - \cos p_x a - \cos p_y a \qquad \text{etc.}
\end{aligned} \tag{3.120}$$

Pairing interaction is assumed to have the following form:

$$V_i(\mathbf{p}, \mathbf{p}') = V_{\mathbf{pp}'}\psi_i(\mathbf{p})\psi_i(\mathbf{p}') \tag{3.121}$$

with $\psi_i(\mathbf{p})$ defined as above and pairing potential

$$V_{\mathbf{pp}'} = -\frac{V_0}{\sqrt{\left(1+\frac{p^2}{p_0^2}\right)\left(1+\frac{p'^2}{p_0^2}\right)}} \tag{3.122}$$

similar to that used in [Nozieres P., Schmitt-Rink S. (1985)] with $p_0 \sim a^{-1}$.

Numerical work required to solve Eqs.(3.117),(3.118) is very heavy even for non impure case [Nozieres P., Schmitt-Rink S. (1985)]. However, it is more or less clear that these equations will produce also the smooth crossover in T_c dependence on disorder, interpolating between the BCS and compact Boson limits discussed above. In isotropic s-wave case T_c will remain practically independent from disorder, i.e. the Anderson theorem remains valid also for compact Boson limit. In case of d-wave pairing the universal dependence of T_c on disorder defined by Eq.(3.100) ceases to be valid in the crossover region from large Cooper pairs to compact Bosons. The physical reason for this is quite clear — depairing mechanism of T_c suppression by disorder ceases to operate with the growth of attractive interaction within pairs, and in the strong coupling region T_c is determined by Bose condensation of pairs in impure system. Qualitative behavior of T_c dependence on disorder is shown in Fig.3.7.

It illustrates the smooth crossover in T_c dependence on normal state resistivity from universal Abrikosov-Gorkov dependence (curve d) to T_c independent on disorder (curve s). Dashed lines correspond to transition region and the values of coupling constant V_0 growing from curve 1 to curve 2. It is clear that for d-wave system belonging to this transitional region we can easily obtain superconducting state persisting for rather large disorder with $\rho > \rho_{AG}$.

Crossover region is qualitatively defined by the simple inequality introduced in Ref. [Pistolesi F., Strinati G.C. (1994)]: $\pi^{-1} < p_F\xi < 2\pi$, where p_F is Fermi momentum and ξ is superconducting coherence length. It appears that high-temperature superconductors lie on the the so-called Uemura plot [Uemura Y.J., et al. (1991)] near the "instability" line $p_F\xi = 2\pi$ [Pistolesi F., Strinati G.C. (1994)]. This can explain deviations of T_c dependence on disorder in these systems from universal Abrikosov-Gorkov curve and their relative stability to disordering, despite the possible d-wave symmetry of the pairing state.

This qualitative picture was essentially confirmed by more detailed calculations performed in [Franz M., et al. (1997)]. By numerical solution of

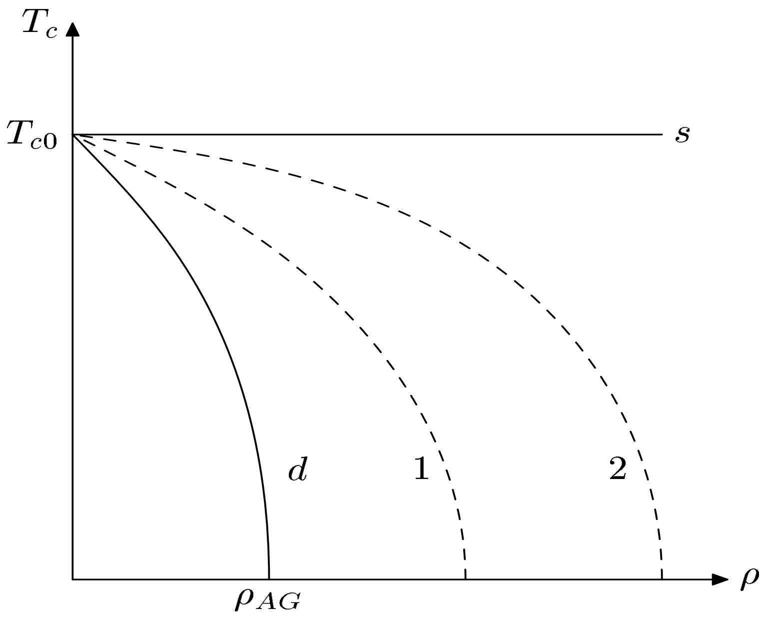

Fig. 3.7 Qualitative dependence of superconducting transition temperature T_c on disorder (residual resistivity ρ). Curve d – universal Abrikosov-Gorkov dependence, defined by Eq.(3.100). Line s – the case of isotropic s–wave pairing. Dashed curves — d–wave pairing in transition region from BSC-pairs to compact Bosons.

Bogoliubov – deGennes (BDG) equations [Gennes de P.G. (1966)] they demonstrated that the standard Abrikosov-Gorkov theory of T_c in disordered d–wave superconductors breaks down in short-coherence length (small size pairs) systems. It was shown that the correct description of such superconductors must allow for the spatial variations of the order parameter, which is strongly suppressed in the vicinity of impurities but mostly unaffected elsewhere [Franz M., Kallin C., Berlinsky A.J. (1996)]. Somehow similar ideas were expressed in Ref.[Zhitomirsky M.E., Walker M.B. (1998)]. Suppression of T_c was found to be significantly weaker than that predicted by Abrikosov-Gorkov relation (3.100) in complete agreement with the picture presented above. It is clearly seen in Fig.3.8 taken from [Franz M., et al. (1997)].

Finally let us mention another situation when localization effects become important and interesting in the case of d-wave pairing — that of localization of BCS-quasiparticles within superconducting gap at relatively

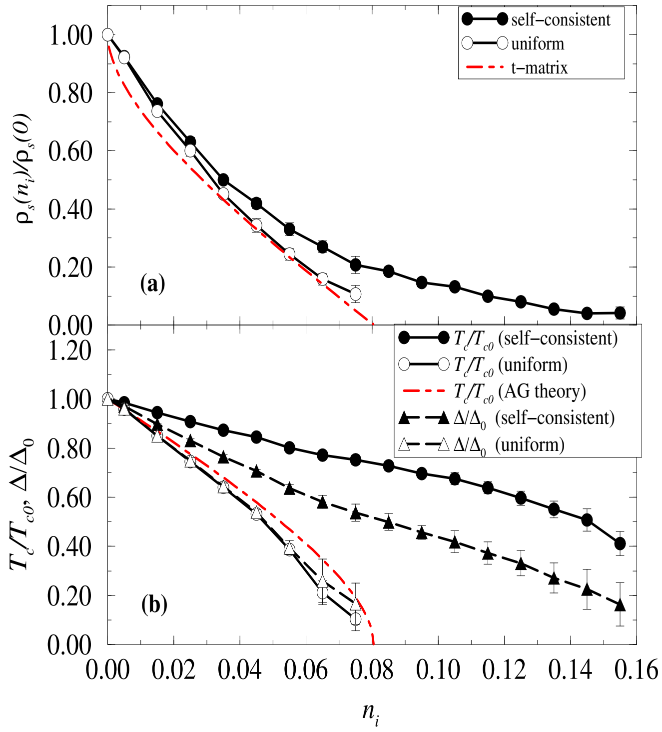

Fig. 3.8 (a) – Normalized zero-temperature superfluid density as a function of impurity concentration: fully self-consistent solution of BDG equations (solid symbols), solution with uniform order parameter (open symbols), and analytic t–matrix solution with unitary scatterers (dash-dotted line). (b) – Normalized critical temperature and average gap. Dash-dotted line is numerical solution of Eq.(3.100).

small disorder [Lee P.A. (1993); Balatsky A.V., Rosengren A., Altshuler B.L. (1994); Nersesyan A.A., Tsvelik A.M., Wenger F. (1994); Nersesyan A.A., Tsvelik A.M., Wenger F. (1995)]. It is known that while in the pure d-wave superconductor density of states close to the Fermi level is linear in energy $N(E) \sim E$ due to the gap nodes at the Fermi surface, the impurity scattering makes it finite at $E = 0$ [Gorkov L.P., Kalugin P.A. (1985)]. In this sense the system becomes similar to the *normal* metal and we can calculate [Lee P.A. (1993)] the low lying quasiparticle contribution to conductivity $\sigma(\omega \to 0)$. This conductivity equals to:

$$\sigma \approx \frac{e^2}{2\pi\hbar}\frac{\xi_0}{a} \tag{3.123}$$

where $\xi_0 = v_F/\pi\Delta_0$ is the superconducting coherence length and a is the lattice spacing (we assume $T = 0$). The surprising thing is that σ *independent of the scattering rate* $1/\tau$, i.e. of disorder. For two-dimensional case (applicable probably for high-T_c systems) we know that all states are localized with localization controlled by dimensionless conductance which now is equal to $g = \sigma/(e^2/2\pi\hbar) = \xi_0/a$. The value of g may be small enough in high temperature superconductors due to the small values of ξ_0, which are typically only slightly larger than the lattice constant. This can make localization effects important with BCS-quasiparticles forming a mobility gap in the vicinity of the Fermi level, leading to anomalies in the low temperature behavior of microwave conductivity and the penetration depth of a d-wave superconductor [Lee P.A. (1993)].

These results were first obtained [Lee P.A. (1993)] for the point-like impurity scattering, later it was shown in Ref.[Balatsky A.V., Rosengren A., Altshuler B.L. (1994)] that the finite range of the impurity potential can lead to the nonuniversal disorder-dependent behavior of conductivity which becomes proportional to the normal state scattering rate. Situation was further complicated by the claim made in Refs.[Nersesyan A.A., Tsvelik A.M., Wenger F. (1994); Nersesyan A.A., Tsvelik A.M., Wenger F. (1995)] that the more rigorous analysis leads to the density of states of the impure d-wave superconductor behaving as $N(E) \sim |E|^\alpha$ with $\alpha > 0$, but dependent on the type of disorder. The renormalization group for the conductivity then apparently leads to some kind of fixed point of intermediate nature, suggesting the finite conductivity in two-dimensions. Opposite results were obtained in an exactly solvable model in Ref.[Ziegler K., Hettler M.H., Hirschfeld P.J. (1996)]. Numerical studies of localization of low-energy quasiparticle states in disordered d–wave superconductors were performed in [Franz M., Kallin C., Berlinsky A.J. (1996)]. All these aspects of disorder and localization for d-wave superconductors deserve further intensive studies.

3.3 Ginzburg-Landau Theory and Anderson Transition

3.3.1 *General Analysis*

The main result of the previous analysis may be formulated as follows. Despite many mechanisms leading to T_c degradation and destruction of superconductivity in strongly disordered systems there seems to be no general rule prohibiting a possibility of a superconducting state in Anderson insulator. Of course we must meet very rigid conditions if we hope to observe this rather exotic state. There is almost no chance to observe it in traditional superconductors but high—T_c systems seem promising. The following analysis will be based on the general assumption that T_c survives in a strongly disordered system or even in Anderson insulator, i.e. that these strict conditions are met. Our aim is to study superconducting properties of such a strongly disordered system to determine specific characteristics which will make this case different from the usual case of "dirty" superconductors. We shall see that even before Anderson transition there are significant deviations from the predictions of standard theory which make strongly disordered system different. So on the practical side our aim is simply to generalize the usual theory of "dirty" superconductors for the case of strong disorder in a sense of the mean free path becoming of the order of interatomic spacing or $l \sim p_F^{-1}$.

To claim that superconductivity is possible close to disorder—induced metal—insulator transition it is not sufficient just to demonstrate the finite values of T_c. Even more important is to show the existence of superconducting response to an external electromagnetic potential $\mathbf{A}$. In general case the analysis of response functions of a superconductor with strong disorder seems to be a difficult task. However, close to T_c significant simplifications take place and actually we only have to show that the free—energy density of the system can be expressed in the standard Ginzburg—Landau form [Ginzburg V.L., Landau L.D. (1950); Gorkov L.P. (1959); Gennes de P.G. (1966)]:

$$F_s = F_n + A|\Delta|^2 + \frac{1}{2}B|\Delta|^4 + C|(\nabla - \frac{2ie}{\hbar c}\mathbf{A})\Delta|^2 \qquad (3.124)$$

where F_n is free energy density of the normal state. Our problem is thus reduced to a microscopic derivation of expressions for the coefficients A, B, and C of Ginzburg—Landau expansion Eq. (3.124) taking into account the possibility of electron localization. This will be the generalization of the famous Gorkov's derivation [Gorkov L.P. (1959)] of similar expressions for the case of "dirty" superconductors. Such analysis was first done by Bulaevskii and Sadovskii [Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)] and later by Kotliar and Kapitulnik [Kapitulnik A., Kotliar G. (1985); Kotliar G., Kapitulnik A. (1986)]. The same results were obtained also by Kravtsov [Kravtsov V.E. (1991)].

Within the BCS model coefficients A and B which determine the transition temperature and the equilibrium value of the order—parameter Δ do not change in comparison with their values found in the theory of "dirty" superconductors, even if the system is close to Anderson transition. This corresponds to the main statement of Anderson theorem. Less trivial is the behavior of the coefficient C, which in fact defines the superconducting response. In the usual theory of "dirty" superconductors [Gorkov L.P. (1959)] this coefficient is proportional to diffusion coefficient of electrons, i.e. to conductivity (at $T = 0$). As the Fermi level approaches the mobility edge conductivity drops to zero. However, we shall see that the coefficient C remains finite in the vicinity of Anderson transition, even in the region of localized states.

Simple scheme for microscopic derivation of Ginzburg—Landau expansion for the free energy is outlined in Appendix C. In fact, to derive Ginzburg—Landau coefficients we must know the two—electron Green's function in the normal state [Gorkov L.P. (1959)]. Let us introduce the following two-particle Matsubara Green's functions in momentum representation [Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$\Psi_E(\mathbf{q},\omega_m,\varepsilon_n) = -\frac{1}{2\pi i}\sum_{\mathbf{p}_+\mathbf{p}'_-} < G(\mathbf{p}_+,\mathbf{p}'_+,-\varepsilon_n+\omega_m)G(-\mathbf{p}'_-,-\mathbf{p}_-,-\varepsilon_n) > \tag{3.125}$$

$$\Phi_E(\mathbf{q},\omega_m,\varepsilon_n) = -\frac{1}{2\pi i}\sum_{\mathbf{p}_+\mathbf{p}'_-} < G(\mathbf{p}_+,\mathbf{p}'_+,-\varepsilon_n+\omega_m)G(\mathbf{p}'_-,\mathbf{p}_-,-\varepsilon_n) > \tag{3.126}$$

where $\mathbf{p}_{+-} = \mathbf{p} \pm \mathbf{q}/2$ and $\omega_m = 2\pi mT$. Graphically these functions are represented in Fig. 3.9. Then Ginzburg—Landau coefficients are defined by (Cf. Appendix C) [Gorkov L.P. (1959); Takagi H., Souda R., Kuroda Y.

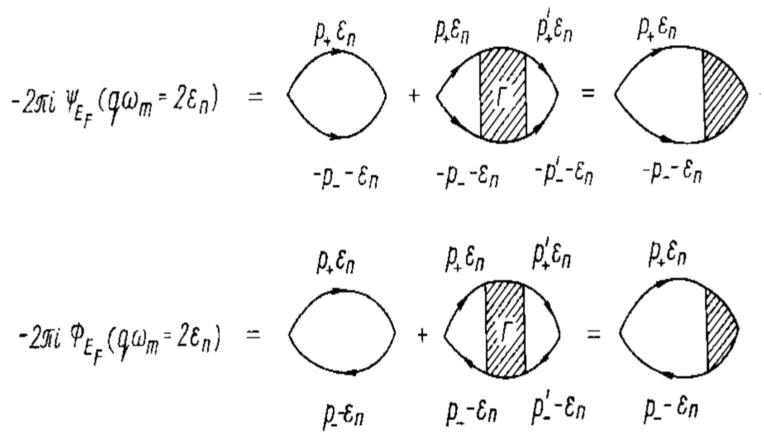

Fig. 3.9 Graphic representation of two—particle Green's functions $\Psi_E(\mathbf{q}\omega_m)$ and $\Phi_E(\mathbf{q}\omega_m)$ (for $\omega_m = 2\varepsilon_n$). There is no summation over ε_n in the loops.

(1982)]:

$$A = \frac{1}{g} + 2\pi i T \sum_{\varepsilon_n} \Psi_E(\mathbf{q} = 0\omega_m = 2\varepsilon_n) \tag{3.127}$$

$$C = i\pi T \sum_{\varepsilon_n} \frac{\partial^2}{\partial q^2} \Psi_E(\mathbf{q}\omega_m = 2\varepsilon_n)|_{q=0} \tag{3.128}$$

Thus the superconducting properties are determined by the Green's function Ψ_E describing the propagation of electronic (Cooper) pair. At the same time we have seen that the Green's function Φ_E determines transport properties of a normal metal and Anderson transition. In case of time–reversal invariance (i.e. in the absence of external magnetic field or magnetic impurities) we have [Yoshioka D., Ono Y., Fukuyama H. (1981)]:

$$\Psi_E(\mathbf{q}\omega_m\varepsilon_n) = \Phi_E(\mathbf{q}\omega_m\varepsilon_n) \tag{3.129}$$

and it is sufficient to know only $\Phi_E(\mathbf{q}\omega_m = 2\varepsilon_n)$ to determine Ginzburg—Landau coefficients.

As a one—electron model of Anderson transition we can take the self—consistent theory of localization which will allow us to perform all calculations explicitly. We only have to formulate the main equations of this theory in Matsubara formalism (finite T) [Bulaevskii L.N., Sadovskii M.V. (1985)]. For small q and ω_m, analogously to Eq. (2.46), we have:

$$\Phi_E(\mathbf{q}\omega_m) = -\frac{N(E)}{i|\omega_m| + D_E(|\omega_m|)q^2} \qquad \omega_m = 2\pi mT \tag{3.130}$$

where the generalized diffusion coefficient $D_E(\omega_m)$ is determined by the self—consistency equation analogous to Eq. (2.58):

$$\frac{D_0}{D_E(\omega_m)} = 1 - \frac{i}{\pi N^2(E)} \sum_{|\mathbf{q}|<k_0} \Phi_E(\mathbf{q}\omega_m) \tag{3.131}$$

In three-dimensional case Eq. (3.131) reduces to (Cf. Eq. (2.78)):

$$\frac{D_E(\omega_m)}{D_0} = 1 - \frac{\lambda}{\lambda_c} + \frac{\pi}{2}\frac{\lambda}{\lambda_c}\left[\frac{D_0}{D_E(\omega_m)}\frac{\omega_m}{2\gamma}\right]^{1/2} \tag{3.132}$$

where we have used the same notations as in our discussion of self—consistent theory of localization. Analogously to Eq. (2.79) and with accuracy sufficient for our aims we can write down the solution of Eq. (3.132) as:

$$D_E(\omega_m) \approx Max\left\{D_E\frac{\omega_m}{\omega_m + 3D_E\omega_0^2(E)/v_F^2}; \quad D_0\left(\frac{\omega_m}{2\gamma}\right)^{1/3}\right\} \tag{3.133}$$

where D_E is the renormalized diffusion coefficient defined in Eq. (2.81) and ω_0 is the fundamental frequency defined by Eq. (2.65), which signals a transition to insulator. Details of calculation of Ginzburg-Landau coefficients are given in Appendix C.

As we have already noted Ginzburg—Landau coefficients A and B are given by the usual expressions valid also for "dirty" superconductors [Gorkov L.P. (1959); Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$A = N(E_F)ln\frac{T}{T_c} \approx N(E_F)\frac{T - T_c}{T_c} \tag{3.134}$$

where T_c is given by the usual BCS relation of Eq. (3.19), and

$$B = \frac{7\zeta(3)}{8\pi^2T_c^2}N(E_F) \tag{3.135}$$

where $\zeta(x)$ is Riemann zeta—function ($\zeta(3) = 1.202...$). These coefficients depend on disorder only through the appropriate disorder dependence of $N(E_F)$ and are valid even in localized phase. This is equivalent to the main statement of Anderson theorem.

Significant changes appear in the gradient term coefficient C. Using Eqs. (3.128)—(3.130) with Eq. (3.133) we can find that in different limiting cases this coefficient can be expressed as (Cf. Appendix C) [Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$C \equiv N(E_F)\xi^2 \approx N(E_F)\begin{cases} \frac{\pi}{8T_c}D_{E_F} & \xi_{loc}(E_F) < (\xi_0 l^2)^{1/3}; \quad E_F > E_c \\ \left(\frac{D_0 l}{T_c}\right)^{2/3} \approx (\xi_0 l^2)^{2/3} & \xi_{loc}(E_F) > (\xi_0 l^2)^{1/3}; \\ \quad E_F \sim E_c & \\ R^2_{loc}(E_F) ln\frac{1.78 D_{E_F}}{\pi T_c R^2_{loc}(E_F)} & R_{loc}(E_F) < (\xi_0 l^2)^{1/3}; \\ \quad E_F < E_c & \end{cases} \tag{3.136}$$

where we have defined the coherence length ξ, and $\xi_0 = 0.18v_F/T_c$ is BCS coherence length, l as usual is the mean free path. Practically the same results were obtained in Refs.[Kapitulnik A., Kotliar G. (1985); Kotliar G., Kapitulnik A. (1986)] using the approach based upon elementary scaling theory of localization, which is as we already noted is equivalent to our use of self—consistent theory of localization. In Ref.[Kravtsov V.E. (1991)] the same results were confirmed using the σ—model approach to localization.

In metallic state, as Fermi level E_F moves towards the mobility edge E_c localization correlation length ξ_{loc} grows and the coefficient C initially drops as the generalized diffusion coefficient D_{E_F}, i.e. as conductivity of a system in the normal state. However, in the vicinity of Anderson transition while $\sigma \to 0$ the drop of C coefficient saturates and it remains finite even for $E_F < E_c$, i.e. in Anderson insulator. With further lowering of E_F into localization region (or with E_c growth with disorder) the C coefficient is being determined by localization radius R_{loc} which diminishes as E_F moves deep into insulating state. However, remembering Eq. (3.20) and Eq. (3.22) we recognize that our analysis is valid only for large enough values of localization length, which satisfy Eq. (3.22). In this sense the last asymptotics in Eq. (3.136) is actually outside these limits of applicability.

The finite value of the coefficient C in Ginzburg—Landau expansion in the vicinity of Anderson transition signifies the existence of superconducting (Meissner) response to an external magnetic field. Accordingly, for $T < T_c$ the system can undergo a transition from Anderson insulator to superconductor. The physical meaning of this result can be understood from the following qualitative picture (Cf. Ref.[Imry Y., Strongin M. (1981)]) where the similar estimates were used for the granular metal). In Anderson insulator all electrons with energies E close to Fermi level are localized in spatial regions of the size of $\sim R_{loc}(E)$. Nearby regions are connected by some tunneling amplitude $\mathcal{V}$ which determines the probability of electron transition between such regions as:

$$P_T \approx 2\pi|\mathcal{V}|^2 N(E) R_{loc}^3(E) \tag{3.137}$$

However, Anderson localization means that

$$|\mathcal{V}| < \frac{1}{N(E)R_{loc}^3(E)} \tag{3.138}$$

and coherent tunneling between states localized in these regions is impossible, so that $P_T < 2\pi N^{-1}(E)R_{loc}^{-3}$. At the same time if conditions given by Eq. (3.20) or Eq. (3.22) are satisfied inside each region $\sim R_{loc}$ Cooper pairs may form and superconducting gap Δ appears in the spectrum. Then a kind of "Josephson" coupling appears between regions of localized states which determines the possibility of *pairs* tunneling:

$$E_J \approx \pi^2[N(E)R_{loc}^3(E)]^2|\mathcal{V}|^2\Delta \tag{3.139}$$

It is easy to see that for

$$\Delta > \frac{2}{\pi}\frac{1}{N(E)R_{loc}^3(E)} \tag{3.140}$$

we have $E_J > P_T$, so that if Eq. (3.20) is satisfied we can get $E_J \gg N^{-1}(E)R^{-3}(E)$ despite of Eq. (3.138) and tunneling of pairs between nearby regions of localized states is possible, even in the absence of single—particle tunneling.

It is convenient to rewrite Eq. (3.136) using the the relation between generalized diffusion coefficient and conductivity like Eq. (2.60) as well as Eqs. (2.74), (2.76). Then using the Ginzburg—Landau expansion and the expressions for its coefficients we can easily find the temperature dependent coherence length $\xi(T)$ [Gennes de P.G. (1966); Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$\xi^2(T) = \frac{T_c}{T_c - T}\begin{cases} \xi_0 l \frac{\sigma}{\sigma+\sigma_c} & \sigma > \sigma^\star \quad (E_F > E_c) \\ (\xi_0 l^2)^{2/3} & \sigma < \sigma^\star \quad (E_F \sim E_c) \end{cases} \tag{3.141}$$

where $\sigma_c = e^2 p_F/(\pi^3\hbar^2)$ and characteristic conductivity scale $\sigma^\star$ is given by

$$\sigma^{\star} \approx \sigma_c (p_F \xi_0)^{-1/3} \approx \sigma_c \left(\frac{T_c}{E_F}\right)^{1/3} \tag{3.142}$$

Thus in the region of very small conductivities $\sigma < \sigma^{\star}$ the scale of $\xi(T)$ is defined not by $\xi \sim \sqrt{\xi_0 l}$ as in the usual theory of "dirty" superconductors [Gorkov L.P. (1959); Gennes de P.G. (1966)] but by the new length $\xi \sim (\xi_0 l^2)^{1/3} \sim (\xi_0/p_F^2)^{1/3}$, which now is the characteristic size of Cooper pair close to Anderson transition. From Eq. (3.141) we can see that $\xi^2(T)$ initially diminishes as we approach metal—insulator transition proportionally to σ as in the case of a "dirty" superconductor. However, already in metallic region for $\sigma < \sigma^{\star}$ it diminishes much slower remaining finite both at the transition itself and below.

In a case if $\omega^{1/3}$—law for a diffusion coefficient at the mobility edge is invalid and we have ω^{δ}—behavior, with some unknown critical exponent δ (which is possible because the modern theory actually cannot guarantee precise values of critical exponents at Anderson transition (remember also possible role of interaction effects!) [Wegner F.J. (1976); Sadovskii M.V. (1986)]) we can easily show in a similar way that for conductivities $\sigma < \sigma^{\star} \approx \sigma_c(p_F\xi_0)^{-\delta}$ the coherence length is defined by $\xi \sim \xi_0^{\frac{1-\delta}{2}} l^{\frac{1+\delta}{2}}$. Qualitatively this leads to the same type of behavior as above.

The superconducting electron density n_s can be defined as [Gennes de P.G. (1966)]:

$$n_s(T) = 8mC\Delta^2(T) = 8mC(-A)/B \tag{3.143}$$

Close to Anderson transition we can estimate:

$$n_s \sim mN(E_F)\xi^2\Delta^2 \sim mp_F(\xi_0/p_F^2)^{2/3}\Delta^2 \sim n(T_c^{1/2}/E_F^2)^{2/3}(T_c - T) \tag{3.144}$$

where $n \sim p_F^3$ is total electron density. If we take here $T \sim 0.5T_c$ i.e. more or less low temperatures we get a simpler estimate:

$$n_s \sim n\left(\frac{T_c}{E_F}\right)^{4/3} \tag{3.145}$$

which is actually valid up to $T = 0$, as we shall see below. From these estimates we can see that only a small fraction of electrons are superconducting in a strongly disordered case. However this confirms a possibility of superconducting response of Anderson insulator.

Characteristic conductivity $\sigma^{\star}$ defined in Eq. (3.142) gives an important conductivity scale at which significant influence of localization effects upon superconducting properties appear [Bulaevskii L.N., Sadovskii M.V. (1985)]. While σ_c is of the order of Mott's "minimal metallic conductivity" [Mott N.F. (1974); Mott N.F, Davis E.A. (1979)] $\sigma^{\star}$ is in general even lower. However, for small enough Cooper pairs (i.e. small ξ_0 which is characteristic of strong coupling and high—T_c superconductors) it is more or less of the order of σ_c. Experimentally it can be defined as a conductivity scale at which significant deviations from predictions of the standard theory of "dirty" superconductors appear under disordering.

We must stress that these results show the possibility of Cooper pairs being delocalized in Anderson insulator, while single—particle excitations of such superconductor are apparently localized, which may lead to some peculiar transport properties of "normal" electrons for $T < T_c$. First attempts to explore this peculiar situation were undertaken in Refs.[Oppermann R. (1987); Oppermann R. (1988); Oppermann R. (1990); Kravtsov V.E., Oppermann R. (1991)].

These results are easily generalized for the case of strongly anisotropic quasi—two—dimensional systems such as high—T_c superconducting oxides. Using the analysis of such systems within the self—consistent theory of localization [Prigodin V.N., Firsov Yu.A. (1984)] we can write down the following Matsubara generalization of Eq. (2.95):

$$\frac{D_j(\omega_m)}{D_j^0} \approx \begin{cases} Max\left[\frac{E_F-E_c}{E_c}; (2\pi E_F w\tau^2)^{-2/3}(\omega_m\tau)^{1/3}\right] & \omega_m \ll w^2\tau \\ 1-\frac{1}{2\pi E_F\tau} ln\left(\frac{1}{\omega_m\tau}\right) \qquad \omega_m \gg w^2\tau \end{cases} \tag{3.146}$$

where $j = \|, \perp$. Now carrying out calculations similar to that of Ref.[Bulaevskii L.N., Sadovskii M.V. (1985)] we obtain for the coefficients of gradient terms in Ginzburg—Landau expansion [Sadovskii M.V. (1989); Aleksashin B.A. et al. (1989)]:

$$C_{\|,\perp} = N(E_F)\xi^2_{\|,\perp} \tag{3.147}$$

where for the coherence lengths $\xi_{\|,\perp}$ we obtain a number of different expressions, depending on the value of the ratio $w^2\tau/2\pi T_c\hbar$ which determines as we shall see the "degree of two—dimensionality" of the problem under study. For the case of $w^2\tau/2\pi T_c\hbar \gg 1$, corresponding to an anisotropic but three—dimensional system, we have:

$$\xi^2_{\parallel,\perp} = \frac{\pi}{8T_c} D^0_{\parallel,\perp} \left(\frac{E_F - E_c}{E_c} \right) \approx \xi^0_{\parallel,\perp} l_{\parallel,\perp} \left(\frac{E_F - E_c}{E_c} \right) \tag{3.148}$$

where $\xi^0_{\parallel} \sim \hbar v_F/T_c$, $\xi^0_{\perp} \sim wa_{\perp}/T_c$, $l_{\parallel} = v_F\tau$ and $l_{\perp} = wa_{\perp}\tau/\hbar$ are the longitudinal and transverse BCS coherence lengths and mean free paths. The above expressions are valid in the conductivity region $\sigma_{\parallel} > \sigma^{\star}$, where

$$\sigma^{\star} \sim \sigma^c_{\parallel} \frac{\xi^0_{\parallel}}{l_{\parallel}} \left(\frac{T_c^2}{E_F w} \right)^{2/3} \tag{3.149}$$

Here $\sigma^c_{\parallel}$ was defined in Eq. (2.94). The condition of $w^2\tau/2\pi T_c\hbar \gg 1$ is equivalent to the requirement:

$$\xi_{\perp} \sim \sqrt{\xi^0_{\perp} l_{\perp}} \gg a_{\perp} \tag{3.150}$$

which clarifies its physical meaning: the transverse size of a Cooper pair must be much greater than interplane lattice spacing. In this case we have just anisotropic three-dimensional superconductivity.

In the immediate vicinity of the Anderson transition, for $\sigma_{\parallel} < \sigma^{\star}$ we have:

$$\xi^2_{\parallel,\perp} \approx (1 - 2^{-5/3})(16\pi^4)^{-1/3}\zeta(5/3) \frac{D_{\parallel,\perp}}{(E_F T_c w)^{2/3}\tau} \approx (\xi^0_{\parallel,\perp})^2 \left(\frac{T_c^2}{E_F w} \right)^{2/3} \tag{3.151}$$

It is easy to see that for $w \sim E_F$ all these expressions naturally go over to those derived above for the three—dimensional case.

For the case of $w^2\tau/2\pi T_c\hbar < 1$ which corresponds to "almost two—dimensional" case of

$$\xi_{\perp} \sim \sqrt{\xi^0_{\perp} l_{\perp}} \leq a_{\perp} \tag{3.152}$$

i.e. of transverse size of Cooper pairs smaller than interplane spacing, we can roughly estimate *:

$$\xi^2_{\parallel,\perp} \approx \left\{ \begin{array}{ll} \frac{D^0_{\parallel,\perp}}{\pi T_c} \frac{E_F - E_c}{E_c} & (\sigma_{\parallel} > \sigma^{\star}) \\ \frac{D^0_{\parallel,\perp}}{(4\pi^2 E_F T_c w)^{2/3}\tau} & (\sigma_{\parallel} < \sigma^{\star}) \end{array} \right\}$$

*Here we just put $w^2\tau/2\pi T_c\hbar = 1$ and use the first expression of (3.146) as only the first term in the sum over Matsubara frequencies determining C (Cf. Appendix C), while for the rest of the sum we use the second expression of (3.146).

$$+(\pi^2/8-1)\frac{D^0_{\parallel,\perp}}{\pi T_c}\left(1-\frac{1}{2\pi E_F\tau}ln\frac{1}{2\pi T_c\tau}\right) \tag{3.153}$$

Essential difference from just anisotropic case of Eq. (3.148) and Eq.(3.151) is the appearance here of a second term of "two-dimensional" type. In purely two—dimensional problem ($w = 0$) we have [Takagi H., Souda R., Kuroda Y. (1982)]:

$$\xi^2_{\parallel} = \frac{\pi D^0_{\parallel}}{8T_c}\left(1-\frac{1}{2\pi E_F\tau}ln\frac{1}{2\pi T_c\tau}\right) \tag{3.154}$$

For high—T_c oxides it is reasonable to estimate $\xi^0_{\parallel} \sim l_{\parallel}$, $T_c \sim w$, $T_c \sim 0.1E_F$, so that $\sigma^{\star} \sim \sigma^c_{\parallel}$, i.e. these systems are always more or less close to the Anderson transition. For $T_c \sim w$ and $\hbar/\tau \sim E_F$ which is characteristic of rather strongly disordered case, we have $w^2\tau/2\pi T_c\hbar < 1$, so that for these systems we can realize almost two—dimensional behavior, though in general high—T_c oxides are apparently an intermediate case between strongly anisotropic three—dimensional and nearly two—dimensional superconductors.

The significant change of Ginzburg—Landau coefficients and the new scale of coherence length close to the Anderson transition lead to an increased width of critical region of thermodynamic fluctuations near T_c [Kapitulnik A., Kotliar G. (1985); Kotliar G., Kapitulnik A. (1986)]. These are well known to be important for any second—order phase transition. The width of the critical region is defined by the so called Ginzburg criterion [Kadanoff L.P. et al. (1967); Patashinskii A.Z., Pokrovskii V.L. (1982)] which may be expressed via the coefficients of Landau expansion. Mean—field approximation for the order parameter in Landau theory is valid (for $d = 3$) for [Kadanoff L.P. et al. (1967); Patashinskii A.Z., Pokrovskii V.L. (1982)]

$$1 \gg \left|\frac{T-T_c}{T_c}\right| \gg \frac{B^2T_c^2}{\alpha C^3} \equiv \tau_G \tag{3.155}$$

where α is defined by $A = \alpha(T-T_c)/T_c$. In case of superconducting transition we have: $\alpha = N(E_F)$, $B \sim N(E_F)/T_c^2$ and $C = N(E_F)\xi^2$. Accordingly, from Eq. (3.155) we get the following estimate for the critical region:

$$\tau_G \sim \frac{1}{N^2(E_F)\xi^6 T_c^2} \sim \left(\frac{E_F}{T_c}\right)^2 \frac{1}{\xi^6 p_F^6} \tag{3.156}$$

In the "pure" limit $\xi = \xi_0 \sim v_F/T_c$ and we get $\tau_G \sim (T_c/E_F)^4$, so that critical region is practically unobservable. In a "dirty" superconductor $\xi \sim \sqrt{\xi_0 l}$ and

$$\tau_G \sim \left(\frac{T_c}{E_F}\right) \frac{1}{(p_F l)^3} \tag{3.157}$$

and again we have $\tau_G \ll 1$. However, for a superconductor close to mobility edge $\xi \sim (\xi_0/p_F^2)^{1/3}$ and from Eq. (3.156) we get [Kapitulnik A., Kotliar G. (1985); Kotliar G., Kapitulnik A. (1986)]:

$$\tau_G \sim 1 \tag{3.158}$$

Note that in fact τ_G may still be small because of numerical constants which we have dropped in our estimates. Anyhow, the critical region in this case becomes unusually wide and superconducting transition becomes similar in this respect to superfluid transition in Helium. Fluctuation effects may thus become observable even in bulk three—dimensional superconductor. In this respect it is interesting to note, that in high-temperature superconductors the critical region is not small and readily observable [Junod A., Erb A., Renner C. (1998)]. Note that in localized phase $\xi \sim R_{loc}$ and $\tau_G \sim [N^2(E_F)R_{loc}^6 T_c^2]^{-1} > 1$ if the condition given by Eq. (3.20) is violated.

Finally we should like to mention that thermodynamic fluctuations lead [Kapitulnik A., Kotliar G. (1985); Kotliar G., Kapitulnik A. (1986)] to an additional mechanism of T_c degradation for a system which is close to Anderson transition. This follows from the general result on the reduction of mean—field transition temperature due to critical fluctuations. If these fluctuations are small (and we can use the so called one—loop approximation) for a three-dimensional system it can be shown that [Kapitulnik A., Kotliar G. (1985); Kotliar G., Kapitulnik A. (1986)]:

$$T_c = T_{c0} - \frac{7\zeta(3)}{16\pi^4 \xi^3 N(E_F)} \tag{3.159}$$

where T_{c0} is the mean—field transition temperature. If we use here our expressions for ξ valid close to metal—insulator transition we easily find for $\sigma > \sigma^{\star}$ [Kotliar G., Kapitulnik A. (1986)]:

$$T_c \approx T_{co}\left[1 - 0.5\left(\frac{\sigma_c}{\sigma}\right)^{3/2}\left(\frac{T_{c0}}{E_F}\right)^{1/2}\right] \tag{3.160}$$

For $\sigma < \sigma^{\star}$ this fluctuation correction saturates as the further drop of coherence length stops there. Obviously higher—order corrections are important here, but unfortunately little is known on the importance of this mechanism of T_c degradation outside the limits of one—loop approximation.

3.3.2 *Upper critical field*

Direct information on the value of $\xi^2(T)$ can be obtained from the measurements of the upper critical field H_{c2} [Gennes de P.G. (1966)]:

$$H_{c2} = \frac{\phi_0}{2\pi\xi^2(T)} \tag{3.161}$$

where $\phi_0 = \pi c\hbar/e$ is superconducting magnetic flux quantum. Using Eq. (3.141) we obtain the following relation between normal state conductivity σ, the slope of the upper critical field at $T = T_c$ given by $(dH_{c2}/dT)_{T_c}$ and the value of electronic density of states at the Fermi level (per one spin direction) $N(E_F)$ [Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)]:

$$-\frac{\sigma}{N(E_F)}\left(\frac{dH_{c2}}{dT}\right)_{T_c} \approx \begin{cases} \frac{8e^2}{\pi^2\hbar}\phi_0 & \sigma > \sigma^{\star} \\ \phi_0\frac{\sigma}{N(E_F)(\xi_0 l^2)^{2/3}T_c} \approx \phi_0\frac{\sigma}{[N(E_F)T_c]^{1/3}} & \sigma < \sigma^{\star} \end{cases} \tag{3.162}$$

For $\sigma > \sigma^{\star}$ the r.h.s. of Eq. (3.162) contains only the fundamental constants. This so called Gorkov's relation [Gorkov L.P. (1959)] is often used to interpret experimental data in "dirty" superconductors. Using it we may find $N(E_F)$ for different degrees of disorder from measurements of $(dH_{c2}/dT)_{T_c}$ and conductivity σ. On the other hand $N(E_F)$ can in principle be determined from independent measurements e.g. of electronic contribution to specific heat. However, our expression for $\sigma < \sigma^{\star}$ which is valid close to metal—insulator transition shows that in this region Gorkov's relation becomes invalid and its use can "simulate" the drop of $N(E_F)$ with the growth of resistivity (disorder). Roughly speaking Eq. (3.162) shows that under the assumption of relatively smooth change of $N(E_F)$ and T_c with disorder the usual growth of $(dH_{c2}/dT)_{T_c}$ with disorder saturates in conductivity region of $\sigma < \sigma^\star$ close to the Anderson transition and the slope of the upper critical field becomes independent of resistivity. Of course this picture may be complicated in real case by T_c degradation and disorder renormalization of the density of states. In particular this stresses the importance of independent measurements of $N(E_F)$.

Note that the qualitative behavior given by Eq. (3.162) is retained also in the case when ω^δ—dependence of diffusion coefficient at the mobility edge (with some arbitrary critical exponent δ), only the expression for $\sigma^\star$ is changed as noted above. Thus this behavior is not related to any specific approximations of self—consistent theory of localization, except the general concept of continuous transition.

For an anisotropic (quasi—two—dimensional) system we have similar relations:

$$\left(\frac{dH_{c2}^{\perp}}{dT}\right)_{T_c} = -\frac{\phi_0}{2\pi\xi_{\parallel}^2 T_c} \tag{3.163}$$

$$\left(\frac{dH_{c2}^{\parallel}}{dT}\right)_{T_c} = -\frac{\phi_0}{2\pi\xi_{\parallel}\xi_{\perp} T_c} \tag{3.164}$$

with $\xi_{\parallel,\perp}$ given above during our discussion after Eq. (3.147). This leads to relations and qualitative behavior similar to Eq. (3.162). However, we would like to note an especially interesting relation for the anisotropy of the slopes of the upper critical field [Sadovskii M.V. (1989)]:

$$\frac{(dH_{c2}^{\parallel}/dT)_{Tc}}{(dH_{c2}^{\perp}/dT)_{Tc}} = \frac{\xi_{\parallel}}{\xi_{\perp}} = \frac{v_F}{wa/\hbar} \tag{3.165}$$

We see that the anisotropy of $(dH_{c2}/dT)_{T_c}$ is actually determined by the anisotropy of the Fermi velocity irrespective of the regime of superconductivity: from the "pure" limit, through the usual "dirty" case, up to the vicinity of the Anderson transition.

The above derivation of C coefficient of Ginzburg—Landau expansion explicitly used the time—reversal invariance expressed by Eq. (3.129). This is valid in the absence of the external magnetic field and magnetic impurities. Accordingly the previous results for the upper critical field are formally valid in the limit of infinitesimal external field and this is sufficient for the demonstration of superconducting (Meissner) response and for the determination of $(dH_{c2}/dT)_{T_c}$, because $H_{c2} \to 0$ as $T \to T_c$. In a finite external field we must take into account its influence upon localization. The appropriate analysis was performed in Refs.[Kuchinskii E.Z., Sadovskii M.V. (1991a); Kuchinskii E.Z., Sadovskii M.V. (1991b)] and with a slightly different method in Ref.[Theumann A., Pires Idiart M.A. (1991)]. The results are essentially similar and below we shall follow Ref.[Kuchinskii E.Z., Sadovskii M.V. (1991a)]. The standard scheme for the analysis of superconducting transition in an external magnetic field [Gorkov L.P. (1959); Gennes de P.G. (1966); Werthamer N.R., Helfand E., Hohenberg P.C. (1966); Saint-James D., Sarma G., Thomas E.J. (1969)] gives the following equation determining the temperature dependence of $H_{c2}(T)$:

$$ln\frac{T}{T_c} = 2\pi T\sum_{\varepsilon_n}\left\{\frac{1}{2|\varepsilon_n| + 2\pi D_2(2|\varepsilon_n|)H/\phi_0} - \frac{1}{2|\varepsilon_n|}\right\} \tag{3.166}$$

where $D_2(2|\varepsilon_n|)$ is the generalized diffusion coefficient in the Cooper channel as defined after Eqs. (2.98) and (2.99). Eq. (3.166) is valid [Gennes de P.G. (1966)] for

$$R_H = \frac{mcv_F}{eH} \gg \xi \tag{3.167}$$

R_H is Larmor radius of an electron in magnetic field, ξ is the coherence length. Note that Eq. (3.166) describes only the orbital motion contribution to H_{c2}. In fact H_{c2} is also limited by the paramagnetic limit [Gennes de P.G. (1966); Saint-James D., Sarma G., Thomas E.J. (1969)]:

$$\frac{1}{2}g_0\mu_B H < \Delta \tag{3.168}$$

where g_0 is the usual g—factor of an electron, μ_B is Bohr magneton. Here disorder may enter apparently only through the appropriate dependence of the gap Δ.

