SUPERCONDUCTIVITY AND LOCALIZATION

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Abstract

We present a review of theoretical and experimental works on the problem of mutual interplay of Anderson localization and superconductivity in strongly disordered systems. Superconductivity occurs close to the metal–insulator transition in some disordered systems such as amorphous metals, superconducting compounds disordered by fast neutron irradiation, etc. High-temperature superconductors are especially interesting from this point of view. Only bulk systems are considered in this review. The superconductor–insulator transition in purely two-dimensional disordered systems is not discussed.

We start with a brief discussion of the modern aspects of localization theory including the basic concept of scaling, self-consistent theory and interaction effects. After that we analyze disorder effects on Cooper pairing and superconducting transition temperature as well as the Ginzburg–Landau equations for superconductors which are close to those for the Anderson transition. A necessary generalization of the usual theory of "dirty" superconductors is formulated which

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allows to analyze anomalies of the main superconducting properties close to the disorder-induced metal–insulator transition. Under very rigid conditions superconductivity may persist even in the localized phase (Anderson insulator).

Strong disordering leads to considerable reduction of superconducting transition temperature T_c and to important anomalies in the behavior of the upper critical field H_{c2}. Fluctuation effects are also discussed. In the vicinity of the Anderson transition, inhomogeneous superconductivity appears due to statistical fluctuations of the local density of states.

We briefly discuss a number of experiments demonstrating superconductivity close to the Anderson transition both in traditional and high-T_c superconductors. In traditional systems superconductivity is in most cases destroyed before the metal–insulator transition. In the case of high-T_c superconductors a number of anomalies show that superconductivity is apparently conserved in the localized phase before it is suppressed by a strong enough disorder.

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1. Introduction

The concept of electron localization [1] is basic for the understanding of electron properties of disordered systems [2,3]. In recent years a number of review papers have appeared, extensively discussing this problem [4–7]. According to this concept introduction of sufficiently strong disorder into a metallic system leads to spatial localization of electronic states near the Fermi level and thus to a transition to dielectric state (Anderson transition). After this transition DC conductivity (at zero temperature, $T = 0$) vanishes, despite the finite value of electronic density of states at the Fermi level (at least in one-electron approximation).

At the same time it is well known that even the smallest attraction of electrons close to the Fermi level leads to formation of Cooper pairs and the system becomes superconducting at sufficiently low temperatures [8,9]. It is known that the introduction of disorder which does not break the time-reversal invariance (normal, nonmagnetic impurities, etc.) does not seriously influence the superconducting transition temperature T_c and superconductivity in general (Anderson theorem) [10–13].

Thus a problem appears of the mutual interplay of these two possible electronic transitions in a disordered system which leads to quite different (even opposite) ground states (insulator or superconductor). This problem is very important both from theoretical and experimental points of view. Actually, superconducting properties of many compounds depend strongly on structural disorder. In this respect we can mention amorphous systems (metallic glasses) and superconductors disordered by different forms of irradiation by high-energy particles (fast neutrons, electrons, heavy-ions, etc.). It appears that in many of these systems superconductivity is realized when the system in normal state is quite close to the metal–insulator transition induced by disorder. In this case many anomalies of superconducting properties appear which cannot be satisfactorily explained within the standard theory of "dirty" superconductors [9–13]. These include rather strong dependence of T_c on disorder in apparent contradiction with Anderson's theorem, as well as some unusual behavior of the upper critical field H_{c2}.

The discovery of high-temperature superconductivity in metallic oxides [14, 15] has lead to the entirely new opportunities in the studies of strong disorder effects in superconductors. Very soon it had been established that high-T_c superconductors are quite sensitive to structural disordering which leads to a rather fast destruction of superconductivity and metal–insulator transition. However, the high values of initial T_c, as well as a small size of Cooper pairs and the quasi-two-dimensional nature of electronic states in these systems are very appropriate for studies of the mutual interplay of localization and superconductivity [16]. It may be stated with some confidence that in these systems superconductivity can be observed even in the region of localization (Anderson insulator).

This review is mainly concerned with theoretical aspects of localization and superconductivity close to the Anderson transition. However, we shall pay some attention to experiments demonstrating the importance of localization phenomena for the correct analysis of superconductivity in strongly disordered systems. Special emphasis will be on the experiments with high-T_c superconductors. We shall limit our discussion to three-dimensional and quasi-two-dimensional (in case of HTSC) systems, practically excluding any discussion of purely two-dimensional systems, which are quite special both in respect to localization and superconductivity. In this case we refer the reader to recent reviews [17–19] which are specifically concerned with the two-dimensional case.

I must stress that the material presented in this review is concerned mainly with the personal interests of its author and I apologize to those people whose important contributions in this field would not be discussed in detail or even would be missed because of the lack of space.

The usual theory of "dirty superconductors" [9–13] is a cornerstone in our understanding of superconducting properties of disordered metals. It is based on the following main statements:

1. As impurity concentration (disorder) grows, a transition takes place from the "pure" limit, when the electron mean-free path l is much larger than the superconducting coherence length ξ_0: $l \gg \xi_0 = \hbar v_F/\pi\Delta_0$ to a "dirty" superconductor with $\xi_0 \gg l \gg \hbar/p_F$ (Here v_F, p_F are the Fermi velocity and momentum, Δ_0 is the zero temperature energy gap). Transition temperature T_c changes only slightly, mainly due to small changes of Debye frequency ω_D and of the pairing constant λ_p, which are due to relatively small changes in the electronic density of states under disordering. Transition from the free electron motion to diffusive one does not change T_c at all (Anderson's theorem). These statements ignore any disorder dependence of microscopic pairing interaction, which is assumed to be some constant as in the simplest BCS model.

2. Superconducting coherence length ξ (at $T = 0$) determining the spatial scale of the superconducting order-parameter (the size of a Cooper pair) diminishes with l so that $\xi \approx \sqrt{\xi_0 l}$ in the limit of $\hbar/p_F \ll l \ll \xi_0$.

3. As ξ diminishes the critical region near T_c where thermodynamic fluctuations are important widens and is of the order of $\tau_G T_c$, where $\tau_G \sim [T_c N(E_F)\xi^3]^{-2}$ is the so-called Ginzburg's parameter ($N(E_F)$ is the electronic density of states at the Fermi level E_F). For "pure" superconductors $\tau_G \sim (T_c/E_F)^4 \ll 1$ and as l drops τ_G grows and ξ drops. However, in the limit of $l \gg \hbar/p_F$ the value of τ_G still remains very small.

Theory of "dirty" superconductors is the basis of our understanding of superconducting properties of many disordered alloys. However, the main results of this theory must be modified for the mean-free path values l of the order of inverse Fermi momentum $\hbar/p_F$ (i.e. of the order of interatomic distance). In three-dimensional systems the growth of disorder leads to destruction of diffusive motion of electrons and transition from extended to localized states at critical disorder determined by $l_c \approx \hbar/p_F$, i.e. to transition to Anderson insulator. This metal–insulator transition is reflected in a continuous drop to zero of the static metallic conductivity (at $T = 0$) as $l \to l_c$. For $l \gg l_c$ conductivity is determined by the usual Drude formula $\sigma_0 \sim l$, while for $l \to l_c$ it drops as $\sigma \sim (l - l_c)^\nu$, where ν is some critical exponent. Transition from diffusion to localization is realized at the conductivity scale of the order of the so-called "minimal metallic conductivity" $\sigma_c \approx (e^2 p_F/\pi^3\hbar^2) \approx (2\text{–}5) \times 10^2\ \Omega^{-1}\,\text{cm}^{-1}$. The usual theory of "dirty" superconductors does not consider localization effects and is valid for conductivities in the interval $(E_F/T_c)\sigma_c \gg \sigma \gg \sigma_c$.