Standard approach of the theory of "dirty" superconductors is based upon the replacement of $D_2(2|\varepsilon_n|)$ in Eq. (3.166) by Drude diffusion coefficient D_0 which is valid for a metal with $l \gg p_F^{-1}$. For a system which is close to the Anderson transition we must take into account both the frequency dependence of diffusion coefficient and the fact that in magnetic field D_2 is not equal to D_1 — the usual diffusion coefficient determining electronic transport. Actually we shall see that the external magnetic field influence upon localization leads to rather small corrections to $H_{c2}(T)$ practically everywhere except the region of localized states [Kuchinskii E.Z., Sadovskii M.V. (1991a)]. Thus we may really neglect this influence as a first approximation as that was done in Refs.[Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)] and start with the replacement of D_2 in Eq. (3.166) by $D_1 = D_E$, where D_E is the frequency dependent generalized diffusion coefficient in the absence of magnetic field. Detailed analysis of Eq. (3.166) can be found in Ref.[Kuchinskii E.Z., Sadovskii M.V. (1991a)].

Summation over Matsubara frequencies in Eq. (3.166) must be cut–off at some frequency of the order of $< \omega >$ – the characteristic frequency of Bose excitations responsible for pairing interaction. It is convenient here to measure the distance from Anderson transition (degree of disorder) via frequency ω_c defined in Eqs. (2.42),(2.80) or Eq. (2.107) . If a system is far from Anderson transition, so that $\omega_c \gg < \omega >$ we can completely neglect the frequency dependence of diffusion coefficient and find the usual results of the theory of "dirty" superconductors:

$$H_{c2}(T) = \frac{4}{\pi^2}\frac{\phi_0 T_c}{D_0} ln\frac{T_c}{T} \qquad T \sim T_c \tag{3.169}$$

$$H_{c2} = \frac{1}{2\gamma}\frac{\phi_0 T_c}{D_0}\left[1 - \frac{1}{24}\left(\frac{4\gamma T}{T_c}\right)^2\right] \qquad T \ll T_c \tag{3.170}$$

where $\gamma = 1.781....$ For the H_{c2} derivative at $T = T_c$ we find from here the first relation of Eq. (3.162), and $H_{c2}(T = 0)$ is conveniently expressed as [Gorkov L.P. (1959); Werthamer N.R., Helfand E., Hohenberg P.C. (1966)]:

$$-\frac{H_{c2}(0)}{T_c(dH_{c2}/dT)_{T_c}} = \frac{\pi^2}{8\gamma} \approx 0.69 \tag{3.171}$$

In this case $H_{c2}(T)$ curve is convex at all temperatures below T_c [Gorkov L.P. (1959); Werthamer N.R., Helfand E., Hohenberg P.C. (1966); Gennes de P.G. (1966); Saint-James D., Sarma G., Thomas E.J. (1969)]. Very close to the Anderson transition, when $\omega_c \ll 2\pi T$, only $\omega^{1/3}$ behavior of diffusion coefficient is important in Eq. (3.166) and it takes the following form [Kuchinskii E.Z., Sadovskii M.V. (1991a)]:

$$ln\frac{T}{T_c} = \sum_{n=0}^{\infty}\left\{[(n+1/2) + (n+1/2)^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)]^{-1} - [n+1/2]^{-1}\right\} \tag{3.172}$$

where $\omega_H = eH/mc$. From here we get:

$$H_{c2}(T) = m\frac{\phi_0}{\pi}\frac{(4\pi)^{2/3}}{c_1}T^{2/3}E^{1/3}ln\frac{T_c}{T} \qquad T \sim T_c \tag{3.173}$$

$$H_{c2}(T) = m\frac{\phi_0}{\pi}(\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}\left[1-\frac{2}{3}c_2\left(\frac{4\gamma T}{T_c}\right)^{2/3}\right] \qquad T \ll T_c \tag{3.174}$$

where $c_1 = \sum_{n=0}^{\infty}(n+1/2)^{-5/3} \approx 4.615$ and $c_2 \approx 0.259$. From these expressions we get:

$$-\frac{1}{N(E)}\left(\frac{dH_{c2}}{dT}\right)_{T_c} = \frac{(4\pi)^{2/3}}{\pi c_1}m\phi_0(E/T_c)^{1/3} = \frac{2\pi}{c_1}\frac{\phi_0}{[N(E)T_c]^{1/3}} \tag{3.175}$$

which makes precise the second relation in Eq. (3.162), while for $H_{c2}(T=0)$ we obtain:

$$-\frac{H_{c2}(0)}{T_c(dH_{c2}/dT)_{T_c}} = \frac{c_1}{(4\gamma)^{2/3}} \approx 1.24 \tag{3.176}$$

As was first noted in Refs.[Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)] this ratio for the system at the mobility edge is significantly larger than its classical value 0.69. In this case $H_{c2}(T)$ curve is concave for all temperatures below T_c [Bulaevskii L.N., Sadovskii M.V. (1985)]. Detailed expressions for the intermediate disorder when $2\pi T \ll \omega_c \ll< \omega >$ can be found in Ref.[Kuchinskii E.Z., Sadovskii M.V. (1991a)].

On Fig.3.10 we present the results of numerical solution of Eq. (3.166) for the different values of characteristic frequency ω_c, i.e. for the different disorder. A smooth crossover from the classical behavior of the theory of "dirty" superconductors [Werthamer N.R., Helfand E., Hohenberg P.C. (1966); Gennes de P.G. (1966); Saint-James D., Sarma G., Thomas E.J. (1969)] to anomalous temperature dependence close to the Anderson transition [Bulaevskii L.N., Sadovskii M.V. (1985)] is clearly seen.

Below the mobility edge (i.e. in Anderson insulator) and for $\omega_c = 1/(2\pi^2 N(E)R_{loc}^3) \ll 2\pi T$, i.e. very close to mobility edge we can again use $\omega^{1/3}$—behavior of diffusion coefficient and find the same temperature dependence of H_{c2} as at the mobility edge itself or just above it. For $2\pi T \ll \omega_c \ll 2\pi T_c$ Eq. (3.166) takes the form [Kuchinskii E.Z., Sadovskii

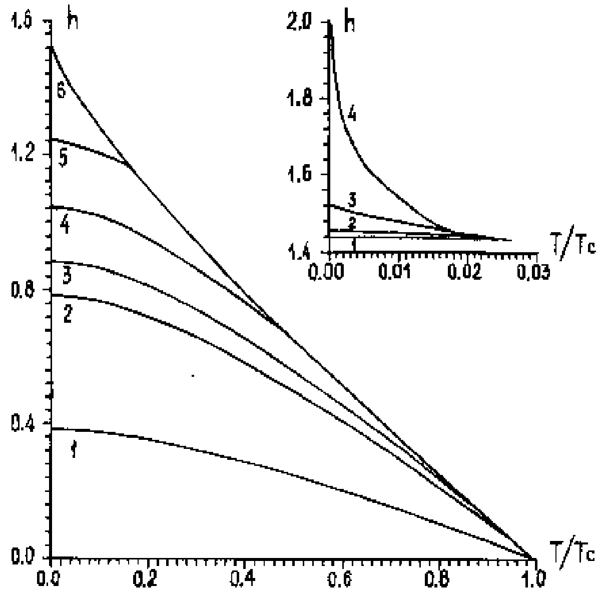

Fig. 3.10 Temperature dependence of the upper critical field H_{c2}. Numerical solution for the dependence of $h = \omega_H/T_c^{2/3}E^{1/3}$ on T/T_c for different values of $\theta = \omega_c/T_c$: 1. $\theta = 100$; 2. $\theta = 10$; 3. $\theta = 2\pi$; 4. $\theta = 3$; 5. $\theta = 1$; 6. $\theta = 0$ (Mobility edge). Metallic state, no magnetic field influence on diffusion. At the insert: Low temperature part of h on T/T_c close to the Anderson transition. 1. Mobility edge ($\theta = 0$) with magnetic field influence on diffusion. 2. Metallic phase ($\theta = 0.1$), no magnetic field influence. 3. Mobility edge ($\theta = 0$), no magnetic field influence. 4. Insulating phase ($\theta = 0.1$), no magnetic field influence. Numerical cut—off was taken at $< \omega > = 100\, T_c$.

M.V. (1991a)]:

$$ln\frac{T}{T_c} = \sum_{n=0}^{n_0-1}\{(n+1/2)[1+(E/\omega_c)^{2/3}(\omega_H/E)]\}^{-1} +$$
$$+\sum_{n=n_0}^{\infty}\{(n+1/2)+(n+1/2)^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)\}^{-1}$$
$$-\sum_{n=0}^{\infty}(n+1/2)^{-1} \qquad (3.177)$$

where $n_0 = \omega_c/4\pi T_c$ corresponds to a change of frequency behavior of diffusion coefficient. Defining $x = \omega_H/\omega_c^{2/3}E^{1/3}$ we can reduce Eq. (3.177) to:

$$ln(T/T_c) = xln(\gamma\omega_c/\pi T_c) + \frac{3}{2}(1+x)ln(1+x) \tag{3.178}$$

which implicitly defines $H_{c2}(T)$ and shows [Kuchinskii E.Z., Sadovskii M.V. (1991a)] that now $H_{c2}(T) \to \infty$ for $T \to 0$ (logarithmic divergence). Numerical solution of Eq. (3.177) is shown at the insert in Fig.3.10. Below we shall see however, that this divergence of H_{c2} is lifted by the inverse influence of magnetic field upon diffusion.

Let us now turn to the problem of magnetic field influence upon diffusion and its consequences for H_{c2} temperature behavior. If we are far from the Anderson transition magnetic field influence is small on parameter $\sim \sqrt{\omega_H/E}$ and its influence upon H_{c2} is insignificant. Close to the transition magnetic field correction may overcome the value of $D(H=0)$ and we have to consider its influence in detail [Kuchinskii E.Z., Sadovskii M.V. (1991a)]. Accordingly we shall limit ourselves with the case of $\omega_c/E \ll (\omega_H/E)^{2/3}$ for which we have already discussed the magnetic field behavior of generalized diffusion coefficient in Cooper channel. It was given in Eq. (2.111) and Eq. (2.112). In this case we have seen that characteristic frequency ω_c is replaced by:

$$\omega_c' = (\varphi\omega_H/E)^{3/2}E \tag{3.179}$$

where $\varphi = W^2/2 \approx 0.18$ (W was defined during our discussion of localization in magnetic field). For $T \sim T_c$ there is no change in the slope of H_{c2} given by Eq. (3.175) as was noted already in Ref.[Bulaevskii L.N., Sadovskii M.V. (1985)]. Here we shall consider the case of $T \ll T_c$.

For $2\pi T > \omega_c'$ in all sums over Matsubara frequencies we can take $D(\omega) \sim \omega^{1/3}$ and actually we can neglect magnetic field influence upon diffusion. In this case $H_{c2}(T)$ behaves like in Eq. (3.174) i.e. as at the mobility edge in the absence of magnetic field effects. For $2\pi T < \omega_c'$ equation for $H_{c2}(T)$ takes the form [Kuchinskii E.Z., Sadovskii M.V. (1991a)]:

$$ln\frac{T}{T_c} = \sum_{n=0}^{n_0-1}[(n+1/2) + (\omega_c/E)^{1/3}(\omega_H/4\pi T)]^{-1} +$$
$$+ \sum_{n=n_0}^{\infty}\{(n+1/2) + (n+1/2)^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)\}^{-1}$$

$$-\sum_{n=0}^{\infty}(n+1/2)^{-1} \qquad (3.180)$$

where $n_0 = \omega_c'/4\pi T$. In this case we find

$$H_{c2}(T) = m\frac{\phi_0}{\pi}(1+\varphi)^{-1/3}(\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}\left[1-\frac{4\gamma}{3\varphi^{1/3}(1+\varphi)}\frac{T}{T_c}\right] \qquad (3.181)$$

Accordingly we have

$$-\frac{H_{c2}(0)}{T_c(dH_{c2}/dT)_{T_c}} = (1+\varphi)^{-1/3}\frac{c_1}{(4\gamma)^{2/3}} \approx 1.18 \qquad (3.182)$$

and the change in comparison with Eq. (3.176) is actually small. However, for $2\pi T < \omega_c'$ the $H_{c2}(T)$ curve becomes convex. The inflexion point can be estimated as $T^\star = \omega_c'/2\pi \approx 0.02T_c$. This behavior is shown in the insert on Fig.3.10.

Consider now insulating region. We shall see that the magnetic field effects on diffusion lead to the effective cut—off of the weak divergence of H_{c2} as $T \to 0$ noted above. Generalized diffusion coefficient D_2 in insulating phase and at low enough frequencies is determined by the following equation [Kuchinskii E.Z., Sadovskii M.V. (1991a)]:

$$2mD_2 = -(\omega_c/E)^{1/3} + (-i\omega/E)^{1/2}(2mD_2)^{-1/2} + \frac{1}{2}W(2\omega_H/E)^{1/2} \quad (3.183)$$

Now we can see that the external field defined by

$$\frac{W}{2}\sqrt{2\omega_H/E} > (\omega_c/E)^{1/3} \qquad (3.184)$$

transfers the system from insulating to metallic state. If the system remains close to mobility edge we can estimate the upper critical field as above by $\omega_H \approx (\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}$ and Eq. (3.184) reduces to:

$$\omega_c \approx \frac{1}{2\pi^2 N(E)R_{loc}^3} < \frac{\pi}{\gamma}(W/\sqrt{2})^3 T_c \approx 0.14T_c \qquad (3.185)$$

and practically in all the interval of localization lengths where according to our main criterion of Eq. (3.20) we can have superconductivity in Anderson insulator *the upper critical field in fact destroys localization* and the system becomes metallic. Accordingly there is no way to observe the divergence of the upper critical field as $T \to 0$ and the $H_{c2}(T)$ curves in "insulating" phase all belong to the region between the curves of $H_{c2}(T)$ at the mobility edge defined in the absence of magnetic field (curve 3 in the insert on Fig.3.10) and at the mobility edge defined in magnetic field (curve 1 in the insert). This result actually shows that it may be difficult to confirm the insulating ground state of strongly disordered superconducting system just applying strong enough magnetic field to destroy superconductivity and perform usual transport measurements at low temperatures.

Note that another mechanism for the change of $H_{c2}(T)$ at low temperatures was proposed by Coffey, Levin and Muttalib [Coffey L., Levin K., Muttalib K.A.(1985)]. They have found the enhancement of H_{c2} at low temperatures due to the magnetic field dependence of the Coulomb pseudopotential $\mu^{\star}$ which appears via the magnetic field dependence of diffusion coefficient. Magnetic field suppression of localization effects leads to the reduction of Coulomb pseudopotential enhancement due to these effects [Anderson P.W., Muttalib K.A., Ramakrishnan T.V (1983)]. Accordingly it is possible to get the enhancement of H_{c2} at low temperatures. Unfortunately very important effects of the frequency dependence of generalized diffusion coefficient were ignored.

Returning to general criteria of validity of Eq. (3.166) we note that the condition of $R_H \gg \xi$ is reduced to $\omega_H \ll T_c^{1/3} E_F^{2/3}$ which is obviously satisfied in any practical case. Note, however, that our estimates for H_{c2} at low temperatures lead to $\omega_H = \Delta_0 (E_F/\Delta_0)^{1/3} > \Delta_0$ which can easily overcome paramagnetic limit. In this case the experimentally observed H_{c2} of course will be determined by paramagnetic limit and anomalous behavior due to localization will be unobservable at low temperatures. At the same time in case of H_{c2} being determined by paramagnetic limit it may become possible to obtain insulating ground state of the system applying the strong enough magnetic field. Note that the effective masses entering to cyclotron frequency and paramagnetic splitting may be actually very different and there may be realistic cases when orbital critical field may dominate at low T. For $T \sim T_c$ H_{c2} is always determined by orbital contribution.

Similar analysis can be performed for the two-dimensional and quasi-two-dimensional cases [Kuchinskii E.Z., Sadovskii M.V. (1993a)], which are important mainly due to quasi-two-dimensional nature of high-temperature superconductors. We shall limit ourselves only to the case of magnetic field perpendicular to highly conducting planes, when the temperature dependence of $H_{c2}(T)$ is again determined by Eq.(3.166) with $D_2(\omega)$ having the meaning of diffusion coefficient in Cooper channel along the plane.

If we neglect the magnetic field influence upon diffusion the frequency dependence of diffusion coefficient in purely two-dimensional case is determined by Eq.(2.116). It is easy to see that the possible anomalies in the temperature behavior of the upper critical field due to the frequency dependence of diffusion coefficient will appear only at temperatures $T \ll \frac{e^{-1/\lambda}}{\tau}$. At higher temperatures we obtain the usual dependence of the "dirty" limit. Accordingly, from Eq.(3.166) we obtain two different types of behavior of $H_{c2}(T)$:

(1) For $T_c \gg \frac{e^{-1/\lambda}}{\tau}$

$$H_{c2}(T) = \frac{4}{\pi^2}\frac{\phi_0}{D_0}T ln\left(\frac{T_c}{T}\right) \text{ for } T \sim T_c \tag{3.186}$$

$$H_{c2}(T) = \frac{1}{2\gamma}\frac{\phi_0 T_c}{D_0}(1 - 2.12\left(\frac{T}{T_c}\right)^2) \text{ for } \frac{e^{-1/\lambda}}{\tau} \ll T \ll T_c \tag{3.187}$$

For $T \ll \frac{e^{-1/\lambda}}{\tau}$ the upper critical field is defined by the equation:

$$ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T}\right) = (1+4\pi\frac{D_0}{\phi_0}\frac{\tau H_{c2}}{e^{-1/\lambda}})ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T_c}(1+4\pi\frac{D_0}{\phi_0}\frac{\tau H_{c2}}{e^{-1/\lambda}})\right) \tag{3.188}$$

from which we can explicitly obtain the dependence of $T(H_{c2})$.
Thus, up to very low temperatures of the order of $\sim \frac{e^{-1/\lambda}}{\tau}$ the upper critical field is determined by Drude diffusion coefficient and we obtain the standard $H_{c2}(T)$ dependence of a "dirty" superconductor. The ratio $-\frac{H_{c2}(T)}{T_c(\frac{dH_{c2}}{dT})_{T_c}}$ for $\frac{e^{-1/\lambda}}{\tau} \ll T \ll T_c$ is equal to the usual value of 0.69. For low temperatures $T \ll \frac{e^{-1/\lambda}}{\tau}$ we obtain significant deviations from the predictions of the usual theory of "dirty" superconductors. $H_{c2}(T)$ dependence acquires the positive curvature and the upper critical field diverges as $T \to 0$. The behavior of the upper critical field for the case of $T_c \gg \frac{e^{-1/\lambda}}{\tau}$ is shown in Fig.3.11, curve 1.

(2) For $T_c \ll \frac{e^{-1/\lambda}}{\tau}$ the upper critical field behavior for any temperature is defined by Eq.(3.188). $H_{c2}(T)$—dependence acquires positive curvature and H_{c2} diverges for $T \to 0$. For small fields $H_{c2} \ll \frac{\phi_0}{D_0}\frac{e^{-1/\lambda}}{\tau}$, i.e. for $T \sim T_c$, Eq.(3.188), gives the explicit expression

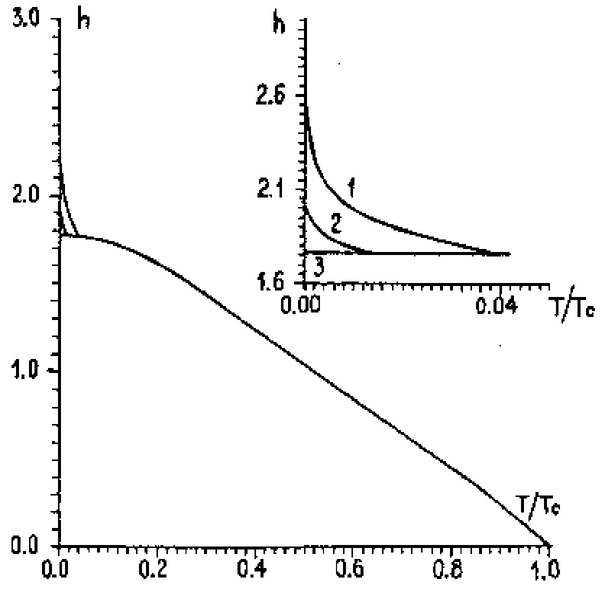

Fig. 3.11 Temperature dependence of the upper critical field for two-dimensional superconductor ($\frac{e^{-1/\lambda}}{\tau T_c} = 0.4, \lambda = 0.1, h = \frac{\omega_H}{\pi\lambda T_c}$). 1—no magnetic field influence upon diffusion, 2—with magnetic field influence upon diffusion, 3—standard theory of "dirty" superconductors.

for H_{c2}:

$$H_{c2} = \frac{1}{4\pi}\frac{\phi_0}{D_0}\frac{e^{-1/\lambda}}{\tau}\frac{ln\left(\frac{T_c}{T}\right)}{ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T}\right)} \tag{3.189}$$

The slope of $H_{c2}(T)$ at superconducting transition is determined by:

$$-\frac{\sigma_0}{N(E)}\left(\frac{dH_{c2}}{dT}\right)_{T_c} = \frac{e^2}{2\pi}\phi_0\frac{e^{-1/\lambda}}{\tau T_c ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T_c}\right)} \tag{3.190}$$

The behavior of the upper critical field for the case of $T_c \ll \frac{e^{-1/\lambda}}{\tau}$ is shown in Fig.3.12, curve 1.

It is clearly seen from Eqs.(2.116) and (2.117) that magnetic field influence upon diffusion becomes relevant only for high enough magnetic fields

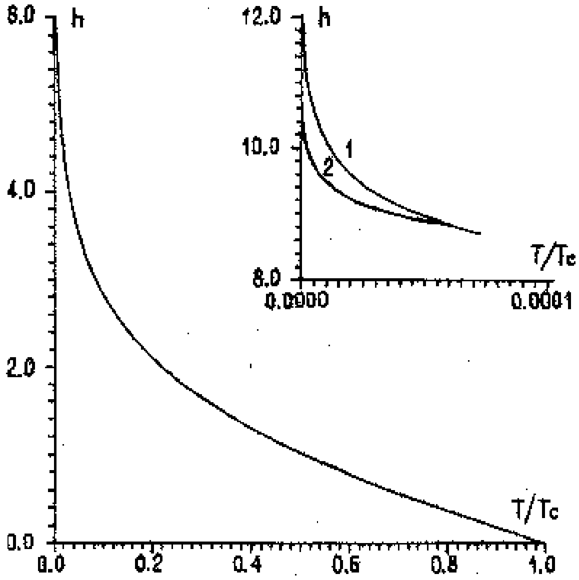

Fig. 3.12 Temperature dependence of the upper critical field for two-dimensional superconductor ($\frac{e^{-1/\lambda}}{\tau T_c} = 4, \lambda = 0.126, h = \frac{\omega_H}{\pi\lambda T_c}$). 1—no magnetic field influence upon diffusion, 2—with magnetic field influence upon diffusion.

$H_{c2} \gg \frac{\phi_0}{D_0}\frac{e^{-1/\lambda}}{\tau}$, i.e. for very low temperatures $T \ll T_c$. If we use Eq.(2.117) in the main equation (3.166), we obtain the following results:

(1) The case of $\frac{e^{-1/\lambda}}{\tau} \ll T_c \ll \frac{1}{\tau}$.
For high enough temperatures $T \gg \frac{e^{-1/\lambda}}{\tau}$ the diffusion coefficient entering Eq.(3.166) coincides with Drude's D_0 and the upper critical field is determined by Eqs.(3.186) and (3.187).
For $\frac{e^{-1/\lambda^2 ln\left(\frac{\gamma^2}{\pi}\frac{1}{\tau T_c}\right)}}{\tau} \ll T \ll \frac{e^{-1/\lambda}}{\tau}$ we obtain:

$$H_{c2}(T) = \frac{1}{2\gamma}\frac{\phi_0 T_c}{D_0}(1 - 3.56\frac{T}{T_c}) \tag{3.191}$$

Eq.(3.191) differs from Eq.(3.187) only by temperature dependent corrections and we can say that the magnetic field influence upon diffusion in this case leads to the widening of the temperature region where we can formally apply the usual theory of "dirty" supercon-

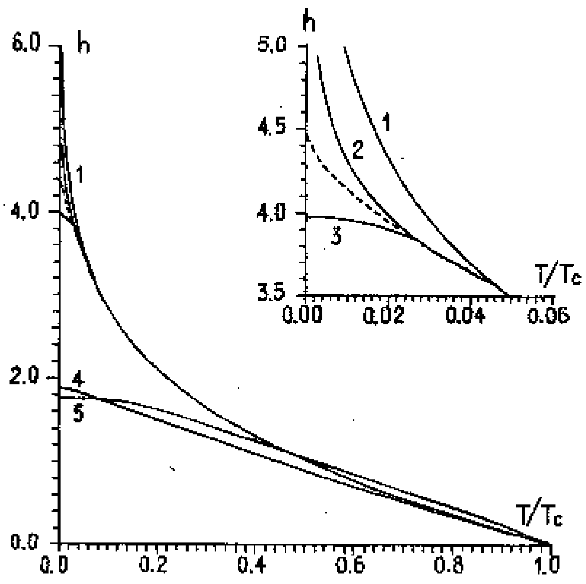

Fig. 3.13 Temperature dependence of the upper critical field for quasi-two-dimensional superconductor ($\frac{e^{-1/\lambda}}{\tau T_c} = 4, \lambda = 0.126, h = \frac{\omega_H}{\pi\lambda T_c}$) for different values of the interplane transfer integral around the critical value of w_c corresponding to Anderson transition at a given disorder. 1—purely two-dimensional behavior ($w = 0$), 2—dielectric side close to Anderson transition ($L = |2ln(w/w_c)| = 0.7$), 3—metallic side close to Anderson transition ($L = 2ln(w/w_c) = 0.7$), 4—metallic state far from Anderson transition ($L = 3$). 5—purely three-dimensional dependence ($\lambda L = 1$). Dashed line represents the behavior at the Anderson transition ($L = 0$).

ductors.

For $T \ll \frac{e^{-1/\lambda^2 ln\left(\frac{\gamma^2}{\pi}\frac{1}{\tau T_c}\right)}}{\tau}$ the upper critical field is defined by:

$$ln\left(\frac{\gamma}{2\pi e}\frac{e^{-1/\lambda^2 lnQ}}{\tau T}\right) = \frac{2\gamma}{Q}\frac{\lambda lnQ}{e^{-1/\lambda^2 lnQ}} ln\left(\frac{\gamma^2}{\pi Q}\frac{1}{\tau T_c}\right) \qquad (3.192)$$

where $Q = \frac{\gamma}{2\pi}\frac{\phi_0}{D_0 H_{c2}}\frac{1}{\tau}$. From Eq.(3.192) we can obtain the explicit dependence $T(H_{c2})$. The upper critical field in this case is slightly concave as in case of Eq.(3.188) where we have neglected the magnetic field influence upon diffusion. However, now we have no divergence of H_{c2} for $T \to 0$ and

$$H_{c2}(T=0) = \frac{\gamma}{2\pi} \frac{\phi_0}{D_0} \frac{1}{\tau} \tag{3.193}$$

In fact the value of $H_{c2}(T=0)$ will be even smaller, because for these values of the field the number of Landau levels below the cutoff will be of the order of unity and we are now outside the limits of applicability of Eqs.(2.114). However, the order of magnitude of $H_{c2}(T=0)$ given by Eq.(3.193) is correct. $H_{c2}(T)$ behavior with the account of magnetic field influence upon diffusion is shown in Fig.3.11, curve 2.

(2) The case of $T_c \ll \frac{e^{-1/\lambda}}{\tau}$.
For small fields $H_{c2} \ll \frac{\phi_0}{D_0}\frac{e^{-1/\lambda}}{\tau}$, i.e. for $T \sim T_c$, magnetic field influence upon diffusion is irrelevant and the upper critical field is determined by Eq.(3.189). For low temperatures $H_{c2}(T)$ is determined by Eq.(3.192), i.e. magnetic field influence upon diffusion liquidates the divergence of the upper critical field as $T \to 0$. The behavior of $H_{c2}(T)$ for $T_c \ll \frac{e^{-1/\lambda}}{\tau}$ is shown in Fig.3.12, curve 2.

It should be noted that the case of $T_c \ll \frac{e^{-1/\lambda}}{\tau}$ is possible only for sufficiently strong disorder. For typical $T_c \sim 10^{-4}E_F$ this case can occur only for $\lambda > 0.2$. Superconducting pairing can exist only in case the condition similar to Eq.(3.20) is satisfied. In two-dimensional case this condition leads to inequality $T_c \gg \lambda\frac{e^{-1/\lambda}}{\tau}$ which makes the region under discussion rather narrow.

Quasi-two-dimensional case was extensively discussed in Ref.[Kuchinskii E.Z., Sadovskii M.V. (1993a)]. Situation here is in many respects similar to that of two-dimensions, e.g. the anomalies in the upper critical field behavior due to the frequency dependence of diffusion coefficient appear only for temperatures $T \ll \frac{e^{-1/\lambda}}{\tau}$, while at higher temperatures $H_{c2}(T)$ is well described by the usual theory of "dirty" superconductors. As the interplane transfer integral w grows the smooth transition from purely two-dimensional behavior to that of three-dimensional isotropic system can be demonstrated.In case of $T_c \gg \frac{e^{-1/\lambda}}{\tau}$ deviations from the usual temperature behavior of H_{c2} is observed only for very low temperatures $T \ll \frac{e^{-1/\lambda}}{\tau}$, while close to T_c there are no significant changes from the standard dependence of $H_{c2}(T)$. For $T_c \ll \frac{e^{-1/\lambda}}{\tau}$ as interplane transfer integral w grows the temperature dependence of $H_{c2}(T)$ changes from purely two-dimensional concave behavior for all temperatures to convex three-dimensional like dependence. In Fig.3.13 we show the typical transformations of $H_{c2}(T)$ behavior as transfer integral w changes driving the system through metal-insulator transiton [Kuchinskii E.Z., Sadovskii M.V. (1993a)]. This clearly demonstrates the sharp anomalies in H_{c2} behavior which can appear due to localization effects.

3.3.3 *Anisotropic pairing*

Up to now in this Chapter we have implicitly assumed that we are dealing with isotropic s–wave pairing. In this section we shall analyze Ginzburg-Landau expansion in disordered system with anisotropic Cooper pairing (both d–wave and anisotropic s–wave.). Strictly speaking, here we shall not deal with localization transition, limiting ourselves to rather standard weak-coupling BCS treatment, when in most cases superconductivity is destroyed long before localization appears (Cf. discussion on d–wave pairing above). However, we shall discover some new ways to increase stability of anisotropic superconducting state to disordering, as well as some extra predictions for anomalous behavior of superconducting properties.

Following Refs.[Borkovski L.S., Hirschfeld P.J. (1994); Fehrenbacher R., Norman M.R. (1994); Posazhennikova A.I., Sadovskii M.V. (1996); Posazhennikova A.I., Sadovskii M.V. (1997a)] and assuming further applications to high-temperature superconductors, we shall analyze two – dimensional electronic system with isotropic Fermi surface and separable pairing potential of the form:

$$g(\mathbf{p},\mathbf{p}') \equiv g(\phi,\phi') = -ge(\phi)e(\phi'), \tag{3.194}$$

where ϕ is a polar angle, determining the electronic momentum direction in the conducting plane, and $e(\phi)$ is given by the following model dependence

$$e(\phi) = \begin{cases} \sqrt{2}cos\,(2\phi) & (\text{d}-\text{ wave }), \\ \sqrt{2}|cos\,(2\phi)\,| & (\text{ anisotropic s}-\text{ wave }). \end{cases} \tag{3.195}$$

The pairing constant g is as usual different from zero in some region of the width of $2 < \omega >$ around the Fermi level ($< \omega >$ – is again some characteristic frequency of the quanta, responsible for the pairing interaction). In this case the superconducting gap (order parameter) takes the form:

$$\Delta(\mathbf{p}) \equiv \Delta(\phi) = \Delta e(\phi), \tag{3.196}$$

and positions of its zeroes for s and d cases just coincide. Note that while the form of ϕ–dependence in (3.195) was chosen rather arbitrarily just to describe the required symmetry, it is precisely this form of angular dependence of the energy gap which is observed (e.g. in ARPES experiments) in high-T_c copper oxides [Randeria M., Campuzano J.C. (1997)].

We shall again examine a superconductor containing "normal" (non-magnetic) impurities, which are chaotically distributed in space with concentration ρ. Following Ref.[Haran G., Nagi A.D.S. (1996)] we consider the square of the scattering amplitude of the impurity in the following form:

$$|V_{imp}(\mathbf{p}, \mathbf{p}')|^2 \equiv |V_{imp}(\phi, \phi')|^2 = |V_0|^2 + |V_1|^2 f(\phi) f(\phi'), \tag{3.197}$$

where V_0 is isotropic point-like scattering amplitude, V_1 is anisotropic scattering amplitude, and $f(\phi)$ is angular-dependent model function (ϕ is a polar angle mentioned above) which describes the type of anisotropic scattering. We consider the scattering to be essentially isotropic and impose the following constraints [Haran G., Nagi A.D.S. (1996)]:

$$|V_1|^2 \leq |V_0|^2;\ < f >= 0;\ < f^2 >= 1, \tag{3.198}$$

where $< ... >$ denotes the averaging over the momentum direction on the Fermi surface (i.e. over the ϕ-angle). Accordingly, the second part in Eq.(3.197) represents the deviations from the isotropic scattering.

The normal and anomalous temperature Green's functions in such a superconductor are [Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)]:

$$G(\varepsilon_n, \mathbf{p}) = -\frac{i\tilde{\varepsilon}_n + \xi_{\mathbf{p}}}{\tilde{\varepsilon}_n^2 + \xi_{\mathbf{p}}^2 + |\tilde{\Delta}(\mathbf{p})|^2}, \tag{3.199}$$

$$F(\varepsilon_n, \mathbf{p}) = \frac{\tilde{\Delta}^*(\mathbf{p})}{\tilde{\varepsilon}_n^2 + \xi_{\mathbf{p}}^2 + |\tilde{\Delta}(\mathbf{p})|^2},$$

where $\varepsilon_n = (2n+1)\pi T$, $\xi-$ is electronic energy with respect to the Fermi level, while

$$\tilde{\varepsilon}_n(\mathbf{p}) = \varepsilon_n + i\rho \int \frac{d\mathbf{p}'}{(2\pi)^2} |V_{imp}(\mathbf{p} - \mathbf{p}')|^2 G(\varepsilon_n, \mathbf{p}'), \tag{3.200}$$

$$\tilde{\Delta}(\mathbf{p}) = \Delta(\mathbf{p}) + \rho \int \frac{d\mathbf{p}'}{(2\pi)^2} |V_{imp}(\mathbf{p} - \mathbf{p}')|^2 F(\varepsilon_n, \mathbf{p}').$$

are Matsubara frequency and gap renormalized by impurity scattering. To determine the transition temperature we can limit ourselves to Eqs.(3.200) linearized over Δ:

$$\tilde{\varepsilon}_n = \varepsilon_n + \rho \frac{N(0)}{2\pi} \int_{-\infty}^{\infty} d\xi \int_0^{2\pi} d\phi' \left\{ |V_0|^2 + |V_1|^2 f(\phi) f(\phi') \right\} \frac{\tilde{\varepsilon}_n}{\tilde{\varepsilon}_n^2 + \xi^2},$$
$$\tilde{\Delta} = \Delta + \rho \frac{N(0)}{2\pi} \int_{-\infty}^{\infty} d\xi \int_0^{2\pi} d\phi' \left\{ |V_0|^2 + |V_1|^2 f(\phi) f(\phi') \right\} \frac{\tilde{\Delta}}{\tilde{\varepsilon}_n^2 + \xi^2}. \tag{3.201}$$

The critical temperature T_c is determined by the linearized gap-equation:

$$\Delta(\mathbf{p}) = -T_c \sum_n \int \frac{d\mathbf{p}'}{(2\pi)^2} g(\mathbf{p}, \mathbf{p}') \frac{\tilde{\Delta}(\mathbf{p}')}{\tilde{\varepsilon}_n^2 + \xi_{\mathbf{p}'}^2} \tag{3.202}$$

Following standard procedure from Eqs.(3.201),(3.202) we obtain the following general equation for the critical temperature T_c:

$$ln\left(\frac{T_c}{T_{c0}}\right) = \left(< e >^2 + < ef >^2 - 1\right) \left[\Psi\left(\frac{1}{2} + \frac{\gamma_0}{2\pi T_c}\right) - \Psi\left(\frac{1}{2}\right)\right] + \tag{3.203}$$
$$< ef >^2 \left[\Psi\left(\frac{1}{2}\right) - \Psi\left(\frac{1}{2} + \frac{\gamma_0}{2\pi T_c}\left(1 - \frac{\gamma_1}{\gamma_0}\right)\right)\right]$$

where T_{c0}- is the transition temperature in the absence of impurities, $\Psi(x)$- is the usual digamma function, $\gamma_0 = \pi\rho V_0^2 N(0)$ and $\gamma_1 = \pi\rho V_1^2 N(0)$ – correspondingly the isotropic and anisotropic impurity scattering rates, while $< ef >^2$ describes the "overlap" between the functions $e(\mathbf{p})$ and $f(\mathbf{p})$.

For the sake of simplicity we take the function $f(\mathbf{p})$ in the form analogous to that in Eq.(3.195):

$$f(\mathbf{p}) \equiv f(\phi) = \sqrt{2} cos(2\phi), \tag{3.204}$$

This corresponds to the maximum overlap for d-case. More general treatment one could find in Ref.[Haran G., Nagi A.D.S. (1996)]. Now the renormalized Eqs.(3.201) can be written as follows:

$$\tilde{\varepsilon}_n = \varepsilon_n + \frac{\gamma_0}{\pi} \int_{-\infty}^{\infty} d\xi \frac{\tilde{\varepsilon}_n}{\tilde{\varepsilon}_n^2 + \xi^2} + \frac{\gamma_1}{2\pi^2} cos(2\phi) \int_{-\infty}^{\infty} d\xi \int_0^{2\pi} d\phi' cos(2\phi') \frac{\tilde{\varepsilon}_n}{\tilde{\varepsilon}_n^2 + \xi^2},$$

$$\tilde{\Delta} = \Delta + \frac{\gamma_0}{\pi}\int_{-\infty}^{\infty} d\xi \frac{\tilde{\Delta}}{\tilde{\varepsilon}_n^2 + \xi^2} + \frac{\gamma_1}{2\pi^2} cos(2\phi)\int_{-\infty}^{\infty} d\xi \int_0^{2\pi} d\phi' cos(2\phi')\frac{\tilde{\Delta}}{\tilde{\varepsilon}_n^2 + \xi^2}. \tag{3.205}$$

From here we obtain the well-known expression for the renormalized frequency in both cases:

$$\tilde{\varepsilon}_n = \varepsilon_n + \gamma_0 sign\varepsilon_n. \tag{3.206}$$

In the case of $d-$wave pairing the gap symmetry in the presence of impurities is not changed and:

$$\tilde{\Delta} = \Delta \frac{|\tilde{\varepsilon}_n|}{|\tilde{\varepsilon}_n| - \gamma_1}. \tag{3.207}$$

In the case of $s-$wave pairing the gap is shifted by a constant, which does not depend on ϕ and γ_1:

$$\tilde{\Delta} = \Delta + \Delta_0 \frac{2\sqrt{2}\gamma_0}{\pi|\varepsilon_n|}. \tag{3.208}$$

Finally T_c-equation for superconductor with $d-$wave pairing is written as:

$$ln\left(\frac{T_c}{T_{c0}}\right) = \Psi\left(\frac{1}{2}\right) - \Psi\left(\frac{1}{2} + \left(1 - \frac{\gamma_1}{\gamma_0}\right)\frac{\gamma_0}{2\pi T_c}\right). \tag{3.209}$$

For superconductor with anisotropic $s-$wave pairing T_c-equation is written as:

$$ln\left(\frac{T_c}{T_{c0}}\right) = \left(1 - \frac{8}{\pi^2}\right)\left[\Psi\left(\frac{1}{2} + \frac{\gamma_0}{2\pi T_c}\right) - \Psi\left(\frac{1}{2}\right)\right]. \tag{3.210}$$

Note that anisotropic scattering rate dependence drops from Eq.(3.210).

The appropriate dependencies of $T_c(\gamma_0/T_{c0})$ are shown in Fig.3.14, for the case of $d-$wave pairing with different values of the normalized anisotropic scattering rate γ_1/γ_0. In the case of $s-$wave pairing the transition temperature T_c is only relatively weakly suppressed with the growth of γ_0/T_{c0}. In the case of $d-$wave pairing T_c suppression is rather fast for small values of γ_1, but the critical value of γ_{0c}/T_{c0} leading to the complete destruction of superconducting state rapidly increases with the growth of the anisotropic scattering rate γ_1/γ_0. In this sense, anisotropy of impurity scattering can, in principle, be another reason (in addition to small size of the pairs) for extra stability of d-wave pairing under disordering, allowing even to conserve finite T_c up to disorder induced metal-insulator transition. However,

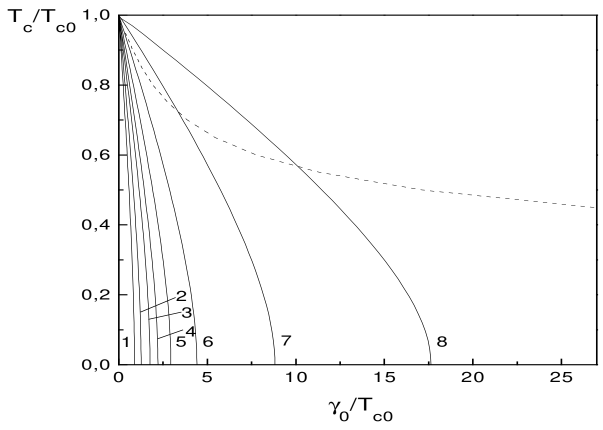

Fig. 3.14 Critical temperature T_c as a function of the normalized isotropic scattering rate γ_0/T_{c0}. The dashed curve represents the s-wave pairing case, the solid curves represent the d-wave pairing case for different values of the normalized anisotropic scattering rate γ_1/γ_0: 1—γ_1/γ_0 =0.0; 2—0.3; 3—0.5; 4—0.6; 5—0.7; 6—0.8; 7—0.9; 8—0.95.

the detailed analysis of this possibility is at present impossible, because of the lack of any analysis of the effects of anisotropic scattering on Anderson transition. Apparently, such analysis can be performed using self-consistent theory of localization. Anisotropic s–wave pairing is practically as stable to disordering as the usual isotropic case analyzed above, so that all qualitative dependencies discussed above for isotropic s–wave case can be applied here.

The gap function as usual can be used as an order parameter in the Ginzburg-Landau expansion for the free-energy density. We consider its amplitude $\Delta(T)$ to be a slowly varying function of the spatial coordinates. Accordingly in momentum space we get the Fourier-component of the order parameter:

$$\Delta(\phi, q) = \Delta_q(T)e(\phi). \tag{3.211}$$

The Ginzburg-Landau expansion for the free energy density difference between superconducting and normal state in the region of small q up to terms quadratic over Δ takes the usual form:

$$F_s - F_n = A|\Delta_q|^2 + q^2 C|\Delta_q|^2 \tag{3.212}$$

and is determined by the diagrams of loop-expansion for the free-energy of electrons moving in the field of superconducting order parameter fluctuations with some small vector q, shown in Fig.C.2 in Appendix C. The only change in comparison with s–wave case is that we must add extra factors of $e(\phi)$ in each vertex to take into account the symmetry of the order parameter. Some details on calculating the impurity vertex $\Gamma_{\mathbf{pp'}}$ in the case of anisotropic impurity scattering, as well as Ginzburg-Landau coefficients for the case of $d-$wave pairing can be found in the Appendix D. Note, that for the $d-$wave superconductors the contribution of diagrams Fig.C.2(b,d) actually vanishes up to terms of the order of q^2, if we don't take into account an anisotropy of impurity scattering. In the case of $s-$wave superconductor all calculations are analogous, we only note that in this a case a dependence on anisotropic scattering rate is absent.

Finally we can express GL-coefficients in the following form:

$$A = A_0 K_A; \qquad C = C_0 K_C, \tag{3.213}$$

where A_0 and C_0 are just the usual GL-coefficients for the case of isotropic s-wave pairing [Gennes de P.G. (1966)] in two – dimensional case:

$$A_0 = N(0)\frac{T - T_c}{T_c}; \qquad C_0 = N(0)\frac{7\zeta(3)}{32\pi^2}\frac{v_F^2}{T_c^2}, \tag{3.214}$$

where $v_F, N(0)$-are electron velocity and normal density of states at the Fermi level, and everything specific to our model is contained in dimensionless combinations K_A and K_C. In the absence of impurities for both models we obtain: $K_A^0 = 1$, $K_C^0 = 1$. For the impure system we get:

(A) d-wave pairing:

$$K_A = \frac{\gamma_0}{4\pi T_c}\int_{-<\omega>}^{<\omega>}\frac{d\xi}{\xi}\int_{-\infty}^{\infty}d\varepsilon\frac{\varepsilon+\xi}{(\varepsilon^2+\gamma_0^2)ch^2\left(\frac{\varepsilon+\xi}{2T_c}\right)} + \tag{3.215}$$

$$\frac{\gamma_1(2\gamma_0+\gamma_1)}{4T_c}\int_{-\infty}^{\infty}d\varepsilon\frac{\varepsilon^2}{(\varepsilon^2+\gamma_0^2)(\varepsilon^2+(\gamma_0-\gamma_1)^2))ch^2\left(\frac{\varepsilon}{2T_c}\right)},$$

$$K_C=\frac{3\pi T_c}{7\zeta(3)\gamma_1}\left\{\frac{2\pi T_c}{\gamma_1}\left[\Psi\left(\frac{1}{2}+\frac{\gamma_0-\gamma_1}{2\pi T_c}\right)-\Psi\left(\frac{1}{2}+\frac{\gamma_0}{2\pi T_c}\right)\right]\right.$$
$$\left.+\Psi'\left(\frac{1}{2}+\frac{\gamma_0-\gamma_1}{2\pi T_c}\right)\right\}; \quad (3.216)$$

(B) anisotropic s-wave pairing:

$$K_A=\frac{\gamma_0}{\pi T_c}\left\{\frac{1}{4}\int_{-<\omega>}^{<\omega>}\frac{d\xi}{\xi}\int_{-\infty}^{\infty}d\varepsilon\frac{\varepsilon+\xi}{(\varepsilon^2+\gamma_0^2)ch^2\left(\frac{\varepsilon+\xi}{2T_c}\right)}\right.$$
$$\left.+\frac{2\gamma_0}{\pi}\int_{-\infty}^{\infty}d\varepsilon\frac{1}{(\varepsilon^2+\gamma_0^2)ch^2\left(\frac{\varepsilon}{2T_c}\right)}\right\}, \quad (3.217)$$

$$K_C=-\frac{3(\pi^2-8)}{28\pi^2\zeta(3)}\Psi''\left(\frac{1}{2}+\frac{\gamma_0}{2\pi T_c}\right)+\frac{24\pi^2}{7\zeta(3)\gamma_0^2}\frac{T_c^2}{(\pi^2-8)}ln\left(\frac{T_c}{T_{c0}}\right)+\frac{6\pi}{7\zeta(3)}\frac{T_c}{\gamma_0}. \quad (3.218)$$

The appropriate dependencies of dimensionless coefficients on disorder parameter γ_0/T_{c0} in the case of $d-$wave pairing and for different values of the normalized anisotropic scattering rate γ_1/γ_0 are shown in Figs.3.15,3.16.

GL-coefficients A and C, as usual, define the temperature dependence of the upper critical magnetic field close to T_c [Gennes de P.G. (1966)]:

$$H_{c2}=\frac{\phi_0}{2\pi\xi^2(T)}=-\frac{\phi_0}{2\pi}\frac{A}{C} \quad (3.219)$$

where $\phi_0=c\pi/e$ – is again the magnetic flux quantum, $\xi(T)$ – is temperature dependent coherence length. Now we can easily find the "slope" of the temperature dependence of $H_{c2}(T)$ near T_c, i.e. the temperature derivative:

$$\left|\frac{dH_{c2}}{dT}\right|_{T_c}=\frac{24\pi\phi_0}{7\zeta(3)v_F^2}T_c\frac{K_A}{K_C} \quad (3.220)$$

In the case of the usual s-wave superconductivity anisotropic scattering does not influence the behavior of the slope of the upper critical field. The appropriate dependencies of dimensionless parameter $h=|dH_{c2}/dT|_{T_c}/|dH_{c2}/dT|_{T_{c0}}$ on disorder γ_0/T_{c0} in the case of $d-$wave pairing for different values of the

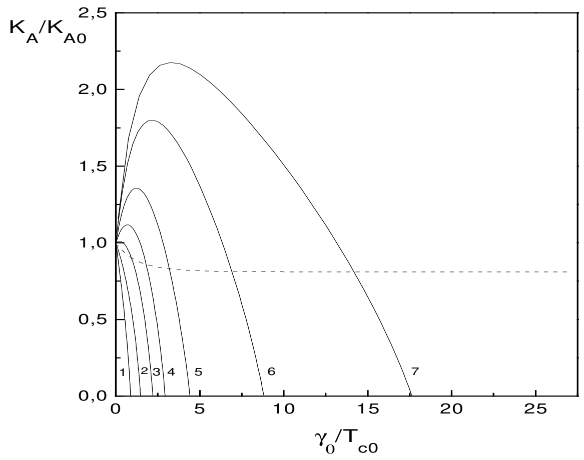

Fig. 3.15 Dependence of dimensionless coefficient K_A/K_{A0} on disorder parameter γ_0/T_{c0}. The dashed curve represents the s-wave pairing case, the solid curves represent the d-wave pairing case for different values of the normalized anisotropic scattering rate γ_1/γ_0: 1—γ_1/γ_0 =0.0; 2—0.4; 3—0.6; 4—0.7; 5—0.8; 6—0.9; 7—0.95.

normalized anisotropic scattering rate γ_1/γ_0 are shown in Fig.3.17. In the case of anisotropic s-wave pairing the slope as usual grows with disorder and in the limit of strong disorder $\gamma_0 \gg T_c$ it crosses over to the linear dependence described by Gorkov's relation (Cf.(3.162)):

$$\frac{\sigma}{N(0)}\left|\frac{dH_{c2}}{dT}\right|_{T_c} = \frac{8e^2}{\pi^2}\phi_0 \tag{3.221}$$

which is characteristic of the impure superconductors with isotropic s-wave pairing. Here $\sigma = N(0)e^2v_F^2/3\gamma_0$ - is residual conductivity in the normal state. It means that strong disordering suppresses gap anisotropy and we obtain a usual limit of the impure superconductor. For the case of d-wave

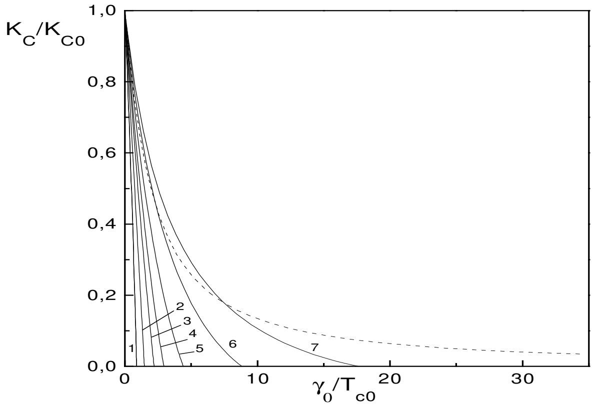

Fig. 3.16 Dependence of dimensionless coefficient K_C/K_{C0} on disorder parameter γ_0/T_{c0}. The dashed curve represents the s-wave pairing case, the solid curves represent the d-wave pairing case for different values of the normalized anisotropic scattering rate γ_1/γ_0: 1—γ_1/γ_0 =0.0; 2—0.4; 3—0.6; 4—0.7; 5—0.8; 6—0.9; 7—0.95.

pairing the slope of H_{c2} drops to zero on the scale $\gamma_0 \sim T_{c0}$ for the small values of the rate γ_1/γ_0. For the values of anisotropic scattering rate on the interval $0.5 \leq \gamma_1/\gamma_0 \leq 0.6$ the behavior of the slope is qualitatively changed: h smoothly but nonlinearly increases with the growth of γ_0/T_{c0}, crosses over the maximum and then has a sharp drop. The interval where the slope grows extends as γ_1 approaches γ_0. In our opinion these sharp anomalies in dependence of the slope of the upper critical field on disorder can be used for determining the pairing type and possible role of anisotropic impurity scattering in "unusual" superconductors. Unfortunately, in case of high-T_c oxides the situation is complicated by the known nonlinearity of temperature dependence of H_{c2}, which is observed in rather wide region

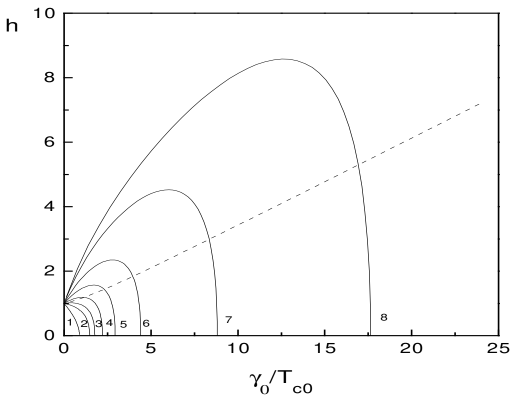

Fig. 3.17 Dependence of normalized slope of the upper critical field $h = \left|\frac{dH_{c2}}{dT}\right|_{T_c} / \left|\frac{dH_{c2}}{dT}\right|_{T_{c0}}$ on disorder parameter γ_0/T_{c0}. The dashed curve represents the s-wave pairing case, the solid curves represent the d-wave pairing case for different values of the normalized anisotropic scattering rate γ_1/γ_0: 1—γ_1/γ_0 =0.0; 2—0.4; 3—0.5; 4—0.6; 5—0.7; 6—0.8; 7—0.9; 8—0.95.

close to T_c and and also by some ambiguity in experimental methods to measure H_{c2}.

3.4 Fluctuation Conductivity Near Anderson Transition

Fluctuation conductivity of Cooper pairs (above T_c) is especially interesting in strongly disordered system because the usual single—particle contribution to conductivity drops to zero as the system moves towards Anderson transition. We shall use the standard approach [Aslamazov L.G., Larkin A.I. (1968)] which takes into account fluctuational Cooper pairs formation above T_c. We assume that it is possible to neglect the so called Maki—Thompson correction which describes the increased one—particle contribution to conductivity due to superconducting fluctuations [Keller J., Korenman V. (1972)]. We expect that these estimates [Bulaevskii L.N., Varlamov A.A., Sadovskii M.V. (1986)] will enable us to find a correct scale of fluctuation conductivity close to mobility edge.

Consider first the averaged fluctuation propagator:

$$L^{-1}(\mathbf{q},\Omega_k) = \lambda^{-1} - \Pi(\mathbf{q},\Omega_k) \tag{3.222}$$

where the polarization operator

$$\begin{aligned}\Pi(\mathbf{q},\Omega_k) &= \sum_{\varepsilon_n}\sum_{\mathbf{pp}'} < G(\mathbf{p}_+\mathbf{p}'_+\varepsilon_n+\Omega_k)G(\mathbf{p}_-\mathbf{p}'_- - \varepsilon_n) >= \\ &= -2i\pi T\sum_{\varepsilon_n}\Phi_E(\mathbf{q},\omega_m=2\varepsilon_n+\Omega_k) \qquad \omega_m = 2\pi m T \end{aligned}\tag{3.223}$$

During our analysis of Ginzburg—Landau coefficients we were interested in $\omega_m = 2\varepsilon_n$, so that one of the Green's functions in Φ_E was automatically retarded, while the other was advanced. Now we need a more general expression of Eq. (3.223) with $\omega_m = 2\varepsilon_n + \Omega_k$. Accordingly, instead of Eq. (3.130) we must use the following expression with additional θ-function:

$$\Phi_E(\mathbf{q},\omega_m = 2\varepsilon_n+\Omega_k) = -\frac{N(E)\theta[\varepsilon_n(\varepsilon_n+\Omega_k)]}{i|2\varepsilon_n+\Omega_k| + iD_E(|2\varepsilon_n+\Omega_k|)q^2} \tag{3.224}$$

where the generalized diffusion coefficient is again determined by Eq. (3.132) and Eq. (3.133). From Eqs. (3.222)—(3.224), performing summation over ε_n we get the following form of fluctuation propagator for small $\mathbf{q}$ ($D_E q^2 < T$):

$$L^{-1}(\mathbf{q},\Omega_k) = -N(E)\left\{ln\frac{T}{T_c} + \Psi\left(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}\right) - \Psi\left(\frac{1}{2}\right) + \eta(|\Omega_k|)q^2\right\} \tag{3.225}$$

where

$$\begin{aligned}\eta(|\Omega_k|) &= 4\pi T\sum_{n=0}^{\infty}\frac{D_E(2\varepsilon_n+|\Omega_k|)}{(2\varepsilon_n+|\Omega_k|)^2} = \\ &= \begin{cases} \frac{D_E}{4\pi T}\Psi'(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}) & \xi_{loc} < (\xi_0 l^2)^{1/3} \quad E > E_c \\ \frac{D_0}{(4\pi T)^{2/3}(2\gamma)^{1/3}}\zeta(\frac{5}{3};\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}) & \xi_{loc} > (\xi_0 l^2)^{1/3}\end{cases}\end{aligned} \tag{3.226}$$

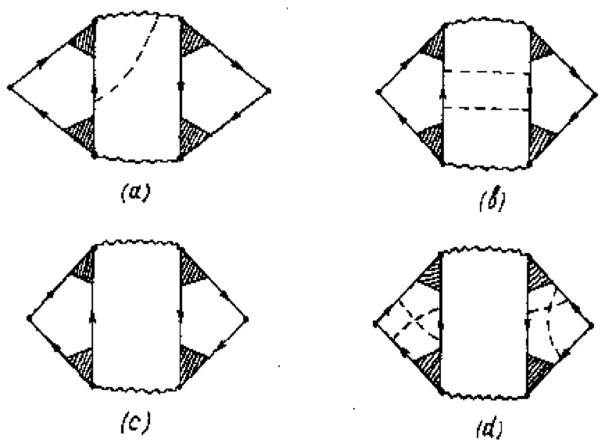

Fig. 3.18 Diagrams for fluctuation conductivity. Wavy lines denote fluctuation propagator, dashed lines—disorder scattering.

It is also useful to know the form of fluctuation propagator for $|\Omega_k| \gg T$. In this case, close to the Anderson transition, we may replace the sum over ε_n in Eq. (3.223) by integral, while far from the transition it can be calculated exactly. As a result we get:

$$L^{-1}(\mathbf{q},\Omega_k) = -N(E)\begin{cases} ln\frac{T}{T_c} + \Psi\left(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}+\frac{D_E q^2}{4\pi T}\right) - \Psi(\frac{1}{2}) \\ \qquad \xi_{loc} < (\xi_0 l^2)^{1/3} \quad E > E_c \\ ln\frac{T}{T_c} + \frac{3}{2}ln\left[\left(\frac{|\Omega|}{4\pi T}\right)^{2/3} + \frac{D_0 q^2}{(4\pi T)^{2/3}(2\gamma)^{1/3}}\right] \\ \qquad \xi_{loc} > (\xi_0 l^2)^{1/3} \end{cases} \tag{3.227}$$

Diagrams determining fluctuation conductivity are shown in Fig.3.18. Contributions of graphs Fig.3.18 (a) and (b) are nonsingular close to T_c because at least one of fluctuation propagators transfers a large momentum of the order of p_F. Thus we have to consider independent contributions formed by three Green's functions entering graphs Fig.3.18(c). We can calculate these contributions using the usual approximations of self—consistent theory of localization taking into account the renormalization of triangular vertices by maximally—crossed graphs [Katsnelson M.I., Sadovskii M.V. (1983); Kotov E.A., Sadovskii M.V. (1985)] (Cf. Eq. (3.41) as in Fig.3.18 (c)). We shall neglect graphs like in Fig.3.18 (d) where the topology of disorder scattering lines is not reduced to the renormalization of triangular vertices. We assume that these approximations are sufficient at least for a qualitative inclusion of localization effects. Note that it is sufficient to calculate the contribution of three Green's functions $\mathbf{B}(\mathbf{q},\Omega_k,\omega_m)$ for small $\mathbf{q}$ and zero external frequency $\omega_m = 0$. It can be easily found differentiating the polarization operator of Eq. (3.223):

$$\mathbf{B}(\mathbf{q},\Omega_k,0) = \mathbf{q}C = -\frac{\partial}{\partial \mathbf{q}}\Pi(\mathbf{q},\Omega_k) \tag{3.228}$$

The contribution of diagram of Fig.3.18 (c) to the operator of electromagnetic response [Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)] is now determined by the following expression:

$$Q_{\alpha\beta} = -\frac{4e^2T}{m^2}\sum_{\Omega_k}\int \frac{d^3\mathbf{q}}{(2\pi)^3}(Cq_\alpha)(Cq_\beta)L(\mathbf{q},\Omega_k)L(\mathbf{q},\Omega_k+\omega_m) \tag{3.229}$$

Close to T_c we can also neglect the dependence of C on Ω_k. Then C reduces to Eq. (3.136) and we have $C = N(E)\xi^2$. Fluctuation propagator analytically continued to the upper halfplane of complex ω takes the usual form:

$$L(\mathbf{q},\omega) = -\frac{1}{N(E)}\frac{1}{\frac{T-T_c}{T_c} - \frac{i\pi\omega}{8T_c} + \xi^2q^2} \tag{3.230}$$

Further calculations can be performed in a standard way and for fluctuation conductivity for $(T-T_c)/T_c \ll 1$ we get the usual result [Aslamazov L.G., Larkin A.I. (1968)]:

$$\sigma_{AL} = \frac{e^2}{32\xi\hbar}\left(\frac{T_c}{T-T_c}\right)^{1/2} \tag{3.231}$$

but with the coherence length ξ being defined as (Cf. Eq.(3.136)):

$$\xi = \begin{cases} \left(\frac{\xi_0 l}{p_F\xi_{loc}}\right)^{1/2} & \xi_{loc} < (\xi_0 l^2)^{1/3} \quad E > E_c \\ (\xi_0 l^2)^{1/3} \sim (\xi_0/p_F^2)^{1/3} & \xi_{loc} > (\xi_0 l^2)^{1/3} \quad E \sim E_c \end{cases} \tag{3.232}$$

From these estimates we can see that as the system approaches the Anderson transition a temperature interval where the fluctuation contribution to conductivity is important widens. Fluctuation Cooper pair conductivity becomes comparable with a single—particle one for $\sigma < \sigma^\star \approx \sigma_c(p_F\xi_0)^{-1/3} \approx \sigma_c(T_c/E_F)^{1/3}$, i.e. close enough to mobility edge. In fact this confirms the above picture of Cooper pairs remaining delocalized while single—particle excitations localize as the system undergoes metal—insulator transition.

It is not difficult to find also the fluctuation contribution to diamagnetic susceptibility [Bulaevskii L.N., Varlamov A.A., Sadovskii M.V. (1986)]. Close to T_c it is determined by a standard expression:

$$\chi_{fl} = -\frac{e^2 T_c}{6\pi c^2}\xi\left(\frac{T_c}{T - T_c}\right)^{1/2} \tag{3.233}$$

where the coherence length is again defined as in Eq. (3.232).

Thus our expressions for fluctuation effects follow more or less obviously from our general picture of Ginzburg—Landau expansion: for the system close to Anderson transition we have only to replace the usual coherence length $\sqrt{\xi_0 l}$ of a "dirty" superconductor by $\xi \sim (\xi_0 l^2)^{1/3} \sim (\xi_0/p_F)^{1/3}$.

3.5 Superconductivity in Anderson Insulator at $T = 0$

We have already considered the superconducting response of a system which is close to Anderson transition within Ginzburg—Landau approximation, i.e. for temperatures $T \sim T_c$. In fact it is not difficult to obtain similar results also for $T = 0$ [Lee P.A., Ma M. (1985)].

Superconducting current density at $T = 0$ is given by [Gennes de P.G. (1966)]:

$$\mathbf{j}_s = -\frac{n_s e^2}{mc}\mathbf{A} \tag{3.234}$$

where n_s is superconducting electron density, $\mathbf{A}$ is vector potential of an external magnetic field. On the other hand, using exact eigenstates representation DeGennes has obtained the following beautiful relation between superconducting response at $T = 0$ and conductivity of a system in the normal state [Gennes de P.G. (1966); Lee P.A., Ma M. (1985)]:

$$\mathbf{j}_s = \left\{\frac{1}{2\pi c}\int d\xi \int d\xi' L(\xi,\xi') Re\sigma(\xi - \xi') - \frac{ne^2}{mc}\right\}\mathbf{A} \tag{3.235}$$

All characteristics of a superconducting state are contained here in the kernel:

$$L(\xi,\xi') = \frac{1}{2}\frac{EE' - \xi\xi' - \Delta_0^2}{EE'(E + E')} \tag{3.236}$$

where $E = \sqrt{\xi^2 + \Delta_0^2}$ and Δ_0 is superconducting gap at $T = 0$. Note that in normal state $\mathbf{j}_s = 0$ and we can rewrite Eq. (3.235) as:

$$\mathbf{j}_s = \frac{1}{2\pi c}\int d\xi \int d\xi' [L(\xi,\xi')|_{\Delta=\Delta_0} - L(\xi,\xi')|_{\Delta=0}] Re\sigma(\xi - \xi')\mathbf{A} \quad (3.237)$$

Taking into account that $L(\xi,\xi')|_{\Delta=\Delta_0} - L(\xi,\xi')|_{\Delta=0}$ for large $|\xi - \xi'|$ drops as $|\xi - \xi'|^{-3}$ it is sufficient to know only the low—frequency response of a system in normal state. In particular, for "pure" system (with no scattering) we have $Re\sigma(\omega) = (ne^2/m)\pi^{-1}\delta(\omega)$ and comparing Eq. (3.234) with Eq. (3.237) it is immediately clear that at $T = 0$ we have $n_s = n$, i.e. in an ideal system all electrons are superconducting.

Close to the Anderson transition we can use the results of elementary scaling theory of localization, e.g. Eq. (2.31) and Eq. (2.33) to write

$$\sigma(\omega) \approx \begin{cases} A\frac{g_c}{\xi_{loc}} & \omega < \omega_c \\ A\frac{g_c}{\xi_{loc}}\left(\frac{\omega}{\omega_c}\right)^{1/3} & \omega > \omega_c \end{cases} \quad (3.238)$$

where $\omega_c \sim [N(E)\xi_{loc}^3]^{-1}$ is defined in Eq. (2.42), g_c is the critical conductance of scaling theory ($g_c \sim 1$), $A \sim 1$. From Eq. (3.236) and Eq. (3.237) it is clear that the main contribution into integral in Eq. (3.237) comes from $|\xi - \xi'| \sim \Delta_0$, so that the value of n_s depends on the relation between Δ_0 and ω_c. For $\Delta_0 < \omega_c$ we have $\sigma(\Delta_0) = Ag_c/\xi_{loc}$ and

$$n_s = A\frac{m}{e^2}\Delta_0\frac{g_c}{\xi_{loc}} \quad (3.239)$$

For $\Delta_0 > \omega_c$ we have $\sigma(\Delta_0) = Ag_c[N(E)\Delta_0]^{1/3}$ and it becomes independent on the further growth of ξ_{loc} in the region of $\xi_{loc} > [N(E)\Delta_0]^{-1/3}$. Accordingly n_s does not vanish at the mobility edge but saturate at

$$n_s = A\frac{m}{e^2}g_c[N(E)\Delta_0]^{1/3} \quad (3.240)$$

In localization region we can write instead of Eq. (3.238)

$$\sigma(\omega) \approx \begin{cases} 0 & \omega < \omega_c \\ Ag_c[N(E)\omega]^{1/3} & \omega > \omega_c \end{cases} \quad (3.241)$$

which again leads to $\sigma(\Delta_0) \approx Ag_c[N(E)\Delta_0]^{1/3}$ and Eq. (3.240) remains valid until $R_{loc} > [N(E)\Delta_0]^{-1/3}$. Thus the density of superconducting electrons n_s remains finite close to Anderson transition both in metallic and insulating states. From Eq. (3.240) it is easy to see that close to Anderson transition

$$\frac{n_s}{n} \sim \left(\frac{\Delta_0}{E_F}\right)^{4/3} \tag{3.242}$$

This coincide with an estimate of Eq. (3.145) based upon Ginzburg—Landau expansion. For typical Δ_0 and E_F only small part ($\sim 10^{-4}$ in traditional superconductors) of conduction electrons form Cooper pairs. The condition of $R_{loc} > [N(E)\Delta_0]^{-1/3} \sim a(E_F/\Delta_0)^{1/3}$ as discussed above defines the size of possible superconducting region in Anderson insulator. This region is of course quite narrow, e.g. if metal—insulator transition takes place with a change of some external parameter x (impurity concentration, pressure, fluence of fast neutrons etc.), so that $R_{loc} \sim a|(x - x_c)/x_c|^{-\nu}$, then for $\nu \approx 1$ and typical $E_F/\Delta_0 \sim 10^4$ we get $|x - x_c| < 0.1x_c$.

These estimates are in complete accordance with the results of our discussion of Ginzburg—Landau approximation [Bulaevskii L.N., Sadovskii M.V. (1984); Bulaevskii L.N., Sadovskii M.V. (1985)] and in fact we now have the complete qualitative picture of superconductivity in Anderson insulator both for $T \sim T_c$ and $T \to 0$, i.e. in the ground state.

Chapter 4

STATISTICAL FLUCTUATIONS OF SUPERCONDUCTING ORDER PARAMETER

The previous discussion of superconductivity in a strongly disordered system was based upon important assumption of the existence of self—averaging superconducting order—parameter Δ. This assumption was first used in the theory of "dirty" superconductors [Abrikosov A.A., Gorkov L.P. (1958); Abrikosov A.A., Gorkov L.P. (1959); Anderson P.W. (1959); Gennes de P.G. (1966)] and also in all early papers on the interplay of localization and superconductivity. It was expected that spatial fluctuations of this order parameter $\Delta(\mathbf{r})$ are actually small and we can always use some disorder averaged parameter $< \Delta(\mathbf{r}) >$. It seems natural for $\sigma \gg \sigma_c$ and it really can be justified in this region as we shall see below. However, close to the mobility edge there are no special reasons to believe in correctness of this assumption. In this case electronic characteristics of the system become strongly fluctuating and we shall see that these lead to the strong spatial (*statistical*) fluctuations of superconducting order parameter, or even to the regime inhomogeneous superconductivity. At the same time we must stress that these fluctuations are in some sense similar to the usual thermodynamic critical fluctuations of the order parameter and become important in some new critical region (we call it statistical critical region) close to T_c. In this sense all the previous analysis is just a kind of statistical mean—field approximation and of course it is a necessary step for further studies taking into account the statistical fluctuations. The importance of these fluctuations is stressed by the fact that the statistical critical region widens (similarly to the usual critical region) as the system goes to the Anderson transition and apparently the role of fluctuations becomes decisive for the physics of the interplay of localization and superconductivity.

4.1 Statistical Critical Region

Here we shall start by a demonstration of the appearance of the new type of fluctuations which are at least of the same importance as the usual critical fluctuations of superconducting order—parameter. We call them statistical fluctuations [Bulaevskii L.N., Sadovskii M.V. (1986)] and their nature is closely connected to the problem of self—averaging properties of this order parameter (i.e. with a possibility of decoupling transforming Eq. (3.12) into Eq. (3.13)). We shall more or less follow Ref.[Bulaevskii L.N., Sadovskii M.V. (1986)], equivalent results were later obtained in Ref.[Ng T.K. (1991)]. Some extra development along similar lines was discussed by [Zhitomirsky M.E., Walker M.B. (1998)].

Let us return to the Eq. (3.7) and analyze the situation in more details. We shall use a simple iteration procedure assuming that fluctuations of the kernel $K(\mathbf{rr}')$ due to disorder are small. Similar approach was first used in Ref.[Caroli C., De Gennes P.G., Matricon J. (1962)]. In this case we can represent $K(\mathbf{rr}')$ and $\Delta(\mathbf{r})$ as

$$K(\mathbf{rr}') = K_0(\mathbf{r}-\mathbf{r}') + K_1(\mathbf{rr}'); \qquad K_0(\mathbf{r}-\mathbf{r}') =< K(\mathbf{rr}') >$$
$$\Delta(\mathbf{r}) =< \Delta > +\Delta_1(\mathbf{r}) \tag{4.1}$$

where $< \Delta >$ is the solution of linearized gap equation with averaged kernel $K_0(\mathbf{r}-\mathbf{r}')$ while $\Delta_1(\mathbf{r})$ is the first order correction over the perturbation defined by $K_1(\mathbf{rr}')$. We have seen that the linearized gap equation Eq. (3.13) with the averaged kernel $K_0(\mathbf{r}-\mathbf{r}')$ determines the standard transition temperature of BCS theory given by Eq. (3.19) which we shall now denote as T_{c0}. In the first order over K_1 there is no correction to T_{c0}: $< K_1 >=0$. In the second order of this perturbation theory we obtain the following change of transition temperature, defined as the temperature of appearance of homogeneous order—parameter:

$$\frac{T_c - T_{c0}}{T_{c0}} = \frac{1}{\lambda_p}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{K_1(\mathbf{q}0)K_1(0\mathbf{q})}{1-K_0(\mathbf{q},T_c)}$$
$$K_0 = \int d\mathbf{r} e^{i\mathbf{qr}}K(\mathbf{r},T_c) \tag{4.2}$$

where

$$K_1(0\mathbf{q}) = K_1(-\mathbf{q}0) = \int d\mathbf{r}\int d\mathbf{r}' e^{i\mathbf{qr}}[K(\mathbf{rr}') - K_0(\mathbf{r}-\mathbf{r}')] =$$

$$= \lambda_p \int_0^{<\omega>} \frac{dE}{E} th \frac{E}{2T_c} \int d\mathbf{r} e^{i\mathbf{qr}} \left[\frac{1}{N(E)} \sum_\mu |\phi_\mu(\mathbf{r})|^2 \delta(E - \varepsilon_\mu) - 1 \right] \qquad (4.3)$$

Here $\lambda_p = gN(E_F)$ and we have used the completeness and orthonormality of exact eigenfunctions $\phi_\mu(\mathbf{r})$. It is obvious that correction to T_{c0} given by Eq. (4.2) is always positive. After averaging Eq.(4.2) over disorder we get the relative change of transition temperature due to fluctuations as

$$\frac{\delta T_c}{T_{c0}} = \left\langle \frac{T_c - T_{c0}}{T_{c0}} \right\rangle = \lambda_p \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{\varphi(\mathbf{q})}{1 - K_0(\mathbf{q}, T_c)}$$

$$\varphi(\mathbf{q}) = \int d\mathbf{r} e^{i\mathbf{qr}} \varphi(\mathbf{r}) \qquad (4.4)$$

where

$$\varphi(\mathbf{r}) = \int_0^{<\omega>} \frac{dE}{E} th \frac{E}{2T_c} \int_0^{<\omega>} \frac{dE'}{E'} th \frac{E'}{2T_c} \left\{ \frac{1}{N(E)} \ll \rho_E(\mathbf{r}) \rho_{E'}(0) \gg^H -1 \right\} \qquad (4.5)$$

and we have introduced the spectral density of Eq. (A.3)

$$\ll \rho_E(\mathbf{r}) \rho_{E'}(\mathbf{r}') \gg^H = \frac{1}{N(E)} < \sum_{\mu\nu} |\phi_\mu(\mathbf{r})|^2 |\phi_\nu(\mathbf{r}')|^2 \delta(E - \varepsilon_\mu) \delta(E' - \varepsilon_\nu) > \qquad (4.6)$$

which is actually a correlation function of *local* densities of states.

Remember now that in a "dirty" system [Saint-James D., Sarma G., Thomas E.J. (1969)]:

$$1 - K_0(\mathbf{q}, T) = 1 - 2\pi T \lambda_p \sum_{\varepsilon_n} \frac{1}{2|\varepsilon_n| + D_E(2|\varepsilon_n|) q^2} \approx$$

$$\approx \lambda_p \left[\frac{T - T_{c0}}{T_{c0}} + \xi^2 q^2 \right] \qquad \varepsilon_n = (2n+1)\pi T \qquad (4.7)$$

where ξ is the coherence length defined previously e.g. in Eq. (3.136). The approximate equality here is valid for $|T - T_{c0}|/T_c \ll 1, \quad \xi^2 q^2 \ll 1$. From Eq. (4.4) and Eq. (4.7) we get the change of transition temperature in the following form:

$$\frac{\delta T_c}{T_{c0}} = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{\varphi(\mathbf{q})}{\xi^2 q^2} \qquad (4.8)$$

Here we must cut—off integration at $q \sim \xi^{-1}$ in accordance with limits of applicability of the last expression in Eq. (4.7). However, the contribution of short—wave fluctuations here may be also important, so that the previous expression should be considered as an order of magnitude estimate.

The Ginzburg—Landau functional expressed via non—averaged order parameter $\Delta(\mathbf{r})$ has the following form [Gennes de P.G. (1966)]:

$$F\{\Delta\} = \int d\mathbf{r}\left\{\frac{N(E_F)}{\lambda_p}|\Delta(\mathbf{r})|^2 - N(E_F)\int d\mathbf{r}' K(\mathbf{r}\mathbf{r}')\Delta(\mathbf{r}')\Delta(\mathbf{r}) + \frac{1}{2}B|\Delta(\mathbf{r})|^4\right\} \tag{4.9}$$

where we have neglected the fluctuations of pairing interaction λ_p and of the coefficient B, which is defined by the standard expression given in Eq. (3.135). Using Eqs. (4.1)—(4.3) we can find Ginzburg—Landau equations which describe the slow changes of $\Delta(\mathbf{r})$:

$$\left\{N(E_F)\frac{T_{c0}-T}{T_{c0}} + \delta A(\mathbf{r}) - B|\Delta(\mathbf{r})|^2 + C\frac{\partial^2}{\partial r^2}\right\}\Delta(\mathbf{r}) = 0 \tag{4.10}$$

where

$$\delta A(\mathbf{r}) = N(E_F)\int_0^{<\omega>}\frac{dE}{E}th\frac{E}{2T_{co}}\left\{\frac{1}{N(E_F)}\sum_\nu|\phi_\nu(\mathbf{r})|^2\delta(E-\varepsilon_\nu) - 1\right\} \tag{4.11}$$

describes the fluctuations of the coefficient A of Ginzburg—Landau expansion and we have neglected the fluctuations of the C coefficient .

Ginzburg—Landau equations with fluctuating coefficients were analyzed for the first time by Larkin and Ovchinnikov [Larkin A.I., Ovchinnikov Yu.N. (1971)]. It was shown that $\delta A(\mathbf{r})$—fluctuations lead to a shift of transition temperature given by Eq.(4.8) and the solution of Eq. (4.10) for the order parameter in the first order over fluctuations has the form of Eq. (4.1) with:

$$\Delta_1(\mathbf{r}) = \int\frac{d^3\mathbf{q}}{(2\pi)^3}\Delta_1(\mathbf{q})e^{i\mathbf{q}\mathbf{r}}$$
$$\Delta_1(\mathbf{q}) = -\frac{<\Delta>}{N(E_F)}\frac{\delta A(\mathbf{q})}{\xi^2q^2+2\tau} \tag{4.12}$$

where $\tau = (T_c - T)/T_c$ is temperature measured relative to the new transition temperature. The mean—square fluctuation of the order—parameter itself is determined from Eq. (4.12) by:

$$\frac{<\Delta^2>}{<\Delta>^2} - 1 = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{\varphi(\mathbf{q})}{[\xi^2 q^2 + 2\tau]^2} \tag{4.13}$$

where $\varphi(\mathbf{q})$ was introduced in Eqs. (4.4), (4.5). It is important to note (as seen from (4.13)) that fluctuations of $\Delta(\mathbf{r})$ as opposed to T_c—shift are determined by small $\mathbf{q}$ behavior of $\varphi(\mathbf{q})$.

We can see now that all the physics of statistical fluctuations is described by the correlation function of local densities of states (or spectral density of Eq. (4.6)). This function was determined above in Eqs. (2.84), (2.85) within self—consistent theory of localization or by Eqs. (2.144), (2.146) which follow from from scaling approach close to the mobility edge.

Using Eq. (2.85) for the metallic state not very close to the mobility edge we can get from Eq. (4.5):

$$\varphi(\mathbf{q}=0) \sim \frac{\xi}{N^2(E_F) D_0^2} \tag{4.14}$$

where $\xi = \sqrt{\xi_0 l}$ and D_0 is the Drude diffusion coefficient. Estimating the T_c—shift from Eq. (4.8) we get:

$$\frac{\delta T_c}{T_{co}} \sim \frac{1}{N^2(E_F) D_0^2 \xi^2} \sim \frac{T_c}{E_F} \frac{1}{(p_F l)^3} \sim \tau_G \tag{4.15}$$

where τ_G is the size of Ginzburg critical region defined by Eq. (3.155). We have seen that in the usual "dirty" superconductor $\tau_G \ll 1$. For the order—parameter fluctuations from Eq. (4.13) we obtain:

$$\frac{<\Delta^2>}{<\Delta>^2} - 1 \approx \frac{1}{8\pi} \frac{\varphi(\mathbf{q}=0)}{\xi^3 \sqrt{2|\tau|}} \approx \left(\frac{\tau_D}{|\tau|}\right)^{1/2} \tag{4.16}$$

From here we can see that the width of the temperature region where statistical fluctuations are important is given by:

$$\tau_D \sim \frac{\varphi^2(0)}{\xi^6} \sim \frac{1}{N^4(E_F) D_0^4 \xi^4} \sim \left(\frac{T_c}{E_F}\right)^2 \frac{1}{(p_F l)^6} \sim \tau_G^2 \tag{4.17}$$

It is obvious that in a"dirty" superconductor we have $\tau_D \ll \tau_G \ll 1$ and statistical fluctuations are absolutely unimportant.

Situation change for a system which is close to the mobility edge. Using Eq. (2.85) with D_0 replaced by $D_0(\omega/\gamma)^{1/3}$ or Eqs. (2.144)—(2.146) we obtain:

$$\varphi(\mathbf{q}) \approx \frac{\gamma^{1/2}}{N^2(E_F)D_0^3T_c}\int\limits_0^{T_c}\frac{d\omega}{\omega^{1/2}}[\omega^2+D_0\gamma^{-2/3}\omega^{2/3}q^4]^{-1/4} \sim \xi^3 ln\frac{1}{\xi q} \quad (4.18)$$

where $\xi \sim (\xi_0 p_F^{-2})^{1/3}$. Similarly we get:

$$\frac{<\Delta^2>}{<\Delta>^2}-1 \approx \int\limits_0^{\xi^{-1}}\frac{\xi^3q^2dq}{(\xi^2q^2+2\tau)^2}ln\frac{1}{q\xi} \sim \frac{1}{\sqrt{|\tau|}}ln\frac{1}{|\tau|} \quad (4.19)$$

From Eq. (4.19) it follows that close to the mobility edge statistical fluctuations become important and even overcome thermodynamic fluctuations due to the logarithmic factor in $\varphi(\mathbf{q})$. Thus in this region we have $\tau_D > \tau_G \sim 1$.

The crossover from the regime of weak statistical fluctuations ($\tau_D \ll \tau_G$) to the strong fluctuation regime occurs at the conductivity scale $\sigma \sim \sigma^\star \approx \sigma_c(p_F\xi_0)^{-1/3}$ which was extensively discussed above. Thus close to the mobility edge the superconducting order—parameter is no more a self—averaging quantity. Here the mean—field theory approach becomes formally invalid due to thermodynamic and also because of statistical fluctuations. Below we shall analyze this situation in more details.

Finally we shall briefly discuss the region of localized states. The appearance here of a singular $\delta(\omega)$—contribution to the correlator of local densities of states given by Eqs. (A8)—(A10) leads to the additional contribution to $\varphi(\mathbf{q})$:

$$\varphi(\mathbf{q}) = \int\limits_0^{<\omega>}\frac{dE}{E^2}\left(th\frac{E}{2T_c}\right)\frac{A_E(\mathbf{q})}{N(E_F)}+\cdots \sim \frac{A_{E_F}}{N(E_F)T_{c0}}+\cdots =$$

$$= \frac{1}{N(E_F)T_c(1+R_{loc}^2q^2)}+\cdots \quad (4.20)$$

Accordingly a new contribution to $\Delta(\mathbf{r})$ fluctuations is given by:

$$\frac{<\Delta^2>}{<\Delta>^2}-1 \approx \frac{1}{N(E_F)T_c}\int\limits_0^{R_{loc}^{-1}}\frac{q^2dq}{(\xi^2q^2+2|\tau|)^2(1+R_{loc}^2q^2)} \sim \frac{1}{N(E_F)T_cR_{loc}^3\tau^2} \quad (4.21)$$

and it grows fast as the localization length R_{loc} diminishes. Using our main criterion of superconductivity in localized phase given by Eq. (3.20) we can see that in all region of possible superconductivity statistical fluctuations of $\Delta(\mathbf{r})$ remain of the order of unity and are important in rather wide temperature interval around T_c.

4.2 Superconducting Transition at Strong Disorder

We consider now superconductivity in systems with strong statistical fluctuations of the "local transition temperature" $T_c(\mathbf{r})$ as described by Eq. (4.10) and Eq. (4.11). In this analysis we shall follow Refs.[Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989)]. For simplicity we assume Gaussian nature of these fluctuations. Note, however, that close to the mobility edge the fluctuations of local density of states become strongly non—Gaussian [Lerner I.V. (1988)] and this can complicate the situation. Unfortunately the importance of this non—Gaussian behavior for superconductivity has not been studied up to now. We shall see that in our model , depending on the degree of disorder, which we shall measure by the ratio τ_D/τ_G , two types of superconducting transition are possible. For τ_D smaller than some critical value $\tau_D^{\star}$ the superconducting transition is the usual second—order phase transition at $T = T_c$. The superconducting order—parameter is in this case equal to zero for $T > T_c$ and is spatially homogeneous over scales exceeding the correlation length $\xi(T)$ below T_c. Statistical fluctuations lead only to a change of critical exponents at the transition [Khmelnitskii D.E. (1975); Lubensky T.C. (1975)]. At $\tau_D > \tau_D^{\star}$ the superconducting state appears in inhomogeneous fashion even if the correlation length of disorder induced fluctuations of $T_c(\mathbf{r})$ is small compared with the superconducting correlation length ξ (microscopic disorder). This case was first analyzed by Ioffe and Larkin [Ioffe L.B., Larkin A.I. (1981)]. Investigating the case of extremely strong disorder they have shown that as the temperature is lowered the normal phase acquires localized superconducting regions (drops) with characteristic size determined by $\xi(T)$. Far from T_c their density is low, but with further cooling the density and dimensions of the drops increase and they begin to overlap leading to a kind of percolative superconducting transition. According to our previous estimates, if we take into account only the fluctuations of local density of states, the parameter τ_D/τ_G increases from very small values to a value greater than unity as the system moves towards the mobility edge. An onset of an inhomogeneous superconducting regime is therefore to be expected as the localization transition is approached.

Our treatment of superconductors with large statistical fluctuations will be based on the Ginzburg—Landau functional:

$$F\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\} = \int d\mathbf{r}\left\{\frac{\mathbf{B}^2(\mathbf{r})}{8\pi} + N(E_F)\left[(\tau + t(\mathbf{r}))|\Delta(\mathbf{r})|^2 + \right.\right.$$
$$\left.\left. +\xi^2\left|\left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}(\mathbf{r})\right)\Delta(\mathbf{r})\right|^2 + \frac{1}{2}\lambda|\Delta(\mathbf{r})|^4\right]\right\} \quad (4.22)$$

where $\mathbf{B} = rot\mathbf{A}$ is magnetic field and we have redefined the coefficient of quartic term as $B = N(E_F)\lambda$. Here $t(\mathbf{r})$ is defined by Eq. (4.11) as $\delta A(\mathbf{r}) = N(E_F)t(\mathbf{r})$ and plays the role of the fluctuation of "local critical temperature", which appears due to fluctuations of local density of states. In general case it also can have contributions from local fluctuations of pairing interaction or other types of microscopic inhomogeneities. As noted above we assume Gaussian statistics of these fluctuations, though real situation close to the mobility edge may be more complicated [Lerner I.V. (1988)]. Given the distribution of $t(\mathbf{r})$, the free energy of the system and the order—parameter correlator are equal to:

$$\mathcal{F}\{t(\mathbf{r})\} = -T lnZ, \quad Z = \int D\{\mathbf{A},\Delta\} exp[-F\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\}/T] \quad (4.23)$$

$$< \Delta(\mathbf{r})\Delta(\mathbf{r}') >= Z^{-1}\int D\{\mathbf{A},\Delta\}\Delta(\mathbf{r})\Delta(\mathbf{r}') exp[-F\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\}/T] \quad (4.24)$$

and must be averaged over the Gaussian distribution of $t(\mathbf{r})$. From our definition of $t(\mathbf{r})$ and using the approach of the previous section, assuming the short—range of fluctuations of local density of states (on the scale of ξ), it is easy to estimate the correlator of $t(\mathbf{r})$ as:

$$< t(\mathbf{r})t(\mathbf{r}') >= \gamma\delta(\mathbf{r}-\mathbf{r}'), \qquad \gamma \approx \tau_D^{1/2}\xi^3 \quad (4.25)$$

Then the probability of a configuration with a given $t(\mathbf{r})$ is given by

$$\mathcal{P}\{t(\mathbf{r})\} = exp\left[-\frac{1}{2\gamma}\int d\mathbf{r} t^2(\mathbf{r})\right] \quad (4.26)$$

The problem reduces thus to calculation of the functional $\mathcal{F}\{t(\mathbf{r})\}$ and correlator $< \Delta(\mathbf{r})\Delta(\mathbf{r}') >$ with subsequent averaging over $\mathcal{P}\{t(\mathbf{r})\}$.