At present the following results are well established for superconductors close to localization transition (i.e. $\sigma \leq \sigma_c$):

1. Assuming independence of the density of states at the Fermi level $N(E_F)$ and of the pairing constant λ_p from the value of the mean-free path l (disorder) we can show that T_c drops as disorder grows due to respective growth of Coulomb pseudopotential μ^*. This effect is due to the growth of retardation effects of Coulomb interaction within the Cooper pair as diffusion coefficient drops close to the Anderson transition [20]. T_c degradation starts even for $\sigma \gg \sigma_c$ and becomes fast for $\sigma < \sigma_c$ [21,22]. The growth of spin fluctuations and changes in the density of states due to interaction effects may also lead to the drop of T_c, though these mechanisms were not analyzed in detail up to now.

2. Close to the Anderson transition the usual expression for superconducting coherence length for a "dirty" limit $\xi = \sqrt{\xi_0 l}$ should be replaced by $\xi \approx (\xi_0 l^2)^{1/3}$ and it remains finite even below the Anderson transition (i.e. in insulating phase) [21–25], signalling the possibility of superconductivity in Anderson insulator. Obviously, these results are valid only in the case of finite T_c close to Anderson transition, which is possible only if very rigid conditions are satisfied.

3. The growth of disorder as the system moves to the Anderson transition leads to the growth of different kinds of fluctuations of the superconducting order-parameter both of thermodynamic nature and due to fluctuations of electronic characteristics of the system.

In our review we shall present an extensive discussion of these and some of the other problems concerning the interplay of superconductivity and localization. However, first of all we shall briefly describe the main principles of the modern theory of electron localization and physics of metal–insulator transition in disordered systems, which will be necessary for a clear understanding of the main problem under discussion. After that we shall give rather detailed presentation of the theoretical problem of superconductivity close to the Anderson transition. Finally, we shall describe the present experimental situation. We shall briefly describe some of the experiments with traditional superconductors, but our main emphasis will be on high-T_c oxides. We shall concentrate on the experiments with high-temperature-superconductors disordered by fast neutron irradiation. We consider this as one of the best methods to introduce disorder in a controlled fashion without any chemical (composition) changes. In this sense our review of experiments is also far from being complete, but we hope that it is full enough to claim that high-T_c systems are especially good for testing some of the main theoretical ideas, expressed throughout this review. Also we believe that better understanding of their properties under disordering may be important for the development of the general theory of high-temperature superconductivity. The preliminary version of this review has been published in Ref. [26].

2. Anderson localization and metal–insulator transition in disordered systems

2.1. Basic concepts of localization

In recent years a number of review papers have appeared dealing with basic aspects of Anderson localization [4–7, 27–29]. Here we shall recall the main points of this theory and introduce the accepted terminology.

In 1958 Anderson [1] showed for the first time that the wave function of a quantum particle in a random potential can qualitatively change its nature if randomness becomes large enough. Usually, when disorder is small, the particle (e.g. electron) is scattered randomly and the wave function changes at the scale of the order of mean free path l. However, the wave function remains as extended plane-wave-like (Bloch wave-like) throughout the system. In the case of a strong enough disorder, the wave function becomes localized so that its amplitude (envelope) drops exponentially with distance from the center of localization $\boldsymbol{r}_0$:

$$|\psi(\boldsymbol{r})| \sim \exp(|\boldsymbol{r} - \boldsymbol{r}_0|/R_{\mathrm{loc}}) , \tag{2.1}$$

where R_{loc} is the localization length. This situation is shown qualitatively in Fig. 1. The physical meaning of Anderson localization is relatively simple: coherent tunneling of electrons is possible

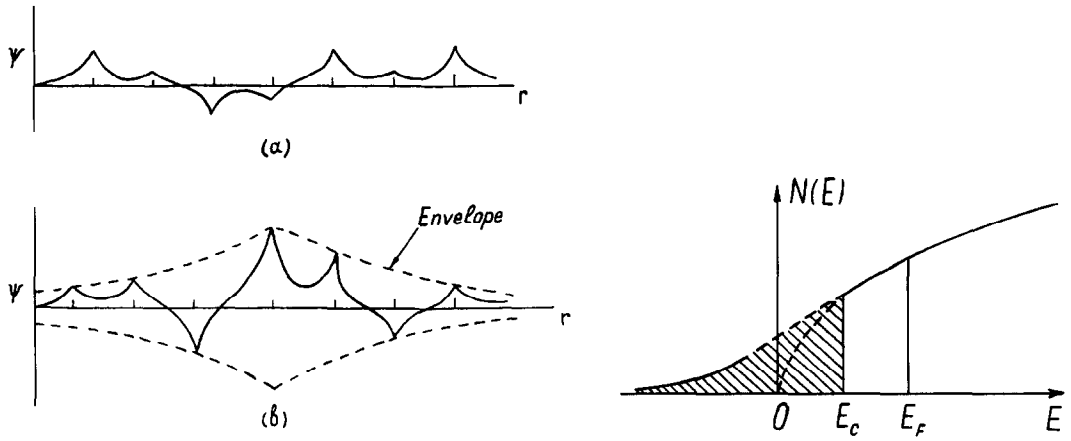

Fig. 1. Electron wave function in a disordered system: (a) extended state, (b) localized state.

Fig. 2. Electron density of states near the band edge in a disordered system. Dashed line is the region of localized states and E_c the mobility edge.

only between energy levels with the same energy (e.g. between equivalent sites in crystalline lattice). However, in case of strong randomness the states with the same energy are too far apart in space for tunneling to be effective.

At small disorder, DC conductivity of a metal at $T = 0$ is determined by the Drude expression:

$$\sigma_0 = (ne^2/m)\tau = (ne^2/p_F)l\,, \tag{2.2}$$

where τ is the mean free time, n the electron density and e its charge. The usual kinetic theory can be applied if

$$p_F l/\hbar \gg 1 \quad \text{or} \quad E_F\tau/\hbar \gg 1 \tag{2.3}$$

which is a condition of weak scattering (disorder). From Eqs. (2.2) and (2.3), taking into account $n = p_F^3/(3\pi^2\hbar^3)$, we can estimate the lower limit of conductivity for which the Drude approximation is still valid:

$$\sigma_0 = \frac{e^2 p_F}{3\pi^2\hbar^2}\left(\frac{p_F l}{\hbar}\right) \gg \frac{e^2 p_F}{3\pi^2\hbar^2}\,. \tag{2.4}$$

The conductivity value:

$$\sigma_c \approx e^2 p_F/3\pi^2\hbar^2 \tag{2.5}$$

is usually called the "minimal metallic conductivity" [2,3]. As disorder grows the mean free path diminishes and becomes of the order of lattice spacing a, so that we reach $p_F l/\hbar \sim 1$, and the usual kinetic theory based upon Boltzmann equation becomes inapplicable. This was first noted by Ioffe and Rege [30], who observed that at such disorder the qualitative form of the wave function must change, transforming from extended to localized accompanied by a metal–insulator transition.

From Eq. (2.5) it is clear that this transition takes place at the conductivity scale of the order of $\sigma_c \sim (2\text{–}5) \times 10^2\,\Omega^{-1}\,\mathrm{cm}^{-1}$ for typical $\hbar/p_F \sim a \sim (2\text{–}3) \times 10^{-8}$ cm.

The qualitative form of the energy spectrum near the band-edge of a disordered system is shown in Fig. 2. When the Fermi level lies in the high-energy region electronic states close to it are slightly distorted plane waves. As Fermi energy moves towards the band edge (or with the growth of disorder) the critical energy E_c (mobility edge) separating extended and localized states crosses the Fermi level. If E_F belongs to the region of localized states the system becomes insulating, conductivity is possible only for $T > 0$ or by exciting the carriers by an alternating electric field. The appearance of these hopping mechanisms of conductivity signals the Anderson transition [2, 3].