We shall limit ourselves to consideration of noninteracting drops and no vortices. Then we can consider the phase of the order—parameter $\Delta(\mathbf{r})$ as nonsingular. After the gauge transformation

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &\to \mathbf{A}(\mathbf{r}) + (c\hbar/2e)\nabla\phi(\mathbf{r}) \\ \Delta(\mathbf{r}) &\to \Delta(\mathbf{r})exp[-i\phi(\mathbf{r})] \end{aligned} \tag{4.27}$$

where $\phi(\mathbf{r})$ is the phase of the order parameter we can use real $\Delta(\mathbf{r})$ and Ginzburg—Landau functional of Eq. (4.22) becomes:

$$\begin{aligned} F\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\} = \int d\mathbf{r} \Bigg\{ \frac{\mathbf{B}^2(\mathbf{r})}{8\pi} &+ N(E_F)\left[(\tau + t(\mathbf{r}))\Delta^2(\mathbf{r})\right. \\ &\left. + \frac{4e^2\xi^2}{c^2\hbar^2}\mathbf{A}^2(\mathbf{r})\Delta^2(\mathbf{r}) + \xi^2(\nabla\Delta(\mathbf{r}))^2 + \frac{1}{2}\lambda\Delta^4(\mathbf{r})\right]\Bigg\} \end{aligned} \tag{4.28}$$

Integration over phase of the order parameter in Eq. (4.23) gives an inessential constant factor to the partition function which we disregard.

To average the logarithm of the partition function Eq. (4.23) over $t(\mathbf{r})$ we can use the so called replica trick [Grinstein G., Luther A. (1976)] widely used in the theory of disordered systems, which permits the averaging to be carried out in explicit form. We express the average free energy Eq.(4.23) of the system in the form:

$$< \mathcal{F} >= -T \lim_{n\to 0} \frac{1}{n}[< Z^n > -1] \tag{4.29}$$

To calculate $< Z^n >$ in accordance with the idea of the replica method, we first assume n to be an arbitrary integer. Expressing Z^n in terms of an n—fold functional integral over the fields of the replicas $\mathbf{A}_\alpha$, $\Delta_\alpha(\mathbf{r})$, $\alpha = 1,...,n$ and carrying out exact Gaussian averaging over $t(\mathbf{r})$, we get

$$< Z^n >= \int D\{\mathbf{A},\Delta\}exp[-S_n\{\mathbf{A}_\alpha,\Delta_\alpha\}] \tag{4.30}$$

$$\begin{aligned} S\{\mathbf{A}_\alpha,\Delta_\alpha\} = \int d\mathbf{r} \Bigg\{ \sum_\alpha^n \frac{\mathbf{B}^2(\mathbf{r})}{8\pi T} + \frac{N(E_F)}{T}\sum_\alpha^n \left[\tau\Delta_\alpha(\mathbf{r})^2\right. \\ \left. + \frac{4e^2\xi^2}{c^2\hbar^2}\mathbf{A}_\alpha^2(\mathbf{r})\Delta_\alpha^2(\mathbf{r}) + \xi^2(\nabla\Delta_\alpha(\mathbf{r}))^2 + \frac{1}{2}\lambda\Delta_\alpha^4(\mathbf{r})\right] - \frac{1}{2}\frac{N(E_F)}{T}\tilde{\gamma}\left[\sum_{\alpha=1}^n \Delta_\alpha^2(\mathbf{r})\right]^2 \Bigg\} \end{aligned}$$

The last expression here represents the "effective action" and parameter $\tilde{\gamma} = \gamma N(E_F)/T_c \approx \tau_D^{1/2} N(E_F)/T_c$ grows with disorder . Note that the random quantities $t(\mathbf{r})$ have already dropped out of these expressions, and that the action $S\{\mathbf{A}_\alpha, \Delta_\alpha\}$ is translationally invariant. For the correlator of Eq. (4.24) we obtain:

$$< \Delta(\mathbf{r})\Delta(\mathbf{r}') >= \lim_{n\to 0} \frac{1}{n} \int D\{\mathbf{A}, \Delta\} exp[-S_n\{\mathbf{A}_\alpha, \Delta_\alpha\}] \sum_{\alpha=1}^{n} \Delta_\alpha(\mathbf{r})\Delta_\alpha(\mathbf{r}') \tag{4.31}$$

where we have symmetrized over the replica indices.

Far from the region of strong fluctuations of the order parameter $|\tau| \gg \tau_D, \tau_G$ the functional integrals in Eq. (4.30) and Eq. (4.24) can be calculated by the saddle—point method. The extrema of the action (in absence of external magnetic field) are determined by classical equations:

$$\left[\tau - \xi^2\nabla^2 + \lambda\Delta_\alpha^2 - \tilde{\gamma}\sum_{\beta=1}^{n}\Delta_\beta^2(\mathbf{r})\right]\Delta_\alpha(\mathbf{r}) = 0 \qquad \mathbf{A}_\alpha = 0 \tag{4.32}$$

The nontrivial fact is that these equations (besides spatially homogeneous and trivial solutions $\Delta_\alpha(\mathbf{r}) = 0$) do have localized solutions with finite action (*instantons*) *. These correspond at $\tau > 0$ to superconducting drops. We shall limit ourselves to a picture of noninteracting drops and consider only instanton solutions above T_c (at $\tau > 0$). We shall be interested only in those solutions that admit analytic continuation as $n \to 0$. We designate them $\Delta_\alpha^{(i)}(\mathbf{r})$, where the superscript i labels the type of solution. These finite action solutions (instantons) of Eq.(4.32) exist for $\tau > 0$ and for $\tilde{\gamma} > \lambda$ (besides the trivial solution $\Delta_\alpha = 0$) and have the following form (Cf. Refs.[Cardy J.L. (1978); Sadovskii M.V. (1979); Sadovskii M.V. (1986)] and Appendix E):

$$\begin{aligned} &\Delta_\alpha^{(i)}(\mathbf{r}) = \Delta_0(\mathbf{r})\delta_{\alpha i}, \qquad i = 1, ..., n \\ &\Delta_0(\mathbf{r}) = \sqrt{\tfrac{\tau}{\tilde{\gamma}-\lambda}}\chi\left[\tfrac{r}{\xi(T)}\right], \qquad \xi(T) = \tfrac{\xi}{\sqrt{\tau}} \end{aligned} \tag{4.33}$$

where the dimensionless function $\chi(x)$ satisfies the condition $d\chi(x)/dx|_{x=0} = 0$ and its asymptotic form: $\chi(x) \sim x^{-1}exp(-x)$ for $x \gg 1$ (for spatial dimension $d = 3$). The qualitative form of this solution is shown in Fig. 4.1.

*Good introduction into the concept and physics of instantons can be found in Refs.[Coleman S. (1979); Rajaraman R. (1982); Vainshtein A.I. et al. (1982)].

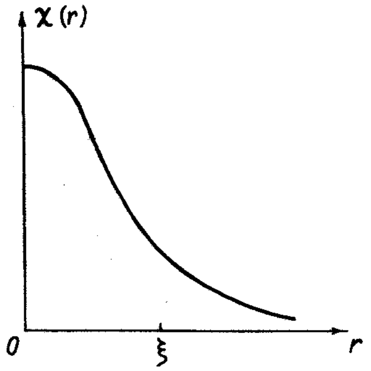

Fig. 4.1 Qualitative form of instanton solution.

The size of instanton (superconducting drop) is effectively determined by coherence length $\xi(T)$. From Eq. (4.33) it is seen that instantons are oriented along axes of replica space (there are n types of instanton solutions) which is due to the "cubic anisotropy" term $\lambda\Delta_\alpha^4$ in the effective action of Eq. (4.30). Index i characterizes the direction in replica space along which the symmetry breaking takes place. For $\lambda \to 0$ the action becomes $O(n)$—symmetric and instantons take the form :

$$\Delta_\alpha(\mathbf{r}) = \Delta_0(\mathbf{r})e_\alpha, \qquad \sum_{\alpha=1}^{n} e_\alpha^2 = 1 \tag{4.34}$$

i.e. are oriented along arbitrary unit vector $\vec{e}$ in replica space. Such instantons earlier were studied in the theory of localization [Cardy J.L. (1978); Sadovskii M.V. (1979); Sadovskii M.V. (1986)].

To find instanton contribution different physical quantities (e.g. partition function or correlators) we must expand the action of Eq. (4.30) up to the terms quadratic in deviations $\varphi_\alpha(\mathbf{r}) = \Delta_\alpha(\mathbf{r}) - \Delta_\alpha^{(i)}(\mathbf{r})$. It can be shown that fluctuations of the fields $\mathbf{A}_\alpha(\mathbf{r})$ can be neglected if we consider non-interacting drops [Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989)]. This quadratic expansion of the effective action near instanton solution takes the form (Cf. analogous treatment in Refs.[Cardy J.L. (1978); Sadovskii M.V. (1979);

Sadovskii M.V. (1986)]):

$$S\{\Delta_\alpha\} = S\{\Delta_\alpha^{(i)}\} + \frac{1}{2}\int d\mathbf{r}\sum_{\alpha,\beta}(\varphi_\alpha \hat{M}_{\alpha\beta}^{(i)}\varphi_\beta) \tag{4.35}$$

where the operator $M_{\alpha\beta}^{(i)}$ on instanton solutions is equal to:

$$\hat{M}_{\alpha\beta}^{(i)} = [\hat{M}_L\delta_{\alpha i} + \hat{M}_T(1-\delta_{\alpha i})]\delta_{\alpha\beta} \tag{4.36}$$

with

$$M_{L,T} = \frac{2N(E_F)}{T}[-\xi^2\nabla^2 + \tau U_{L,T}(\mathbf{r})] \tag{4.37}$$

where

$$\begin{aligned} U_L(\mathbf{r}) &= 1 - 3\chi^2[r/\xi(T)] \\ U_T(\mathbf{r}) &= 1 - (1-\lambda/\tilde{\gamma})^{-1}\chi^2[r/\xi(T)] \end{aligned} \tag{4.38}$$

The value of Gaussian functional integral over φ_α is determined by the spectra of eigenstates of operators M_L and M_T. Actually, to calculate this integral we have to expand φ_α over normalized eigenfunctions of these operators:

$$\begin{aligned} \varphi_\alpha(\mathbf{r}) &= \sum_k c_k\varphi_{k\alpha}(\mathbf{r}) \\ \sum_\beta M_{\alpha\beta}^i\varphi_{k\beta} &= \varepsilon_k\varphi_{k\alpha} \end{aligned} \tag{4.39}$$

Inserting this to (4.35) gives:

$$S\{\Delta_\alpha\} = S\{\Delta_\alpha^{(i)}\} + \frac{1}{2}\sum_k c_k^2\varepsilon_k \tag{4.40}$$

Then the Gaussian integral over φ_α is calculated with the following change of variables:

$$\int D\{\varphi_\alpha\}... = \prod_k \int \frac{dc_k}{(2\pi)^{1/2}}... \tag{4.41}$$

and its value is obviously defined by eigenvalues of $M^{(i)}$. Detailed analysis can be found in Refs.[Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989)]. The qualitative form of the spectra of M_L and M_T is shown in Fig. 4.2.

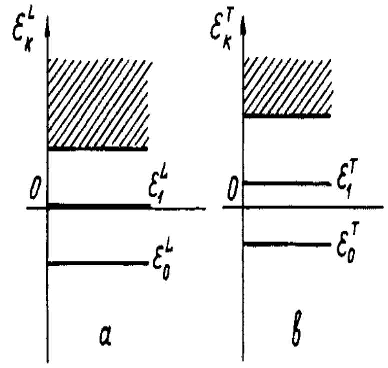

Fig. 4.2 Qualitative structure of eigenvalues of M_L (a) and M_T (b) operators. $\varepsilon_1^L = 0$ — translation zero—mode $\varepsilon_0^T \to 0$ for $\lambda \to 0$—transforms to "rotation" zero—mode. The continuous part of the spectrum is shaded.

Operator M_L always possesses an eigenvalue $\varepsilon_1^L = 0$—the so called translation zero—mode, connected with translation symmetry: instanton center may be placed anywhere in space, the action does not change . However, this is not a lowest eigenvalue of M_L, there is always a negative eigenvalue $\varepsilon_0^L < \varepsilon_1^L = 0$. It can be shown rigorously that it is the only negative eigenstate of M_L [Zinn-Justin J. (1981)]. Operator M_T possesses also a single negative eigenvalue $\varepsilon_0^T < 0$ [Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989)], however this eigenvalue tends to zero for $\lambda \to 0$ becoming the "rotation" zero—mode, reflecting the arbitrary "direction" of instanton in replica space in the absence of cubic anisotropy in the action [Cardy J.L. (1978); Sadovskii M.V. (1979); Sadovskii M.V. (1986)]. For $\lambda = \lambda^\star = 2/3\tilde{\gamma}$ we have $M_L = M_T$ and the spectra of both operators coincide.

Including the contributions of instantons oriented along all the axes in replica space we obtain the following one—instanton contribution to the partition function entering Eq. (4.29) [Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989)]:

$$< Z^n >= n\Omega \left(\frac{J_L}{2\pi}\right)^{d/2} [Det' M_L]^{-\frac{1}{2}} [Det M_T]^{\frac{1-n}{2}} exp\{-S_0(\tau)\} \qquad (4.42)$$

where Ω is the system volume,

$$J_L = \frac{1}{d}\int dr \left(\frac{\partial \Delta_0}{\partial r}\right)^2 = \frac{T}{2N(E_F)}\frac{S_0(\tau)}{\xi^2} \tag{4.43}$$

and the action at the instanton is given by:

$$S_0(\tau) = \mathcal{A}\frac{\xi^3\tau^{1/2}}{\gamma - \lambda T/N(E_F)} \tag{4.44}$$

where $\mathcal{A} \approx 37.8$ is a numerical constant [Lifshits I.M., Gredeskul S.A., Pastur L.A. (1982)]. The prime on $DetM_L$ means that we must exclude the zero—eigenvalue $\varepsilon_1^L = 0$ from the product of eigenvalues determining this determinant. The condition of applicability of the saddle—point approximation looks like $S_0(\tau) \gg 1$, and in fact all our analysis is valid outside the critical regions both for thermodynamic and statistical fluctuations.

In the limit of $n \to 0$ the total cancellation of imaginary contributions appearing due to negative eigenvalues takes place in Eq. (4.42) and using Eq. (4.29) we get for $\tilde{\gamma} > 3/2\lambda$ the following *real* contribution to the free energy:

$$\mathcal{F} = -\rho_s(\tau)T\Omega \tag{4.45}$$

where the density of superconducting "drops"

$$\rho_s(\tau) = \left[\frac{T}{4\pi N(E_F)}S_0(\tau)\right]^{3/2}\xi^{-3}\left[\frac{DetM_T}{Det'M_L}\right]^{1/2} exp\{-S_0(\tau)\} \tag{4.46}$$

Thus for $\tilde{\gamma} > 3/2\lambda$ even for $T > T_c$ the superconducting "drops" (instantons) appear in the system which directly contribute to the equilibrium free energy. This contribution given by Eqs. (4.45)— (4.46) exists along the usual thermodynamic fluctuations. The condition of $\tilde{\gamma} > 3/2\lambda$ defines critical disorder $\tau_D > \tau_D^\star > \tau_G$, and this inhomogeneous picture of superconducting transition appears only for the case of sufficiently strong statistical fluctuations. The knowledge of qualitative structure of spectra of eigenvalues of M_L and M_T allows to analyze different asymptotics of Eq. (4.45) [Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989)]. For $\tilde{\gamma}S_0(\tau) \ll \lambda \ll \lambda^\star$ we get:

$$\rho_s(\tau) \approx \xi^{-3}(T)\left(\frac{\lambda}{\tilde{\gamma}}\right)^{1/2} S_0^{3/2}(\tau)exp[-S_0(\tau)] \tag{4.47}$$

For $\lambda \to \lambda^\star$ we obtain:

$$\rho_s(\tau) \approx \xi^{-3}(T)\left(\frac{\lambda^\star}{\lambda} - 1\right)^{3/2} S_0^{3/2}(\tau) exp[-S_0(\tau)] \tag{4.48}$$

Thus the density of superconducting "drops" $\rho_s(\tau)$ vanishes as $\lambda \to \lambda^\star$, they are destroyed by thermodynamic fluctuations.

For the order—parameter correlator of Eq. (4.24) we get the following result:

$$< \Delta(\mathbf{r})\Delta(\mathbf{r}') > \approx \rho_s(\tau) \int d\mathbf{R_0}\Delta_0(\mathbf{r} + \mathbf{R_0})\Delta_0(\mathbf{r}' + \mathbf{R_0}) \tag{4.49}$$

The integration over instanton center $\mathbf{R_0}$ here means in fact averaging over different positions of "drops". Note that over large distances this correlator decreases like $exp[-|\mathbf{r} - \mathbf{r}'|/\xi(T)]$ and does not contain the usual Ornstein—Zernike factor $|\mathbf{r} - \mathbf{r}'|^{-1}$.

To understand the physical picture obtained above on more elementary level note, that for a given configuration of $t(\mathbf{r})$, defining local values of transition temperature, superconducting drops may appear only in the regions where this temperature is higher. Let us number these regions by index i. In each of these regions, the order parameter is defined by nontrivial localized solution $\Delta_d^{(i)}$ of Ginzburg-Landau equations, so that the contribution of such drop to partition function is given by:

$$N^i\{t(\mathbf{r})\} exp\left(-\frac{E_d^{(i)}}{T}\right)$$
$$E_d^{(i)}\{t(\mathbf{r})\} = F\{0, \Delta_d^{(i)}(\mathbf{r})\} \tag{4.50}$$

where $E_d^{(i)}$ is the energy of the drop, defined in fact by (4.28). Preexponential factor N^i is determined by configurations of $\Delta(\mathbf{r})$ which are close to classic configuration given by $\Delta_d^{(i)}(\mathbf{r})$. Summing over configurations with arbitrary number ob drops and neglecting their interaction with each other, we can obtain the following form of partition function in (4.23):

$$Z = Z_0\left[1 + \sum_i N^{(i)} exp\left(-\frac{E_d^{(i)}}{T}\right) + \right.$$
$$\left. + \frac{1}{2!}\sum_{ij} N^{(i)}N^{(j)} exp\left(-\frac{E_d^{(i)} + E_d^{(j)}}{T}\right) + ...\right] =$$

$$= Z_0 exp\left[\sum_i N^{(i)} exp\left(-\frac{E_d^i}{T}\right)\right] \tag{4.51}$$

where Z_0 is partition function of the system without drops. Substituting (4.51) into (4.23) and performing the averaging of the free energy over random configurations of $t(\mathbf{r})$ with (4.26), we obtain:

$$\mathcal{F} = -\frac{T}{N}\int D\{t(\mathbf{r})\}\sum_i N^{(i)}\{t(\mathbf{r})\} exp\left(-\frac{F_d^{(i)}\{t(\mathbf{r})\}}{T}\right) \tag{4.52}$$

where N is some normalization factor, while F_d is the free energy of a single drop:

$$F_d\{t(\mathbf{r})\} = E_d\{t(\mathbf{r})\} - T\ln P\{t(\mathbf{r})\} \tag{4.53}$$

Main contribution to the functional integral in (4.52) comes from configurations $t_0(\mathbf{r})$ corresponding to the extremum of (4.53):

$$\begin{aligned} t_0(\mathbf{r}) &= -\tilde{\gamma}\Delta_d^2(\mathbf{r}) \\ \tilde{\gamma} &= \gamma N(E_F)/T_c \end{aligned} \tag{4.54}$$

Note that $t_0(\mathbf{r})$ is negative, which corresponds to higher value of local transition temperature in the region where drop appears. Using (4.54) in Ginzburg-Landau equations following from (4.22) we immediately obtain nonlinear equation, determining $\Delta_d(\mathbf{r})$ of superconducting drop, which in fact is equivalent to Eq. (4.32) with (4.33). Superconducting drops are localized on the length scale of the order of $\xi(T)$. Instanton solutions, along with (4.54), define the free energy (action) of the drop given in (4.44). In fact, these simple relations define the free energy (4.52) with exponential accuracy. Rather complicated instanton formalism used above is necessary only to calculate preexponential factors.

We have found the free—energy of inhomogeneous superconducting state in the temperature region $\tau \gg \tau_D$, where the "drop" concentration is exponentially small and the picture of noninteracting "drops" is valid. They give exponentially small contribution to the specific heat and diamagnetic susceptibility. The characteristic size of "drops" is determined by $\xi(T)$ and as $T \to T_c$ the "drops" grow and begin to overlap leading to a percolative superconducting transition. Thus for $\tau_D > \tau_D^\star > \tau_G$ superconductivity first appears in isolated "drops". This is similar to the picture of decay of a metastable state in case of the first—order phase transitions [Langer J.S.

(1967)]. However, in this latter case instantons give imaginary contribution to the free energy determining the decay rate of a "false" equilibrium state (critical bubble formation). Here instanton contributions lead as was noted above to real free energy and "drops" appear in the true equilibrium state.

It is more or less obvious that between isolated "drops" a kind of Josephson coupling may appear and lead to rather complicated phase diagram of the system in external magnetic field, e.g. including the "superconducting glass" phase [John S., Lubensky T.C. (1985); John S., Lubensky T.C. (1986)]. The existence of inhomogeneous regime of superconductivity will obviously lead to the rounding of BCS—like singularities of the density of states and superconductivity may become gapless. Note that diffusion—enhanced Coulomb interactions can also lead to the gaplessness of strongly disordered superconductors via Coulomb—induced inelastic scattering [Browne D.A., Levin K., Muttalib K.A. (1987)]. Fluctuation conductivity in a similar inhomogeneous superconducting state was studied in Ref.[Char K., Kapitulnik A. (1988)]. Note the closely related problem of strongly disordered superfluids [Ma M., Halperin B.I., Lee P.A. (1986); Fisher D.S., Fisher M.P.A. (1988)]. Some results of this work may be quite useful for the case of strongly disordered superconductors, though the limitations of this analogy are also important.

A major unsolved problem here is the possible influence of statistical fluctuations of gradient term coefficient in Ginzburg—Landau expansion which has been neglected above, or the equivalent problem (Cf. Eq.(3.143)) for superconducting electron density n_s. This problem was briefly considered for the case of weak disorder in Ref.[Spivak B., Zyuzin A. (1988)]. It was shown that:

$$< (\delta n_s/n_s)^2 > \sim (\xi_0 p_F^2 l)^{-1} \sim \frac{e^4}{g(\xi)} \tag{4.55}$$

where $g(\xi) = \sigma\xi$ is the conductunce of metallic sample with the size of the order of superconducting coherence length $\xi = \sqrt{\xi_0 l}$. Extrapolating this estimate up to the Anderson transition using $\xi = (\xi_0/p_F^2)^{1/3}$ we get:

$$< (\delta n_s/n_s)^2 > \sim \frac{e^4}{\sigma^2(\xi_0/p_F^2)^{2/3}} \tag{4.56}$$

Obviously we get $< (\delta n_s/n_s)^2 > \geq 1$ for $\sigma \leq \sigma^\star$ so that statistical fluctuations of n_s become important close to the Anderson transition in the same region we have discussed above. This further complicates the picture of superconducting transition and can also be very important for the possible anomalous behavior of H_{c2} which was recently studied on the similar lines in Ref.[Spivak B., Zhou Fei (1995)]. Some qualitative conjectures for the case of $< (\delta n_s/n_s)^2 > \geq 1$ were formulated in Ref.[Spivak B., Kivelson S.A. (1991)], where it was argued that in this case there will occur regions in the sample with locally *negative* values of superfluid density. This is equivalent to a negative sign of a Josephson coupling betweenthe "drops". In this sense, the disordered superconductor is unlike a Bose liquid. This leads to an important prediction that in a small superconducting ring, if there is a segment with negative n_s, the ground state of the ring will spontaneously break the time—reversal invariance. The ground state will have nonzero supercurrent and magnetic flux (or rather random, trapped fluxes in the ground state) and will be two—fold degenerate. At longer times the symmetry will be restored due to thermal activation of macroscopic quantum tunneling between the two states, but according to Ref.[Spivak B., Kivelson S.A. (1991)] it can be expected that for dirty metal rings with conductance of the order of $e^2/\hbar$ there will be "roughly 50% chance that the ground state will break time—reversal symmetry". By the way this means that in the presence of disorder there may be no way to distinguish between an anyon superconductor [Laughlin R.B. (1988)] and a conventional superconductor. Of course we must stress that these speculations are entirely based upon a simple extrapolation of Eq.(4.55) to the vicinity of metal—insulator transition and there is no complete theory of statistical fluctuations of gradient term in this region at the moment.

The physical picture qualitatively very similar to that described above was obtained recently for $T = 0$ in [Ghosal A., Randeria M., Trivedi N. (1998)], where the effects of non-magnetic impurities on two-dimensional s–wave superconductors were studied beyond the weak disorder by numerical solution of Bogoliubov – de Gennes (BDG) equations. It was shown that the local pairing amplitude develops a broad distribution with significant weight near zero with increasing disorder. However, the density of states continued to show a finite spectral gap. This persistence of the spectral gap was shown to arise from the break up of the system into superconducting "islands". Superfluid density and off-diagonal correlations showed substantial reduction at high disorder.

Chapter 5

SUPERCONDUCTIVITY IN STRONGLY DISORDERED METALS: EXPERIMENT

Our review of experiments on strongly disordered superconductors will be in no sense exhaustive. This is mainly a theoretical book and the author is in no way an expert on experiment. However, we shall try to illustrate the situation with the interplay of Anderson localization and superconductivity in *bulk* (three—dimensional) superconductors, both traditional and high—temperature. Again we must stress that we practically exclude any discussion of numerous data on thin films which are to be described by two—dimensional theories. In this case we just refer to existing reviews [Ramakrishnan T.V. (1989); Belevtsev B.I. (1990); Liu Y., Goldman A.M. (1994)]. Here we shall confine ourselves to a limited number of the experiments , which we consider most interesting from the point of view of illustration of some of the ideas expressed above, just to convince the reader, that previous discussion, while purely theoretical, has something to do with the real life. More than anywhere else in this book our choice of material is based on personal interests of the author, or his direct involvement in the discussion of experiments. We shall not deal with the general problem of disorder influence upon superconductivity, but shall consider only the systems which remain superconducting close to the disorder–induced metal–insulator transition.

5.1 Traditional Superconductors

There exists a number of strongly disordered systems which remain superconducting close to the metal—insulator transition induced by disorder.

The drop of T_c with conductivity decrease from the value of the order of

$10^4 Ohm^{-1}cm^{-1}$ was observed in amorphous alloys of $GeAl$ [Nichida N. et al. (1982)], $SiAu$ [Furubayashi T. (1985)] and $MoRe$ [Tenhover M., Johnson W.L., Tsuei C.C. (1981)], in Chevrel phase superconductors disordered by fast neutron irradiation, such as $Pb_{1-x}U_xMo_6S_8$ [Alekseevskii N.E. et al. (1983)], $SnMo_6S_8$ [Davydov S.A. et al. (1983)], Mo_6Se_8 [Davydov S.A., Arkhipov V.E., Goshchitskii B.N. (1986)], in amorphous InO_x [Fiory A.T., Hebard A.F. (1984)], in $BaPb_{1-x}Bi_xO_3$ in the concentration interval $0.25 < x < 0.30$ [Batlogg B. (1984)] and in metallic glass $Zr_{0.7}Ir_{0.3}$ [Poon S.J. et al. (1985)]. In all of these systems superconducting transition is observed apparently not very far from the metal—insulator transition. For a number of these systems, such as $Pb_{1-x}U_xMo_6S_8$, $SnMo_6S_8$, Mo_6Se_8, $Zr_{0.7}Ir_{0.3}$ and $BaPb_{0.75}Bi_{0.25}O_3$ and some others a characteristic strongly negative temperature resistivity coefficient has been observed. Note, however, that this fact alone in no way indicates that a specimen is on one side or the other of the metal—insulator transition. The drop of T_c close to the mobility edge apparently was also observed in As_2Te_3 [Berman I.V. et al. (1986)]. In all of these systems T_c apparently vanishes before metal—insulator transition. Below we present some of the data on these and other similar systems.

On Fig.5.1 we show the dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in $SnMo_5S_6$ (Chevrel phase superconductor) on the fluence of fast neutron irradiation (the number of neutrons which passed through a crossection of a sample during irradiation)[Davydov S.A. et al. (1983)]. In the region of large fluences (large disorder), when the system becomes amorphous, characteristic values of conductivity in the normal state are of the order of $\sim 10^3 Ohm^{-1}cm^{-1}$, which is not far from the values of "minimal metallic conductivity" $\sigma_c \sim 5\ 10^2\ Ohm^{-1}cm^{-1}$, which define the conductivity scale of disorder induced metal—insulator transition. The negative temperature coefficient of resistivity was observed in this conductivity range. The experimental data on T_c decrease with the growth of resistivity in this system were rather well fitted in Ref.[Bulaevskii L.N., Sadovskii M.V. (1985)] using the $\mu^\star$ dependence on resistivity given by Eq. (3.63). A clear tendency for $|dH_{c2}/dT|_{T_c}$ saturation with disorder is also observed. Analogous dependence of T_c and $|dH_{c2}/dT|_{T_c}$ on the resistivity in the normal state for Mo_6Se_8 disordered by fast neutrons is shown in Fig.5.2[Davydov S.A., Arkhipov V.E., Goshchitskii B.N. (1986)]. Here superconductivity exists up to conductivities $\sigma \sim 250\ Ohm^{-1}cm^{-1}$. Further disordering (irradiation)

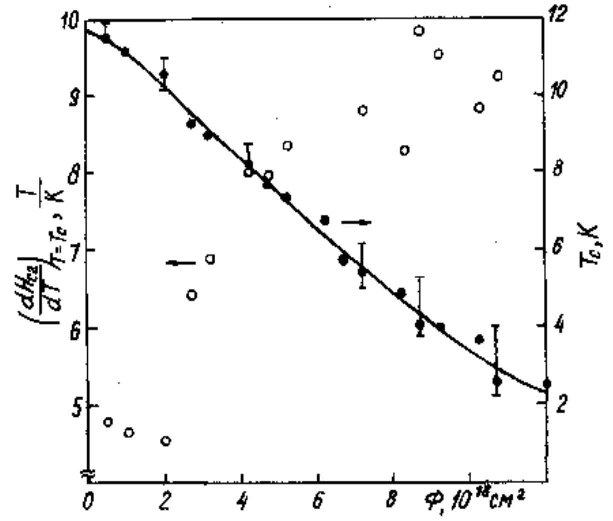

Fig. 5.1 Fluence dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in $SnMo_5S_6$.

leads to the destruction of superconducting state and metal—insulator transition (an unlimited growth of resistivity with decrease of T, with variable—range hopping conduction [Mott N.F. (1974); Mott N.F, Davis E.A. (1979)] is observed). The slope of the upper critical field $|dH_{c2}/dT|_{T_c}$ also has a tendency to saturate with the growth of resistivity. Standard interpretation of such behavior of $|dH_{c2}/dT|_{T_c}$ was based upon the use of Gorkov's relation (Cf. first relation in Eq. (3.162)) and lead to the conclusion of $N(E_F)$ decrease under disordering. In fact, we have seen that no such conclusion can be reached for systems with conductivities $\sigma < 10^3 Ohm^{-1}cm^{-1}$, because such saturation behavior may be a natural manifestation of the approaching metal—insulator transition. Similar dependencies were observed in other Chevrel phase superconductors [Alekseevskii N.E. et al. (1983); Goshchitskii B.N., Arkhipov V.E., Chukalkin Yu.G. (1987); Aleksandrov A.S. et al.(1989)].

In Fig.5.3 we show the dependence of conductivity and T_c on the parameter $p_F l/\hbar$ in amorphous InO_x alloy [Fiory A.T., Hebard A.F. (1984)]. In Fig.5.4 from Ref.[Hebard A.F., Paalanen M. (1984)] the data on the temperature dependence of H_{c2} in amorphous In/InO_x (bulk) films are presented for different degrees of disorder. We can see that in the low

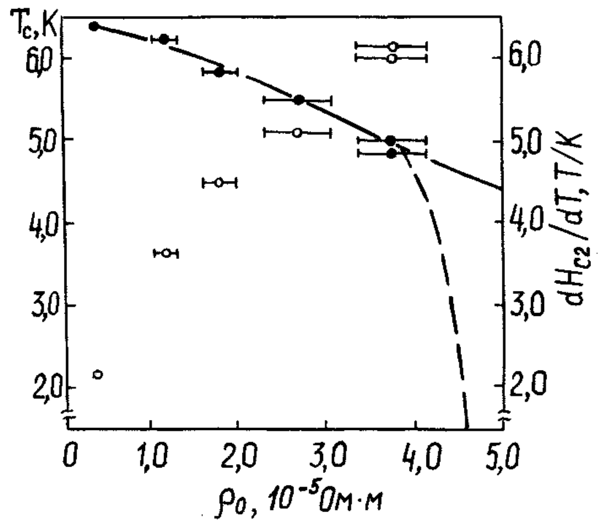

Fig. 5.2 Resistivity dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in Mo_6Se_8.

temperature region $H_{c2}(T)$ deviates from the standard temperature dependence, apparently confirming the qualitative form predicted above for systems which are close to Anderson transition. The same system was also studied in Ref.[Shahar D., Ovadyahu Z. (1992)]. In Fig.5.5 we show the dependence of two characteristic energies on disorder which in the opinion of the authors of Ref.[Shahar D., Ovadyahu Z. (1992)] demonstrate the narrow region of coexistence of superconductivity and insulating state. In Fig.5.6 we show the dependencies of localization length and superconducting coherence length on disorder according to Ref.[Shahar D., Ovadyahu Z. (1992)]. It demonstrates the qualitative agreement with our general criterion of coexistence of superconductivity and localization—localization length must be larger or at least of the order of the size of the Cooper pair.

Very impressive are the data for amorphous $Si_{1-x}Au_x$ alloy [Nichida N. et al. (1982); Furubayashi T. (1985); Furubayashi T. et al. (1986)]. In Fig.5.7 [Furubayashi T. (1985)] the data on T_c and conductivity dependence on the gold concentration x are shown. In Fig.5.8 $H_{c2}(T)$ dependence for this system is shown for different alloy compositions [Furubayashi T. (1985)]. From these data it is clearly seen that T_c vanishes *before* metal—insulator transition. The metal—insulator transition itself is *continuous*,

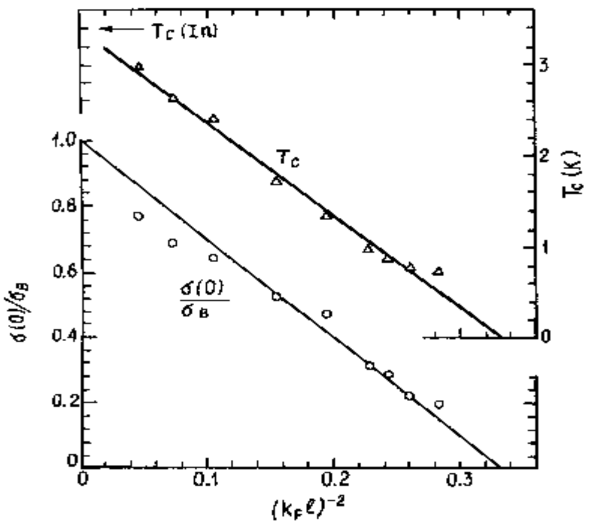

Fig. 5.3 Conductivity σ and T_c dependence on the parameter $p_F l/\hbar$ in amorphous InO_x. σ_B is estimated Drude conductivity.

conductivity vanishes linearly with the decrease of gold concentration and the values of conductivity significantly less than the estimated "minimal metallic conductivity" are definitely observed. The system remains superconducting even for such low conductivity values. The slope of $H_{c2}(T)$ at $T = T_c$ is practically constant , despite the change of conductivity (disorder) in rather wide range. This behavior apparently cannot be explained only by the appearance of correlation pseudogap in the density of states observed in Ref.[Furubayashi T. et al. (1986)], which becomes significant only very close to metal—insulator transition. Low temperature deviation from standard convex dependence on T is also clearly seen. In Fig.5.9 from Ref.[Furubayashi T. et al. (1986)] we show the temperature dependencies of resistivity and superconducting energy gap (determined by tunneling) of a sample with $x = 0.21$. It nicely demonstrates superconducting transition in the system which is very close to disorder induced metal—insulator transition. Note, that according to Ref.[Furubayashi T. et al. (1986)] the superconducting energy gap in this sample is substantially broadened which may indicate the growth of statistical gap fluctuations due to the same fluctuations of the local density of states. These data are in obvious qualitative correspondence with the general theoretical picture described throughout

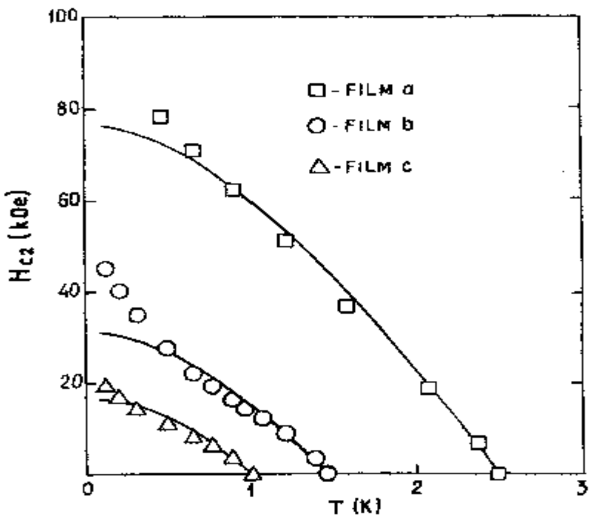

Fig. 5.4 $H_{c2}(T)$ in amorphous films of In/InO_x. Lines show standard theoretical dependencies.

this book.

We can see that in systems which are superconducting close to the disorder induced metal—insulator (Anderson) transition T_c decreases rather fast and practically in all reliable cases vanishes before the transition to insulating state. At the same time the temperature dependence of H_{c2} is not described by the standard theory of "dirty" superconductors both with respect to $(dH_{c2}/dT)_{T_c}$ behavior and at low temperatures, where the upward deviations from the standard dependence are readily observed. This confirms most of our theoretical conclusions.

Some indications of a possible superconducting state in the insulating phase of granular Al and $Al - Ge$ were observed in Refs.[Chui T. et al. (1981); Shapira Y., Deutscher G. (1983)]. Obviously, the granular systems are more or less outside the scope of our analysis. However, we should like to mention that the strong smearing of BCS—like density of states and the gapless regime of superconductivity was observed (via tunneling measurements) in Refs.[Dynes R.C. et al. (1984); White A.E., Dynes R.C., Garno J.P. (1986)], close to the metal—insulator transition in these systems. This may confirm our picture of statistical fluctuation smearing of the density of states. Note, that more recent work on granular Al [Miller T.A. et al.

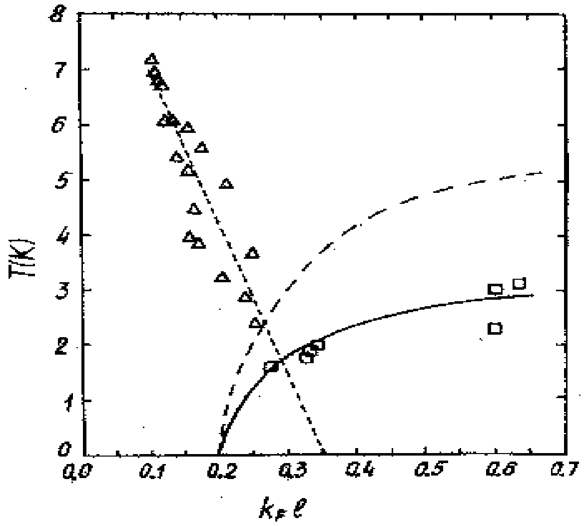

Fig. 5.5 The dependence of activation energy of hopping conductivity (triangles) and superconducting transition temperature T_c (squares) in amorphous films of In/InO_x on disorder parameter $p_F l/\hbar$ as determined from room-temperature conductivity and Hall measurements.Long-dashed line represents $\Delta = 1.76 T_c$ following the BCS gap formula.The short-dashed line best fits the insulating data points with $(p_F l/\hbar)_c \approx 0.35$ —the critical disorder of metal-insulator transition. A narrow region of superconductivity within insulating phase can be inferred from these data.

(1988)] apparently exclude the possibility of superconductivity in the insulating phase. In this work a small amount of Bi was added to granular Al in order to enhance spin-orbit scattering, which leads to *antilocalization* effect [Altshuler B.L. et al. (1982)]. This shifts both metal—insulator and to the *same extent* the superconducting transition, with the preservation of a narrow range of concentration on the metallic side where the material is not fully superconducting. The fact that the superconducting transition shifts with metal—insulator transition demonstrates that its position is determined by the vicinity of the metal—insulator transition, and that it is the impending transition to the insulating state which inhibits superconductivity. Similar conclusions on superconductivity vanishing at the point of metal—insulator transition were reached for amorphous Al_xGe_{1-x} [Lesueur J., Dumoulin L., Nedellec P. (1988)] and amorphous $Ga - Ar$ mixtures [Zint Th., Rohde M., Micklitz H. (1990)]. This later case is particularly interesting because it has been shown that conductivity exponent

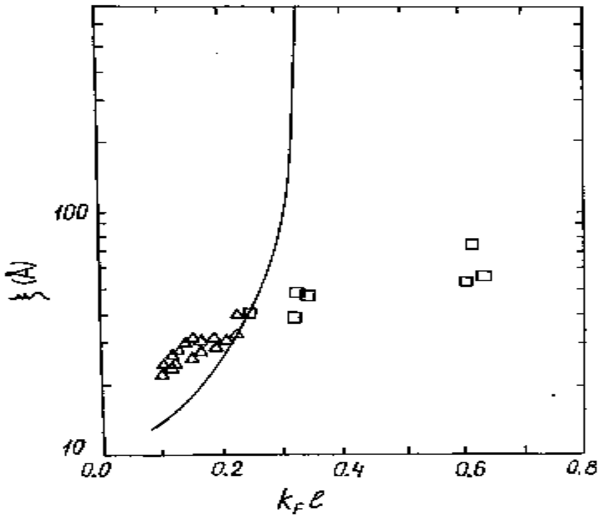

Fig. 5.6 Disorder dependence of localization length (full curve) and superconducting coherence length in amorphous In/InO_x films. Squares represent superconducting ξ for metallic films while triangles refer to insulating samples.

at metal—insulator transition here is $\nu \approx 0.5$ which places this system to a different universality class than those discussed above and similar to that observed in some doped uncompensated semiconductors like $Si : P$ [Milligan R.F. et al. (1985)]. Usual interpretation of this difference is based upon the importance of interaction effects in these systems [Castellani C., Kotliar G., Lee P.A. (1987)], though no theory at present produces the value of $\nu = 0.5$. Starting with the value of T_c of amorphous Ga ($T_c = 7.6K$), T_c decreases rather slowly with decreasing Ga volume fraction v, until one enters the critical region near $v_c \approx 0.145$. Further approach to v_c leads to a rapid decrease of T_c. Taking McMillan formula Eq.(3.25) for T_c (with $\omega_{log}/1.20 = 320K$ and $\lambda = 0.45$) and assuming negligible Coulomb repulsion $\mu^\star$ for pure amorphous Ga the increase of $\mu^\star$ on the approach of metal—insulator transition can be determined from the experimental data for T_c. This increase is approximately given by $\mu^\star \sim (v - v_c)^{-0.33}$. From this it is easy to see that $T_c \to 0$ for $v \to v_c$, so that these data does not indicate the survival of superconductivity beyond metal—insulator transition. These results are not surprising since we have seen the existence of strong mechanisms of T_c degradation close to disorder induced metal—insulator

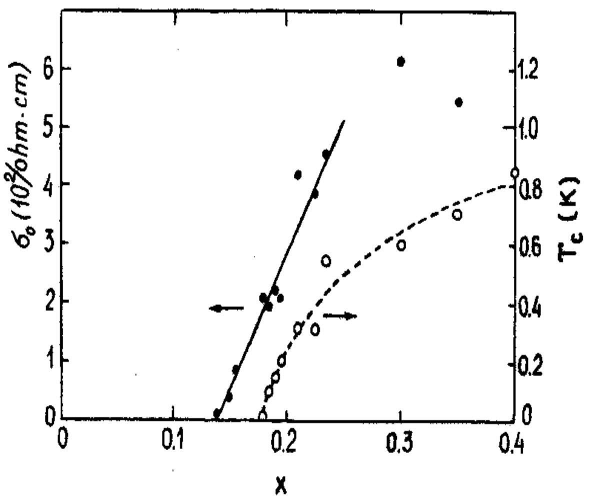

Fig. 5.7 Conductivity σ and T_c dependence on gold concentration in amorphous $Si_{1-x}Au_x$ alloy.

transition.

In Ref.[Sugawara K., et al. (1993)] the critical temperature and the coherence length in the layered superconductor $2H - NbSe_{2-x}S_x$ with $x = 0 - 0.6$ were investigated. T_c decreases with the increase of residual resistivity in general agreement with predictions of Ref.[Bulaevskii L.N., Sadovskii M.V. (1985)]. Coherence lengths both along and across the layers were estimated from the measurements of the upper critical field and compared with our predictions for quasi-two-dimensional case made in Ref.[Aleksashin B.A. et al. (1989)] with rather good agreement found for $x < 0.6$, the discrepancy at $x = 0.6$ being attributed to to the change of the crystal structure.