One of the main problems is the qualitative behavior of conductivity when the Fermi level E_F crosses the mobility edge E_c (at $T = 0$). While Mott assumed the discontinuous drop of conductivity from σ_c to zero [2, 3] modern approach [4–6, 29] based mainly on the scaling theory to localization [31] demonstrates a *continuous* transition. Experiments at low temperatures clearly confirm this type of behavior [6], and σ_c acts as a characteristic conductivity scale close to transition. Static conductivity of a metal at $T = 0$ close to Anderson transition within this approach is written as

$$\sigma = A\frac{e^2}{\hbar\xi_{loc}} \approx \sigma_c \left|\frac{E_F - E_c}{E_c}\right|^{(d-2)\nu}, \tag{2.6}$$

where A is a numerical constant, d the space dimension, and $\sigma_c \approx Ae^2/(\hbar a^{d-2})$. Here we introduced the correlation length of scaling theory diverging at the transition

$$\xi_{loc} \approx \frac{\hbar}{p_F}\left|\frac{E_F - E_c}{E_c}\right|^{-\nu}. \tag{2.7}$$

Critical exponent ν determines this divergence. In the one-electron approximation and in the absence of magnetic scattering, $\nu \sim 1$ [6, 7, 29, 32]. In the region of localized states (i.e. for $E_F < E_c$) ξ_{loc} coincides with the localization length of electrons R_{loc}. In the metallic region, ξ_{loc} determines the effective size of a sample at which "ohmic" behavior appears, i.e. conductivity becomes independent of the sample size [6, 33]. "Minimal metallic conductivity" σ_c determines, as we noted, the conductivity scale close to a transition.

In the vicinity of Anderson transition, conductivity acquires an important frequency dependence [51, 34]. For $E_F = E_c$, i.e. at the transition we have

$$\sigma(\omega) \approx \sigma_c(i\omega\tau)^{(d-2)/d} \tag{2.8}$$

which is also valid close to the transition (from either side) for frequencies $\omega \gg \omega_c \sim [N(E_F)\xi_{loc}^d]^{-1}$. For $d = 3$ this is sometimes referred to as Gotze's [35] law $\omega^{1/3}$, although this particular derivation was later acknowledged to be wrong [36].

The spatial dimension $d = 2$ is the so-called "lower critical dimensionality" [4–7]. For $d = 2$ all electronic states are localized for infinitesimal disorder [31], and there is no Anderson transition.

Quasi-two-dimensional systems are especially interesting, mainly because most of the high-T_c oxides demonstrate strongly anisotropic electronic properties. Here we shall make the simplest estimates for such systems on the line of Ioffe–Regel approach. Consider a system made of

highly-conducting "planes" where the current carriers are "nearly free", while the interplane tunneling is possible only due to some small transfer integral $w \ll E_F$ (E_F is the Fermi energy of the two-dimensional gas within the plane). Conductivity within the plane is determined for small disorder as

$$\sigma_{\parallel} = e^2 D_{\parallel} N(E_F) , \tag{2.9}$$

where $D_{\parallel} = v_F^2 \tau/2$, $N(E_F) = m/(\pi a_{\perp} \hbar^2)$, $a_{\perp}$ is the interplane spacing, which is noticeably larger than the interatomic distance within the plane. Interplane conductivity is given by

$$\sigma_{\perp} = e^2 D_{\perp} N(E_F) , \tag{2.10}$$

where $D_{\perp} = (wa_{\perp})^2 \tau/\hbar^2$. The appropriate mean free paths are $l_{\parallel} = v_F \tau$, $l_{\perp} = wa_{\perp} \tau/\hbar$, Ioffe–Regel criterion for a quasi-two-dimensional system can be written as

$$l_{\perp} = wa_{\perp} \tau/\hbar \sim a_{\perp} \tag{2.11}$$

which is equivalent to $w\tau/\hbar \sim 1$ – the condition of breaking of coherent tunneling between the planes. Elementary estimate shows that this corresponds to

$$\sqrt{\sigma_{\parallel} \sigma_{\perp}} \sim e^2/\sqrt{2}\pi\hbar a \sim \sigma_c , \tag{2.12}$$

where a is the interatomic distance *within the planes*. In the isotropic case this reduces to Eq. (2.5). For a strongly anisotropic system, when $\sigma_{\parallel} \gg \sigma_{\perp}$ it is clear that Eq. (2.12) can be satisfied even for $\sigma_{\parallel} \gg \sigma_c$, because of small values of $\sigma_{\perp}$. Formally, for $\sigma_{\perp} \to 0$, the critical value of $\sigma_{\parallel}$ diverges, which reflects, at this elementary level, the tendency towards complete localization in the purely two-dimensional case.

The important property of the energy spectrum in the region of localized states is its local discreteness. As we noted above, the physical meaning of localization itself leads to a picture of close energy levels being far apart in space, despite the continuous nature of the average density of states. Due to exponential decay of the localized wave functions it leads to the absence of tunneling [1]. The energy spacing between levels of electrons localized within a sphere of radius of the order of $R_{loc}(E)$ can be estimated [2, 3] as

$$\delta_{E_F} \approx [N(E_F) R_{loc}^d]^{-1} . \tag{2.13}$$

As the metallic system moves toward the Anderson transition, i.e. as the mean free path drops to interatomic distances and conductivity becomes less than $\sim 10^3\, \Omega^{-1}\, cm^{-1}$, there appear the well-known anomalies like the negative temperature coefficient of resistivity [30, 37]. These anomalies are apparently closely connected with the localization phenomena [6].

So far we discussed the Anderson transition, neglecting electron interactions. Its importance in the problem of metal–insulator transitions in disordered systems was known for a long time [2]. In recent years, there has been considerable progress in the general approach to the theory of "dirty" metals, based on the analysis of interference of impurity scattering and Coulomb interactions [38–40]. Later, we shall review its implications for the general picture of Anderson transition. Apparently, the continuous nature of metal–insulator transition is not changed though interaction leads to a number of specific effects, e.g. in the behavior of the density of states at the Fermi level, as well as to the growth of magnetic (spin) fluctuations. Here we shall briefly describe the concept of

"soft" Coulomb gap appearing below the transition in the region of localized states [41–44]. Coulomb interaction between localized electrons can be estimated as $e^2/\varepsilon R_{\rm loc}$, and it is obviously important if lthis energy is comparable with the local level spacing $[N(E_{\rm F})R_{\rm loc}^3]^{-1}$ (for three-dimensional system). As a result, a Coulomb pseudogap appears at the Fermi level with the width:

$$\Delta_{\rm C} \approx (\varepsilon^3/\varepsilon^{3/2})[N(E_{\rm F})]^{1/2} , \tag{2.11}$$

where ε is the dielectric constant. We shall see later that close to the Anderson transition $\varepsilon \approx 4\pi e^2 N(E_{\rm F})R_{\rm loc}^2$ and accordingly,

$$\Delta_{\rm C} \approx [N(E_{\rm F})R_{\rm loc}^3]^{-1} \approx \delta_{E_{\rm F}} \tag{2.15}$$

so that in this case Coulomb effects are comparable with the effects of the discreteness of energy spectrum in the localized phase. At the moment there is no complete theory connecting the localization region with the metallic phase within the general approaches of interaction theory.

2.2. *Elementary scaling theory of localization*

The behavior of the electronic system close to the Anderson transition can be described by a scaling theory similar to that used in the theory of critical phenomena [45–47]. The main physical idea of this approach is based upon a series of scale transformations from smaller to larger "cells" in coordinate space with the appropriate description of a system by transformed parameters of the initial Hamiltonian. These transformations usually constitute the renormalization group. In the theory of critical phenomena, this approach is usually motivated by the growth of correlation length of the order-parameter fluctuations near the critical point [45]. This is analogous to the growth of the localization length on approaching the mobility edge from the Anderson insulator.

The accepted scaling approach to the localization problem was proposed by Abrahams et al. [31]. In this theory localization is described in terms of *conductance* g as a function of the sample size L. For a small disorder ($p_{\rm F}l/\hbar \gg 1$) the system is in a metallic state and conductivity σ is determined by Eq. (2.2) and is independent of the sample size if this size is much larger than the mean free path, $L \gg l$. Conductance is determined in this case just by Ohm's law and for a d-dimensional hypercube we have

$$g(L) = \sigma L^{d-2} . \tag{2.16}$$

If electronic states near the Fermi level are localized, conductivity of an infinite system at $T = 0$ is zero and matrix elements for transitions between different electronic states drop exponentially on distances of the order of $R_{\rm loc}$. Then it can be expected that for $L \gg R_{\rm loc}$, the effective conductance becomes exponentially small:

$$g(L) \sim \exp(-L/R_{\rm loc}) . \tag{2.17}$$

Elementary scaling theory of localization assumes that in the general case the conductance of a hypercube of size L satisfies the simplest differential equation of a renormalization group:

$$\mathrm{d}\ln g(L)/\mathrm{d}\ln L = \beta_d(g(L)) . \tag{2.18}$$

The most important assumption here is the dependence of $\beta_d(g)$ only on one variable g (one parameter scaling). Then the qualitative behavior of β_d can be analyzed in the simplest possible way

by interpolating between limiting forms given by Eqs. (2.16) and (2.17). For the metallic phase (large g), we get from Eqs. (2.16) and Eq. (2.18)

$$\lim_{g\to\infty} \beta_d(g) \to d - 2 \,. \tag{2.19}$$

For insulator ($g \to 0$), it follows from Eqs. (2.18) and (2.17) that

$$\lim_{g\to 0} \beta_d(g) \to \ln(g/g_c) \,. \tag{2.20}$$

Assuming the existence of two perturbation expansions over the "charge" g in the limits of weak and strong "couplings" we can write the corrections to Eqs. (2.20) and (2.19) in the following form:

$$\beta_d(g \to 0) = \ln(g/g_c)(1 + bg + \cdots) \,, \tag{2.21}$$

$$\beta_d(g \to \infty) = d - 2 - (\alpha/g) + \cdots, \quad \alpha > 0 \,. \tag{2.22}$$

Following these *assumptions* and supposing now a monotonous and continuous form of $\beta_d(g)$, it is easy to plot it qualitatively for all g, as shown in Fig. 3. All the previous equations are written for dimensionless conductance, which is measured in natural units of $e^2/\hbar \approx 2.5 \times 10^{-4}\,\Omega^{-1}\,\mathrm{cm}^{-1}$. We see that $\beta_d(g)$ definitely has no zeros for $d < 2$. If expansion Eq. (2.22) is valid there is no zero for $d = 2$ also. For $d > 2$, function β_d must have a zero: $\beta_d(g_c) = 0$. It is clear that $g_c \sim 1$ and no form of perturbation theory is valid near that zero. The existence of a zero of $\beta_d(g)$ corresponds to existence of an unstable fixed point of Eq. (2.18). The state of a system is supposedly determined by disorder at microscopic distances of the order of interatomic spacing a, i.e. by $g_0 = g(L = a)$. Using g_0 as an initial value and integrating Eq. (2.18) it is easy to find that for $g_0 > g_c$ conductivity $\sigma_L = g(L)L^{2-d}$ tends for $L \to \infty$ to a constant (metallic) value. For $g < g_c$ in the limit of $L \to \infty$ we get insulating

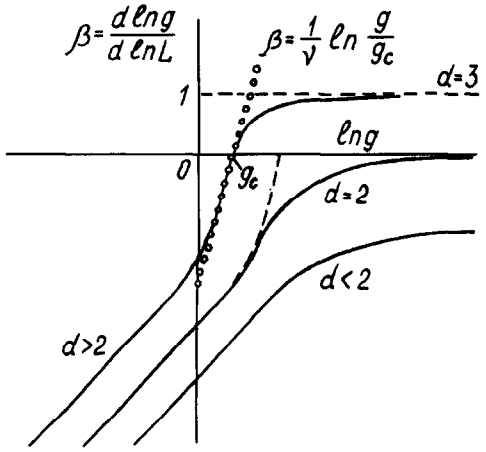

Fig. 3. Qualitative form of $\beta_d(g)$ for different d. Dashed line shows the behavior necessary to get discontinuous drop of conductivity at the mobility edge for $d = 2$.

behavior. Using for $g \sim g_c$ an approximation (shown with circles in Fig. 3)

$$\beta_d(g) \approx (1/\nu)\ln(g/g_c) \approx (1/\nu)(g - g_c)/g_c\,, \tag{2.23}$$

we obtain from Eq. (2.18) for $g_0 > g_c$ the following behavior of conductivity for $L \to \infty$:

$$\sigma \approx A\frac{e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(\ln\frac{g_0}{g_c}\right)^{(d-2)\nu} \approx A\frac{e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(\frac{g_0 - g_c}{g_c}\right)^{(d-2)\nu}, \tag{2.24}$$

where $A = \text{const.}$ and we have explicitly introduced the conductivity scale of the order of σ_c (cf. Eq. (2.5)). We see that the existence of a fixed point leads to the existence of a mobility edge, and behavior of $\beta_d(g)$ close to its zero determines the critical behavior at the Anderson transition. Under these assumptions conductivity continuously goes to zero for $g_0 \to g_c$, and the value of $\sigma_c \approx e^2/(\hbar a^{d-2})$ is the characteristic scale of conductivity at the metal–insulator transition. To get a discontinuous drop of conductivity at the mobility edge, $\beta_d(g)$ must be nonmonotonic as shown by the dashed line for $d = 2$ in Fig. 3. This behavior seems more or less unphysical.

Integrating Eq. (2.18) with $\beta_d(g)$ from Eq. (2.23) with initial $g_0 < g_c$ gives

$$g(L) \approx g_c \exp\left\{-A|\ln(g_0/g_c)|^{\nu}\frac{L}{a}\right\}. \tag{2.25}$$

From this it is clear (cf. Eq. (2.7)) that

$$R_{\text{loc}} \sim a|(g_0 - g_c)/g_c|^{-\nu} \tag{2.26}$$

and ν is the critical exponent of the localization length. For $d = 2$ we have $\beta_d(g) < 0$ in the whole interval of g, so that $\sigma_{L\to\infty} \to 0$ for any initial value of g and there is no mobility edge and all states are localized.

For $d > 2$, limiting ourselves by those terms of the perturbation expansion in g^{-1} shown in Eq. (2.22) we can solve $\beta_d(g_c) = 0$ to find

$$g_c = \alpha/(d-2)\,. \tag{2.27}$$

We can see that for $d \to 2$ the mobility edge goes to infinity which corresponds to complete localization in the two-dimensional case. Now we have

$$\beta_d(g \sim g_c) \approx (d-2)(g_0 - g_c)/g_c \tag{2.28}$$

and for the critical exponent of localization length we get (cf. Eq. (2.23))

$$\nu = 1/(d-2)\,. \tag{2.29}$$

which may be considered as the first term of the ε-expansion near $d = 2$ (where $\varepsilon = d - 2$), i.e. near the "lower critical dimension" for localization [31, 82, 48]. Note that the expansion of Eq. (2.22) can be reproduced in the framework of standard perturbation theory over impurity scattering [49, 50]. For $d = 3$ this gives $\alpha = \pi^{-3}$ (cf. Ref. [6]).

Let us now define the correlation length of a localization transition as

$$\xi_{\text{loc}} \sim a|(g_0 - g_c)/g_c|^{-\nu}\,. \tag{2.30}$$

For $g_0 < g_c$ this length coincides with the localization length R_{loc}. It is easy to see that Eq. (2.24) can be written as [51]

$$\sigma \approx A g_c (e^2/\hbar \xi_{loc}^{d-2}) . \tag{2.31}$$

It follows that for $g > g_c$ correlation length ξ_{loc} determines the behavior of conductivity close to the mobility edge, when this length becomes much larger than the interatomic distance and mean free path.