The interesting new high-pressure metastable metallic phase of an amorphous alloy $Cd_{43}Sb_{57}$ exhibiting the gradual metal—insulator transition during the slow decay of this phase at room temperature and atmospheric pressure has been studied in Refs.[Teplinskii V.M., Gantmakher V.F., Barkalov O.I. (1992); Gantmakher V.F. et al. (1992)]. Authors claim that during this decay the system remains homogeneous while going from metallic to insulating phase. At the same time the metallic phase is superconducting with $T_c \approx 5K$ and remains such up to metal—insulator transition. Close to it su-

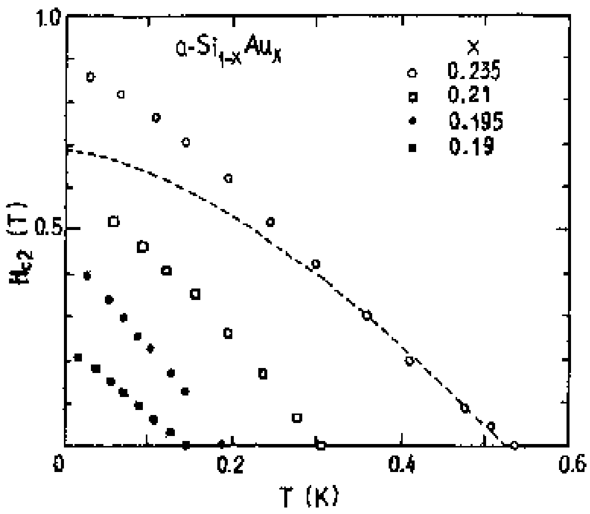

Fig. 5.8 $H_{c2}(T)$ in amorphous $Si_{1-x}Au_x$ alloy.

perconducting transition becomes smeared, while incomplete transition persists even in the insulating state. While these data are reminiscent of data on quench-condensed films of Sn and Ga[Markiewicz R.S. (1988)], which were interpreted as reentrant superconductivity due to sample inhomogeneities, it is stressed in Refs.[Teplinskii V.M., Gantmakher V.F., Barkalov O.I. (1992); Gantmakher V.F. et al. (1992)] that in this new system situation is different and we are dealing with intrinsically inhomogeneous superconductors state discussed in Refs.[Bulaevskii L.N., Sadovskii M.V. (1986); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987); Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989)]. From our point of view further studies of this system are necessary in order to show unambiguously the absence of structural inhomogeneities. Also rather peculiar characteristic of this system is almost complete independence of the onset temperature of superconducting transition on disorder.

Now we shall try to make more detailed comparison of some of these experimental results with certain theoretical predictions discussed earlier in the book. To be precise, we shall use the approach to the mechanism of T_c–degradation proposed in Ref.[Kuchinskii E.Z., Sadovskii M.V., Erkabaev M.A. (1997)] and based upon density of state effects ("Coulomb"

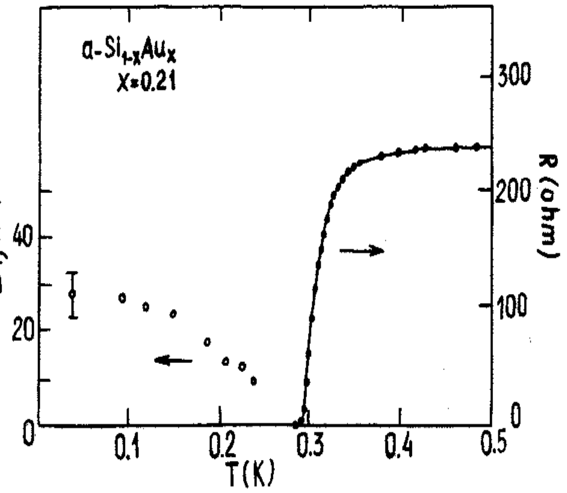

Fig. 5.9 Temperature dependencies of superconducting energy gap Δ and of the resistance R for amorphous $Si_{0.79}Au_{0.21}$.

pseudogap). We shall see that within this approach it is possible to obtain rather good agreement with experiments on amorphous alloys of InO_x [Fiory A.T., Hebard A.F. (1984)], Nb_xSi_{1-x} [Hertel G. et al. (1983); Bishop D.J., Spencer E.G., Dynes R.C. (1985)], Au_xSi_{1-x} [Nichida N. et al. (1982); Furubayashi T. (1985); Nichida N., et al. (1985)].

As we have seen above the authors of Ref.[Fiory A.T., Hebard A.F. (1984)] had presented the results of the measurements of disorder parameter $(p_Fl)^{-1}$ for the amorphous alloy of InO_x, as well as the data for T_c and static conductivity close to the metal- insulator transition. According to our work[Kuchinskii E.Z. et al. (1995); Kuchinskii E.Z., Erkabaev M.A. (1997)] the static conductivity close to the metal-insulator transition can be expressed as:

$$\sigma = \sigma_0[(p_Fl)W_c(\mu) - 1], \tag{5.1}$$

where σ_0 - is some characteristic scale of conductivity close to the metal-insulator transition, $W_c(\mu)$ - the value of disorder parameter (depending on the Coulomb constant) corresponding to the point of metal- insulator transition. Approximating the experimental data for InO_x by Eq.(5.1) allows us to estimate the characteristic conductivity scale σ_0 and also, from the value of W_c, the Coulomb constant μ. Satisfactory correlation (Cf. the insert in Fig.3.4) are obtained for the following values: $\sigma_0 \simeq 324.95$ $\Omega^{-1} \cdot cm^{-1}$, and $W_c \simeq 0.606$ giving $\mu \simeq 1.0$. Fig.5.10 demonstrates the comparison of our results with experimental data on T_c dependence on static conductivity σ for the amorphous InO_x using the value of $T_{co} = 3.41$ K, $\omega_D = 112\ K$ and $E_F = 9.98 \cdot 10^4\ K$, $[\omega_D/E_F] \simeq 1.1 \cdot 10^{-3}$ - for pure In and the given above values of σ_0 and μ, which allow us to estimate the pairing constant λ. Satisfactory agreement is obtained for $\lambda \simeq 0.45$. Dashed curves correspond to the values of $\lambda \simeq 0.4$ and 0.5.

Let us discuss now the results for T_c and static conductivity dependence on the Si content for amorphous alloys of Nb_xSi_{1-x} [Hertel G. et al. (1983); Bishop D.J., Spencer E.G., Dynes R.C. (1985)] and Au_xSi_{1-x} [Nichida N. et al. (1982); Furubayashi T. (1985); Nichida N., et al. (1985)] close to the metal-insulator transition. Assuming that the disorder parameter in this case is just proportional to Si concentration, so that $(p_Fl)^{-1} \sim 1 - x$, we can express Eq.(5.1) for static conductivity in the following form:

$$\sigma = \sigma_0 \frac{x - x_c}{1 - x}, \tag{5.2}$$

where x_c - the critical concentration of Nb or Au at the point of metal-insulator transition. Approximating the experimental data for conductivity in Nb_xSi_{1-x} Au_xSi_{1-x} by Eq.(5.2) allows us to estimate σ_0 and critical concentration x_c. Satisfactory correlation (Cf. inserts in Fig.5.11 and Fig.5.12) is obtained for:
Nb_xSi_{1-x}: $\sigma_0 \simeq 1963.9\ \Omega^{-1} \cdot cm^{-1}$, $x_c \simeq 0.115$;
Au_xSi_{1-x}: $\sigma_0 \simeq 2782.13\ \Omega^{-1} \cdot cm^{-1}$, $x_c \simeq 0.14$.

Fig.5.11 and Fig.5.12 present the comparison our results with the experimental data for T_c - dependence on conductivity σ for amorphous Nb_xSi_{1-x} and Au_xSi_{1-x}, using for the pure Nb: $T_{co} = 9.26\ K$, $\omega_D = 276\ K$ and $E_F = 6.18 \cdot 10^4\ K$, $[\omega_D/E_F] \simeq 3.0 \cdot 10^{-3}$; while for Au_xSi_{1-x} we assume $T_{co} = T_{cmax} \simeq 0.86\ K$, $\omega_D = 170\ K$ and $E_F = 6.42 \cdot 10^4\ K$, $[\omega_D/E_F] \simeq 0.9 \cdot 10^{-3}$ with the mentioned above values of σ_0, which allows us to estimate the pairing constant λ. Assuming for these systems the Coulomb constant $\mu \simeq 1$, the satisfactory agreement is obtained in case of Nb_xSi_{1-x} for $\lambda \simeq 0.54$ and for Au_xSi_{1-x} with $\lambda \simeq 0.62$. Note that correlation pseudogap in the density of states was actually observed for Au_xSi_{1-x}

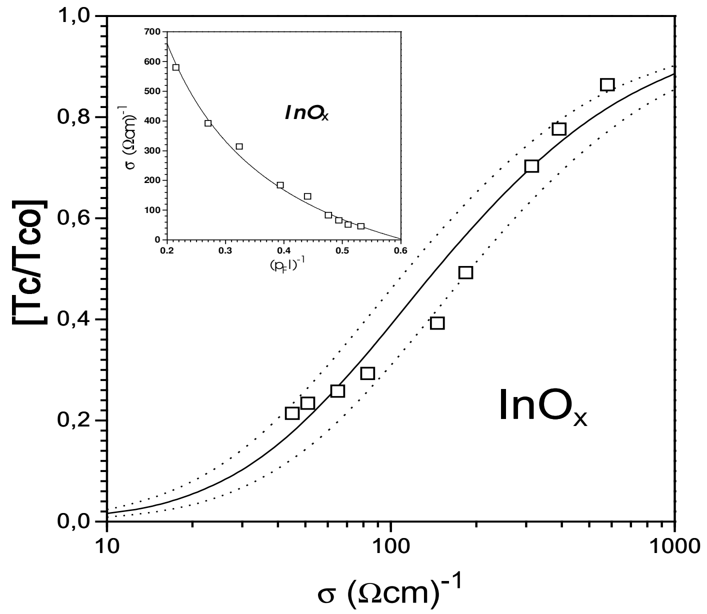

Fig. 5.10 T_c dependence on static conductivity σ for amorphous alloy of InO_x. At the insert we show our approximation of the data on static conductivity σ as a function of disorder parameter $(p_F l)^{-1}$.

in Ref.[Furubayashi T. et al. (1986)].

Surely the results presented above are essentially based upon the simplest BCS-model and are probably oversimplified. More rigorous approach to calculations of T_c must be based upon Eliashberg equations and realistic models of electron-phonon interaction[Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)]. Especially this is important in case of large enough values of λ, which demonstrate the possibility of superconductivity persist-

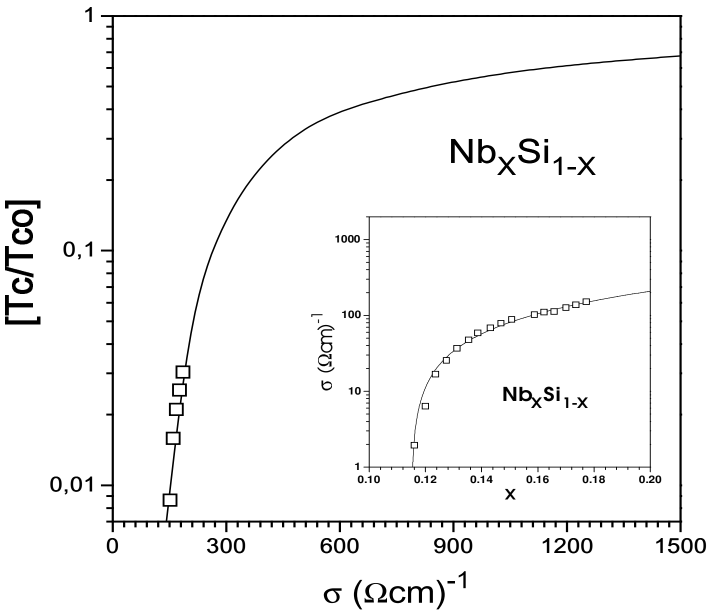

Fig. 5.11 T_c dependence on static conductivity σ for amorphous alloy of Nb_xSi_{1-x}. At the insert we show our approximation of the data on static conductivity σ as a function of Nb concentration.

ing in the insulating phase. At the same time, in present analysis we were not concerned with the problem of the genesis of the initial value of T_{c0} in pure system, but were studying only the T_c dependence on disorder. In this sense our results may be also qualitatively valid also in the case of strong-coupling superconductivity. We must also note that the more rigorous analysis is also needed taking into account disorder effects in the matrix element of Coulomb repulsion, which lead to the additional mechanism of

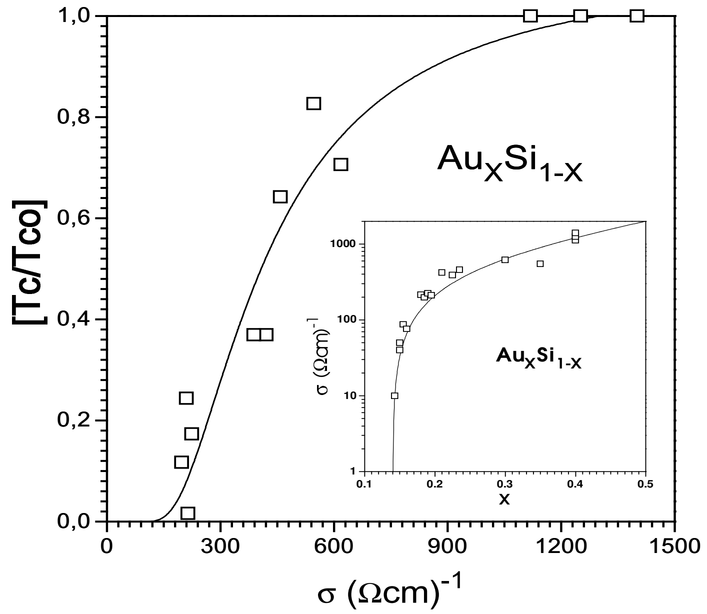

Fig. 5.12 T_c dependence on static conductivity σ for amorphous alloy of Au_xSi_{1-x}. At the insert we show our approximation of the data on static conductivity σ as a function of Au concentration.

T_c degradation T_c and were extensively discussed in Chapter 3. Here we have only taken into account pseudogap effects in the density of states. It is possible that rather satisfactory agreement with experiments can signify the dominating role of pseudogap formation effects in the problem of T_c degradation under disordering, which was claimed (on the level of small corrections) already in Ref.[Belitz D. (1989)].

Our general conclusion is that in most cases of traditional superconducting systems we can not find unambiguous demonstration of the possibility of superconductivity in insulating state induced by disorder. At the same time we can see rather rich variety of data on superconductivity close to metal–insulator transition which stimulate further studies. Some of the anomalies of superconducting behavior discussed above can be successfully explained by theories presented in this book, while the other require further theoretical investigations.

5.2 High—T_c Superconductors

Very soon after the discovery of high—temperature oxide superconductors [Bednorz J.G., Müller K.A. (1986); Bednorz J.G.,Müller K.A. (1988)] it was recognized that localization effects has an important role to play in these systems. There are many sources of disorder in these systems and the low level of conductivity indicates from the very beginning their closeness to Anderson transition. In the field where there are hundreds of papers published on the subject it is impossible to review or even to quote all of them. More or less complete impression of the status of high—T_c research can be obtained from Conference Proceedings [Proc. M2S-HTSC]. It must be stressed that even now, after more than ten years of intensive studies, there is a lot of contradicting data and explanations of different anomalies in these systems. Here we shall start with early papers dealing with disordering by fast neutron irradiation which we consider probably the "purest" method to introduce disorder into the system (allowing to neglect the complicated problems associated with chemical substitutions). Also historically it was apparently the earliest method used to study disorder effect in high—T_c superconductors in a controllable way [Voronin V.I. et al. (1987); Goshchitskii B.N., Kozhevnikov V.L., Sadovskii M.V. (1988)]. After that we shall briefly discuss some of the more modern papers devoted to studies of high—T_c systems disordered by impurities and some other types of irradiation.

There are several reasons for localization effects to be important in high—T_c oxides:

- *Intrinsic Disorder.* Practically all samples of high—T_c superconductors contain intrinsic disorder, which is mainly due to different degrees of oxygen non-stoichiometry, impurities, lattice defects or even macroscopic inhomogeneities, associated with complicated processes of phase separation [Proc. M2S-HTSC].

- *Two—Dimensionality.* All the known high—T_c systems (with $T_c > 30K$) are strongly anisotropic or quasi-two-dimensional conductors. We have seen above that for such systems it is natural to expect the strong enhancement of localization effects due to the special role of spatial dimensionality $d = 2$: in purely two-dimensional case localization appears for infinitely small disorder [Abrahams E. et al. (1979); Sadovskii M.V. (1981); Lee P.A., Ramakrishnan T.V. (1985); Sadovskii M.V. (1986)]. The inplane conductivity scale for metal—insulator transition in such systems as given by Eq. (2.12) or Eq. (2.94) is larger than in isotropic case. Reasonable estimates show that the values of inplane "minimal metallic conductivity" may exceed $10^3 Ohm^{-1} cm^{-1}$. While due to continuous nature of Anderson transition there is no rigorous meaning of minimal metallic conductivity, these estimates actually define the scale of conductivity near the metal—insulator transition caused by disorder. Then it is clear that most of the real samples of high—T_c superconductors are quite close to Anderson transition and even the very slight disordering is sufficient to transform them into Anderson insulators [Aleksashin B.A. et al. (1989)].
- *"Marginal" or non Fermi Liquid.* During our discussion of interaction effects we have seen that there are serious reasons to believe that importance of localization effects in high—T_c oxides may be actually due to more fundamental reasons connected with anomalous electronic structure and interactions in these materials. The concept of "marginal" Fermi liquid [Varma C.M. et al. (1989)] leads to extreme sensitivity of such system to disordering and the appearance of localized states around the Fermi level at rather weak disorder [Kotliar G., Varma C.M. (1990)]. "Only Fermi-liquids are metals" [Varma C.M. (1997)].

On the other hand high—T_c systems are especially promising from the point of view of the search for superconductivity in the Anderson insulator:

- High transition temperature T_c itself may guarantee the survival of superconductivity at relatively high disorder.
- Due to small size of Cooper pairs in high—T_c systems in combination with high—T_c (large gap !) we can easily satisfy the main

criterion for superconductivity in localized phase as given by Eq. (3.20). This fact may help to overcome the obvious difficulties due to almost certain d–wave symmetry of pairing in these systems.

- Being narrow band systems as most of the conducting oxides high—T_c systems are promising due to low values of the Fermi energy E_F which leads to less effective T_c degradation due to localization enhancement of Coulomb pseudopotential $\mu^{\star}$.

Anomalous transport properties of high—T_c oxides in normal phase are well known [Iye Y. (1992)]. Experimentally there are two types of resistivity behavior of good single-crystals of these systems. In highly conducting ab plane of $YBa_2Cu_3O_{7-\delta}$ and other oxides resistivity of a high quality single-crystal always shows the notorious linear—T behavior (by "good" we mean the samples with resistivity $\rho_{ab} < 10^3\ Ohm\ cm$). However, along orthogonal c direction the situation is rather curious: most samples produce semiconductor-like behavior $\rho_c \sim 1/T$, though some relatively rare samples (apparently more pure) show metallic-like $\rho_c \sim T$ (with strong anisotropy $\rho_c/\rho_{ab} \approx 10^2$ remaining) [Iye Y. (1992); Forro L. et al. (1992)]. Metallic behavior in c direction was apparently observed only in the best samples of $YBa_2Cu_3O_{7-\delta}$ and almost in no other high—T_c oxide. In Fig.5.13 taken from Ref.[Ito T. et al. (1991)] we show the temperature dependence of ρ_c in a number of high—T_c systems. It is seen that $\rho_c(T)$ changes between metallic and semiconducting behavior depending on whether the resistivity is below or above the Ioffe—Regel limit defined for quasi-two-dimensional case by Eq. (2.12). Rather strange is the absence of any obvious correlation between the behavior of ρ_c and T_c.

This unusual behavior leads us to the idea that most of the samples of high—T_c systems which are studied in the experiment are actually already in localized phase due to internal disorder which is always present. Surely, we realize that such a drastic assumption contradicts the usual expectations and propose it just as an alternative view open for further discussion. The attempted justification of this idea may be based upon the quasi-two-dimensional nature of these systems or on marginal Fermi liquid effects. In this case a simple conjecture on the temperature behavior of resistivity of single-crystals can be made which qualitatively explains the observations [Sadovskii M.V. (1990); Kotliar G. et al. (1991)]. In case of localized states at the Fermi level and for finite temperatures it is important to compare localization length R_{loc} with diffusion length due to inelastic scattering

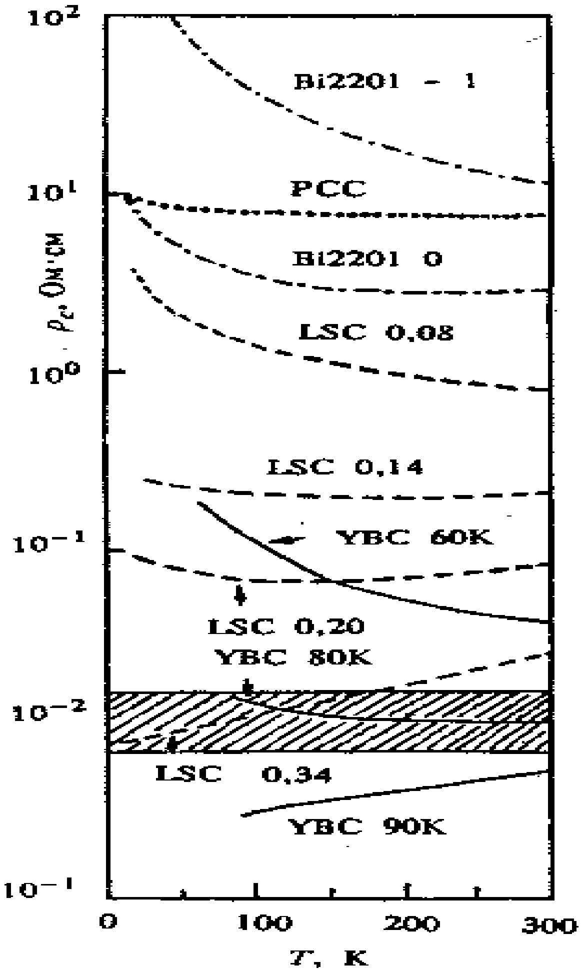

Fig. 5.13 Temperature dependence of ρ_c for different high—T_c cuprates. The dashed region indicates the resistivity range corresponding to Ioffe—Regel limit.

$L_\varphi \approx \sqrt{D\tau_\varphi}$, where D is diffusion coefficient due to elastic scattering on disorder, while τ_φ is phase coherence time determined by inelastic processes. For $T > 0$ this length L_φ effectively replaces the sample size L in all expressions of scaling theory of localization when $L \gg L_\varphi$, because on distances larger than L_φ all information on the nature of wave functions (e.g. whether they are localized or extended) is smeared out. Taking into account the usual low—temperature dependence like $\tau_\varphi \sim T^{-p}$ (where p is some integer, depending on the mechanism of inelastic scattering) this can lead to a non—trivial temperature dependence of conductivity, in particular to a possibility of a negative temperature coefficient of resistivity [Imry Y. (1980)]. Similar expressions determine the temperature dependence of conductivity also for the localized phase until $L_\varphi < R_{loc}$. In this case electrons do not "feel" being localized and conductivity at high enough T will show metallic like behavior. For localization to be important we must go to low enough temperatures, so that L_φ becomes greater than R_{loc}. If disordered high—T_c superconductors are in fact Anderson insulators with very anisotropic localization length, $R^{ab}_{loc} \gg R^c_{loc}$ and both localization lengths diminish as disorder grows, L_φ is also anisotropic and we can have three different types of temperature behavior of resistivity [Sadovskii M.V. (1990)]:

(1) Low T or strong disorder, when we have

$$L^{ab}_\varphi \approx \sqrt{D_{ab}\tau_\varphi} \gg R^{ab}_{loc} \qquad L^c_\varphi \approx \sqrt{D_c\tau_\varphi} \gg R^c_{loc} \tag{5.3}$$

This gives semiconductor-like behavior for both directions.

(2) Medium T or medium disorder, when

$$L^{ab}_\varphi < R^{ab}_{loc} \qquad L^c_\varphi > R^c_{loc} \tag{5.4}$$

and metallic behavior is observed in ab plane, while semiconducting temperature dependence of resistivity is observed along c axis.

(3) High T or low disorder, when

$$L^{ab}_\varphi < R^{ab}_{loc} \qquad L^c_\varphi < R^c_{loc} \tag{5.5}$$

and metallic behavior is observed in both directions.

Here we do not speculate on the inelastic scattering mechanisms leading to the concrete temperature behavior in high—T_c oxides, in particular on linear T behavior in ab plane or $1/T$ behavior in c direction. Unfortunately too little is known on these mechanisms [Iye Y. (1992)] to be able to make quantitative estimates on the different types of behavior predicted above. Of course detailed studies of such mechanisms are necessary to prove the proposed idea and to explain the temperature dependence of resistivity in high-T_c systems on its basis. However, most of the experimental data as we shall see below at least do not contradict the idea of possibility of Anderson localization in disordered high—T_c cuprates.

Now let us consider the experiments on controllable disordering of high—temperature superconductors. Already the first experiments on low temperature ($T = 80K$) fast neutron irradiation of ceramic samples of high—T_c systems [Aleksashin B.A. et al. (1988); Davydov S.A. et al. (1989a); Davydov S.A. et al. (1989b); Goshchitskii B.N. et al. (1989a); Goshchitskii B.N. et al. (1989b); Goshchitskii B.N. (1989d)] has shown that the growth of structural disorder leads to a number of drastic changes in their physical properties:

- continuous metal—insulator transition at very slight disordering,
- rapid degradation of T_c,
- apparent coexistence of hopping conductivity and superconductivity at intermediate disorder,
- approximate independence of the slope of H_{c2} at $T \sim T_c$ on the degree of disorder,
- anomalous *exponential* growth of resistivity with defect concentration.

These anomalies were later confirmed on single-crystals and epitaxial films [Goshchitskii B.N. et al. (1989c); Goshchitskii B.N. et al. (1990); Valles J.M. et al. (1989); Karkin A. et al. (1991)], and were interpreted [Aleksashin B.A. et al. (1989); Sadovskii M.V. (1989)] using the ideas of possible coexistence of Anderson localization and superconductivity.

In Fig.5.14 we show data [Aleksashin B.A. et al. (1989)] on the dependence of the superconducting transition temperature and resistivity (at $T = 100K$, i.e. just before superconducting transition) on fast neutron fluence for $YBa_2Cu_3O_{6.95}$. In all high—T_c compounds introduction of defects leads to strong broadening of superconducting transition. The derivative $(dH_{c2}/dT)_{T_c}$ in ceramic samples measured at the midpoint of the superconducting transition does not appreciably change as ρ_{100K} grows by an order magnitude. In Fig.5.15[Aleksashin B.A. et al. (1989)] we show the temperature dependence of resistivity for samples of $YBa_2Cu_3O_{6.95}$ and $La_{1.83}Sr_{0.17}CuO_4$ for different degrees of disorder. In all these materials the $\rho(T)$ curves vary in the same way. In the fluence range $\Phi > 10^{19} cm^{-2}$, where superconductivity is absent, $\rho(T)$ follows a dependence which is characteristic of conductivity via localized states [Mott N.F. (1974);

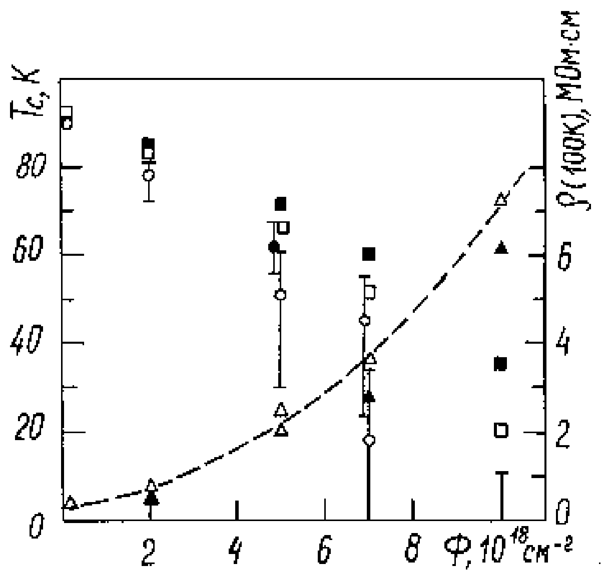

Fig. 5.14 Dependence of the superconducting transition temperature and resistivity (at $T = 100K$) on neutron fluence for ceramic $YBa_2Cu_3O_{6.95}$. Different notations correspond to different methods of measurement and also evolution after annealing at $300K$.

Mott N.F, Davis E.A. (1979)]:

$$\rho(T) = \rho_0 exp(Q/T^{1/4}) \qquad Q = 2.1[N(E_F)R_{loc}^3]^{-1/4} \tag{5.6}$$

as shown in Fig.5.16. (Mott's variable-range hopping conduction).

The most striking anomaly of resistivity behavior of all high—T_c systems under disordering is nonlinear, practically *exponential* growth of resistivity at fixed temperature (e.g. $\rho(T = 100K)$) with fluence, starting from the low fluences $\Phi < 7\ 10^{18} cm^{-2}$, including superconducting samples [Aleksashin B.A. et al. (1988); Aleksashin B.A. et al. (1989); Davydov S.A. et al. (1989a); Davydov S.A. et al. (1989b); Goshchitskii B.N. et al. (1989a); Goshchitskii B.N. et al. (1989b)]. These data are shown in Fig.5.17 [Aleksashin B.A. et al. (1989)] for the dependence of $\rho(T = 80K)$ on Φ obtained from measurements made directly during the process of irradiation. For comparison the similar data for $SnMo_6S_8$ are shown which do not demonstrate such an anomalous behavior, its resistivity is just proportional to Φ and saturates at large fluences. We can apparently relate this exponential

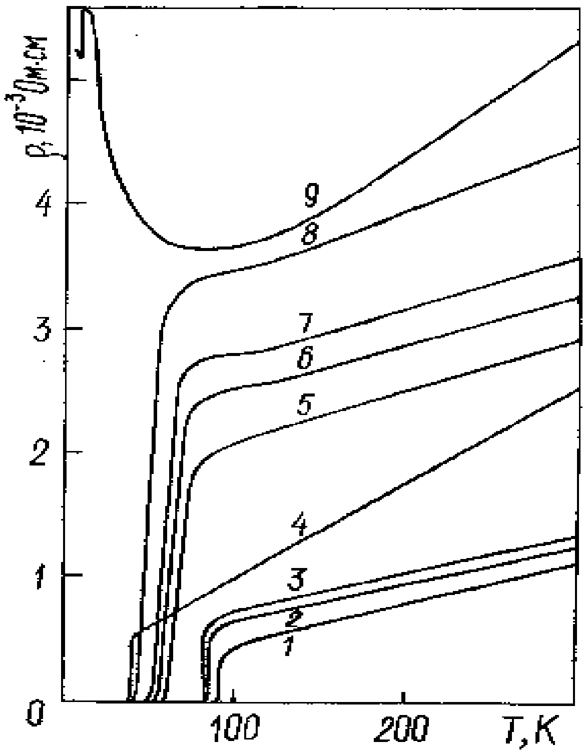

Fig. 5.15 Temperature dependence of resistivity ρ for ceramic samples of $YBa_2Cu_3O_{6.95}$ (curves 1—3 and 5—8) and $La_{1.83}Sr_{0.17}CuO_4$ (curves 4, 9) irradiated at $T = 80K$ with different fluences: 1—$\Phi = 0$; 3, 6, 8 — $\Phi = 2.5$ and $7 10^{18} cm^{-2}$ plus annealing for 2 hours at $300K$; 2, 5, 7 — irradiated with $\Phi = 2.5$ and $7\ 10^{18} cm^{-2}$ plus annealing for 2 weeks at $300K$; 4 — $\Phi = 0$; 9 — $\Phi = 5\ 10^{18} cm^{-2}$ plus annealing for 2 hours at $300K$.

growth of ρ with the increase of Φ (i.e. of defect concentration) in all high—T_c systems to localization, which already appears for very small degrees of disorder in samples with high values of T_c. As we have seen in samples with much reduced or vanishing T_c localization is observed directly via Mott's hopping in the temperature behavior of resistivity given by Eq. (5.6). From these results it follows that the electronic system of high—T_c superconductors is very close to the Anderson transition. The observed variation of ρ as a function both of fluence and of temperature can be described by the

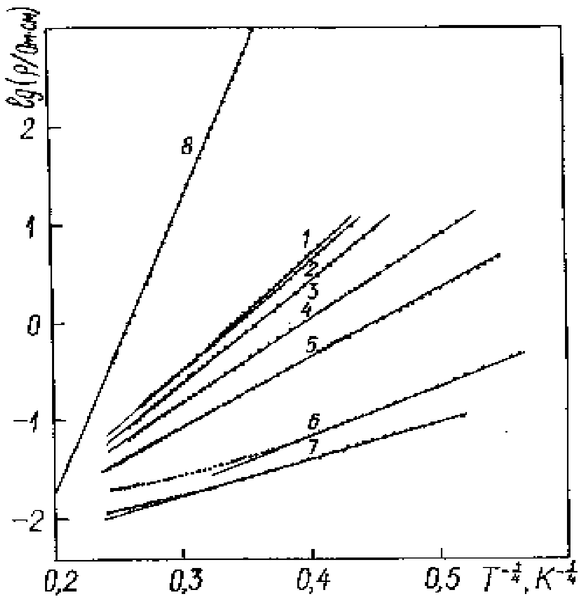

Fig. 5.16 Dependence of $ln\rho$ on $T^{-1/4}$ for $YBa_2Cu_3O_{6.95}$ irradiated with a fluence of $\Phi = 1.2\ 10^{19}cm^{-2}$ at $T = 80K$ (curve 1), and after 20-minute annealing at $T = 150K(2); 200K(3); 250K(4); 300K(5)$ and two weeks annealing at $T = 300K(7)$. Similar dependencies for $La_{1.83}Sr_{0.17}CuO_4$ for $\Phi = 2\ 10^{19}cm^{-2}$ annealed for 2 hours at $300K(6)$ and for La_2CuO_4 for $\Phi = 2\ 10^{19}cm^{-2}$ annealed for 2 hours at $300K(8)$.

following empirical formula [Aleksashin B.A. et al. (1988)]:

$$\rho(T,\Phi) = (a + cT)exp(b\Phi/T^{1/4}) \tag{5.7}$$

Identifying the exponential factors in Eq. (5.6) and Eq. (5.7) it is possible to obtain a fluence dependence of localization length (Cf. Ref.[Aleksashin B.A. et al. (1989)] and below).

Detailed neutron diffraction studies of structural changes in irradiated samples were also performed [Aleksashin B.A. et al. (1988); Aleksashin B.A. et al. (1989); Voronin V. et al. (1991)]. These investigations has shown definitely that there are no oxygen loss in $YBa_2Cu_3O_{6.95}$ during low temperature irradiation. Only some partial rearrangement of oxygens between positions $O(4)$ and $O(5)$ in the elementary cell occur as radiation-induced defects are introduced. In addition, in all high—T_c compounds the Debye— Waller factors grow and the lattice parameters a, b, c increase

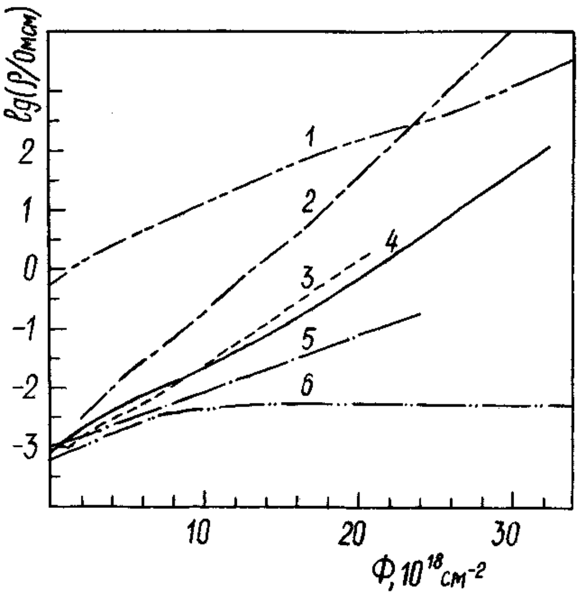

Fig. 5.17 Dependence of $ln\rho$ on fluence Φ during irradiation at $T = 80K$: 1—La_2CuO_4; 2—$YBa_2Cu_3O_{6.95}$; 3—single crystalline ρ_{ab} in $YBa_2Cu_3O_{6.95}$; 4—$La_{1.83}Sr_{0.17}CuO_4$; 5—$Bi - Sr - Ca - Cu - O$; 6— $SnMo_6Se_8$.

slightly [Aleksashin B.A. et al. (1989); Voronin V. et al. (1991)]. The growth of Debye—Waller factors reflect significant atomic shifts, both static and dynamic, from their regular positions, which induce a random potential. This disorder is pretty small from the structural point of view, the lattice is only slightly distorted. However, we have seen that this small disorder is sufficient to induce metal—insulator transition and complete degradation of superconductivity. At the same time this relatively small amount of structural disorder can support the idea of essentially homogeneous nature of disordering, which is important for direct application of the most of theoretical schemes discussed above*. The absence of oxygen loss implies that there is no significant change in concentration of carriers and we have really disorder—induced metal—insulator transition. This is also confirmed by other methods [Goshchitskii B.N. (1989d); Podlesnyak A. et al. (1991)].

*In fact our theories neglected all kinds of macroscopic inhomogeneities, which may be present in these samples either due to irradiation, or phase separation (which may be an intrinsic property of copper oxides).

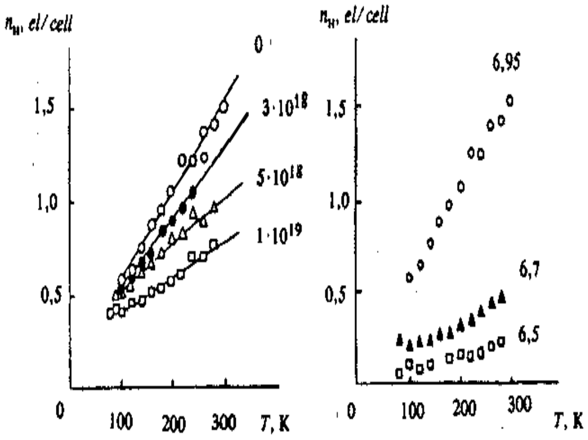

Fig. 5.18 Temperature dependence of Hall concentration for the irradiated (left) and oxygen deficient (right) ceramic samples of $YBa_2Cu_3O_{7-\delta}$.

In Fig.5.18 we show the data [Goshchitskii B.N. (1989d); Goshchitskii B.N. et al. (1990)] on temperature dependence of the Hall concentration of ceramic samples of irradiated and oxygen deficient $YBa_2Cu_3O_{7-\delta}$. It is seen that disordering weakens the anomalous temperature dependence of Hall effect, but Hall concentration n_H at low T practically does not change in striking difference with data on oxygen deficient samples, where n_H drops several times. This also confirms the picture of mainly disorder—induced metal—insulator transition in radiation disordering experiments. Similar Hall data were obtained on epitaxial films [Valles J.M. et al. (1989)] and single-crystals [Karkin A. et al. (1991)]. Qualitatively the same resistivity behavior as discussed above was also obtained in the experiments on radiation disordering of single-crystals [Goshchitskii B.N. et al. (1989c); Goshchitskii B.N. et al. (1990)] and epitaxial films [Valles J.M. et al. (1989)]. Electrical resistivities of $YBa_2Cu_3O_{7-\delta}$ single crystals were measured at $T = 80K$ directly during irradiation by fast neutrons. The data are shown in Fig.5.19. We can see that ρ_{ab} increases exponentially with Φ (defect concentration) starting from the smallest doses, while ρ_c grows slower and only for $\Phi > 10^{19} cm^{-2}$ they grow with the same rate. At large fluences both ρ_{ab} and ρ_c demonstrate [Davydov S.A. et al. (1990a)] Mott's hopping $ln\rho_{ab,c} \sim T^{-1/4}$. Similar data of Ref.[Valles J.M. et al. (1989)] show $ln\rho \sim T^{-1/2}$ characteristic of the Coulomb gap. We do not know the

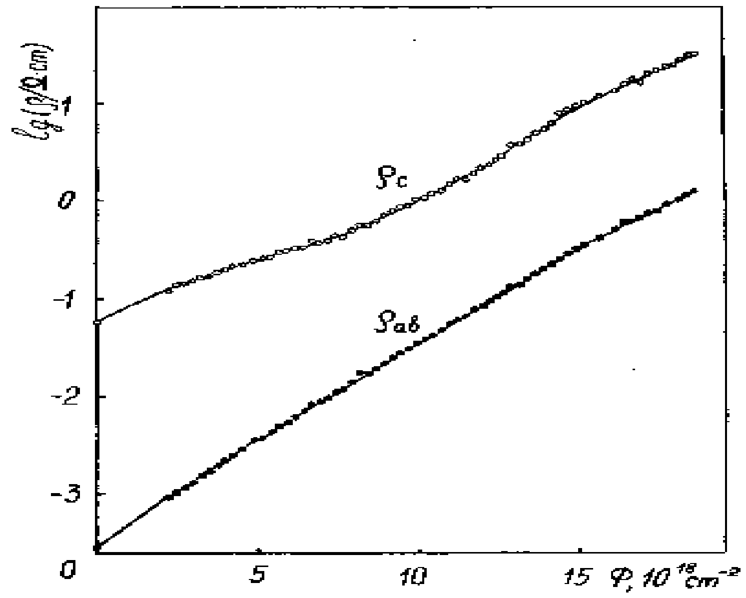

Fig. 5.19 Fluence dependence of ρ_{ab} and ρ_c at $T = 80K$ during fast neutron irradiation.

reasons for this discrepancy between single-crystalline and epitaxial films data (note that another method of disordering by 1MeV Ne^+ ions was used in Ref.[Valles J.M. et al. (1989)]). Anisotropy ρ_c/ρ_{ab} at $T = 80K$ drops rapidly (to the values ~ 30 for $\Phi = 10^{19}cm^{-2}$) and then practically does not change and "residual" anisotropy of the order of its room—temperature value in initial samples remains. This means that temperature dependence of anisotropy weakens in the disordered samples. Note, that unfortunately only the single-crystals with "semiconducting" temperature dependence of resistivity along c axis were investigated in these experiments up to now.

Remarkable results were obtained in Refs.[Ando Y., et al. (1995); Boebinger G.S., et al. (1996)] where $61T$ pulsed magnetic field was used to suppress superconductivity in $La_{2-x}Sr_xCuO_4$ single crystals with different doping. It was found that both in-plane resistivity ρ_{ab} and c-axis resistivity ρ_c *diverge* for $T/T_c \to 0$, for all underdoped samples with $x < 0.16$. Some of these data are shown in Figs.5.20, 5.21, 5.22. Low-temperature data, shown in Fig.5.22 demonstrate logarithmic divergence of ρ_{ab} for $x = 0.13$ sample as $T \to 0$ [Ando Y., et al. (1995)]. Thus, in underdoped samples

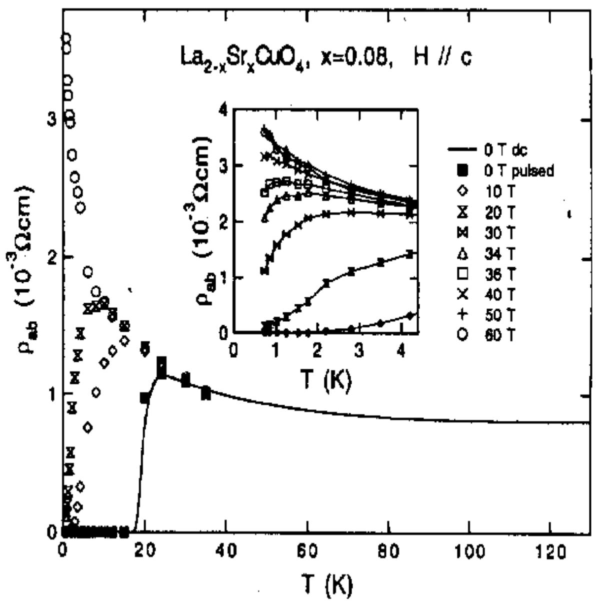

Fig. 5.20 Temperature dependence of ρ_{ab} in $0, 1, 20$ and $60T$ for $La_{2-x}Sr_xCuO_4$ with $x = 0.08$. The insert contains the low temperature data.

a superconductor to insulator transition occurs upon applying a magnetic field, with no evidence of metallic phase at low temperatures. The $x = 0.13$ data of Fig.5.23 are particularly striking, because ρ_{ab} shows linear-T behavior above $150K$, which extrapolates to a zero intercept at $T = 0$, while ρ_{ab} crosses over to logarithmic insulating behavior at low temperatures, similar to what may be expected for two-dimensional case. Normal state anisotropy ratio of resistivities becomes temperature independent over the entire range of this "insulating" behavior. Optimally doped and overdoped samples show metallic behavior which is clearly seen even in ρ_c data shown in Fig.5.21. This lead the authors of Ref.[Boebinger G.S., et al. (1996)] to propose the qualitative "phase" diagram shown in Fig.5.23. These results indicate insulating ground state of the system without superconductivity at least for all uderdoped samples. It must be stressed, that this insulating behavior is not due to Cooper pair localization, because Cooper pairs would not exist above H_{c2}. Apparently, single-particle states are localized.