Let us consider the three-dimensional case in more details. Integrating Eq. (2.18) with $\beta_3(g) = 1 - g_c/g$ where $g_c = \alpha$ gives $g(L) = (\hbar/e^2)\sigma_L L = (\hbar/e^2)\sigma + g_c$ so that for a finite sample close to the mobility edge ($\xi_{loc} \gg l$) we obtain

$$\sigma_L = \sigma + (e^2 g_c/\hbar L) , \tag{2.32}$$

where in correspondence with Eq. (2.31)

$$\sigma \approx A g_c \frac{e^2}{\hbar \xi_{loc}} . \tag{2.33}$$

It follows that for $L \gg \xi_{loc} \gg l$ conductivity $\sigma_L \to \sigma$ while for $l \ll L \ll \xi_{loc}$ conductivity σ_L and the appropriate diffusion coefficient, determined by Einstein relation $\sigma = e^2 D N(E_F)$ are given by

$$\sigma_L \approx e^2 g_c/\hbar L , \tag{2.34}$$

$$D_L \approx [g_c/N(E_F)](1/\hbar L) , \tag{2.35}$$

respectively, where $N(E_F)$ is the electron density of states at the Fermi level. Thus, in this latest case, conductivity is not Ohmic, i.e. diffusion of electrons is "non-classical" [20, 6]. From this discussion it is clear that the characteristic length ξ_{loc} in the metallic region determines the scale at which conductivity becomes independent of the sample size. Close to the mobility edge when $\xi_{loc} \to \infty$ only the samples with growing sizes $L \gg \xi_{loc}$ can be considered as macroscopic. These considerations allow us to understand the physical meaning of the diverging length ξ_{loc} of scaling theory in the metallic region [33]. Close to mobility, ξ_{loc} is considered as the only relevant length in the problem (with an exception of a sample size L) and the scaling hypothesis is equivalent to the assumption

$$g(L) = f(L/\xi_{loc}) , \tag{2.36}$$

where $f(x)$ is some universal (for a given dimensionality d) function. In the metallic region, for $L \gg \xi_{loc} \gg l$ it is obvious that $f(x) \sim x^{d-2}$ which reproduces Eq. (2.31).

For finite frequencies ω of an external electric field a new length appears in the system [34]:

$$L_\omega = [D(\omega)/\omega]^{1/2} , \tag{2.37}$$

where $D(\omega)$ is the frequency dependent diffusion coefficient. L_ω is a length of electron diffusion during one cycle of an external field. Close to the mobility edge ξ_{loc} is large and for $L_\omega < \xi_{loc}, L$ and L_ω become the relevant length scale. In general, for finite ω localization transition is smeared, a sharp transition is realized only for $L^{-1} = L_\omega^{-1} = 0$. Thus, for the finite frequency case the scaling hypothesis of Eq. (2.36) can be generalized as [34]

$$g(L, \omega) = f(L/\xi_{loc}, L_\omega/\xi_{loc}) , \tag{2.38}$$

where g denotes a real part of conductance. In the metallic phase, for $L \gg \xi_{\rm loc}$ we have $g \sim L^{d-2}$ so that

$$\sigma(\omega) = \frac{e^2}{\hbar} L^{2-d} f\left(\frac{L}{\xi_{\rm loc}}, \frac{L_\omega}{\xi_{\rm loc}}\right) \to \frac{e^2}{\hbar} \xi_{\rm loc}^{2-d} f\left(\infty, \frac{L_\omega}{\xi_{\rm loc}}\right) \equiv \frac{e^2}{\hbar \xi_{\rm loc}^{d-2}} F\left(\frac{\xi_{\rm loc}}{L_\omega}\right). \tag{2.39}$$

For small frequencies, when $L_\omega \gg \xi_{\rm loc}$, we can write down the universal function $F(x)$ as $F(x) \approx Ag_c + Bx^{d-2}$ which reproduces Eq. (2.31) and the small frequency corrections found earlier in [49]. For $L_\omega \ll \xi_{\rm loc}$ i.e. for high frequencies or close to the mobility edge, the relevant length is L_ω and the frequency dependent part of conductivity is dominating. In particular, at the mobility edge itself the length $\xi_{\rm loc}$ drops out and must cancel in Eq. (2.38) which leads to

$$\sigma(\omega, E_F = E_c) \sim L_\omega^{2-d} \sim [\omega/D(\omega)]^{(d-2)/2}. \tag{2.40}$$

On the other hand, according to Einstein relation we must have $\sigma(\omega) \sim D(\omega)$. Accordingly, from $[\omega/D(\omega)]^{(d-2)/2} \sim D(\omega)$ we get at the mobility edge

$$\sigma(\omega, E_F = E_c) \sim D(\omega) \sim \omega^{(d-2)/d}. \tag{2.41}$$

For $d = 3$ this leads [51,35] to $\sigma(\omega) \sim D(\omega) \sim \omega^{1/3}$. The crossover between different types of frequency dependence occurs for $L_\omega \sim \xi_{\rm loc}$ which determines the characteristic frequency [34]:

$$\omega_c \sim \frac{1}{\hbar \xi_{\rm loc}^d N(E_F)}. \tag{2.42}$$

The $\omega^{(d-2)/d}$ behavior is realized for $\omega \gg \omega_c$, while for $\omega \ll \omega_c$ we get small corrections of the order of $\sim\omega^{(d-2)/2}$ to Eq. (2.31).

Finally, we must stress that for finite temperatures there appear *inelastic* scattering processes which destroy the phase correlations of wave functions at distances greater than a characteristic length of the order of $L_\varphi = \sqrt{D\tau_\varphi}$, where D is the diffusion coefficient due to *elastic* scattering processes considered above and τ_φ is the "dephasing" time due to inelastic processes [39]. For $T > 0$ this length L_φ effectively replaces the sample size L in all expressions of scaling theory when $L \gg L_\varphi$, because on distances larger than L_φ all information on the nature of wave functions (e.g. whether they are localized or extended) is smeared out. Taking into account the usual low-temperature dependence like $\tau_\varphi \sim T^{-p}$ (where p is some integer, depending on the mechanism of inelastic scattering) this can lead to a nontrivial temperature dependence of conductivity, in particular, to the possibility of a negative temperature coefficient of resistivity of "dirty" metals [33] which are close to the localization transition. It is important to stress that similar expressions determine the temperature dependence of conductivity also for the localized phase until $L_\varphi < R_{\rm loc}$. Only for $L_\varphi > R_{\rm loc}$ the localized nature of the wave functions reveals itself in the temperature dependence of conductivity and the transition to exponentially activated hopping behavior takes place, which becomes complete for $T < [N(E_F)R_{\rm loc}^d]^{-1}$.

2.3. *Self-consistent theory of localization*

2.3.1. *Isotropic systems*

It is obvious that the qualitative scaling picture of Anderson transition described in the previous section requires microscopic justification. At the same time, we need a practical method of explicit

calculations for any physical characteristic of the electronic system close to the mobility edge. Here we shall briefly describe the main principles of the so-called self-consistent theory of localization which while leaving aside some important points, leads to an effective scheme for analysis of the relevant physical characteristics important for us. This approach, first formulated by Gotze [52,35] was later further developed by Vollhardt and Wolfle and others [53–56,32,7].

Complete information concerning the Anderson transition and transport in a disordered system is contained in the two-particle Green's function

$$\Phi^{RA}_{pp'}(E\omega\boldsymbol{q}) = -\frac{1}{2\pi i}\langle G^{R}(\boldsymbol{p}_{+}\boldsymbol{p}'_{-}E+\omega)G^{A}(\boldsymbol{p}'_{-}\boldsymbol{p}_{-}E)\rangle\,, \tag{2.43}$$

where $\boldsymbol{p}_{+-} = \boldsymbol{p} \pm \frac{1}{2}\boldsymbol{q}$, in most cases below E just coincides with the Fermi energy E_F. Angular brackets denote averaging over disorder. Graphically this Green's function is shown in Fig. 4. It is well known that this Green's function is determined by the Bethe–Salpeter equation also shown graphically in Fig. 4 [57,58,53]:

$$\Phi^{RA}_{pp'}(E\boldsymbol{q}\omega) = G^{R}(E+\omega\boldsymbol{p}_{+})G^{A}(E\boldsymbol{p}_{-})\left\{-\frac{1}{2\pi i}\,\delta(\boldsymbol{p}-\boldsymbol{p}') + \sum_{p''} U^{E}_{pp''}(\boldsymbol{q}\omega)\Phi^{RA}_{p''p'}(E\boldsymbol{q}\omega)\right\}, \tag{2.44}$$

where $G^{R,A}(E\boldsymbol{p})$ is the averaged retarded (advanced) one-electron Green's function, while the irreducible vertex part $U^{E}_{pp'}(\boldsymbol{q}\omega)$ is determined by the sum of all diagrams which cannot be cut over two electron lines (cf. Fig. 4).