Very similar data were obtained independently in Ref.[Karpinska K., et al. (1996); Malinowski A. et al. (1997)] on the same system. Here for low-x

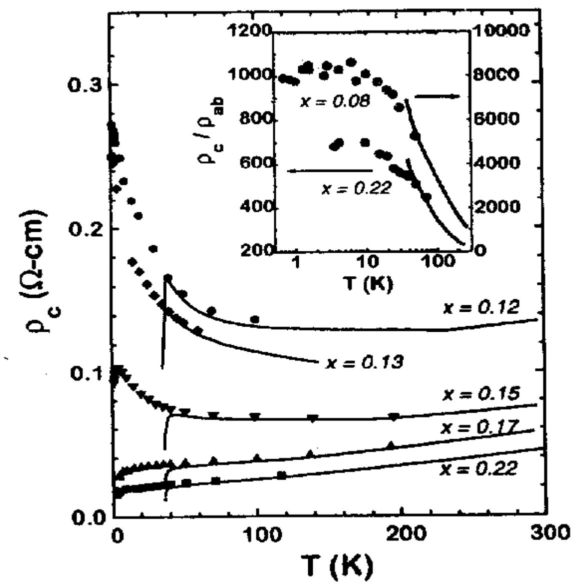

Fig. 5.21 ρ_c for five samples of $La_{2-x}Sr_xCuO_4$ for different x. The insert shows the low-temperature saturation of the anisotropy ratio.

samples the data for resistivity were well fitted to Mott's $T^{-1/4}$–hopping law, demonstrating apparently more or less usual localization, after suppression of superconductivity. The same type of behavior was obtained in the experiments with La-doped $Bi_2Sr_2CuO_y$ ($T_c = 13K$) in pulsed $61T$ magnetic fields in Ref. [Ando Y., et al. (1996)] and also in Zn-doped single crystals of $YBa_2Cu_3O_{7-\delta}$ by suppressing superconductivity with magnetic fields of up to $18T$ [Segawa K., Ando Y. (1998)]. In La-doped $Bi_2Sr_2CuO_y$ single crystals and $La_{2-x}Sr_xCuO_4$ thin films normal state Hall coefficient was also measured in [Ando Y., et al. (1997)] after suppressing superconductivity by high enough magnetic field. In contrast to data above T_c, Hall coefficient below $10K$ shows little temperature dependence in all samples, irrespective of whether ρ_{ab} exhibits insulating or metallic behavior. In Ref.[Gantmakher V.F. et al (1997)] the low temperature resistivity of single crystals of $YBa_2Cu_3O_{6+x}$ with $x = 0.37$ was measured after applying magnetic field and also reducing the hole concentration varying the oxygen content. The temperature dependence of in-plane resistivity ρ_{ab} was fitted

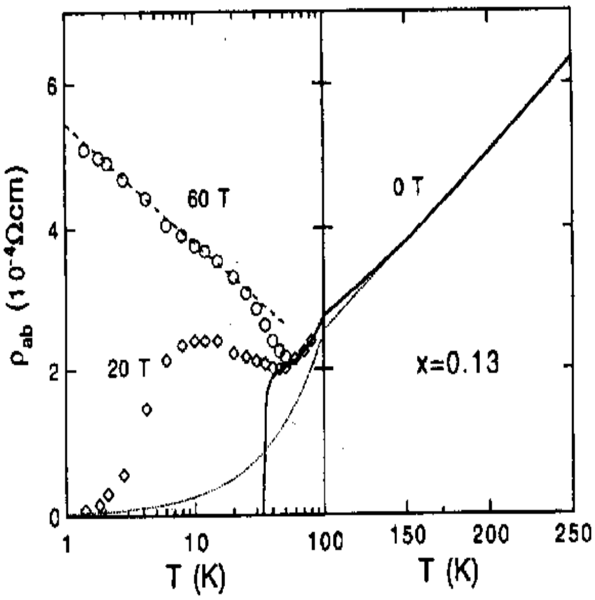

Fig. 5.22 In-plane resistivity ρ_{ab} for the $x = 0.13$ sample of $La_{2-x}Sr_xCuO_4$ plotted vs linear T at high temperature and $\ln T$ at low temperature. The dotted line is the extrapolated linear-T dependence.

by logarithmic law in the temperature range of almost two decades. However, a different fit by a power law typical for a three-dimensional metal near the metal - insulator transition, given e.g. by Eq.(2.163), was also rather good. The c-axis conductivity followed the power law and could not be fitted by logarithm. It was concluded that there were both superconductor - normal metal and normal metal - superconductor transition in this system, distanced in oxygen content by $\delta x \approx 0.025$.

The nature of insulating states in these experiments remains unclear. Logarithmic temperature behavior at low temperatures suggests two – dimensional (weak) localization of carriers in CuO_2 planes. However, objections to this interpretation were raised in [Ando Y., et al. (1995)], based on magnetoresistance and ρ_c behavior. More detailed comparison with predictions of weak localization theory in [Castellani C., Schwab P., Grilli M. (1998)] shows, that it is difficult to get quantitative fit of experimental data to this theory. At the same time, the obsevation of hopping conduction in low-doped system in [Karpinska K., et al. (1996);

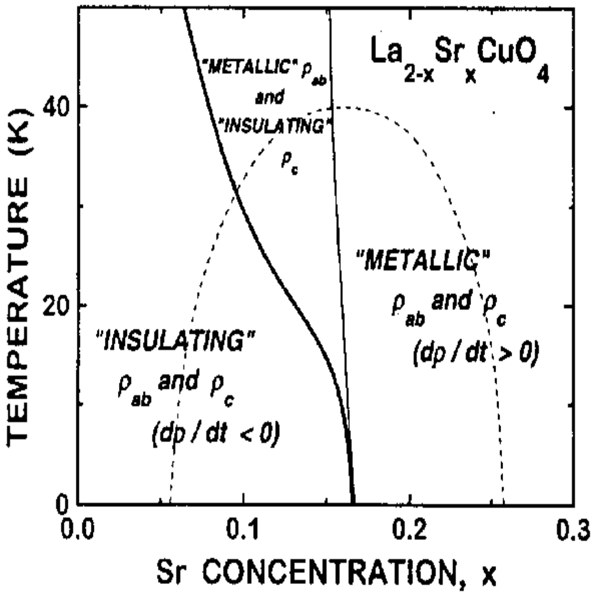

Fig. 5.23 Schematic diagram of the behavior of the normal state resistivity in the absense of superconductivity in $La_{2-x}Sr_xCuO_4$ versus Sr concentration. The dashed line shows $T_c(x)$ in zero magnetic field.

Malinowski A. et al. (1997)] clearly shows, that localization (probably even strong!)is at work here. Apparently, we must somehow take into account complicated effects of electron-electron interactions [Varma C.M. (1997)], including possible non-Fermi-liquid behavior. Still these data are at least in qualitative agreement of general predictions discussed in this book.

The upper critical fields of $YBa_2Cu_3O_{7-\delta}$ single-crystals (determined from standard resistivity measurements) for different degrees of disorder are shown in Fig.5.24 [Goshchitskii B.N. et al. (1990)]. Temperature dependence of H_{c2} in disordered samples is essentially nonlinear, especially for samples with low T_c, which strongly complicates the estimates of the slope of the upper critical field close to T_c. The estimated from high-field regions temperature derivative of $H_{c2}^{\perp}$ (field along the c axis) increases with disorder. However, similar derivative of $H_{c2}^{\|}$ (field along ab plane) drops in the beginning and then does not change. Anisotropy of H_{c2} decreases with disorder and in samples with $T_c \sim 10K$ the ratio of $(H_{c2}^{\|})'/(H_{c2}^{\perp})'$ is close to unity. According to Eq. (3.165) this means the complete isotropisation

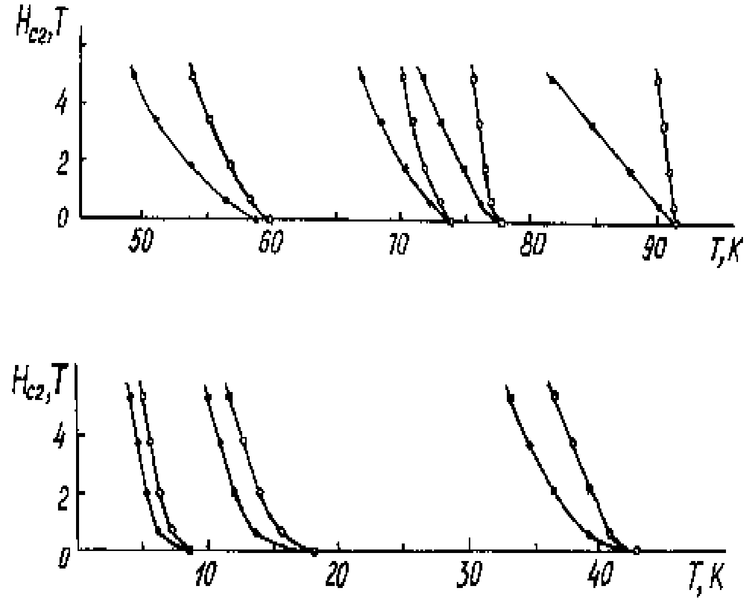

Fig. 5.24 Temperature dependence of $H_{c2}^{||}$ (upper curves) and $H_{c2}^{\perp}$ (lower curves) for the single-crystals of $YBa_2Cu_3O_{7-\delta}$ with different degrees of disorder.

of the Cooper pairs. This is illustrated by Fig.5.25 [Davydov S.A. (1991)]. The remaining anisotropy of resistivity may be connected with some kind of planar defects in the system.

In Ref.[Osofsky M.S. et al. (1993)] the authors presented rather unique data on the temperature dependence of the upper critical field of high-temperature superconductor $Bi_2Sr_2CuO_y$ in wide temperature interval from $T_c \approx 19K$ to $T \approx 0.005T_c$, which has shown rather anomalous dependence with positive curvature at any temperature. It was noted in Ref.[Osofsky M.S. et al. (1993)] that this type of behavior is difficult to explain within any known theory. In fact, it is sharply different from the standard behavior of BCS-model. It was demonstrated in Refs.[Kuchinskii E.Z., Sadovskii M.V. (1993b); Kuchinskii E.Z., Sadovskii M.V. (1994)] that the observed dependence of $H_{c2}(T)$ can be satisfactorily explained by localization effects in two-dimensional (quasi-two-dimensional) model in the limit of sufficiently strong disorder. Measurements of H_{c2} in Ref.[Osofsky M.S. et al. (1993)] were performed on epitaxially grown films of $Bi_2Sr_2CuO_y$,

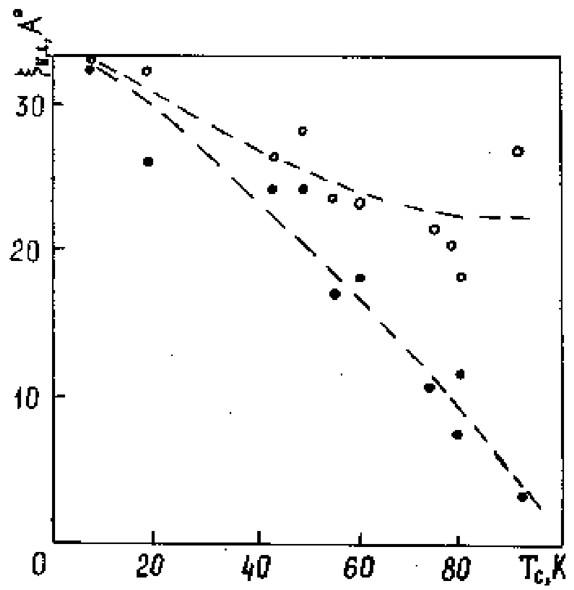

Fig. 5.25 The dependence of coherence lengths determined from H_{c2} behavior under disordering on the critical temperature T_c: $\xi_{||}$—circles; $\xi_{\perp}$—black circles.

however it is quite possible that the films were disordered enough, which can be guessed from rather wide ($\sim 7K$) superconducting transition. Unfortunately the relevant data, in particular on the value of conductivity of the films studied, were absent. This gives us some grounds to try to interpret the data obtained in Ref.[Osofsky M.S. et al. (1993)] in the framework of rather strong disorder the effects of which are obviously enhanced by the quasi-two-dimensional nature of high-temperature superconductors. The general discussion of the temperature dependence of the upper critical field in two-dimensional and quasi-two-dimensional case with strong localization effects was presented above in Chapter 3. Note that we analyzed there the case of magnetic field perpendicular to the highly conducting planes, which is precisely the case of Ref.[Osofsky M.S. et al. (1993)]. We have seen [Kuchinskii E.Z., Sadovskii M.V. (1993a)] that the anomalies of the upper critical field due to the frequency dependence of diffusion coefficient appear only for temperatures $T \ll \frac{e^{-1/\lambda}}{\tau}$. For higher temperatures we obtained the usual behavior as in the theory of "dirty" superconductors. Also we

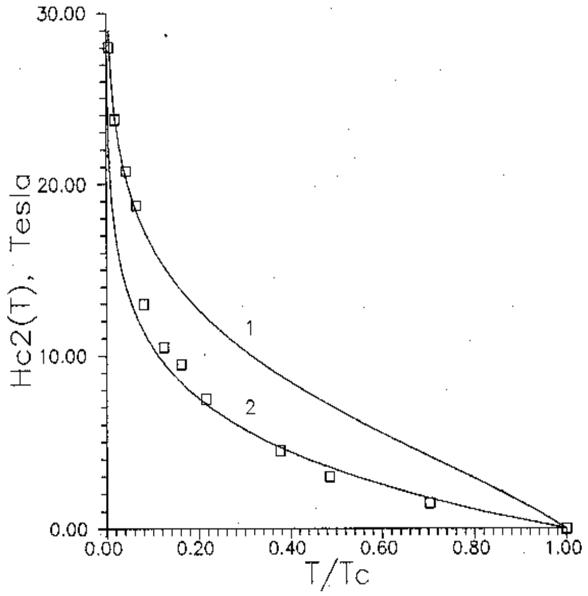

Fig. 5.26 Temperature dependence of the upper critical field: theoretical curve (1) is given for the case of $\frac{e^{-1/\lambda}}{T_c\tau}=2$, $\lambda=0.18$, while curve (2) is for $\frac{e^{-1/\lambda}}{T_c\tau}=20$, $\lambda=0.032$. Squares represent the experimental data for $Bi_2Sr_2CuO_y$.

have noted [Kuchinskii E.Z., Sadovskii M.V. (1993a)] that superconductivity survives in a system with finite localization length if $T_c \gg \lambda\frac{e^{-1/\lambda}}{\tau}$, which is equivalent to our criteria of the smallness of Cooper pair size compared with localization length. This latter length is exponentially large in two- dimensional systems with small disorder ($\lambda \ll 1$). The most interesting (for our aims) limit of relatively strong disorder is defined by $T_c \ll \frac{e^{-1/\lambda}}{\tau}$, so that in fact we are dealing with pretty narrow region of λ's when $\lambda\frac{e^{-1/\lambda}}{\tau} \ll T_c \ll \frac{e^{-1/\lambda}}{\tau}$. In this case we have seen that the upper critical field is determined by Eq.(3.188) :

$$ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T}\right)=(1+4\pi\frac{D_0}{\phi_0}\frac{\tau H_{c2}}{e^{-1/\lambda}})ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T_c}(1+4\pi\frac{D_0}{\phi_0}\frac{\tau H_{c2}}{e^{-1/\lambda}})\right) \tag{5.8}$$

($\gamma=1.781$) from which we can directly obtain the $T(H_{c2})$—dependence. The appropriate behavior of the upper critical field for two sets of parameters is shown in Fig.5.26. The dependence of H_{c2} on temperature demonstrates positive curvature and H_{c2} diverges for $T \to 0$. We have seen that this weak (logarithmic) divergence is connected with our neglect of the magnetic field influence upon diffusion. Taking this influence into account we can suppress this divergence of H_{c2} as $T \to 0$. This is the main effect of broken time reversal invariance and it is clear that it is important only for extremely low temperatures [Kuchinskii E.Z., Sadovskii M.V. (1993a)]. For the quasi-two-dimensional case on the dielectric side of Anderson's transition, but not too very close to it, the behavior of diffusion coefficient is quite close to that of purely two-dimensional case, so that the upper critical field can be analyzed within two-dimensional approach. Close to the transition (e.g. over interplane transfer integral) both for metallic and insulating sides and for parameters satisfying the inequality $\lambda\frac{e^{-1/\lambda}}{\tau} \ll T_c \ll \frac{e^{-1/\lambda}}{\tau}$, the temperature dependence of H_{c2} is in fact again very close to those in purely two-dimensional case considered above[Kuchinskii E.Z., Sadovskii M.V. (1993a)]. Some deviations appear only in a very narrow region of very low temperatures [Kuchinskii E.Z., Sadovskii M.V. (1993a)]. In Fig.5.26 we also show the experimental data for H_{c2} from Ref.[Osofsky M.S. et al. (1993)]. Theoretical curve (1) is given for the parameters which lead to rather good agreement with experiment in the low temperature region. The curve (2) corresponds to parameters giving good agreement in a wide temperature region except the lowest temperatures. The cyclotron mass m was always assumed to be equal that of the free electron. In general we observe satisfactory agreement between theory and experiment. Unfortunately, the values of the ratio $\frac{e^{-1/\lambda}}{T_c\tau}$ for the second curve, while corresponding to quite reasonable values of λ, lead to nonrealistic (too small) values of $T_c\tau$, which are rather doubtful for the system with relatively high T_c. For the first curve situation is much better though the electron damping on the scale of T_c is still very large which corresponds to strong disorder. Note however, that the detailed discussion of these parameters is actually impossible without the knowledge of additional characteristics of the films studied in Ref.[Osofsky M.S. et al. (1993)]. In particular it is quite interesting to have an independent estimate of λ. We also want to stress relatively approximate nature of these parameters due to our two- dimensional idealization. More serious comparison should be done using the expressions of Ref.[Kuchinskii E.Z., Sadovskii M.V. (1993a)] for the quasi-two-dimensional case, which again requires the additional information on the system, in particular, the data on the anisotropy of electronic properties. In our opinion the relatively good agreement of experimental data of Ref.[Osofsky M.S. et al. (1993)] with theoretical dependencies obtained for the two-dimensional (quasi-two- dimensional) case of disordered system with Anderson localization illustrates the importance of localization effects for the physics of high-temperature superconductors. However, we must note that the similar anomalies of the temperature dependence of the upper critical field were also observed in Ref.[Mackenzie A.P. et al. (1993)] for the single crystals of the overdoped $Tl_2Ba_2CuO_{6+\delta}$ which authors claim to be extremely clean, so that apparently no explanation based upon strong localization effects can be used. Similar data were recently obtained for thin films of underdoped $YBa_2(Cu_{0.97}Zn_{0.03})_3O_{7-\delta}$ with pretty low transition temperatures [Walker D.J.C. et al. (1995)]. These films again seem to be disordered enough to call localization effects as a possible explanation of unusual positive curvature of $H_{c2}(T)$ dependence for all temperatures.

Under irradiation, localized moment contribution appears in the magnetic susceptibility of high—T_c oxides [Aleksashin B.A. et al. (1988); Aleksashin B.A. et al. (1989)]. In the temperature range from T_c to $300K$ $\chi(T)$ is satisfactorily described by Curie—Weiss type dependence: $\chi(T) = \chi_0 + C/(T - \Theta)$. The value of χ_0 and the Curie constant C as a function of fluence Φ are given in Fig.5.27. The value of C is proportional to fluence. Note that the threefold larger slope of $C(\Phi)$ in $YBa_2Cu_3O_{6.95}$ as compared with $La_{1.83}Sr_{0.17}CuO_4$ is an evidence that this Curie-law temperature dependence is associated with localized moments forming on Cu (there are three times more coppers in the elementary cell of Y compound than in La compound).

The data presented above show that electronic properties of high—T_c systems are quite different under disordering from that of traditional superconductors [Goshchitskii B.N., Arkhipov V.E., Chukalkin Yu.G. (1987); Aleksandrov A.S. et al.(1989)] or even some closely related metallic oxides [Karkin A. et al. (1991); Davydov S.A. et al. (1990b)]. We associate these anomalies with the closeness of the Anderson transition and believe that real samples of high—T_c systems which always possess some noticeable disorder may well be already in the state of the Anderson insulator. However, we must stress that it is quite difficult to decide from the experiments described above the precise position of the Anderson transition on disorder scale.

Additional information on this problem may be obtained from experiments on NMR relaxation rate $R = 1/T_1$ in disordered state, using the ap-

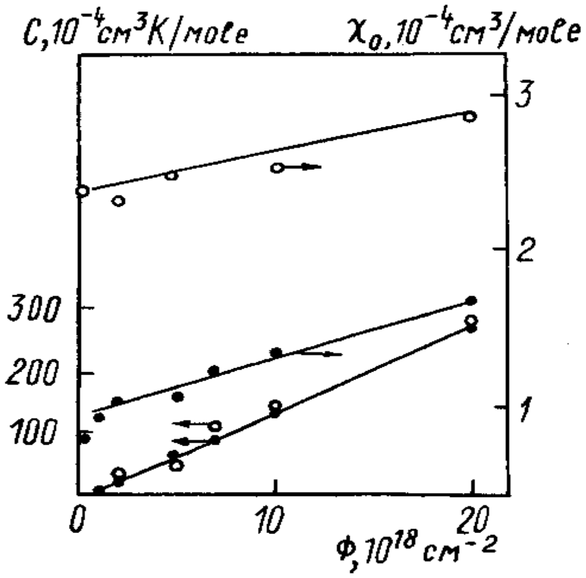

Fig. 5.27 Dependence of the Curie constant C and the temperature-independent part χ_0 of magnetic susceptibility on neutron fluence Φ for $La_{1.83}Sr_{0.17}CuO_4$ (black circles) and $YBa_2Cu_3O_{6.95}$ (circles).

proach proposed rather long ago by Warren [Warren W.W. (1971)] and later quantified theoretically in Refs.[Götze W., Ketterlee W. (1983); Sadovskii M.V. (1985)]. As we have demonstrated above the approach of the system to localization transition leads to the significant enhancement of nuclear spin relaxation rate in comparison with the usual Korringa-like relaxation in metals. In metallic phase, according to (2.121), this enhancement factor η can be introduced as:

$$\eta = \frac{T_1^K}{T_1} \approx 1 + \frac{\pi}{3}\frac{\sigma_c^2}{\sigma(\sigma + \sigma_c)} \tag{5.9}$$

where Korringa relaxation rate can be written from (2.122) as:

$$R_K = \frac{1}{T_1^K} = 2A^2 T\pi \frac{\chi_s^2}{2\mu_B^2} \tag{5.10}$$

where $\chi_s = 2\mu_B N(E_F)^2$ is the spin susceptibility of conduction electrons. In localized phase η must drop with disorder according to (2.123). The study of NMR relaxation rate on ^{89}Y nuclei in radiationally disordered $YBa_2Cu_3O_{6.95}$ (which opposite to Cu nuclei demonstrate Korringa behavior) was performed in Refs. [Verkhovskii S.V. et al. (1992); Zhdanov Yu.I. et al. (1992)]. Disorder (fluence) dependence of enhancement factor η was estimated form experimental data on Knight shift and NMR relaxation using the obvious relation, following from (5.9),(5.10):

$$\eta = \frac{R(\Phi)}{R(\Phi = 0)} \left(\frac{\chi_s(\Phi = 0)}{\chi_s(\Phi)} \right)^2 \tag{5.11}$$

where the last factor, containing χ_s, takes into account the change in the electronic density of states under disordering. In fact the Knight shift data of Refs.[Verkhovskii S.V. et al. (1992); Zhdanov Yu.I. et al. (1992)] strongly indicate some kind of the pseudogap opening at the Fermi level of strongly disordered oxides with rather significant drop of the density of states at the Fermi level with disordering. Similar conclusions follow from tunneling experiments of Ref.[Srikanth H. et al. (1992)] on a number of oxides disordered by doping. In this case we are most probably dealing with correlation (Coulomb) pseudogap discussed above, which is due to the effects of electron-electron interactions. However, similar behavior may be due to a smearing of some kind of Van Hove singularity close to the Fermi level. In Fig.5.28 we show the data for the enhancement factor η, obtained by [Zhdanov Yu.I. et al. (1992)]. We can clearly see a *maximum* of the enhancement factor which according to our discussion after Eq.(2.123) may indicate the Anderson transition somewhere in the fluence interval $\Phi = (1-2)10^{19} cm^{-2}$. In metallic region these data are rather well described by (5.9) with reasonable values of σ_c and σ values taken from the data on the same samples at $T > 100K$ [Verkhovskii S.V. et al. (1992); Zhdanov Yu.I. et al. (1992)]. Unfortunately the number of samples in these experiments was too limited to place the transition point more precisely, while superconductivity disappears exactly in this interval. In this sense we still have no direct proof of coexistence of superconductivity and localization in disordered high—T_c oxides. However the method used in Refs.[Verkhovskii S.V. et al. (1992); Zhdanov Yu.I. et al. (1992)] seems to be very promising.

Using the experimental data on electrical resistivity of disordered samples of $YBa_2Cu_3O_{6.95}$ and the relations given by Eq. (5.6) and Eq. (5.7) (assuming that exponentials there are identical) we can calculate the change of localization length R_{loc} as a function of fluence [Aleksashin B.A. et

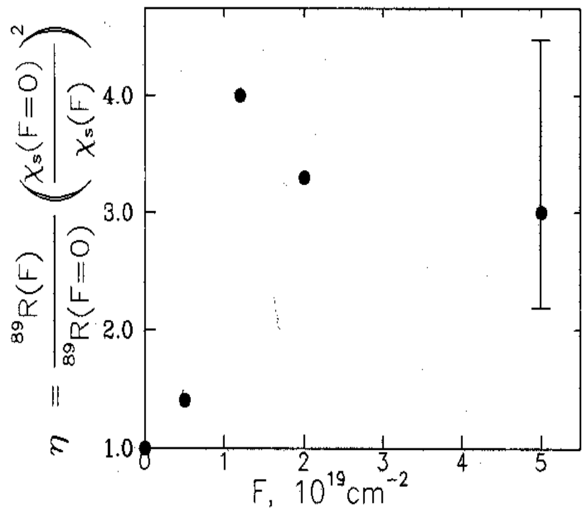

Fig. 5.28 Enhancement factor η of NMR relaxation rate (5.11) as a function of the fast-neutron fluence at $T = 300K$ form the measurements of ^{89}Y relaxation in radiationally disordered $YBa_2Cu_3O_{6.9}$.

al. (1989); Davydov S.A. et al. (1989a); Davydov S.A. et al. (1989b); Goshchitskii B.N. et al. (1989a); Goshchitskii B.N. et al. (1989b)]. This dependence is shown on Fig.5.29 along with fluence dependence of T_c. It can be seen that superconductivity is destroyed when localization length R_{loc} becomes smaller than $\sim 30\mathring{A}$, i.e. it becomes of the order or smaller than a typical size of the Cooper pair in this system (Cf. Fig.5.25) in complete accordance with our basic criterion of Eq. (3.20). We can estimate the minimal value of R_{loc} for which superconductivity can still exist in a system of localized electrons via Eq.(3.20) [Aleksashin B.A. et al. (1989)] taking the free-electron value of $N(E_F) \approx 5\ 10^{33}(ergcm^3)^{-1}$ (for carrier concentration of $\sim 6\ 10^{21}cm^{-3}$) and the gap value $\Delta \sim 5T_c$, corresponding to very strong coupling [Goshchitskii B.N., Kozhevnikov V.L., Sadovskii M.V. (1988)]. We obtain the result shown in Fig.5.29. In any case we can see that criterion of Eq. (3.20) ceases to be fulfilled for $\Phi \sim (5-7)10^{18}cm^{-2}$ in rather good agreement with the experiment.

In the absence of accepted pairing mechanism for high temperature superconductors it is very difficult to speculate on the reasons for T_c degradation in these systems. If we assume that the main mechanism of T_c degra-

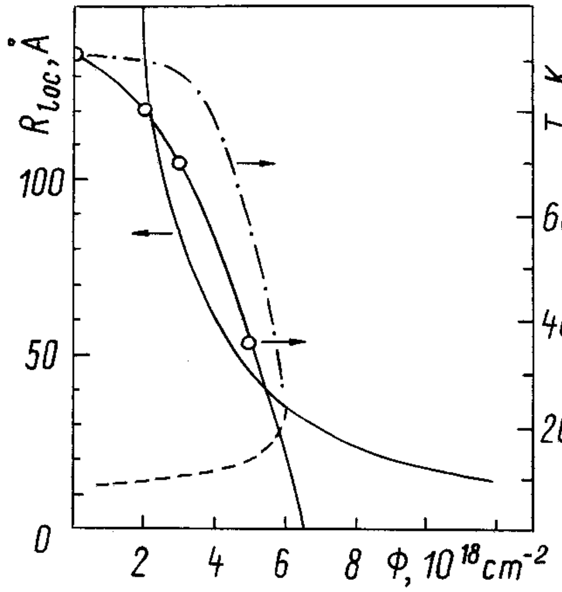

Fig. 5.29 Dependence of T_c on fluence for $YBa_2Cu_3O_{6.95}$ (circles). The solid curve is the localization length calculated from hopping conductivity. Dashed curve defines the minimum localization length at which superconductivity can exist at given T_c. Dashed-dotted curve is theoretical fit using expressions described in the text.

dation is connected with the growth of Coulomb effects during disordering, as discussed above, we can try to use appropriate expressions to describe the experimental data. Assuming superconductivity in the localized phase we can use Eq. (3.69), estimating R_{loc} as above from empirical relation Eq. (5.7) and Eq. (5.6) (or directly expressing the parameters entering Eq. (3.69) via experimental dependence of resistivity on fluence as described by Eq. (5.7) [Aleksashin B.A. et al. (1989)]). The results of such a fit (with the assumption of $\mu \approx 1$) are also shown in Fig.5.29. The agreement is also rather satisfactory, the more rapid degradation of T_c for small degrees of disorder can be related to additional contributions to Coulomb repulsion within Cooper pairs neglected in the derivation of Eq. (3.69). Surely we do not claim that this is a real explanation of T_c degradation in disordered high temperature superconductors. However, note its relation to localized moment formation under disordering which leads to the usual Abrikosov-Gorkov mechanism of depairing due to spin-flip scattering on magnetic impurities. According to Mott [Mott N.F. (1971)] (Cf. also Refs.[Kamimura H. (1980); Sadovskii M.V. (1986)]) the appearance of localized moments may be related to the presence of localized states (single occupied states below the Fermi level as briefly discussed above). We can then estimate the value of the effective magnetic moment (in Bohr magnetons) in unit cell as [Aleksashin B.A. et al. (1989)]:

$$\mu R_{loc}^{-3}\Omega_0 = p_{theor}^2 \tag{5.12}$$

where Ω_0 is the volume of a unit cell. For large degrees of disorder ($\Phi = 2\ 10^{19} cm^{-2}$) and $R_{loc} \approx 8\AA$ with $\mu \approx 1$ we obtain $p_{theor}^2 = 0.66$ for $YBa_2Cu_3O_{6.95}$ in full agreement with experiment. However, for smaller fluences p_{theor} is considerably smaller than the experimental value. Note, though , that the estimate of Eq. (5.12) is valid only for small enough values of R_{loc}, i.e. when the Fermi level is well inside the localized region. On the other hand, the accuracy with which the Curie constant is determined in weakly disordered samples is considerably less than in strongly disordered case. Of course, the other mechanisms of local moment formation, which were discussed above and can become operational even before the metal—insulator transition can be important here.

Of course, a plenty of works on localization effects in high-T_c oxides are being done using disorder induced by different types of chemical substitutions in these systems. Of these we shall rather arbitrarily quote Refs.[Ellman B. et al. (1989); Infante C. et al. (1990); Jayaram B., Lanchester P.C., Weller M.T. (1991); Mandrus D. et al. (1991); Cieplak M. et al. (1992)], which demonstrate the data quite similar, though not necessarily identical, to those described above on different types of systems and obtained by different experimental methods. We note that the effects of chemical disorder are almost always complicated by the inevitable changes of carrier concentration due to doping effects. Still all these data indicate that superconductivity in high-T_c systems is realized close to disorder induced metal—insulator transition, so that these systems provide us with plenty of possibilities to study experimentally the general problems discussed in our book. More details on chemical substitutions can be found in the extensive review paper of [Agarwal S.K., Narlikar A.V. (1994)].

Special attention should be payed to studies of angle resolved photoemission in Co and Ni doped single-crystals of $Bi_2Sr_2CaCu_2O_{8+y}$ [Quittmann C. et al. (1994); Beschoten B., et al. (1996)]. Doping $Bi_2Sr_2CaCu_2O_{8+y}$

with Co causes superconductor to insulator transition (Co doping decreases T_c and causes increase in residual resistivity). The changes in temperature behavior of resistivity from metallic to insulating like correlate with the disappearance of the dispersing band-like states in angle-resolved photoemission. Authors believe that Anderson localization caused by the impurity potential of the doped Co atoms provides a consistent explanation of all experimental features and T_c reduction is not caused by magnetic impurity pairbreaking effects, but by spatial localization of carriers with superconducting ground state being formed out of spatially localized carriers. Similar data were also obtained for some exceptional (apparently strongly disordered) samples of undoped $Bi_2Sr_2CaCu_2O_{8+y}$ [Ma Jian et al. (1994)]. Of course it will be very interesting to make similar type of experiments on neutron irradiated samples where we are dealing with pure disorder. Coexistence of superconductivity and localization was also claimed to follow from transport measurements on $Bi_2Sr_2(Ca_zPr_{1-z})Cu_2O_{8+y}$ [Beschoten B., et al. (1996)].

Superconductor-insulator transition was observed in Zn-substituted single crystal of $YBa_2Cu_3O_{7-y}$ and $La_{2-x}Sr_xCuO4$ [Fukuzumi Y. et al. (1996)]. The primary effect of Zn was to produce a large residual resistivity. In the underdoped regime (low carrier density in CuO_2 plane) only a few percent Zn was sufficient for resistivity to reach the critical value near the universal two-dimensional resistance $h/4e^2$ and to induce a superconductor – insulator transition. Note, that T_c suppression rate with disorder in this work was found in general to be more slow, than predicted by d–wave pairing theory [Radke R.J. et al. (1993)].

Based upon all of these data we can come to an apparently wrong conclusion that studies of disorder effects in high—T_c superconductors more or less exclude any possibility of d–wave pairing in these systems [Sadovskii M.V. (1997)]. We have seen in Section 3.2.7 that this kind of pairing is extremely sensitive to disordering (if we analyze it within the standard BCS approach) and is completely suppressed, roughly speaking, at the disorders measured by the energy scale $1/\tau \sim T_{c0}$, which is at least an order of magnitude smaller than the disorder necessary to induce the metal-insulator transition which can be estimated as $1/\tau \sim E_F$. Quite similar conclusios were, in fact, reached after rather extensive theoretical analysis in Ref.[Radke R.J. et al. (1993)]. However, nowadays the experimental evidence for d–wave pairing in these systems is almost compelling [Wollman D.A. et al. (1994);

Tsuei C.C. et al. (1994); Annett J., Goldenfeld N., Leggett A.J. (1996); Proc. M2S-HTSC]. In fact, modern ARPES data [Randeria M., Campuzano J.C. (1997)] for the anisotropic energy gap can even be nicely fitted to simplest angular dependence of the gap assumed e.g. in (3.195),(3.196), while Josephson data definitely signal the change of the sign of the gap [Tsuei C.C. et al. (1994); Annett J., Goldenfeld N., Leggett A.J. (1996)], as in the first expression in (3.195). This poses probably one of the most difficult theoretical problems with our understanding of high—T_c superconductors. There may be several ways out of this dilemma, some were discussed in Section 3.2.7 above. One possible (and in our opinion most probable) solution is that high-temperature superconductors are actually in the transition region between BCS-like systems with very large Cooper pairs to superconductors with very compact Cooper pairs (Bosons) [Posazhennikova A.I., Sadovskii M.V. (1997b); Franz M., et al. (1997)]. The other way to conserve finite T_c in systems with d–wave pairing up to disorder-induced metal-insulator transition is to assume that strong enough anisotropic disorder scattering operates in all of these systems [Haran G., Nagi A.D.S. (1996); Posazhennikova A.I., Sadovskii M.V. (1997a)], despite the fact that very different types of disorder were studied experimentally.

Here we shall discuss some of attempts to understand data on disordered high—T_c superconductors within the picture of d–wave pairing. In Ref.[Giapintzakis J., et al. (1994)] the authors carried out direct resistivity measurements of an untwinned single-crystal of $YBa_2Cu_3O_{7-\delta}$ in a high-voltage electron microscope to determine how the superconducting transition temperature T_c and resistivity change with with low-temperature electron irradiation. They concluded that point defects introduced by oxygen displacements from the CuO_2 planes make the dominant contribution to the irradiation-induced suppression of T_c. The data on T_c suppression were in general agreement with the picture of anisotropic pairing, but the rate of T_c degradation with the growth of disorder (resistivity) was significantly slower, than predictions of BCS-like theory for d–wave pairing, given by (3.100) and (3.101). In a later work [Giapintzakis J., Kirk M.A., Ginsberg D.M. (1995)] the same authors has given more data on T_c suppression in the same system. They have achieved rather good fit of their data to Abrikosov-Gorkov dependence for d–wave depairing (3.100) as a function of residual resistivity, but only using rather unrealistic values of plasma frequency ω_p in (3.101). The use of the values of ω_p determined from independent optical experiments leads to (3.100) predicting significantly faster T_c suppression, than observed in the experiment.

Somehow similar and even more detailed study of T_c suppression by disorder was performed in Ref.[Tolpygo S.K. et al. (1996)], where electron irradiated films of $YBa_2Cu_3O_{6+x}$ with different oxygen content were studied. Here also the comparison with T_c suppression by other in-plane defects such as Zn substitution for Cu, Pr substitution for Y, and radiation defects in $Bi-2201$ and $Bi-2212$ was made. The authors claimed that T_c suppression by defects occured in a universal way which was independent of the T_{c0}, carrier concentration, and number of CuO_2 planes per unit cell. This universal dependence could be described by pair-breaking theory for normal scattering in d–wave superconductors (3.100),(3.101), but requiring the pair-breaking rate be a factor of 3 *smaller* than suggested by the transport data. Thus again the unexpected stability of d–wave pairing to disorder was clearly demonstrated.

In a recent paper by [Lin J.Y. et al. (1999)] a study was made of impurity effects on the superconducting transition temperature T_c and the upper critical field H_{c2} in electron irradiated $YBa_2Cu_3O_y$ with in-plane oxygen defects and $YBa_2(Cu_{1-x}Zn_x)_3O_y$. T_c decreased slower in irradiated $YBa_2Cu_3O_y$ than in Zn-doped system (it is known for a long time, that Zn is especially effective in pair-breaking in copper oxides.). The authors were able to fit their T_c data on Zn-doped system to Abrikosov-Gorkov dependence of (3.100) (compare, however, the data of Ref.[Fukuzumi Y. et al. (1996)]), while data for irradiated samples were well fitted to generalized pair-breaking dependence of (3.209) [Haran G., Nagi A.D.S. (1996); Posazhennikova A.I., Sadovskii M.V. (1997a)] with unusually high anisotropy of disorder scattering $\gamma_1/\gamma_0 \approx 0.54$. The data on H_{c2} demonstrated the *drop* of the slope $(dH_{c2}/dT)_{T_c}$ with disorder growth, as predicted by [Posazhennikova A.I., Sadovskii M.V. (1996); Posazhennikova A.I., Sadovskii M.V. (1997a)], and nicely fitted to appropriate theoretical predictions of [Won H., Maki K. (1998)], similar to that of Ref. [Posazhennikova A.I., Sadovskii M.V. (1997a)]. This seems to first confirmation of the drastic difference in disorder dependence of the slope of H_{c2} in d–wave case in comparison with even anisotropic s–wave pairing, predicted in Refs. [Posazhennikova A.I., Sadovskii M.V. (1996); Posazhennikova A.I., Sadovskii M.V. (1997a)]. However, it should be taken into account, that reliable data on the slope of H_{c2} in copper oxides are rather difficult to obtain, mainly due to nonlinear dependence of H_{c2} on temperature close to T_c, which is observed in most experiments. Also these new data are in some contradiction with earlier studies quoted above. As to a good fit of T_c dependence on disorder, obtained in [Lin J.Y. et al. (1999)] using (3.209), it must be noted that the origin of scattering anisotropy in this case remains completely unclear. As we have already noted, scattering anisotropy really can explain the slower suppression rate of T_c with disorder in d-wave pairing systems, but it seems rather artificial explanation in view of rather universal problem of unusual stability of T_c in a wide range of copper oxides with completely different types of disorder, as discussed above. In this sense we are in need of more general explanation, e.g. like slower suppression rate of T_c in systems with "small" pairs [Posazhennikova A.I., Sadovskii M.V. (1997b); Franz M., et al. (1997)].

We shall limit ourselves to this discussion of localization effects in high temperature superconductors. Our conclusion is that these effects are extremely important in these systems and some of the anomalies can be successfully described by theoretical ideas formulated in this book. At the same time, these descriptions were only of qualitative nature and neglected many important aspects of the problem. We must stress that much additional work is needed both theoretical and experimental to clarify the general picture of disorder effects in high-T_c superconductors and we can expect that the future progress, especially with the quality of samples, may provide some new and exciting results.

Chapter 6

CONCLUSION

We conclude our book trying to formulate the basic unsolved problems. From the theoretical point of view probably the main problem is to formulate the theory of superconducting pairing in strongly disordered system along the lines of the general theory of interacting Fermi systems. This problem is obviously connected with the general theory of metal–insulator transition, which as we mentioned during our brief discussion above is rather far from its final form. Nevertheless, there were several attempts to analyze superconducting transition within this framework [Finkelstein A.M. (1987); Ma M., Fradkin E. (1986); Kirkpatrick T.R., Belitz D. (1991); Kirkpatrick T.R., Belitz D. (1992); Belitz D., Kirkpatrick T.R. (1994)]. In all cases, the authors limited themselves to certain universality classes within the general renormalization group approach of interaction theory of metal–insulator transition. Ref.[Finkelstein A.M. (1987)] dealt only with two–dimensional problem, while Refs.[Ma M., Fradkin E. (1986); Kirkpatrick T.R., Belitz D. (1991); Kirkpatrick T.R., Belitz D. (1992); Belitz D., Kirkpatrick T.R. (1994)] also considered the bulk case. These papers have demonstrated a large variety of possible behavior of superconductivity under disordering, from disorder–induced (triplet) superconductivity [Kirkpatrick T.R., Belitz D. (1991)] to a complete destruction of it close [Finkelstein A.M. (1987); Ma M., Fradkin E. (1986)] or even long before the metal–insulator transition [Kirkpatrick T.R., Belitz D. (1992)]. Our point of view is that at the moment it is rather difficult to make any *general* conclusions from the results of these approaches. In particular, we do not believe that the present status of these theories is sufficient to prove or disprove the general possibility of superconductivity in Anderson insulators. However, it is obvious that further theoretical progress in the problem of T_c behavior under disordering will be largely possible only within this general approach. In this sense our simplified discussion of Coulomb effects and other mechanisms of T_c degradation in this book is only of qualitative nature. Still, more general approaches apparently do not change our qualitative conclusions. These problems become even more complicated if we address ourselves to the case of high temperature superconductors, where we do not know precisely the nature of pairing interaction even in regular systems, as well as probably unusual role of interelectron interactions (non Fermi-liquid behavior etc.).

Concerning the semiphenomenological approach to the theory of superconductivity close to the Anderson transition we must stress the necessity of further investigation of the region of strong statistical fluctuations with the aim of more detailed study of their influence upon different physical properties, like e.g. the upper critical field, density of states, nuclear relaxation etc. Obviously, all of them may be significantly changed in comparison with predictions of what we called the statistical mean—field theory. Especially important are further studies of rather exotic predictions of random fluxes in the ground state [Spivak B., Kivelson S.A. (1991)].

Despite our explicit limitation to a discussion of superconductivity in *bulk* disordered superconductors we have to mention the extremely interesting problem of universal conductivity at the superconductor—insulator transition at $T = 0$ in two-dimensional systems which attracted much attention recently [Fisher M.P.A., Grinstein G., Girvin S.M. (1990); Cha Min-Chul et al. (1991); Girvin S.M. et al. (1992); Liu Y., Goldman A.M. (1994)]. It was argued that the transition between the insulating and superconducting phases of disordered two-dimensional system at zero temperature is of continuous quantum nature, but the system behaves like a normal metal right at the transition, i.e. the conductivity has a finite, nonzero value. This value is *universal* and, apparently, equal to $(2e)^2/h$ (with $2e$ being the Cooper pair charge). There is strong experimental evidence [Hebard A.F., Paalanen M. (1984); Hebard A.F., Paalanen M.A. (1985); Haviland D.B., Liu Y., Goldman A.M. (1989); Lee S.J., Ketterson J.B. (1990); Wang T. et al. (1991); Liu Y., Goldman A.M. (1994)] that a variety of systems (metallic films, high-T_c films, etc.) show the onset of superconductivity to occur when their sheet resistance falls below a value close to $h/4e^2 \approx 6.45k\Omega$. The theoretical analysis here is based upon Boson (Cooper pairs) approach to superconductivity and the main conclusion is that in contrast to the case of localization of fermions in two dimensions, bosons exhibit a superconductor to insulator transition (as disorder grows) with the value of conductivity at the critical point being independent of microscopic details. A major theoretical problem arises to describe a crossover to such behavior e.g. in quasi–two–dimensional case of BCS superconductivity as interplane coupling goes to zero.