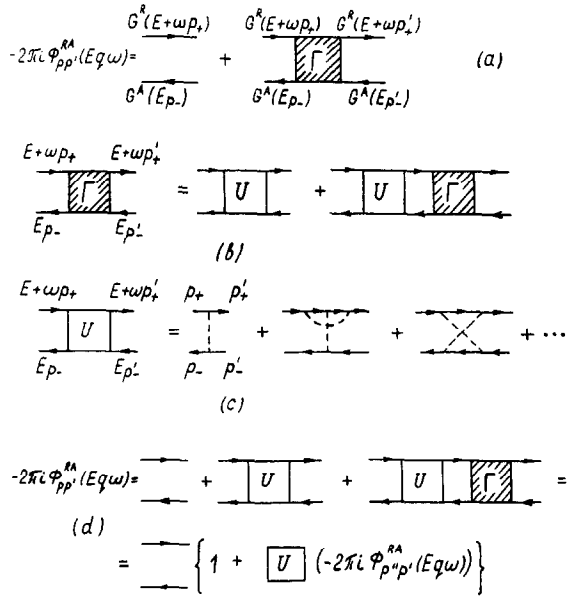

Fig. 4. Graphical representation of (a) a two-electron Green's function $\Phi^{RA}_{pp'}(E\boldsymbol{q}\omega)$, (b) the equation for the full vertex part $\Gamma^{E}_{pp'}(\boldsymbol{q}\omega)$, (c) typical diagrams for the irreducible vertex $U^{E}_{pp'}(\boldsymbol{q};\omega)$; (d) the Bethe–Salpeter equation. Dashed line denotes interaction $U_0(\boldsymbol{p}-\boldsymbol{p}') = \rho|V(\boldsymbol{p}-\boldsymbol{p}')|^2$, where ρ is the density of scatterers and $V(\boldsymbol{p}-\boldsymbol{p}')$ the Fourier transform of a single scatterer potential.

In fact, the two-particle Green's function of Eq. (2.43) contains some extra information and for the complete description of Anderson transition it is sufficient to know the two-particle Green's function summed over $\boldsymbol{pp}'$ [53]:

$$\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) = -\frac{1}{2\pi \mathrm{i}} \sum_{\boldsymbol{pp}'} \langle G^{\mathrm{R}}(\boldsymbol{p}_+ \boldsymbol{p}'_+ E + \omega) G^{\mathrm{A}}(\boldsymbol{p}'_- \boldsymbol{p}_- E) \rangle \,. \tag{2.45}$$

Using the Bethe–Salpeter equation, Eq. (2.44), and exact Ward identities we can obtain a closed equation for $\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega)$ [53, 32, 7], and for small ω and $\boldsymbol{q}$ the solution of this equation has a typical diffusion-pole form

$$\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) = -N(E) \frac{1}{\omega + \mathrm{i} D_E(\boldsymbol{q}\omega) q^2} \,, \tag{2.46}$$

where $N(E)$ is the electron density of states at energy E and the *generalized* diffusion coefficient $D_E(\boldsymbol{q}\omega)$ is expressed through the so-called relaxation kernel $M_E(\boldsymbol{q}\omega)$:

$$D_E(\boldsymbol{q}\omega) = \mathrm{i} \frac{2E}{dm} \frac{1}{M_E(\boldsymbol{q}\omega)} = \frac{v_{\mathrm{F}}^2}{d} \frac{\mathrm{i}}{M_E(\boldsymbol{q}\omega)} \,, \tag{2.47}$$

where v_{F} is the Fermi velocity of an electron. The retarded density–density response function at small ω and $\boldsymbol{q}$ is given by

$$\chi^{\mathrm{R}}(\boldsymbol{q}\omega) = \omega \Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) + N(E) + \mathrm{O}(\omega, q^2) \tag{2.48}$$

or from Eq. (2.46)

$$\chi^{\mathrm{R}}(\boldsymbol{q}\omega) = N(E) \frac{\mathrm{i} D_E(\boldsymbol{q}\omega) q^2}{\omega + \mathrm{i} D_E(\boldsymbol{q}\omega) q^2} \,. \tag{2.49}$$

For the relaxation kernel $M_E(\boldsymbol{q}\omega)$ (or for the generalized diffusion coefficient) a self-consistency equation can be derived, which is actually the main equation of the theory [53, 29, 32]. The central point in this derivation is some approximation for the irreducible vertex part $U^E_{\boldsymbol{pp}'}(\boldsymbol{q}\omega)$ in the Bethe–Salpeter equation. The approximation of Vollhardt and Wolfle is based upon the use for $U^E_{\boldsymbol{pp}'}(\boldsymbol{q}\omega)$ of the sum of "maximally-crossed" graphs shown in Fig. 5. This series is easily summed and we get the so-called "Cooperon" [49, 53]:

$$U^{EC}_{\boldsymbol{pp}'}(\boldsymbol{q}\omega) = \frac{2\gamma\rho V^2}{D_0(\boldsymbol{p} + \boldsymbol{p}')^2 + \mathrm{i}\omega} \,, \tag{2.50}$$

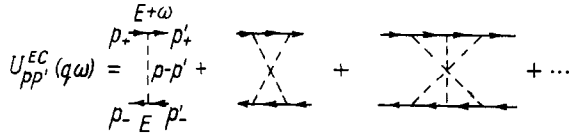

Fig. 5. "Maximally-crossed" diagrams for the irreducible vertex part of Bethe–Salpeter equation ("Cooperon").

where

$$D_0 = E/md\gamma = v_F^2\tau/d \tag{2.51}$$

is the classical (bare) diffusion coefficient determining the Drude conductivity (Eq. (2.2)). For point scatterers randomly distributed with spatial density ρ (V is the scattering amplitude) we have

$$\gamma = 1/2\tau = \pi\rho V^2 N(E_F) . \tag{2.52}$$

These "maximally crossed" diagrams lead to the following quantum correction to the diffusion coefficient:

$$\frac{\delta D(\omega)}{D_0} = -\frac{1}{\pi N(E)} \sum_{|k|<k_0} \frac{1}{-i\omega + D_0 k^2} . \tag{2.53}$$

Appropriate correction to the relaxation kernel can be expressed via the correction to diffusion coefficient as

$$\delta M_E(\omega) = -i\frac{2E_F}{dm}\frac{\delta D(\omega)}{D(\omega)^2} = -\frac{M_E(\omega)}{D(\omega)}\delta D(\omega) . \tag{2.54}$$

Considering the usual Drude metal as the zeroth approximation we get

$$\delta M_E(\omega) = -(M_0/D_0)\delta D(\omega) . \tag{2.55}$$

The central point of the self-consistent theory of localization [52] reduces to the replacement of the Drude diffusion coefficient D_0 in the diffusion pole of Eq. (2.53) by the generalized one $D(\omega)$. Using this relation in Eq. (2.55) we obtain the main equation of the self-consistent theory of localization determining the relaxation kernel $M(0\omega)$ (for $\boldsymbol{q} = 0$) [53, 32]:

$$M_E(\omega) = 2i\gamma\left\{1 + \frac{1}{\pi N(E)} \sum_{|k|<k_0} \frac{i}{\omega + (2E/dm)(k^2/M_E(\omega))}\right\} \tag{2.56}$$

or the equivalent equation for the generalized diffusion coefficient itself:

$$\frac{D_0}{D_E(\omega)} = 1 + \frac{1}{\pi N(E)} \sum_{|k|<k_0} \frac{1}{-i\omega + D_E(\omega)k^2} . \tag{2.57}$$

Cut-off in momentum space in Eqs. (2.53), (2.56), (2.57) is determined by the limit of applicability of diffusion-pole approximation of Eq. (2.46) or Eq. (2.50) [7]:

$$k_0 \approx \mathrm{Min}\{p_F, l^{-1}\} . \tag{2.58}$$

Close to the mobility edge, $p_F \sim l^{-1}$. Note that from here on we are generally using natural units with Planck constant $\hbar = 1$; however, in some of the final expressions we shall write $\hbar$ explicitly.