Very interesting problems arise with respect to the study of disorder effects in anomalous "pseudogap" state observed in underdoped high—T_c cuprates [Randeria M. (1997)]. There is almost no theoretical and very few experimental studies of disordering in this peculiar state.

So we are not short of theoretical problems in this important field of research. As to the experiments, certainly too much is still to be done for unambiguous demonstration of exotic possibility of superconductivity of Anderson insulators.

Appendix A

Spectral Densities and Criterion for Localization

Convenient formalism to consider general properties of disordered system is based upon exact eigenstate representation for an electron in a random field created by disorder. These eigenstates $\phi_\nu(\mathbf{r})$ are formally defined by the Schrödinger equation:

$$H\phi_\nu(\mathbf{r}) = \varepsilon_\nu \phi_\nu(\mathbf{r}) \tag{A.1}$$

where H is one—particle Hamiltonian of disordered system under consideration, ε_ν are exact eigenvalues of electron energy in a random potential. Obviously $\phi_\nu(\mathbf{r})$ and ε_ν are dependent on locations of scatterers $\mathbf{R}_n$ for a given realization of random field.

Let us define two—particle spectral densities [Berezinskii V.L., Gorkov L.P. (1979); Sadovskii M.V. (1986)]:

$$\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F = \frac{1}{N(E)} < \sum_{\nu\nu'} \phi^\star_\nu(\mathbf{r})\phi_{\nu'}(\mathbf{r})\phi^\star_{\nu'}(\mathbf{r}')\phi_\nu(\mathbf{r}')\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'}) > \tag{A.2}$$

$$\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^H = \frac{1}{N(E)} < \sum_{\nu\nu'} |\phi_\nu(\mathbf{r})|^2|\phi_{\nu'}(\mathbf{r}')|^2\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'}) > \tag{A.3}$$

where angular brackets denote averaging over disorder and

$$N(E) = < \sum_\nu |\phi_\nu(\mathbf{r})|^2 \delta(E-\varepsilon_\nu) > \tag{A.4}$$

is one—electron (average) density of states. Obviously Eq. (A.3) is just a correlation function of *local* densities of states in a disordered system. Spectral density given by Eq. (A.2) determines electronic transport [Berezinskii V.L., Gorkov L.P. (1979)]. The following general properties are easily verified using the completeness and orthonormality of $\phi(\mathbf{r})$ functions:

$$\int d\mathbf{r} \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F = \delta(\omega) \qquad \int d\omega \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F = \delta(\mathbf{r}-\mathbf{r}') \tag{A.5}$$

or for the Fourier—components:

$$\ll \rho_E\rho_{E+\omega} \gg_{\mathbf{q}=0} = \delta(\omega) \qquad \int d\omega \ll \rho_E\rho_{E+\omega} \gg_{\mathbf{q}} = 1 \tag{A.6}$$

and $\ll \rho_E\rho_{E+\omega} \gg_{\mathbf{q}} \geq 0$. From general definitions given in Eqs. (A.2) and (A.3) it is clear that:

$$\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}) \gg^F = \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}) \gg^H \tag{A.7}$$

i.e. these spectral densities coincide for $\mathbf{r} = \mathbf{r}'$.

Terms with $\varepsilon_\nu = \varepsilon_{\nu'}$ are in general present in Eqs. (A.2) and (A.3). However, if these states are extended the appropriate wave—functions $\phi_\nu(\mathbf{r})$ are normalized on the total volume Ω of the system and these contributions to Eqs. (A.2) and (A.3) are proportional to to Ω^{-1} and vanish as $\Omega \to \infty$. Things change if states are localized. In this case states are normalized on finite volume of the order of $\sim R^d_{loc}$. This leads to the appearance of $\delta(\omega)$—contribution to spectral densities:

$$\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^{F,H} = A_E(\mathbf{r}-\mathbf{r}')\delta(\omega) + \rho_E^{F,H}(\mathbf{r}-\mathbf{r}'\omega) \tag{A.8}$$

or in momentum representation:

$$\ll \rho_E\rho_{E+\omega} \gg^{F,H}_{\mathbf{q}} = A_E(\mathbf{q})\delta(\omega) + \rho_E^{F,H}(\mathbf{q}\omega) \tag{A.9}$$

where the second terms are regular in ω. This singular behavior was proposed as a general criterion for localization [Berezinskii V.L., Gorkov L.P. (1979)]. It is easy to show that

$$A_E(\mathbf{r}-\mathbf{r}') = \frac{1}{N(E)} < \sum_\nu \delta(E-\varepsilon_\nu)|\phi_\nu(\mathbf{r})|^2|\phi_\nu(\mathbf{r}')|^2 > \tag{A.10}$$

$$A_E = A_E(\mathbf{r}-\mathbf{r}')|_{\mathbf{r}=\mathbf{r}'} \sim R^{-d}_{loc}$$

$A_E(\mathbf{r}-\mathbf{r}')$ represents the so called inverse participation ratio [Thouless D.J. (1974); Wegner F.J (1980)]. Roughly speaking its value at $\mathbf{r}=\mathbf{r}'$ is inversely proportional to the number of atomic orbitals which effectively form quantum state ν. These $\delta(\omega)$–singularities in spectral densities signal nonergodic behavior of the system in localized state. This leads to a difference between so called adiabatic and isothermal response functions [Kubo R. (1957); Katsnelson M.I., Sadovskii M.V. (1984); Sadovskii M.V. (1986)]. The intimate connection between localization and nonergodic behavior was already noted in the first paper by Anderson [Anderson P.W. (1958)].

From general properties given by Eqs. (A.5) and (A.6) for $\mathbf{q}\to\mathbf{0}$ in localization region we have [Berezinskii V.L., Gorkov L.P. (1979)]:

$$\ll \rho_E\rho_{E+\omega} \gg^F_{\mathbf{q}} \approx [1-R^2_{loc}q^2]\delta(\omega)+\cdots \tag{A.11}$$

where

$$R^2_{loc}=\frac{1}{2dN(E)}\int d^drr^2<\sum_\nu\delta(E-\varepsilon_\nu)|\phi_\nu(\mathbf{r})|^2|\phi_\nu(0)|^2> \tag{A.12}$$

defines the localization length. Delocalization leads to smearing of $\delta(\omega)$—singularity for finite q.

Spectral densities of Eqs. (A.2) and (A.3) can be expressed via two—particle Green's functions [Sadovskii M.V. (1986)]. Using nonaveraged retarded and advanced Green's functions:

$$G^R(\mathbf{r}\mathbf{r}'E)=G^{A\star}(\mathbf{r}\mathbf{r}'E)=\sum_\nu\frac{\phi_\nu(\mathbf{r})\phi^\star_\nu(\mathbf{r}')}{E-\varepsilon_\nu+i\delta} \tag{A.13}$$

we immediately get from Eqs. (A.2) and (A.3):

$$\begin{aligned}\ll \rho_E(\mathbf{r})\rho_{E'}(\mathbf{r}') \gg^F = \frac{1}{2\pi^2N(E)}Re\,\{&<G^R(\mathbf{r}\mathbf{r}'E')G^A(\mathbf{r}'\mathbf{r}E)>\\ &-<G^{R,A}(\mathbf{r}\mathbf{r}'E')G^{R,A}(\mathbf{r}'\mathbf{r}E)>\}\end{aligned} \tag{A.14}$$

$$\begin{aligned}\ll \rho_E(\mathbf{r})\rho_{E'}(\mathbf{r}') \gg^H = \frac{1}{2\pi^2N(E)}Re\,\{&<G^R(\mathbf{r}\mathbf{r}E')G^A(\mathbf{r}'\mathbf{r}'E)>\\ &-<G^{R,A}(\mathbf{r}\mathbf{r}E')G^{R,A}(\mathbf{r}'\mathbf{r}'E)>\}\end{aligned} \tag{A.15}$$

In momentum representation Eq. (A.14) is equivalent to:

$$\ll \rho_E\rho_{E+\omega} \gg^F_{\mathbf{q}} = \frac{1}{\pi N(E)}Im\,\{\Phi^{RA}_E(\omega\mathbf{q})-\Phi^{RR}_E(\omega\mathbf{q})\} \tag{A.16}$$

where

$$\Phi_E^{RA(R)}(\mathbf{q}\omega) = -\frac{1}{2\pi i}\sum_{\mathbf{pp}'} < G^R(\mathbf{p}_+\mathbf{p}'_+E+\omega)G^{A(R)}(\mathbf{p}'_-\mathbf{p}_-E) > \quad \text{(A.17)}$$

and $\mathbf{p}_{+-} = \mathbf{p} \pm 1/2\mathbf{q}$. It can be shown [Vollhardt D., Wölfle P. (1980); Vollhardt D., Wölfle P. (1982)] that $\Phi_E^{RR(AA)}(\omega\mathbf{q})$ are nonsingular for small ω and q. Accordingly $\delta(\omega)$—singularity signalling localization can appear only from the first term in Eq. (A.16).

The appearance of $\delta(\omega)$ singularities in (A.8) and (A.9) immediately leads to $1/\omega$–singularity in $\Phi_E^{RA}(\mathbf{q}\omega)$ in the localized phase [Sadovskii M.V. (1986); Suslov I.M. (1995)]:

$$\Phi_E^{RA}(\mathbf{q}\omega) = -\frac{N(E)}{\omega}A_E(\mathbf{q}) + \Phi_{reg}^{RA}(\mathbf{q}\omega) \quad \text{(A.18)}$$

where the second term in the right-hand side is some non-singular contribution. Comparison of (A.18) with (2.46) shows that in the localized phase $D_E(\mathbf{q}\omega) \sim \omega$, as the slower ω-dependence will destroy the $1/\omega$–singularity in (A.18) , while the faster will lead to disappearance of $\mathbf{q}$–dependence of the singular part of (2.46),(A.18). Thus in fact we have the following form of the generalized diffusion coefficient in the localized phase:

$$D_E(\mathbf{q}\omega) = (-i\omega)d(\mathbf{q}) \quad \text{(A.19)}$$

where in $d(\mathbf{q})$ the limit of $\omega \to 0$ is assumed. Thus from the general localization criterion of Berezinskii and Gorkov it follows that $D_E(\mathbf{q}\omega \to 0) \to 0$ for *all* values of $\mathbf{q}$ simultaneously, leading to conclusions and criticism following (2.86).

Appendix B

Linearized Gap Equation in Disordered System

Let us consider the derivation of linearized gap equation Eq. (3.56) used to determine T_c [Dolgov O.V. (1977); Dolgov O.V., Sadovskii M.V. (1984)]. Equation for Gorkovs's anomalous Green's function in an inhomogeneous disordered system (before any averaging procedure) at $T = T_c$ takes the following form:

$$(\varepsilon_n^2 + \hat{\varepsilon}_{\mathbf{r}}^2)F(\mathbf{rr}'\varepsilon_n) = -T_c \sum_m V(\mathbf{rr}'\varepsilon_n - \varepsilon_m)F(\mathbf{rr}'\varepsilon_m) \tag{B.1}$$

where $\varepsilon_n = (2n+1)\pi T_c$ and $V(\mathbf{rr}'\varepsilon_n - \varepsilon_m)$ is an effective interelectron potential, $\hat{\varepsilon}_{\mathbf{r}}$ is one–electron energy operator (energy zero is at the Fermi energy). Define

$$\Delta(\mathbf{rr}') = -2\hat{\varepsilon}_{\mathbf{r}} cth \frac{\hat{\varepsilon}_{\mathbf{r}}}{2T_c} T_c \sum_n F(\mathbf{rr}'\varepsilon_n) \tag{B.2}$$

and assume the following relation between $\Delta(\mathbf{rr}')$ and $F(\mathbf{rr}'\varepsilon_n)$:

$$F(\mathbf{rr}'\varepsilon_n) = \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_{\mathbf{r}}^2} T_c \sum_m V(\mathbf{rr}'\varepsilon_n - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_{\mathbf{r}}^2} \hat{Q}(\mathbf{rr}'\varepsilon_m)\Delta(\mathbf{rr}') \tag{B.3}$$

where $\hat{Q}$ is some unknown operator. Then after substitution of Eq. (B.3) into Eq. (B.2) we get a BCS—like equation for T_c:

$$\Delta(\mathbf{rr}') = -\hat{U}(\mathbf{rr}') \frac{th \frac{\hat{\varepsilon}_{\mathbf{r}}}{2T_c}}{2\hat{\varepsilon}_{\mathbf{r}}} \Delta(\mathbf{rr}') \tag{B.4}$$

where the operator of "effective" interaction is defined by:

$$\hat{U}(\mathbf{rr}') = 2\hat{\varepsilon}_{\mathbf{r}} cth\left(\frac{\hat{\varepsilon}_{\mathbf{r}}}{2T_c}\right) T_c \sum_n \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_{\mathbf{r}}^2} T_c \sum_m V(\mathbf{rr}'\varepsilon_n - \varepsilon_m) \times$$
$$\times \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_{\mathbf{r}}^2} \hat{Q}(\mathbf{rr}'\varepsilon_m) 2\hat{\varepsilon}_{\mathbf{r}} cth \frac{\hat{\varepsilon}_{\mathbf{r}}}{2T_c} \quad \text{(B.5)}$$

From Eqs. (B.1)–(B.3) we obtain the following equation for $\hat{Q}$ (we drop $\mathbf{rr}'$ for brevity):

$$\hat{Q}(\varepsilon_n) = 1 - T_c \sum_m V(\varepsilon_n - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}^2} \hat{Q}(\varepsilon_m) +$$
$$+ 2\hat{\varepsilon} cth\left(\frac{\hat{\varepsilon}}{2T_c}\right) T_c \sum_{n'} \frac{1}{\varepsilon_{n'}^2 + \hat{\varepsilon}^2} T_c \sum_m V(\varepsilon_{n'} - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}^2} \hat{Q}(\varepsilon_m) \quad \text{(B.6)}$$

In case of weak coupling in the lowest order over interaction in Eq. (B.6) we can leave only the first term $\hat{Q}(\varepsilon_n) = 1$. Then Eq. (B.5) reduces to

$$\hat{U}(\mathbf{rr}') = 2\hat{\varepsilon}_{\mathbf{r}} cth\left(\frac{\hat{\varepsilon}_{\mathbf{r}}}{2T_c}\right) T_c \sum_n \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_{\mathbf{r}}^2} T_c \sum_m V(\mathbf{rr}'\varepsilon_n - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_{\mathbf{r}}^2} 2\hat{\varepsilon}_{\mathbf{r}} cth \frac{\hat{\varepsilon}_{\mathbf{r}}}{2T_c} \quad \text{(B.7)}$$

and Eq. (B.4) completely determines T_c.

Using the usual definition of superconducting gap:

$$\Delta(\mathbf{rr}'\varepsilon_n) = T_c \sum_m V(\mathbf{rr}'\varepsilon_n - \varepsilon_m) F(\mathbf{rr}'\varepsilon_m) = -(\varepsilon_n^2 + \hat{\varepsilon}_{\mathbf{r}}^2) F(\mathbf{rr}'\varepsilon_n) \quad \text{(B.8)}$$

it is easy to get:

$$\Delta(\mathbf{rr}'\varepsilon_n) = \hat{Q}(\mathbf{rr}'\varepsilon_n)\Delta(\mathbf{rr}') \quad \text{(B.9)}$$

so that $\Delta(\mathbf{rr}')$ represents the energy gap in the absence of frequency dispersion, while $\hat{Q}$ describes the frequency dependence of the energy gap.

Cooper pairing takes place in the states which are time–reversed, thus in the exact eigenstate representation of an electron in disordered system we have:

$$\Delta(\mathbf{rr}') = \sum_\nu \Delta_\nu \phi_\nu^\star(\mathbf{r}')\phi_\nu(\mathbf{r}) \quad \text{(B.10)}$$

and Eq. (B.4) gives

$$\Delta_\nu = -\sum_{\nu\nu'} \frac{1}{2\varepsilon_{\nu'}} th \frac{\varepsilon_{\nu'}}{2T_c} U_{\nu\nu'} \Delta_{\nu'} \tag{B.11}$$

where the kernel

$$U_{\nu\nu'} = \int d\mathbf{r} \int d\mathbf{r}' \phi^\star_\nu(\mathbf{r}) \phi^\star_{\nu'}(\mathbf{r}') \hat{U}(\mathbf{r}\mathbf{r}') \phi_\nu(\mathbf{r}') \phi_\nu(\mathbf{r}) \tag{B.12}$$

has the form of "Fock" matrix element of an effective interaction. From Eq. (B.7) we have:

$$U_{\nu\nu'} = T_c^2 \frac{2\varepsilon_\nu \varepsilon_{\nu'}}{th\frac{\varepsilon_\nu}{2T_c} th\frac{\varepsilon_{\nu'}}{2T_c}} \sum_n \sum_m \frac{1}{\varepsilon_n^2 + \varepsilon_\nu^2} \frac{1}{\varepsilon_m^2 + \varepsilon_{\nu'}^2} \times$$
$$\times \int d\mathbf{r} \int d\mathbf{r}' \phi^\star_\nu(\mathbf{r}) \phi^\star_{\nu'}(\mathbf{r}') V(\mathbf{r}\mathbf{r}'\varepsilon_n - \varepsilon_m) \phi_\nu(\mathbf{r}') \phi_{\nu'}(\mathbf{r}) \tag{B.13}$$

It is convenient to rewrite Eq. (B.11) introducing summation over states belonging to some surface of constant energy with subsequent integration over energies:

$$\Delta_\nu = -\int_{-\infty}^{\infty} dE' \frac{1}{2E'} th \frac{E'}{2T_c} \sum_{\nu' \in E'} \mathcal{N}(E') U_{\nu\nu'(E)} \Delta_{\nu'(E')} \tag{B.14}$$

where $\mathcal{N}(E) = \sum_\nu \delta(E - \varepsilon_\nu)$.

Consider now averaging of the gap equation. Define

$$\Delta(E) = \frac{1}{N(E)} < \sum_\nu \Delta_\nu \delta(E - \varepsilon_\nu) > \tag{B.15}$$

i.e. the gap averaged over disorder and a surface of constant energy $E = \varepsilon_\nu$. Here as usual we denote $N(E) =< \mathcal{N}(E) >$. Suppose now that $\Delta_\nu = \Delta(\varepsilon_\nu) = \Delta(E = \varepsilon_\nu)$, i.e. that Δ_ν depends only on energy $E = \varepsilon_\nu$, but not on the quantum numbers ν. This is similar to the usual assumption of $\Delta(\mathbf{p})$ depending only on $|\mathbf{p}|$ in a homogeneous and isotropic system [Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982)].

After the usual decoupling used e.g. in transforming Eq. (3.12) into Eq. (3.13), i.e. assuming the self—averaging of the gap, we obtain the following linearized gap equation determining T_c:

$$\Delta(E) = -\int_{-\infty}^{\infty} dE' K(E,E') \frac{1}{2E'} th \frac{E'}{2T_c} \Delta(E') \tag{B.16}$$

where

$$K(E,E') = \frac{1}{N(E)} < \sum_{\nu\nu'} U_{\nu\nu'} \delta(E-\varepsilon_\nu)\delta(E'-\varepsilon_{\nu'}) >=$$
$$= T_c^2 \sum_n \sum_m \left[\frac{2E}{th(E/2T_c)} \frac{1}{\varepsilon_n^2+E^2}\right] \left[\frac{2E'}{th(E'/2T_c)} \frac{1}{\varepsilon_m^2+E'^2}\right] \times$$
$$\times \int d\mathbf{r} \int d\mathbf{r}' V(\mathbf{r}-\mathbf{r}'\varepsilon_n-\varepsilon_m) \ll \rho_E(\mathbf{r})\rho_{E'}(\mathbf{r}') \gg^F \tag{B.17}$$

where we have again introduced Gorkov–Berezinskii spectral density defined in Eq. (A.2). Effective interaction can be written as:

$$V(\mathbf{r}-\mathbf{r}'\varepsilon_n-\varepsilon_m) = V_p(\mathbf{r}-\mathbf{r}'\varepsilon_n-\varepsilon_m) + V_c(\mathbf{r}-\mathbf{r}'\varepsilon_n-\varepsilon_m) \tag{B.18}$$

i.e. as the sum of some kind of Boson–exchange attractive interaction V_p and Coulomb repulsion V_c, which leads to:

$$K(E,E') = K_p(E,E') + K_c(E,E') \tag{B.19}$$

Assuming $V_c(\mathbf{r}-\mathbf{r}'\varepsilon_n-\varepsilon_m) = v(\mathbf{r}-\mathbf{r}')$, i.e. static approximation for Coulomb repulsion, we obtain:

$$K_c(E,E') = \int d\mathbf{r} \int d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \ll \rho_E(\mathbf{r})\rho_{E'}(\mathbf{r}') \gg^F \tag{B.20}$$

which coincides with Eq. (3.34) used above in our analysis of Coulomb repulsion within Cooper pairs in disordered systems. Above we have used the approximation of Eq. (3.53) to model K_p due to electron–phonon pairing mechanism (or similar model for some kind of excitonic pairing). In this case Eq. (B.16) reduces to Eq. (3.56).

Note that $V_c(\mathbf{r}-\mathbf{r}'\varepsilon_n-\varepsilon_m)$ may be taken also as dynamically screened Coulomb interaction. Then we must use the appropriate expressions for dielectric function $\epsilon(\mathbf{q}\omega_m)$ which may be found using the self–consistent theory of localization [Katsnelson M.I., Sadovskii M.V. (1983); Katsnelson M.I., Sadovskii M.V. (1984)]. Then after some tedious calculations we can get the expressions for $K_c(E,E')$ which for small $|E-E'|$ practically coincide with those used by us above for the case of static short–range interaction [Dolgov O.V., Sadovskii M.V. (1984)].

Appendix C

Derivation of Ginzburg-Landau Coefficients

Here we present some details on the derivation of Ginzburg-Landau expansion for the free-energy of a superconductor. Consider an electron propagating in a random field of superconducting order-parameter fluctuations $\Delta_{\mathbf{q}}$ with small wave-vector $\mathbf{q}$ above T_c. Let us calculate a correction to the free-energy due to these fluctuations. In general case it is expressed via the average finite temperature scattering σ-matrix [Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)]:

$$\Delta F = -Tln < \sigma >, \tag{C.1}$$

where

$$\sigma = T_{\tau} exp \left\{ - \int_0^{1/T} H_{int}(\tau) d\tau \right\}, \tag{C.2}$$

and τ - imaginary (Matsubara) time. The interaction Hamiltonian in our case describes the system of free electrons in the field of fluctuating "sources" of Cooper pairs:

$$H_{int}(\tau) = \sum_{\mathbf{p}} \left[\Delta_{\mathbf{q}} a^{+}_{\mathbf{p}_+}(\tau) a^{+}_{-\mathbf{p}_-}(\tau) + \Delta^{*}_{\mathbf{q}} a_{-\mathbf{p}_-}(\tau) a_{\mathbf{p}_+}(\tau) \right], \tag{C.3}$$

where $\mathbf{p}_{\pm} = \mathbf{p} \pm \mathbf{q}/2$, $\quad a^{+}_{\mathbf{p}}, a_{\mathbf{p}}$ – creation and annihilation operators of electrons with momenta $\mathbf{p}$. Next we consider only static fluctuations, neglecting τ – dependence of $\Delta_{\mathbf{q}}$.

Free-energy correction is determined by connected (loop) diagrams from perturbation theory expansion of $< \sigma >$ [Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)]:

$$\Delta F = -T\left\{< \sigma >_c -1\right\}, \tag{C.4}$$

where $< \sigma >_c -1$ up to a fourth order in H_{int} is given by:

$$< \sigma >_c -1 = \frac{1}{2!}\int_0^{1/T} d\tau_1 \int_0^{1/T} d\tau_2 \left\langle T_\tau \left(H_{int}(\tau_1)H_{int}(\tau_2)\right)\right\rangle +$$
$$\frac{1}{4!}\int_0^{1/T} d\tau_1 ... \int_0^{1/T} d\tau_4 \left\langle T_\tau \left(H_{int}(\tau_1)...H_{int}(\tau_4)\right)\right\rangle \tag{C.5}$$

Calculating these contributions via Wick-theorem [Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965)] we get the following expression for free-energy of an electron in the field of fluctuating pairs:

$$\Delta F \simeq -|\Delta_{\mathbf{q}}|^2 T \sum_{\mathbf{p}} \sum_n G(\epsilon_n \mathbf{p}_+)G(-\epsilon_n - \mathbf{p}_-)$$
$$+\frac{T}{2}|\Delta_{\mathbf{q}}|^4 \sum_{\mathbf{p}} \sum_n G^2(\epsilon_n \mathbf{p}_+)G^2(-\epsilon_n - \mathbf{p}_-). \tag{C.6}$$

We can now obtain Ginzburg-Landau expansion for the free-energy difference between superconducting and normal state of the system if we rewrite (C.6) in such a way that the coefficient A of $|\Delta_{\mathbf{q}}|^2$ for $\mathbf{q} = 0$ be negative for $T < T_c$ and zero at $T = T_c$ [Gennes de P.G. (1966)]. To guarantee this we can subtract from the right-hand side of (C.6) an additional term equal to the $|\Delta_{\mathbf{q}}|^2$ correction to the free-energy at $T = T_c$ and $\mathbf{q} = 0$. As a result we get the Ginzburg-Landau expansion in the following form:

$$F_s - F_n = -|\Delta_{\mathbf{q}}|^2 T \sum_{\mathbf{p}} \sum_n G(\epsilon_n \mathbf{p}_+)G(-\epsilon_n - \mathbf{p}_-) +$$
$$|\Delta_{\mathbf{q}}|^2 T_c \sum_{\mathbf{p}} \sum_n G(\epsilon_n \mathbf{p})G(-\epsilon_n - \mathbf{p}) + \frac{T_c}{2}|\Delta_{\mathbf{q}}|^4 \sum_{\mathbf{p}} \sum_n G^2(\epsilon_n \mathbf{p})G^2(-\epsilon_n - \mathbf{p}) \tag{C.7}$$

Here we have taken into account that the coefficient B is finite for $T = T_c$, so that it can be taken at $T = T_c$, and neglect its dependence on $\mathbf{q}$.

Using the standard BCS equation for T_c we can rewrite (C.7) as:

$$F_s - F_n = \frac{1}{g}|\Delta_{\mathbf{q}}|^2 - |\Delta_{\mathbf{q}}|^2 T \sum_{\mathbf{p}} \sum_n G(\epsilon_n \mathbf{p}_+)G(-\epsilon_n - \mathbf{p}_-)$$

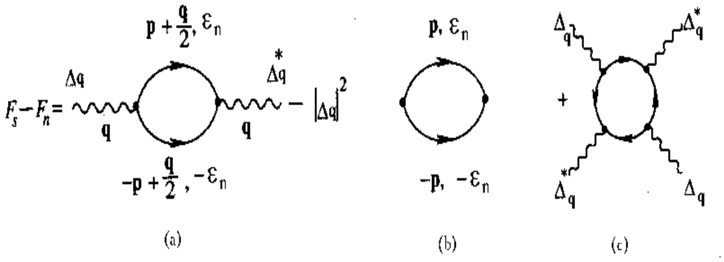

Fig. C.1 Diagrams determining Ginzburg-Landau expansion of a superconductor without impurities.

$$+\frac{T_{c0}}{2}|\Delta_{\mathbf{q}}|^4\sum_{\mathbf{p}}\sum_{n}G^2(\epsilon_n\mathbf{p})G^2(-\epsilon_n{-}\mathbf{p}). \tag{C.8}$$

Now it is clear that the Ginzburg-Landau expansion can be derived from diagram expansion shown in Fig.C.1. Expressions given above assume s–wave pairing, in case of anisotropic pairing we have to add to the vertexes extra factors of $e(\phi)$, defined e.g. in (3.195), to take into account the symmetry of the order parameter ($e(\phi) = 1$ for isotropic s–wave pairing). Subtraction of the second diagram, corresponding to the usual T_c–equation, guarantees correct behavior of the coefficient A.

In the case of an impure superconductor these diagram expansion is obviously generalized as shown in Fig.C.2. From here we can easily obtain all the expressions like (3.134), (3.135) used in the main text.

Now we give some details of the calculation of the expressions (3.134), (3.135), (3.136) for the GL-coefficients. Using (3.127), (3.129), (3.130), we obtain:

$$A = \frac{1}{g} - 2N(E_F)\sum_{n\geq 0}^{n\star}\frac{1}{2n+1} = \frac{1}{g} - N(E_F)\ln 1.13\frac{<\omega>}{T} = N(E_F)\ln\frac{T}{T_c} \tag{C.9}$$

where $n^\star = \frac{<\omega>}{2\pi T}$ has been introduced to cut off the logarithmic divergence, taking into account that electron attraction exists in the energy region of the order of $2 < \omega >$ around the Fermi level. The generalized diffusion coefficient $D_E(\omega_m)$ does not contribute (due to $q = 0$ in (3.127)). This is a reflection of the Anderson theorem – disorder influences T_c only through changes in the density of states $N(E_F)$.

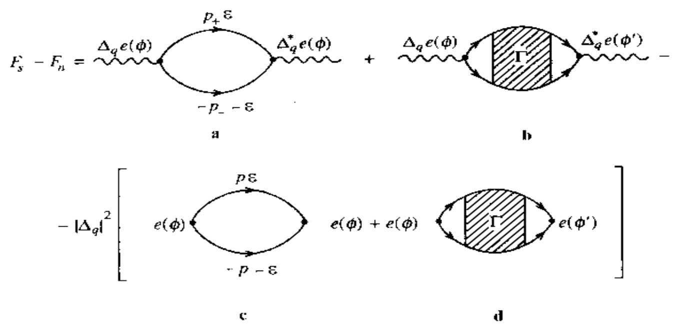

Fig. C.2 Diagrams determining Ginzburg-Landau expansion of a superconductor with impurities. Only diagrams up to a second order in $\Delta_{\mathbf{q}}$ are shown. Γ – vertex-part of impurity scattering in the "ladder" approximation. Diagrams (c,d) are calculated for $q = 0$ and $T = T_c$. Extra factors of $e(\phi)$ are shown at the vertexes, taking into account the symmetry of the order-parameter.

We shall calculate the coefficient B, neglecting the weak dependence on $\mathbf{q}$. Then it can be seen from that the contribution of the diagrams in Figs.C.3b and C.3c is small in comparison with that of Fig.C.3a. The "triangular" vertex can be found in the self-consistent theory of localization as described in [Katsnelson M.I., Sadovskii M.V. (1983); Katsnelson M.I., Sadovskii M.V. (1984)]. We have

$$\gamma(\mathbf{q}=0, \omega_m = 2\varepsilon_n) \approx 1 + \frac{1}{2\tau|\varepsilon_n|} \tag{C.10}$$

where the first term takes account of "high" frequencies, while the next is a diffusion contribution. Then, from Fig.C.3a we obtain:

$$B \approx N(E_F)T \sum_{\varepsilon_n} \int_{-\infty}^{\infty} d\xi_{\mathbf{p}} \gamma^4(\mathbf{q}=0, \omega_m = 2\varepsilon_n) G^2(\varepsilon_n \xi_{\mathbf{p}}) G^2(-\varepsilon_n \xi_{\mathbf{p}})$$
$$= N(E_F)T \frac{\pi}{2} \sum_{n \geq 0} \frac{1}{\varepsilon_n^3} = \frac{7\zeta(3)}{8\pi^2 T} N(E_F) \tag{C.11}$$

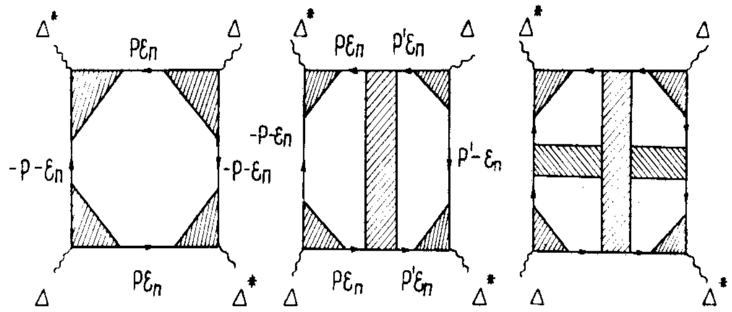

Fig. C.3 Diagrams determining the coefficient B.

where

$$G(\varepsilon_n \xi_{\mathbf{p}}) = \frac{1}{i\varepsilon_n - \xi_{\mathbf{p}} + \frac{i}{2\tau}\frac{\varepsilon_n}{|\varepsilon_n|}} \tag{C.12}$$

is the usual approximation for one-particle Green's function in "dirty" system.

Consider now derivation of the coefficient C. Using (3.128), (3.129), (3.130), we find for the metallic region ($E_F > E_c$), not very close to the mobility edge, when $D_E(\omega_m) = D_{E_F}$,

$$C = -i\pi T N(E_F) \sum_{\varepsilon_n} \frac{\partial^2}{\partial q^2} \left. \frac{1}{2i|\varepsilon_n| + iD_{E_F} q^2} \right|_{q=0}$$

$$= \pi T N(E_F) D_{E_F} \sum_{\varepsilon_n} \frac{1}{2\varepsilon_n^2} = \frac{N(E_F) D_{E_F}}{\pi T} \sum_{n \geq 0} \frac{1}{(2n+1)^2} = \frac{\pi}{8T} N(E_F) D_{E_F} \tag{C.13}$$

Analogously, for an insulator ($E_F < E_c$), but also not very close to the mobility edge, when (Cf.(3.133))

$$D_E(\omega_m) = D_{E_F} \frac{\omega_m}{\omega_m + 3D_F \omega_0^2 / v_F^2} \tag{C.14}$$

we obtain

$$C = \frac{\pi}{2} T N(E_F) \sum_{\varepsilon_n} \frac{1}{\varepsilon_n^2} D_{E_F}(2|\varepsilon_n|)$$

$$= \frac{N(E_F)D_{E_F}}{2\pi T}\sum_{\varepsilon_n}\frac{1}{(2n+1)^2+(2n+1)3D_{E_F}\omega_0^2/2\pi T v_F^2}$$
$$= \frac{N(E_F)v_F^2}{3\omega_0^2}\left[\Psi(\frac{1}{2}+\frac{3D_{E_F}\omega_0^2}{4\pi T v_F^2})-\Psi(\frac{1}{2})\right]$$
$$= N(E_F)R_{loc}^2\left[\Psi(\frac{1}{2}+\frac{D_{E_F}}{4\pi T R_{loc}^2})-\Psi(\frac{1}{2})\right]$$
$$\approx N(E_F)R_{loc}^2\ln\frac{1.78D_{E_F}}{\pi T R_{loc}^2} \qquad (C.15)$$

where the approximate expression is valid for $D_{E_F}R_{loc}^2 \gg 4\pi T$.

In the vicinity of the mobility edge, both for a metal and an insulator, we have (Cf.(3.133)):

$$D_{E_F}(\omega_m) \approx (\pi/2)^{2/3}D_0(\omega_m\tau)^{1/3} = (\pi/2\sqrt{3})^{2/3}(D_0 l)^{2/3}\omega_m^{1/3} \qquad (C.16)$$

so that

$$C = \frac{\pi}{2}TN(E_F)\sum_{\varepsilon_n}\frac{1}{\varepsilon_n^2}D_{E_F}(2|\varepsilon_n|) = \frac{\pi}{2}\left(\frac{\pi}{\sqrt{6}}\right)^{2/3}(D_0 l)^{2/3}TN(E_F)\sum_{\varepsilon_n}\frac{1}{|\varepsilon_n|^{5/3}}$$
$$= \frac{1}{6^{1/3}}\left(\frac{D_0 l}{T}\right)^{2/3}N(E_F)\sum_{n\geq 0}\frac{1}{(2n+1)^{5/3}}$$
$$= \frac{1}{6^{1/3}}\left(1-\frac{1}{2^{5/3}}\right)\zeta(\frac{5}{3})N(E_F)\left(\frac{D_0 l}{T}\right)^{2/3} \qquad (C.17)$$

Expression (C.17) dominates over (C.13) for

$$D_{E_F}/T_c \approx D_0 l/R_{loc}T_c \leq D_0^{2/3}(l/T_c)^{2/3} \qquad (C.18)$$

which defines the limits of applicability given in (3.136).

Appendix D

Anisotropic scattering

Here we shall present some technical details concerning the analysis of anisotropic impurity scattering. The Bethe-Salpeter equation for the vertex part takes the form (for brevity we drop some of the arguments):

$$\Gamma_{\mathbf{pp}'} = U(\mathbf{p},\mathbf{p}') + \sum_{\mathbf{p}''} U(\mathbf{p},\mathbf{p}'')G^R(\mathbf{p}'')G^A(\mathbf{p}'')\Gamma_{\mathbf{p}''\mathbf{p}'}, \qquad \text{(D.1)}$$

where $U(\mathbf{p},\mathbf{p}')$-is irreducible vertex function. We take $U(\mathbf{p},\mathbf{p}')$ in the following form ("ladder" approximation):

$$U(\mathbf{p},\mathbf{p}') = \rho V_0^2 + \rho V_1^2 f(\mathbf{p})f(\mathbf{p}'). \qquad \text{(D.2)}$$

where the notations and definitions are the same as in (3.197) and (3.198). Then Eq.(D.1) can be written as:

$$\Gamma_{\mathbf{pp}'} = \rho V_0^2 + \rho V_1^2 f(\mathbf{p})f(\mathbf{p}') + \rho V_0^2 \Psi(\mathbf{p}') + \rho V_1^2 f(\mathbf{p})\Phi(\mathbf{p}') \qquad \text{(D.3)}$$

where

$$\Psi(\mathbf{p}') = \sum_{\mathbf{p}''} G^R(\mathbf{p}'')G^A(\mathbf{p}'')\Gamma_{\mathbf{p}''\mathbf{p}'}, \qquad \text{(D.4)}$$

$$\Phi(\mathbf{p}') = \sum_{\mathbf{p}''} f(\mathbf{p}'')G^R(\mathbf{p}'')G^A(\mathbf{p}'')\Gamma_{\mathbf{p}''\mathbf{p}'}.$$

From Eq.(D.3) one can obtain the following equations for $\Psi(\mathbf{p}')$ and $\Phi(\mathbf{p}')$:

$$\begin{cases} \Psi(\mathbf{p}') = \rho V_0^2 I_1 + \rho V_1^2 f(\mathbf{p}')I_2 + \rho V_0^2 I_1 \Psi(\mathbf{p}') + \rho V_1^2 I_2 \Phi(\mathbf{p}'), \\ \Phi(\mathbf{p}') = \rho V_0^2 I_2 + \rho V_1^2 f(\mathbf{p}')I_3 + \rho V_0^2 I_2 \Psi(\mathbf{p}') + \rho V_1^2 I_3 \Phi(\mathbf{p}'), \end{cases} \qquad \text{(D.5)}$$

where

$$I_1 = \sum_{\mathbf{p}} G^R(\mathbf{p})G^A(\mathbf{p}),$$
$$I_2 = \sum_{\mathbf{p}} f(\mathbf{p})G^R(\mathbf{p})G^A(\mathbf{p}), \tag{D.6}$$
$$I_3 = \sum_{\mathbf{p}} f^2(\mathbf{p})G^R(\mathbf{p})G^A(\mathbf{p}).$$

Solving system (D.5), one can find the appropriate expressions for $\Psi(\mathbf{p}')$ and $\Phi(\mathbf{p}')$ and hence the expression for the vertex part:

$$\Gamma_{\mathbf{pp}'} = \frac{\rho V_0^2(1-\rho V_1^2 I_3 + \rho V_1^2 f(\mathbf{p}')I_2) + \rho V_1^2(f(\mathbf{p})f(\mathbf{p}')(1-\rho V_0^2 I_1) + \rho V_0^2 f(\mathbf{p})I_2)}{(1-\rho V_0^2 I_1)(1-\rho V_1^2 I_3) - \rho V_0^2 \rho V_1^2 I_2^2} \tag{D.7}$$

Unfortunately, similar analysis has not been performed yet within the formalism of self-consistent theory of localization (i.e. using less trivial form of irreducible vertex), so that the effects of anisotropic scattering upon Anderson transition are at present unclear.

Now let us consider some details on calculating Ginzburg-Landau coefficients for the system with anisotropic impurity scattering. We can easily see that the contribution of the diagram Fig.C.2(a) is

$$-\frac{T}{(2\pi)^2}\Delta_q^2 \sum_n \int d\mathbf{p} 2cos^2(2\phi)G(\varepsilon_n \mathbf{p}_+)G(-\varepsilon_n \mathbf{p}_-) =$$
$$-\Delta_q^2 TN(0)\sum_n \int \frac{d\xi}{\tilde{\varepsilon}_n^2 + \xi^2} + \Delta_q^2 q^2 \frac{N(0)\pi v_F^2 T_c}{8} \sum_n \frac{1}{|\tilde{\varepsilon}_n|^3}. \tag{D.8}$$

The contribution of the diagram Fig.C.2(c) is

$$-\frac{T}{(2\pi)^2}\Delta_q^2 \sum_n \int d\mathbf{p} 2cos^2(2\phi)G(\varepsilon_n \mathbf{p})G(-\varepsilon_n \mathbf{p}) = -\Delta_q^2 T_c N(0)\sum_n \int \frac{d\xi}{\tilde{\varepsilon}_n^2 + \xi^2}. \tag{D.9}$$

The contribution of the diagram with diffusion propagator Fig.C.2(b) is

$$-T\sum_n \sum_{\mathbf{pp}'} \sqrt{2}cos(2\phi)G^R(\mathbf{p}_+)G^A(\mathbf{p}_-)\Gamma_{\mathbf{pp}'}\sqrt{2}cos(2\phi')G^R(\mathbf{p}'_+)G^A(\mathbf{p}'_-). \tag{D.10}$$

Taking into account (D.6) and (D.7) we get from here

$$-TN(0)\pi\gamma_1 \sum_n \left[\frac{1}{|\tilde{\varepsilon}_n|(|\tilde{\varepsilon}_n| - \gamma_1)} - \frac{v_F^2(2|\tilde{\varepsilon}_n| - \gamma_1)q^2}{8|\tilde{\varepsilon}_n|^3(|\tilde{\varepsilon}_n| - \gamma_1)^2} \right]. \tag{D.11}$$

Note that in the absence of anisotropic scattering for the case of $d-$pairing the contribution of diagrams Fig.C.2(c) actually vanishes up to terms of the order of q^2.

In the same way we get the appropriate contribution of the diagram Fig.C.2(d)

$$-TN(0)\pi\gamma_1 \sum_n \frac{1}{|\tilde{\varepsilon}_n|(|\tilde{\varepsilon}_n| - \gamma_1)}. \tag{D.12}$$

Finally we get the expression for $F_s - F_n$ and Ginzburg-Landau coefficients cited in Section 3.3.3.

Appendix E

Instanton solution

The existence of instanton solutions of Eq.(4.32) of the form (4.33) can be demonstrated using very simple mechanical analogy [Sadovskii M.V. (1986)]. Substituting (4.33) into (4.32) we obtain the following dimensionless nonlinear differential equation for $\chi(t)$ (which we write for the general d–dimensional case):

$$\frac{d^2\chi(t)}{dt^2} + \frac{d-1}{t}\frac{d\chi(t)}{dt} = \chi(t) - \chi^3(t) \tag{E.1}$$

This can be considered as an equation of motion of a particle of unit mass in a potential $U(\chi)$, shown in Fig.E.1:

$$U(\chi) = -\frac{1}{2}\chi^2 + \frac{1}{4}\chi^4 \tag{E.2}$$

which is also subjected to a time-dependent frictional force. Clearly, we are seeking solutions that satisfy the initial conditions:

$$\chi(t=0) = const; \qquad \left.\frac{d\chi}{dt}\right|_{t=0} = 0 \tag{E.3}$$

The asymptotic form of the solutions of (E.1) at "time" $t \gg 1$ can be easily determined by linearizing this equation near the extremal values of $U(\chi)$. An instanton with minimal action corresponds to the motion when our "particle" rolls down an incline of $U(\chi)$ and *stops* at the point $\chi = 0$ for $t \to \infty$. The asymptotic form of this solution at $t \gg 1$ is:

$$\chi(t) \sim t^{(1-d)/2} exp(-t) \tag{E.4}$$

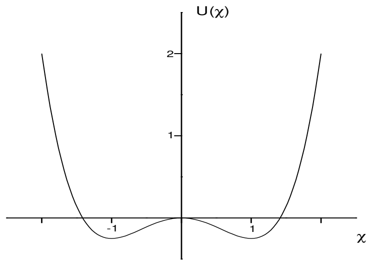

Fig. E.1 Potential energy corresponding to the equation of motion (E.1). The "particle" moves downhill with friction diminishing with "time" as $\sim 1/t$.

while its general form was shown in Fig.4.1. Of course, this general form can be determined by numerical solution of Eq.(E.1). All other types of motion, following from (E.1), lead either to infinite action solutions or instantons with larger actions.

More general and rigorous mathematical analysis using methods developed e.g. in Refs.[Makhankov V.G. (1977); Coleman S., Glaser V., Martin A. (1978)] shows that the finite action solutions of the type we are seeking exist for Eq.(E.1) only for $d < 4$.