Conductivity can be expressed as [53, 32]

$$\sigma(\omega) = (ne^2/m)[i/(\omega + M_E(\omega))] \to e^2 D_E(\omega)N(E) \quad \text{for } \omega \to 0 , \tag{2.59}$$

where we have used $n/N(E) = 2E/d$. It is clear that for the metallic phase $M_E(\omega \to 0) = i/\tau_E$, where τ_E is the generalized mean free time. Far from the Anderson transition (for weak disorder) $\tau_E \approx \tau$ from Eqs. (2.52) and (2.59) reduces to the standard Drude expression.

If the frequency behavior of relaxation kernel leads to the existence of a limit $\lim_{\omega\to 0}\omega M_E(\boldsymbol{q}\omega)$ a singular contribution appears in Eq. (2.46) for $\omega\to 0$ [35, 7]:

$$\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega)\approx -\frac{N(E)}{\omega}\frac{1}{1-(2E/md)(q^2/\omega M_E(\boldsymbol{q}\omega))}\approx -\frac{N(E)}{\omega}\frac{1}{1+R_{\mathrm{loc}}^2 q^2}\,, \tag{2.60}$$

where we have defined

$$R_{\mathrm{loc}}^2(E)=-(2E/md)\lim_{\omega\to 0}(1/\omega M_E(\omega))\,. \tag{2.61}$$

According to the general criterion of localization [59, 7] (cf. Appendix A) this behavior corresponds to the region of localized states. Using Eq. (A.16) we immediately obtain from Eq. (2.60) the singular contribution to Gorkov–Berezinskii spectral density (cf. Eqs. (A.8), (A.9)):

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_q^F=(1/\pi N(E))\,\mathrm{Im}\,\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega)=A_E(\boldsymbol{q})\delta(\omega)\,, \tag{2.62}$$

where

$$A_E(\boldsymbol{q})=1/(1+R_{\mathrm{loc}}^2(E)q^2)\to 1-R_{\mathrm{loc}}^2(E)q^2\quad\text{for } q\to 0\,. \tag{2.63}$$

From the above expression and from Eq. (A.11) we can see that $R_{\mathrm{loc}}(E)$ as defined in Eq. (2.61) is actually the localization length. It is useful to define a characteristic frequency [53]:

$$\omega_0^2(E)=-\lim_{\omega\to 0}\omega M_E(\omega)>0 \tag{2.64}$$

so that

$$R_{\mathrm{loc}}(E)=\sqrt{(2E/md)}(1/\omega_0(E))\,. \tag{2.65}$$

Thus, the localization transition is signalled by the divergence of the relaxation kernel for $\omega\to 0$ [53], so that two characteristic types of its behavior for $\boldsymbol{q}=0$ and $\omega\to 0$ appear:

$$M_E(0\omega)\approx\begin{cases}\dfrac{\mathrm{i}}{\tau_E} & \text{for } E\geq E_{\mathrm{c}}\,,\\[2ex] \dfrac{\mathrm{i}}{\tau_E}-\dfrac{\omega_0^2(E)}{\omega} & \text{for } E\leq E_{\mathrm{c}}\,.\end{cases} \tag{2.66}$$

The frequency $\omega_0(E)$ is in some crude sense analogous to the order parameter in the usual theory of phase transitions. It appears in the localized phase signalling about the Anderson transition.

From Eq. (A.16), for $\omega\to 0$ and $\boldsymbol{q}=0$, neglecting the nonsingular contribution from $\mathrm{Im}\,\Phi_E^{\mathrm{RR}}(\boldsymbol{q}\omega)$ we can get an explicit expression for the Berezinskii–Gorkov spectral density which is valid for small ω and $\boldsymbol{q}$ [60, 7]:

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_q^F=\begin{cases}\dfrac{1}{\pi}\dfrac{D_E q^2}{\omega^2+(D_E q^2)^2} & \text{(Metal)}\,,\\[2ex] A_E(\boldsymbol{q})\delta(\omega)+\dfrac{1}{\pi}\dfrac{D_E q^2}{\omega^2+[\omega_0^2(E)\tau_E+D_E q^2]^2} & \text{(Insulator)}\,,\end{cases} \tag{2.67}$$

where we have introduced the renormalized diffusion coefficient, determined by relaxation time τ_E:

$$D_E = \frac{2E}{dm}\tau_E = \frac{1}{d}v_F^2\tau_E. \tag{2.68}$$

Substituting Eq. (2.66) into the equation (Eq. (2.56)) for self-consistency we can obtain equations for τ_E and $\omega_0(E)$ [54, 55, 7] and thus determine all the relevant characteristics of the system. For $d > 2$, Eqs. (2.56) and (2.57) do really describe a metal–insulator transition [54, 55, 7, 29]. For $d = 2$ all the electronic states are localized [53].

Below we present some of the results of this analysis which will be important for the following. For $2 < d < 4$ a correlation length similar to that of Eqs. (2.7) and (2.30) appears:

$$\xi_{loc}(E) \sim \frac{1}{p_F}\left|\frac{E - E_c}{E_c}\right|^{-\nu} \quad \text{for } E \sim E_c\,, \tag{2.69}$$

where $\nu = 1/(d-2)$. The position of the mobility edge is determined by the condition:

$$\left.\frac{E}{\gamma}\right|_{E=E_c} = \frac{d}{\pi(d-2)} \tag{2.70}$$

which follows if we assume the cut-off $k_0 = p_F$ in Eqs. (2.56) and (2.57). Static conductivity in the metallic phase ($E > E_c$) is given by (cf. Eq. (2.31))

$$\sigma = \frac{\sigma_0}{[p_F\xi_{loc}(E)]^{d-2}}\,, \tag{2.71}$$

where $\sigma_0 = (ne^2/m)\tau$ is the usual Drude conductivity. In particular, for $d = 3$

$$\left.\frac{E}{\gamma}\right|_{E=E_c} = p_F l|_{E=E_c} = \frac{3}{\pi} \tag{2.72}$$

in complete accordance with Ioffe–Regel criterion, and

$$\sigma = \frac{\sigma_0}{p_F\xi_{loc}(E)}\,. \tag{2.73}$$

Critical exponent $\nu = 1$. The mean free path which follows from Eq. (2.72) corresponds to Drude conductivity:

$$\sigma_c = \frac{ne^2}{m}\tau\bigg|_{E=E_c} = \frac{e^2p_F}{3\pi^2\hbar^2}\left(\frac{p_F l}{\hbar}\right)\bigg|_{E=E_c} = \frac{e^2p_F}{\pi^3\hbar^2} \tag{2.74}$$

which is equivalent to the elementary estimate of Eq. (2.5).

Eq. (2.73) can also be rewritten as [22]

$$\sigma = \sigma_0\left\{1 - \frac{\sigma_c}{\sigma_0}\right\} = \sigma_0 - \sigma_c\,, \tag{2.75}$$

where the Drude conductivity σ_0 is now the measure of disorder. It is obvious that for small disorder (large mean free path) $\sigma_0 \gg \sigma_c$, Eq. (2.75) reduces to $\sigma \approx \sigma_0$. As disorder grows (mean free path drops) conductivity $\sigma \to 0$ for $\sigma_0 \to \sigma_c$.

In the dielectric phase ($E < E_c$) we have $\xi_{loc}(E) = R_{loc}(E)$ and finite $\omega_0^2(E)$ from Eq. (2.64) which tends to zero as $E \to E_c$ from below. This frequency determines the dielectric function of the insulating phase [7]:

$$\varepsilon(\omega \to 0) = 1 + \frac{\omega_p^2}{\omega_0^2(E)} = 1 + \kappa_D^2 R_{loc}^2(E) \sim \left| \frac{E - E_c}{E_c} \right|^{-2\nu}, \tag{2.76}$$

where $\omega_p^2 = 4\pi ne^2/m$ is the square of the plasma frequency, $\kappa_D^2 = 4\pi e^2 N(E)$ is the square of the inverse screening length of a metal.

Thus the main results of the self-consistent theory of localization coincide with the main predictions of elementary scaling theory of localization. Vollhardt and Wolfle have shown [54, 32] that equations of this theory and especially the main differential equation of renormalization group Eq. (2.18) for conductance may be explicitly derived from self-consistency equations (Eqs. (2.56) and (2.57)) reformulated for a finite system by introduction of low-momentum cut-off at $k \sim 1/L$, where L is the system size.