Bibliography

Abrahams E. (1992) J.Phys.Chem.Solids **53**, 1487

Abrahams E., Anderson P.W., Lee P.A. (1981) Phys. Rev. **B24**, 6783

Abrahams E., Anderson P.W., Licciardello D.C., Ramakrishnan T.V. (1979) Phys. Rev. Lett. **42**, 673

Abrahams E., Lee P.A. (1986) Phys. Rev. **33**, 683

Abrahams E., Ramakrishnan T.V. (1980) J. Non—Cryst. Sol. **35**, 15

Abrikosov A.A., Campuzano J.C., Gofron K. (1993) Physica **C214**, 73

Abrikosov A.A., Gorkov L.P. (1958) Zh. Eksp. Teor. Fiz. (JETP) **35**, 1158

Abrikosov A.A., Gorkov L.P. (1959) Zh. Eksp. Teor. Fiz. (JETP) **36**, 319

Abrikosov A.A., Gorkov L.P., Dzyaloshinskii I.E. (1965) "Methods of Quantum Field Theory in Statistical Physics", Pergamon Press, Oxford

Abrikosov A.A., Gorkov L.P. (1960) Zh.Eksp.Teor.Fiz. (JETP) **39**, 1781

Agarwal S.K., Narlikar A.V. (1994) Prog. Crystal Growth and Charact. **28**, 219

Aleiner I., Altshuler B.L., Gershenzon M. (1998) Preprint cond-mat/9808053; See also Phys. Rev. Lett. **82**, 3190 (1999)

Aleksandrov A.S., Arkhipov V.E., Goshchitskii B.N., Elesin V.F. (1989) " Influence of Irradiation on Physical Properties of Perspective Ordered Superconductors", Energoatomizdat, Moscow (in Russian).

Aleksashin B.A., Berger I.F., Verkhovskii S.V., Voronin V.I., Goshchitskii B.N., Davydov S.A., Karkin A.E., Kozhevnikov V.L., Mirmelstein A.V., Mikhalyov K.N., Parkhomenko V.D., Cheshnitskii S.M. (1998) Pis'ma Zh. Eksp. Teor. Fiz. (JETP Lett.) **47**, 193; Physica **C153-155**, 339; Series of Preprints "Problems of High—Temperature Superconductivity", No. 1, USSR Ac. Sci. Ural Branch, Syktyvkar 1988.

Aleksashin B.A., Voronin V.P., Verkhovskii S.V., Goshchitskii B.N., Davydov S.A., Zhdanov Yu.I., Kar'kin A.E., Kozhevnikov V.L., Mirmelstein A.V., Mikhalev K.N., Sadovskii M.V., Serikov V.V., Cheshnitskii S.M. (1989) Zh. Eksp. Teor. Fiz. **95**, 678 (Sov. Phys.—JETP **68**, 382)

Alekseevskii N.E., Mitin A.V., Samosyuk V.N., Firsov V.I. (1983) Zh. Eksp. Teor.

Fiz. (JETP) **85**, 1092
Allen P.B., Dynes R.C. (1975) Phys. Rev. **B12**, 905
Altshuler B.L, Aronov A.G. (1979) Zh. Eksp. Teor. Phys. (JETP) **77**, 2028
Altshuler B.L., Aronov A.G. (1983a) Pis'ma Zh. Eksp. Teor. Fiz. (JETP Letters) **37**, 349
Altshuler B.L., Aronov A.G. (1983b) Sol. State Comm. **46**, 429
Altshuler B.L., Aronov A.G. (1985) In "Electron-Electron Interactions in Disordered Systems". Ed. by A. L. Efros and M. Pollak. Elsevier Science Publishers B.V., p. 1
Altshuler B.L., Aronov A.G., Lee P.A. (1980) Phys. Rev. Lett. **44**, 1288
Altshuler B.L., Kravtsov V.E., Lerner I.V. (1986) Zh. Eksp. Teor. Fiz. (JETP) **91**, 2276
Altshuler B.L., Aronov A.G., Khmelnitskii D.E., Larkin A.I. (1982) In "Quantum Theory of Solids". Ed. by I. M. Lifshits. Mir Publishers, Moscow, p. 130
Altshuler B.L, Khmelnitskii D., Larkin A.I., Lee P.A. (1980) Phys.Rev. **B22**, 5142
Anderson Localization (1982). Ed. by Y. Nagaoka and H. Fukuyama. Springer Verlag, Berlin–NY
Anderson P.W. (1958) Phys. Rev. **109**, 1492
Anderson P.W. (1959) J. Phys. Chem. Solids **11**, 26
Anderson P.W., Muttalib K.A., Ramakrishnan T.V. (1983) Phys. Rev. **B28**, 117
Ando Y., Boebinger G.S., Passner A., Kimura T., Kishio K. (1995) Phys.Rev.Lett. **75**, 4662
Ando Y., Boebinger G.S., Passner A., Wang N.L., Geibel C., Steglich F. (1996) Phys.Rev.Lett. **77**, 2065
Ando Y., Boebinger G.S., Passner A., Wang N.L., Geibel C., Steglich F., Trofimov I.E., Balakirev F.F. (1997) Phys.Rev. **B56**, 8530R
Annett J., Goldenfeld N., Leggett A.J. (1996) In "Physical Properties of High Temperature Superconductors", Vol.5. Ed. by D.M.Ginsberg. World Scientific, Singapore; Preprint cond-mat/9601060
Aslamazov L.G., Larkin A.I. (1968) Fiz. Tverd. Tela (Sov. Phys.—Solid State) **10**, 1104
Bardeen J., Cooper L.N., Schrieffer J.R. (1957) Phys. Rev. **106**, 162
Balatsky A.V., Rosengren A., Altshuler B.L. (1994) Phys. Rev. Lett. **73**, 720
Batlogg B. (1984) Physica **B126**, 275
Bednorz J.G., Müller K.A. (1986) Zs. Phys. **64**, 189
Bednorz J.G., Müller K.A. (1988) Rev. Mod. Phys. **60**, 585
Belevtsev B.I. (1990) Uspekhi Fiz. Nauk **160**, 65
Belitz D. (1984) Sol. State Comm. **52**, 989
Belitz D. (1985)J. Phys. **F15**, 2315
Belitz D. (1987a) Phys. Rev. **B35**, 1636
Belitz D. (1987) Phys. Rev. **B35**, 1651
Belitz D. (1989) Phys. Rev. **B40**, 111
Belitz D., Gold A., Gotze W. (1981) Zs. Phys. **B44**, 273

Belitz D., Kirkpatrick T.R. (1989) Nucl. Phys. **B316**, 509
Belitz D., Kirkpatrick T.R. (1990) Physica **A167**, 259
Belitz D., Kirkpatrik T.R. (1991) Phys. Rev. **B44**, 955
Belitz D., Kirkpatrick T.R. Rev. Mod. Phys. (1994) **66**, 261
Berezinskii V.L., Gorkov L.P. (1979) Zh. Eksp. Teor. Fiz. (JETP) **77**, 2498
Beschoten B., Sadewasser S., Guntherodt G., Quittmann C. (1996) Phys.Rev.Lett. **77**, 1837
Betbeder-Matibet O., Nozieres P. (1966) Ann. Phys. **37**, 17
Berman I.V., Brandt N.B. et al. (1986) Pis'ma Zh. Eksp. Teor. Fiz. (JETP Lett.) **43**, 48
Bickers N., Scalapino D., White S.R. (1989) Phys. Rev. Lett. **62**, 961
Bishop D.J., Spencer E.G., Dynes R.C. (1985) Solid State Electr. **28**, 73
Boebinger G.S., Ando Y., Passner A., Kimura T., Okuya M., Shimoyama J., Kishio K., Tamasaku K., Ichikawa N., Uchida S. (1996) Phys.Rev.Lett. **77**, 5417
Borkovski L.S., Hirschfeld P.J. (1994) Phys.Rev. **B49**, 15404
Browne D.A., Levin K., Muttalib K.A. (1987) Phys. Rev. Lett. **58**, 156
Bulaevskii L.N., Sadovskii M.V. (1984) Pis'ma Zh. Eksp. Teor. Fiz. **39**, 524 (JETP Letters, **39**, 640)
Bulaevskii L.N., Sadovskii M.V. (1985) J. Low-Temp. Phys. **59**, 89
Bulaevskii L.N., Sadovskii M.V. (1986) Pis'ma Zh. Eksp. Teor. Fiz. **43**, 76 (JETP Letters **43**, 99)
Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1987) Zh. Eksp. Teor. Fiz. **92**, 380 (Sov. Phys.—JETP **65**, 380)
Bulaevskii L.N., Panyukov S.V., Sadovskii M.V. (1989) In "Problems of Theoretical Physics and Astrophysics"(Volume devoted to the 70th anniversary of V. L. Ginzburg), Nauka, Moscow, p. 120
Bulaevskii L.N., Varlamov A.A., Sadovskii M.V. (1986) Fiz. Tverd. Tela (Sov. Phys.—Solid State)**28**, 1799
Cai W., Ting C.S. (1985a) J. Phys. **C18**, 1827
Cai W., Ting C.S. (1985b) J. Phys. **C18**, 1843
Caroli C., De Gennes P.G., Matricon J. (1962) J. Phys. Rad. **23**, 707
Cardy J.L. (1978) J. Phys. **C11**, L321
Castellani C., DiCastro C., Lee P.A., Ma M. (1984) Phys. Rev. **B30**, 527
Castellani C., Kotliar G., Lee P.A. (1987) Phys. Rev. Lett. **59**, 323
Castellani C., Schwab P., Grilli M. (1998) Preprint cond-mat/9812368
Cha Min-Chul, Fisher M.P.A., Girvin S.M. Wallin M., Young A.P. (1991) Phys. Rev. **44**, 6883
Chakravarty S. Yin L., Abrahams E. (1998) Phys. Rev. Lett. **58**, R559
Char K., Kapitulnik A. (1998) Zs. Phys. **72**, 253
Cieplak M., Guha S., Kojima H., Lindenfeld P., Gang Xiao, Xiao J.Q., Chien C.L. (1992) Phys. Rev. **B46**, 5536
Chui T., Deutscher G., Lindenfeld P., McLean W.L. (1981) Phys. Rev. **B23**, 6172
Coleman S. (1979) In "The Whys of Subnuclear Physics", p. 805. Ed. by

A.Zichichi. Plenum Press, NY

Coleman S., Glaser V., Martin A. (1978) Comm.Math.Phys. **58**, 211

Coffey L., Levin K., Muttalib K.A. (1985) Phys. Rev. **B32**, 4382

Dagotto E., Nazarenko A., Moreo A. (1995) Phys. Rev. Lett. **74**, 310

Davydov S.A., Arkhipov V.E., Goshchitskii B.N. (1986) Fiz. Metal. Metalloved. (Phys. Metals—Metallogr.) **62**, 1130

Davydov S.A., Arkhipov V.E., Voronin V.I., Goshchitskii B.N. (1983) Fiz. Metal. Metalloved. (Phys. Metals—Metallogr.) **55**, 931

Davydov S.A., Goshchitskii B.N., Karkin A.E., Mirmelstein A.V., Voronin V.I., Sadovskii M.V., Kozhevnikov V.L., Verkhovskii S.V., Cheshnitskii S.M. (1989a) In "Advances in Superconductivity" Ed. by Kitazawa and Ischiguro. Springer Verlag, Tokyo, p. 463

Davydov S.A., Goshchitskii B.N., Karkin A.E., Mirmelstein A.V., Sadovskii M.V., Voronin V.I., Kozhevnikov V.L., Cheshnitskii S.M., Aleksashin B.A., Mikhalyov K.N., Serikov V.V., Verkhovskii S.V., Zhdanov Yu.I. (1989b) Int. J. Mod. Phys. **3**, 87

Davydov S.A., Karkin A.E., Mirmelstein A.V., Goshchitskii B.N., Sadowski W., Walker E. (1990a) Proc. Int. Workshop on "Effects of Strong Disordering in HTSC" (Zarechny, USSR), Moscow, p. 118.

Davydov S.A., Goshchitskii B.N., Karkin A.E., Mirmelstein A.V., Voronin V.I., Parkhomenko V.D. (1990b) Int. J. Mod. Phys. **4**, 1531

Davydov S.A. (1991) Thesis. Institute for Metal Physics, USSR Ac. Sci. Ural Branch, Sverdlovsk

Dobrosavljevič V., Kirkpatrick T.R., Kotliar G. (1992) Phys. Rev. Lett. **69**, 1113

Dolgov O.V. (1977) P. N. Lebedev Physical Institute Preprint No. 92

Dolgov O.V., Sadovskii M.V. (1984) Unpublished calculations

Dynes R.C., Garno J.P., Hertel G.B., Orlando T.P. (1984) Phys. Rev. Lett. **53**, 2437

Ebisawa H., Fukuyama H., Maekawa S. (1985) J. Phys. Soc. Japan **54**, 4735

Edwards S.F. Phil. Mag. (1958) **3**, 1020

Efetov K.B. (1983) Adv. Phys. **32**, 53

Efetov K.B., Larkin A.I., Khmelnitskii D.E. (1980) Zh. Eksp. Teor. Fiz. (JETP) **79**, 1120

Efros A.L., Shklovskii B.I. (1975) J. Phys. **C8**, L49

Efros A.L. (1976) J. Phys. **C9**, 2021

Efros A.L., Shklovskii B.I. (1985) In "Electron—Electron Interactions in Disordered Systems". Ed. by A. L. Efros and M. Pollak. Elsevier Science Publishers B.V., 1985, p. 409

Efros A.L. (1978) Uspekhi Fiz. Nauk **126**, 41, (1978)

Ellman B., Jaeger H.M., Katz D.P., Rosenbaum T.F., Cooper A.S., Espinosa G.P. (1989) Phys. Rev. **B39**, 9012

Fehrenbacher R., Norman M.R. (1994) Phys. Rev. **B50**, 3495

Finkelstein A.M. (1983) Zh. Eksp. Teor. Fiz. (JETP)**84**, 168

Finkelstein A.M. (1984a) Zh. Eksp. Teor. Fiz. (JETP) **86**, 367

Finkelstein A.M. (1984b) Zs. Phys. **B56**, 189

Finkelstein A.M. (1987) Pis'ma Zh. Eksp. Teor. Fiz. **45**, 37 (1987) (JETP Letters, **45**, 46 (1987))

Finkelstein A.M. (1990) Soviet Scientific Reviews—Phys. Rev. Ed. by I.M.Khalatnikov. Vol. 14, p. 1, Harwood Academic Publishers, NY

Fiory A.T., Hebard A.F. (1984) Phys. Rev. Lett. **52**, 2057

Fisher D.S., Fisher M.P.A. (1988) Phys. Rev. Lett. **61**, 1847

Fleishman L., Anderson P.W. (1980) Phys. Rev. **B21**, 2366

Fleurov V.N., Kondratenko P.S., Kozlov A.N. (1980) J. Phys. **F10**, 1953

Forro L., Ilakovac V., Cooper J.R., Ayache C., Henry J.Y. (1992) Phys. Rev. **B46**, 6626

Franz M., Kallin C., Berlinsky A.J. (1996) Phys.Rev. **B54**, R6897

Franz M., Kallin C., Berlinsky A.J., Salkola M.I. (1997) Phys.Rev. **B56**, 7882

Friedel J. (1989) J. Phys. **C1**, 7757

Fukuyama H. (1985a) J. Phys. Soc. Japan **54**, 2092

Fukuyama H. (1985b) J. Phys. Soc. Japan **54**, 2393

Fukuzumi Y., Mizuhashi K., Takenaka K., Uchida S. (1996) Phys.Rev.Lett. **76**, 684

Furubayashi T., Nichida N., Yamaguchi M., Morigaki K., Ishimoto H. (1985) Solid State Comm. **55**, 513

Furubayashi T., Nichida N., Yamaguchi M., Morigaki K., Ishimoto H. (1986) Solid State Comm. **58**, 587

Fulde P., Luther A. (1968) Phys. Rev. **170**, 570

Gan Zi-Zhao, Lee P.A. (1986) Phys.Rev. **B33**, 3595

Gantmakher V.F., Zverev V.N., Teplinskii V.M., Barkalov O.I. (1992) Pis'ma Zh. Eksp. Teor. Fiz. **56**, 311

Gantmakher V.F., Kozeeva L.P., Lavrov A.N., Pushin D.A., Shovkin D.V., Tsyndynzhapov G.E. (1997) Pis'ma Zh. Eksp. Teor. Fiz. **65**, 834

Georges A., Kotliar G., Krauth W., Rozenberg M. (1996) Rev.Mod.Phys. **68**, 13

Gennes de P.G. (1966) Superconductivity of Metals and Alloys. W.A.Benjamin. NY

Ghosal A., Randeria M., Trivedi N. (1998) Phys. Rev. Lett. **81**, 3940 (1998)

Giapintzakis J., Ginsberg D.M., Kirk M.A., Ockers S. (1994) Phys.Rev.**B50**, 15967

Giapintzakis J., Kirk M.A., Ginsberg D.M. (1995) Preprint; Czech. J. Phys. **46**, Suppl.3, 1203

Ginzburg V.L., Landau L.D. (1950) Zh. Eksp. Teor. Phys. (JETP) **20**, 1064 (1950)

Girvin S.M., Wallin M., Cha Min-Chul, Fisher M.P.A., Young A.P. (1992) Prog. Theor. Phys. Suppl. No. 107, 135

Golubev D.S., Zaikin A.D. (1998) Phys.Rev.Lett. **81**, 1074

Gorkov L.P. (1958) Zh. Eksp. Teor. Fiz. (JETP) **34**, 735

Gorkov L.P. (1959) Zh. Eksp. Teor. Fiz. (JETP) **37**, 1407

Gorkov L.P., Larkin A.I., Khmelnitskii D.E. (1979) Pis'ma Zh. Eksp. Teor. Fiz. (JETP Letters) **30**, 248

Gorkov L.P., Kalugin P.A. (1985) Pisma Zh.Eksp.Teor.Fiz. **41**, 208; (JETP Lett.**41**, 253)

Goshchitskii B.N., Arkhipov V.E., Chukalkin Yu.G. (1987) Soviet Scientific Reviews –Phys. Rev. Ed. by I. M. Khalatnikov. Vol. 8, p. 519, Harwood Academic Publ. NY

Goshchitskii B.N., Kozhevnikov V.L., Sadovskii M.V. (1988) Int. J. Mod. Phys. **B2**, 1331

Goshchitskii B.N., Davydov S.A., Karkin A.E., Mirmelstein A.V., Sadovskii M.V., Voronin V.I. (1989a) In "High Temperature Superconductivity from Russia", Ed. by A. I. Larkin, N. V. Zavaritskii. World Scientific, Singapore, p. 270

Goshchitskii B.N., Davydov S.A., Karkin A.E., Mirmelstein A.V., Sadovskii M.V., Voronin V.I. (1989b) Physica **C162-164**, 1019

Goshchitskii B.N., Davydov S.A., Karkin A.E., Mirmelstein A.V., Voronin V.I., Chebotaev N.M., Samokhvalov A.A. (1989c) Physica **C162-164**, 1023

Goshchitskii B.N., Davydov S.A., Karkin A.E., Mirmelstein A.V. (1989d) Physica **C162-163**, 997

Goshchitskii B.N., Davydov S.A., Karkin A.E., Mirmelstein A.V., Sadovskii M.V., Voronin V.I., Aleksashin B.A., Verkhovskii S.V., Serikov V.V., Zhdanov Yu.I. (1990) In "High Temperature Superconductivity", Ed. by V. L. Aksenov, N. N. Bogolubov, N. M. Plakida. World Scientific, Singapore, p.104

Götze W. (1979) J. Phys. **C12**, 219

Götze W. (1981) Phil. Mag. **B43**, 219 (1981)

Götze W., Ketterlee W. (1983) Zs. Phys. **B54**, 49

Grinstein G., Luther A. (1976) Phys. Rev. **B13**, 1329

Haran G., Nagi A.D.S. (1996) Phys.Rev. **54**, 15463

Haviland D.B., Liu Y., Goldman A.M. (1989) Phys. Rev. Lett. **62**, 2180

Hebard A.F., Paalanen M. (1984) Phys. Rev. **30**, 4063

Hebard A.F., Paalanen M.A. (1985) Phys. Rev. Lett. **54**, 2155

Hertel G., Bishop D.J., Spencer E.G., et al. (1983) Phys.Rev.Lett. **50**, 743

Imry Y. (1980) Phys. Rev. Lett. **44**, 469

Imry Y., Ovadyahu Z. (1982) Phys.Rev.Lett. **49**, 841

Imry Y., Strongin M. (1981) Phys. Rev. **B24**, 6353

Infante C., El Mously M.K., Dayal R., Husain M., Siddiqi S.A., Ganguly P. (1990) Physica **A167**, 640

Ioffe A.F., Regel A.R. (1960) Progr. Semiconductors **4**, 237

Ioffe L.B., Larkin A.I. (1981) Zh. Eksp. Teor. Fiz. **81**, 707 (Sov. Phys.—JETP **54**, 378)

Ito T., Takagi H., Ishibashi S., Ido T., Uchida S. (1991) Nature **350**, 596

Iye Y. (1992) In "Physical Properties of High Temperature Supeconductors". Ed. by D. M. Ginsberg. World Scientific, Singapore

Jayaram B., Lanchester P.C., Weller M.T. (1991) Phys. Rev. **B43**, 5444

John S., Lubensky T.C. (1985) Phys. Rev. Lett. **55**, 1014

John S., Lubensky T.C. (1986) Phys. Rev. **B34**, 4815

Junod A., Erb A., Renner C. (1998) Proc. ASC'98, Crete 1998; (Physica C – in press)

Kadanoff L.P. et al. (1967) Rev. Mod. Phys. **39**, 395

Kamimura H. (1980) Phil. Mag. **24**, 935

Karkin A., Davydov S., Mirmelstein A., Goshchitskii B., Chebotaev N., Samokhvalov A., Sadowski W., Walker E. (1991) Physica **C185-189**, 1269

Karpinska K., Malinowski A., Cieplak M.Z., Guha S., Gershman S., Kotliar G., Skoskiewicz T., Plesiewicz W., Berkowski M., Lindenfeld P. (1996) Phys.Rev.Lett. **77**, 3033

Katsnelson M.I., Sadovskii M.V. (1983) Fiz. Tverd. Tela **25**, 3372 (Sov. Phys.—Solid State **25**, 3372)

Katsnelson M.I., Sadovskii M.V. (1984) Zh. Eksp. Teor. Fiz. **87**, 523 (Sov. Phys.—JETP **60**, 300)

Kapitulnik A., Kotliar G. (1985) Phys. Rev. Letters **54**, 473

Keck B., Schmid A. (1976) J. Low Temp. Phys. **24**, 611

Keller J., Korenman V. (1972) Phys. Rev. **B5**, 4367

Khmelnitskii D.E. (1975) Zh. Eksp. Teor. Fiz. **68**, 1960 (Sov. Phys.—JETP **41**, 981)

Kirkpatrick T.R., Belitz D. (1989) Phys. Rev. bf 40, 5227

Kirkpatrik T.R., Belitz D. (1990) Phys. Rev. **B41**, 11082

Kirkpatrick T.R., Belitz D. (1991) Phys. Rev. Lett. **66**, 1533

Kirkpatrick T.R., Belitz D. (1992) Phys. Rev. Lett. **68**, 3232

Fisher M.P.A., Grinstein G., Girvin S.M. (1990) Phys. Rev. Lett. **64**, 587

Kotliar G. (1987) In "Anderson Localization", Ed. by T. Ando and H. Fukuyama. Springer Verlag, Berlin—NY, p. 107

Kotliar G., Kapitulnik A. (1986) Phys. Rev. **B33**, 3146

Kotliar G., Varma C.M. (1990) Physica **A167**, 288

Kotliar G., Abrahams E., Ruckenstein A.E., Varma C.M., Littlewood P.B., Schmitt-Rink S. (1991) Europhys. Lett. **15**, 655 (1991)

Kotov E.A., Sadovskii M.V. (1983) Zs. Phys. **B51**, 17 (1983)

Kotov E.A., Sadovskii M.V. (1985) Fiz. Metal. Metalloved. (Phys. Metals—Metallogr.) **60**, 22

Kotov E.A. (1988) Fiz. Metal. Metalloved. (Phys. Metals — Metallogr.) **66**, 436

Kotov E.A. (1989) Fiz. Metal. Metalloved. (Phys. Metals — Metallogr.) **67**, 859

Kravchenko S.V., Mason W.E., Bowker G.E., Furneaux J.E., Pudalov V.M. D'Iorio M. (1995) Phys. Rev **51**, 7038

Kravchenko S.V., Simonian D., Sarachik M.P., Mason W.E., Furneaux J.E. (1996) Phys.Rev.Lett. **77**, 4938

Kravtsov V.E. (1991) Zs. Phys. **B82**, 205 (1991)

Kravtsov V.E., Lerner I.V. (1985) Zh. Eksp. Teor. Fiz. (JETP) **88**, 1281

Kravtsov V.E., Oppermann R. (1991) Phys. Rev. **B43**, 10865

Kubo R. (1957) J. Phys. Soc. Japan **12**, 570

Kuchinskii E.Z., Erkabaev M.A. (1997) Fiz.Tverd.Tela (Solid State Phys.) **39**, 412

Kuchinskii E.Z., Sadovskii M.V. (1991a) Sverkhprovodimost' (Superconductivity: Physics, Chemistry, Technology) **4**, 2278

Kuchinskii E.Z., Sadovskii M.V. (1991b) Physica **C185-189**, 1477

Kuchinskii E.Z., Sadovskii M.V. (1993a) Sverkhprovodimost' (Superconductivity: Physics, Chemistry, Technology) **6**, 1119

Kuchinskii E.Z., Sadovskii M.V. (1993b) Pisma ZhETF **58**, 878; (JETP Lett., **58**, 819)

Kuchinskii E.Z., Sadovskii M.V. (1994) Physica **C235-240**, 2621

Kuchinskii E.Z., Sadovskii M.V., Erkabaev M.A. (1997) Zh. Eksp. Teor. Fiz. **112**, 192; (JETP **85**, 104)

Kuchinskii E.Z., Sadovskii M.V., V. G. Suvorov, M. A. Erkabaev. (1995) Zh. Eksp. Teor. Fiz. **107**, 2027 (JETP **80**, 1122)

Kuroda Y., Varma C.M. (1990) Phys. Rev. **B42**, 8619

Langer J.S. (1967) Ann. Phys. **41**, 108

Landau L.D., Lifshits E.M. (1976) Statistical Physics. Part I. "Nauka", Moscow

Larkin A.I., Ovchinnikov Yu.N. (1971) Zh. Eksp. Teor. Fiz. (JETP) **61**, 1221

Laughlin R.B. (1988) Science **242**, 525

Lee P.A. (1980) J. Non-Cryst. Solids **35**, 21

Lee P.A. (1982) Phys. Rev. **B26**, 5882

Lee P.A. (1993) Phys. Rev. Lett. **71**, 1887

Ma M., Lee P.A. (1985) Phys. Rev. **B32**, 5658

Lee P.A., Ramakrishnan T.V. (1985) Rev. Mod. Phys. **57**, 287

Lee S.J., Ketterson J.B. (1990) Phys. Rev. Lett. **64**, 3078

Lerner I.V. (1988) Phys. Lett. **A133**, 253

Lifshits I.M., Gredeskul S.A., Pastur L.A. (1982) "Introduction to the Theory of Disordered Systems", Nauka, Moscow (in Russian)

Lesueur J., Dumoulin L., Nedellec P. (1988) Solid State Comm. **66**, 723

Li Qiming, Soukoulis C.M., Economou E.N. (1989) Phys. Rev. **B40**, 2825

Lin J.Y., Chen S.J., Chen S.Y., Chang C.F., Yang H.D., Tolpygo S.K., Gurvich M., Hsu Y.Y., Ku H.C. (1999) Phys. Rev. **59**, 6047

Liu Y., Goldman A.M. (1994) Mod. Phys. Lett. **B8**, 277

Lubensky T.C. (1975) Phys. Rev. **B11**, 3573

Ma Jian, Quittmann C., Kelley R.J., Margaritondo G., Onellion M. (1995) Phys.Rev. **B51**, 3832

Ma M., Fradkin E. (1986) Phys. Rev. Lett. **56**, 1416

Ma M., Halperin B.I., Lee P.A. (1986) Phys. Rev. **B34**, 3136

Ma Shang-keng. (1976) "Modern Theory of Critical Phenomena", W. A. Benjamin. London

Mackenzie A.P., Julian S.R., Lonzarich G.G., Carrington A., Hughes S.D., Liu R.S., Sinclair D.C. (1993) Phys. Rev. Lett. **71**, 1238

Maekawa S., Ebisawa H., Fukuyama H. (1985) Prog. Theor. Phys.–Suppl. No. 84, 154

Makhankov V.G. (1977) Phys.Lett. **A61**, 431

Malinowski A., Cieplak M.Z., van Steenbergen A.S., Perenboom J.A., Karpinska

K., Berkowski M., Guha S., Lindenfeld P. (1997) Phys. Rev. Lett. **79**, 495
Mandrus D., Forro L., Kendziora C., Mihaly L. (1991) Phys. Rev. **B44**, 2418
Markiewicz R.S. (1988) Phys. Rev. **B37**, 644
Markowitz D., Kadanoff L.P. (1963) Phys.Rev. **131**, 563
McKane A.J., Stone M. (1981) Ann. Phys. **131**, 36
W. L. McMillan. (1968) Phys. Rev. **167**, 331
McMillan W.L. (1985) Phys. Rev. **B24**, 2739
McMillan W.L., Mochel J. (1981) Phys.Rev.Lett. **46**, 556
Migdal A.B. (1983) "Theory of Finite Fermi Systems", Nauka, Moscow (in Russian)
Miller T.A., Kunchur M., Zhang Y.Z., Lindenfeld P., McLean W.L. (1988) Phys. Rev. Lett. **61**, 2717
Milligan R.F., Rosenbaum T.F, Bhatt R.N., Thomas G.A. (1985) In "Electron-Electron Interactions in Disordered Systems". Ed. by A. L. Efros and M. Pollak. Elsevier Science Publishers B.V., p. 231
Mohanty P., Jariwala E.M., Webb R.A. (1997) Phys.Rev.Lett. **77**, 3366
Monthoux P., Balatsky A.V., Pines D. (1991) Phys.Rev.Lett. **67**, 3448
Monthoux P., Balatsky A.V., Pines D. (1992) Phys. Rev. **B46**, 14803
Monthoux P., Pines D. (1993) Phys. Rev. **B47**, 6069
Monthoux P., Pines D. (1994) Phys. Rev. **B49**, 4261
Mooij J.H. (1973) Phys. Stat. Sol. (a), **17**, 521
Mott N.F. (1971) Phil. Mag. **24**, 935
Mott N.F. (1974) Metal—Insulator Transitions. Taylor and Francis, London
Mott N.F., Davis E.A. (1979) Electron Processes in Non-Crystalline Materials. Clarendon Press, Oxford
Mühlschlegel B., Scalapino D.J., Denton R. (1972) Phys. Rev. **B6**, 1767
Myasnikov A.V., Sadovskii M.V. (1982) Fiz. Tverd. Tela **24**, 3569 (Sov. Phys.—Solid State Phys. **24**, 2033)
Narat A. (1967) In "Hyperfine Interactions". Ed. by A.J.Freeman, R.B.Frankel, Academic Press, NY
Nersesyan A.A., Tsvelik A.M., Wenger F. (1994) Phys. Rev. Lett. **72**, 2628
Nersesyan A.A., Tsvelik A.M., Wenger F. (1995) Nucl. Phys. **B438**, 561
Newns D.M., Pattnaik P.C., Tsuei C.C. (1991) Phys.Rev. **B43**, 3075
Ng T.K. (1991) Phys. Rev. **B43**, 10204
Ng T.K. (1992) Phys. Rev. **B45**, 10830
Nielsen P.E., Taylor P.L. (1992) Phys. Rev. **B10**, 4061
Nichida N., Yamaguchi M., Furubayashi T., Morigaki K., Ishimoto H., Oho K. (1982) Solid State Comm. **44**, 305
Nichida N., Furubayashi T., Yamaguchi M., Morigaki M., Ishimoto H. (1985) Solid State Electr. **28**, 81
Novokshonov S.G., Groshev A.G. (1998) Zh.Eksp.Teor.Fiz. (JETP) **114**, 711
Nozieres P., Schmitt-Rink S. (1985) J.Low.Temp.Phys. **59**, 195
Ohkawa F.J. (1982) In "Anderson Localization". Ed. by Y. Nagaoka and H. Fukuyama. Springer Verlag, Berlin–NY, p. 113

Ono Y., Yoshioka D., Fukuyama H. (1981) J. Phys. Soc. Japan **50**, 2465
Oppermann R. (1987) Nucl. Phys. **B280**, 753
Oppermann R. (1988) Physica **C153**, 711
Oppermann R. (1990) Physica **A167**, 301
Oppermann R., Wegner F.J. (1979) Zs. Phys. **B34**, 327
Osofsky M.S., Soulen R.J., Wolf S.A., Broto J.M., Rakoto H., Ousset J.C., Coffe G., Askenazy S., Pari P., Bozovič I., Eckstein J.N., Virshup G.F., Ginzton E.L. (1993) Phys.Rev.Lett., **71**, 2315
Patashinskii A.Z., Pokrovskii V.L. (1982) Fluctuation Theory of Phase Transitions. Nauka, Moscow (in Russian)
Phillips P., Wan Y., Martin I., Knysh S., Dalidovich D. (1998) Nature **395**, 253 (1998)
Pippard A.B. (1955) Phil. Mag. **46**, 1104
Pistolesi F., Strinati G.C. (1994) Phys.Rev. **49**, 6356
Podlesnyak A., Kozhevnikov V., Mirmelstein A., Allenspach P., Mesot J., Staub U., Furrer A. (1991) Physica **C185-189**, 817
Poon S.J., Drehma A.J., Wang K.M., Clegg A.W. (1985) Phys. Rev. **B31**, 3100
Popovic D., Fowler A.B., Washburn S. (1997) Phys.Rev.Lett. **79**, 1543
Posazhennikova A.I., Sadovskii M.V. (1996) Pis'ma Zh. Eksp. Teor. Fiz. **65**, 258 (JETP Lett. **65**, 358)
Posazhennikova A.I., Sadovskii M.V. (1997a) Zh.Eksp.Teor.Fiz. **112**, 2124 (JETP **85**, 1162)
Posazhennikova A.I., Sadovskii M.V. (1997b) Pis'ma Zh.Eksp.Teor.Fiz. **65**, 258 (JETP Lett. **65**, 270)
Pudalov V.M. (1997) Pis'ma Zh. Eksp. Teor. Fiz. (JETP Lett.) **66**, 168
Prigodin V.N., Firsov Yu.A. (1984) J. Phys. **C17**, L979
Proc. M2S-HTSC - Int. Conf. High Temperature Superconductivity—Materials and Mechanisms of Superconductivity, M2S-HTSC (I), (II), (III), (IV), (V)]. Physica **153-155** (1988); Physica **C162-164** (1990); Physica **C185-189** (1991); Physica **C235-240** (1994); Physica **C282-287** (1997)
Quittmann C., Almeras P., Ma Jian, Kelley R.J., Berger H., Margiritondo G., Onellion M. (1994) Univ. Wisconsin Preprint
Quittmann C., Almeras P., Ma Jian, Kelley R.J., Berger H., Xuyeu Cai, Margaritondo G., Onellion M. (1996) Phys.Rev. **B53**, 6819
Radtke R.J., Levin K., Schüttler H.-B., Norman M.R. (1993) Phys. Rev. **B48**, 653
Rajaraman R. (1982) “Solitons and Instantons”. North-Holland, Amsterdam-NY
Ramakrishnan T.V. (1989) Physica Scripta **T27**, 24
Randeria M. (1997) Varenna Lectures, Preprint cond-mat/9710223
Randeria M., Campuzano J.C. (1997) Varenna Lectures , Preprint cond-mat/9709107
Rojo A.G., Balseiro C.A. (1998) J. Phys.–Cond. Matter **10**, 7587
Sadovskii M.V. (1979) Fiz. Tverd. Tela **21**, 743 (Sov. Phys.—Solid State **21**, 435)
Sadovskii M.V. (1981) Uspekhi Fiz. Nauk **133**, 223 (Sov. Phys.—Uspekhi **24**, 96)

Sadovskii M.V. (1985) Thesis. P. N. Lebedev Phys. Inst., Moscow
Sadovskii M.V. (1986) Soviet Scientific Reviews—Phys.Rev. Ed. by I. M. Khalatnikov. Vol. 7, p. 1, Harwood Academic Publ. NY
Sadovskii M.V. (1989) In "Frontiers in Condensed Matter Theory". Ed. by M. Lax, L. P. Gorkov, J. L. Birman. Ann. NY Acad. Sci. Vol. 581, AIP Conference Proceedings, Vol. 213, p. 207
Sadovskii M.V. (1990) Proc. Int. Workshop on "Effects of Strong Disordering in HTSC" (Zarechny, USSR), Moscow, p. 32
Sadovskii M.V. (1993) In "Studies of High Temperature Superconductors", Ed. by A. V. Narlikar. Vol.11, p. 131, Nova Science Publishers Inc. NY
Sadovskii M.V. (1995) Sverkhprovodimost' (Superconductivity: Physics, Chemistry, Technology) **8**, 337
Sadovskii M.V. (1997) Physics Reports **282**, 225
Saint-James D., Sarma G., Thomas G. (1969) "Type II Superconductivity", Pergamon Press, Oxford
Scalapino D.J., Loh E., Hirsch J.E. (1987) Phys.Rev. **35**, 6694
Schäfer L., Wegner F.J. (1980) Zs. Phys. **B38**, 113
Schäfer L., Pruisken A.M.M. (1982) Nucl. Phys. **B200**, 20
Segawa K., Ando Y. (1998) Preprint cond-mat/9811278
Shahar D., Ovadyahu Z. (1992) Phys. Rev. **B46**, 10917
Shapira Y., Deutscher G. (1983) Phys. Rev. **B27**, 4463
Shapiro B. (1982) Phys. Rev. **B25**, 4266
Shapiro B., Abrahams E. (1981a) Phys. Rev. **B24**, 4889
Shapiro B., Abrahams E. (1981b) Phys. Rev. **B24**, 4025
Shastry B.S., Abrahams E. (1994) Phys.Rev.Lett. **72**, 1933
Shklovskii B.I., Efros A.L. (1979) "Electronic Properties of Doped Semiconductors". Nauka, Moscow (in Russian)
Si Qimiao, Varma C.M. (1998) Phys. Rev. Lett. **81**, 4951 (1998)
Spivak B., Kivelson S.A. (1991) Phys. Rev. **B43**, 3740
Spivak B., Zhou Fei. (1995) Phys. Rev. Lett. **74**, 2800
Spivak B., Zyuzin A. (1988) Pis'ma Zh. Exp. Teor. Fiz. (JETP Letters) **47**, 221
Srikanth H., Rajeev K.P., Shivashankar G.V., Raychaudhuri A.K. (1992) Physica **C192**, 87
Sugawara K., Yokota K., Takemoto J., Tanokura Y., Sekine T. (1993) J.Low.Temp.Phys. **91**, 39
Suslov I.M. (1995) Zh. Eksp. Teor. Fiz. (JETP) **108**, 1686
Suslov I.M. (1998) Uspekhi Fiz. Nauk. **168**, 503 (Physics–Uspekhi **41**, 441)
Takagi H., Kuroda Y. (1982) Solid State Comm. **41**, 643
Takagi H., Souda R., Kuroda Y. (1982) Prog. Theor. Phys. **68**, 426
Tenhover M., Johnson W.L., Tsuei C.C. (1981) Solid State Comm. **38**, 53
Teplinskii V.M., Gantmakher V.F., Barkalov O.I. (1992) Zh. Eksp. Teor. Fiz. (JETP) **101**, 1698
Ting C.S. (1982) Phys. Rev. **B26**, 678
Theumann A., Pires Idiart M.A. J. Phys.–Cond. Matter. **3**, 3765

Thouless D.J. (1974) Phys. Reports **13**, 93

Tolpygo S.K., Lin J.Y., Gurvich M., Hou S.Y., Phillips J.M. (1996) Phys.Rev. **53**, 12454, 12462

Toyota N., Inoue A., Matsumoto K., Fukase T., Matsumoto T. (1984) J. Phys. Soc. Japan **53**, 924

Tsuei C.C. et al. (1994) Phys. Rev. Lett. **73** 593

Uemura Y.J., et al. (1991) Phys.Rev.Lett. **66**, 2665

Vainshtein A.I., Zacharov V.I., Novikov V.A., Shifman M.A. (1982) Uspekhi Fiz. Nauk **136**, 553

Valles J.M., White A.E., Short K.T., Dynes R.C., Garno J.P., Levi A.F.J., Anzlowar M., Baldwin K. (1989) Phys. Rev. **39**, 11599

Varma C.M. (1990) Int. Conf. on Superconductivity–ICSC. (Bangalore, India). Eds. S.K.Joshi, C.N.R.Rao, S.V.Subramanyam. World Scientific, Singapore, p. 396

Varma C.M. (1997) Phys. Rev. Lett. **79**, 1535

Varma C.M., Littlewood P.B., Schmitt-Rink S., Abrahams E., Ruckenstein A.E. (1989) Phys. Rev. Lett. **63**, 1996

Verkhovskii S.V., Zhdanov Yu.I., Bogdanovich A.M., Aleksashin B.A., Lavrentiev V.V., Mikhalyev K.N., Sadovskii M.V., Serikov V.V. (1992) Apll. Magn. Resonance **3**, 649

Vollhardt D., Wölfle P. (1980) Phys. Rev. **B22**, 4666

Vollhardt D., Wölfle P. (1982) Phys. Rev. Lett. **48**, 699

Vollhardt D., Wölfle P. (1990) In “Electronic Phase Transitions”, Ed. by W. Hanke and Yu. V. Kopaev. North-Holland, Amsterdam

Vollhardt D. (1993) In “Correlated Electron Systems”. Ed. by V.J.Emery, World Scientific, Singapore, p.57

Vonsovsky S.V., Izyumov Yu.A., Kurmaev E.Z. (1982) “Superconductivity of Transition Metals, Their Alloys and Compounds”, Springer Verlag, Berlin—NY

Voronin V.I., Goshchitskii B.N., Davydov S.A., Karkin A.E., Mirmelstein A.V., Parkhomenko V.D., Kozhevnikov V.L., Cheshnitskii S.M. (1987) In “Novel Superconductivity”, Ed. by S. A. Wolf and V. Z. Kresin. Plenum Press, N.Y., p. 875

Voronin V., Mirmelstein A., Karkin A., Davydov S., Berger I., Goshchitskii B. (1991) Physica **C185-189**, 877

Walker D.J.C., Laborde O., Mackenzie A.P., Julian S.R., Carrington A., Loram J.W., Cooper J.R. (1995) Preprint

Wang T., Beauchamp K.M., Berkely D.D., Johnson B.R., Liu J.X., Zhang J., Goldman A.M. (1991) Phys. Rev. **B43**, 8623

Warren W.W. (1971) Phys. Rev. **B3**, 3708 (1971)

Wegner F.J. (1976) Zs. Phys. **B25**, 327

Wegner F.J. (1979a) Zs. Phys. **B35**, 207

Wegner F.J. (1979b) Renormalization Group and the Anderson Model of Disordered System. Heidelberg Univ. Preprint No. 30

Wegner F.J. (1980) Zs. Phys. **B36**, 209 (1980)

Wegner F.J. (1982) In "Anderson Localization", p.8, Ed. by Y. Nagaoka and H. Fukuyama. Springer Verlag, Berlin—NY

Werthamer N.R., Helfand E., Hohenberg P.C. (1966) Phys. Rev. **147**, 295

White A.E., Dynes R.C., Garno J.P. (1986) Phys. Rev. **B33**, 3549

Wilson K.G., Kogut J. (1974) Phys. Reports **12**, 75

Wölfle P., Bhatt R.N. (1984) Phys. Rev. **30**, 3542

Wölfle P., Vollhardt D. (1982) In "Anderson Localization", p.68, Ed. by Y. Nagaoka and H. Fukuyama. Springer Verlag, Berlin—NY

Yoshioka D., Ono Y., Fukuyama H. J. (1981) Phys. Soc. Japan **50**, 3419

Wollman D.A. et al. (1993) Phys. Rev. Lett. **72**, 2134

Won H., Maki K. (1998) Physica **B244**, 66

Zhdanov Yu.I., Bogdanovich A.M., Aleksashin B.A., Mikhalyev K.N., Lavrentiev V.V., Verkhovskii S.V., Serikov V.V., Sadovskii M.V. (1993) Zh. Eksp. Teor. Fiz. **103**, 1762 (JETP **76**, 868)

Zhitomirskii M.E., Walker M.B. (1998) Phys. Rev. Lett. **80**, 5413

Ziegler K., Hettler M.H., Hirschfeld P.J. (1996) Phys. Rev. Lett. **77**, 3013

Zinn-Justin J. (1981) Phys. Reports **70**, 109

Zint Th., Rohde M., Micklitz H. (1990) Phys. Rev. **B41**, 4831