The results considered up to now are valid for $\omega \to 0$. Self-consistent theory of localization allows to study the frequency dependence of conductivity (generalized diffusion coefficient) [32]. At finite frequency the main Eq. (2.57) for the generalized diffusion coefficient for $d = 3$ can be rewritten as [36, 32]

$$\frac{D_E(\omega)}{D_0} = 1 - \left(\frac{E_c}{E}\right)^{1/2} + \frac{\pi}{2}\left(\frac{E_c}{E}\right)^{1/2} \left\{ -\frac{i\omega}{2\gamma} \frac{D_0}{D_E(\omega)} \right\}^{1/2} \tag{2.77}$$

which can be solved explicitly. For the level of accuracy we are aiming, this solution may be written as

$$D_E(\omega) \approx \begin{cases} D_E\,, & \omega \ll \omega_c, \quad E \geq E_c \quad \text{(Metal)}\,, \\ D_0 \left(-\dfrac{i\omega}{2\gamma} \right)^{1/3}, & \omega \gg \omega_c \qquad\quad \text{(Metal and insulator)}\,, \\ D_E \dfrac{-i\omega}{-i\omega + (3D_E/v_F^2)\omega_0^2(E)}\,, & \omega \ll \omega_c, \quad E < E_c \quad \text{(Insulator)}\,, \end{cases} \tag{2.78}$$

where (cf. Eq. (2.42)):

$$\omega_c \sim 2\gamma [p_F \xi_{loc}]^{-d} \sim \frac{1}{N(E)\xi_{loc}^d}\,. \tag{2.79}$$

Here the renormalized diffusion coefficient

$$D_E = \frac{D_0}{p_F \xi_{loc}(E)}\,. \tag{2.80}$$

At the mobility edge itself $\xi_{loc}(E = E_c) = \infty$, so that $\omega_c = 0$ and we get the $\omega^{1/3}$-behavior (cf. Eq. (2.41)):

$$D_E(\omega) = D_0(-i\omega/2\gamma)^{1/3} . \tag{2.81}$$

Note that ω_c is in fact determined by $D_E(\omega_c) \sim D_E \sim D_0(\omega_c/2\gamma)^{1/3}$. The meaning of the limit $\omega \to 0$ used above (cf. e.g. Eq. (2.66)) is just that $\omega \ll \omega_c$. In particular, the expression Eq. (2.67) for the Gorkov–Berezinskii spectral density is valid only for $\omega \ll \omega_c$. For $\omega_c \leq \omega \leq 2\gamma$, using Eq. (2.81) in Eq. (2.46) we get from Eq. (A.16):

$$\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_q^F = \frac{\sqrt{3}}{2\pi} \frac{\alpha^{2/3}\omega^{1/3}q^2}{\omega^2 + \alpha^{2/3}\omega^{4/3}q^2 + \alpha^{4/3}\omega^{2/3}q^4} , \tag{2.82}$$

where $\alpha = D_0 v_F/2\gamma = D_0 l \sim [N(E)]^{-1}$, where the last estimate is for $l \sim p_F^{-1}$. Eq. (2.82) is valid also at the mobility edge itself where $\omega_c = 0$. Obviously, the correct estimate can be obtained from Eq. (2.67) by a simple replacement $D_E \to D_0(\omega/\gamma)^{1/3}$. It should be noted that the self-consistent theory approach to the frequency dependence of conductivity is clearly approximate. For example, it is unable to reproduce the correct $\mathrm{Re}\,\sigma(\omega) \sim \omega^2 \ln^4 \omega$ dependence for $\omega \to 0$ in the insulating state [3]. This is apparently related to its inability to take the correct account of the locally discrete nature of energy levels in Anderson insulators (cf. below). However, this is unimportant for our purposes while the general nature of frequency dependence at the mobility edge is apparently correctly reproduced.

In the following analysis we will also need a correlator of *local* densities of states defined in Eq. (A.3). This correlator can be expressed via the two-particle Green's function as in Eq. (A.15). For small ω and q and far from the Anderson transition (weak disorder), neglecting the nonsingular contribution from the second term of Eq. (A.15) we can estimate the most important contribution to that correlator from the diagram shown in Fig. 6 [62]. The same contribution comes from the diagram which differs from that in Fig. 6 by the direction of electron lines in one of the loops. Direct calculation gives

$$\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_q^H \sim \frac{N(E)}{\gamma^2}(\rho V^2)^2 \,\mathrm{Re} \int \mathrm{d}^d \boldsymbol{Q} \frac{1}{-i\omega + D_0 \boldsymbol{Q}^2} \frac{1}{-i\omega + D_0(\boldsymbol{Q} + \boldsymbol{q})^2}$$

$$\sim \frac{1}{N(E)} \mathrm{Re} \frac{1}{D_0^{d/2}} \frac{1}{(-i\omega + D_0 q^2)^{2-d/2}} . \tag{2.83}$$

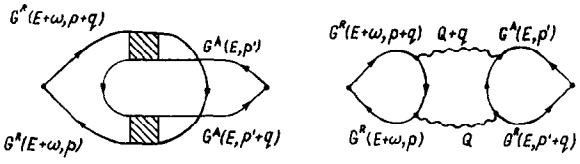

Fig. 6. Two equivalent forms of the diagram for the correlator of local density of states. Wavy lines denote the diffusion propagator, i.e. the sum of ladder diagrams.

A similar result for this correlator for some special model was first obtained by Oppermann and Wegner [63]. From Eq. (2.83) for $d = 3$ we find

$$\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_q^H \sim \frac{1}{N(E) D_0^{3/2}} \left\{ \frac{D_0 q^2}{\omega^2 + (D_0 q^2)^2} + [\omega^2 + (D_0 q^2)^2]^{-1/2} \right\}^{1/2} . \tag{2.84}$$

It is obvious that for the estimates close to the mobility edge we can in the spirit of self-consistent theory of localization replace D_0 in Eqs. (2.83) and (2.84) by the generalized diffusion coefficient $D(\omega)$. In particular, for a system at the mobility edge ($\omega_c = 0$)$D_0 \to D_0(\omega/\gamma)^{1/3}$ in Eq. (2.84).

Surely, the self-consistent theory of localization is not free from some difficulties. Apparently the main problem is an uncontrollable nature of the self-consistency procedure itself. These are discussed in Refs. [7, 29] in more detail. Here we shall concentrate only on some problems relevant for the future discussion. From the definition of the generalized diffusion coefficient in Eq. (2.47) it is clear that it may be a function of both ω and $\boldsymbol{q}$, i.e. it can also possess spatial dispersion. Self-consistent theory of localization deals only with the limit of $D_E(\boldsymbol{q} \to 0\omega)$. At present it is not clear whether we can in any way introduce spatial dispersion into the equations of self-consistent theory. Using scaling considerations the q dependence of $D_E(\boldsymbol{q}\omega \to 0)$ can be estimated as follows [6, 101]. We have seen above that for the system of finite size of $L \ll \xi_{\text{loc}}$ elementary scaling theory of localization predicts the L-dependent diffusion coefficient $D_E \approx (g_c/N(E))/L^{d-2}$ (cf. Eq. (2.35) for $d = 3$). From simple dimensional considerations we can try the replacement $L \to q^{-1}$ and get

$$D_E(\omega \to 0\boldsymbol{q}) \approx \begin{cases} D_E & \text{for } q\xi_{\text{loc}} \ll 1 , \\ \alpha q^{d-2} & \text{for } q\xi_{\text{loc}} \gg 1 , \end{cases} \tag{2.85}$$

where $\alpha \sim g_c/N(E) \sim D_0 l$ and $E \sim E_c$, $l^{-1} \sim p_F$. Obviously, an attempt to incorporate such q-dependence into the equations of self-consistent theory of localization (like Eqs. (2.56) and (2.57)) will radically change its structure. At the same time the L-dependence like $D_E \sim \alpha/L^{d-2}$ (for $L \ll \xi_{\text{loc}}$) can be directly derived from Eq. (2.57) as equations of elementary scaling theory are derived from it [54, 32, 29]. Thus the foundations for simple replacements like $L \to q^{-1}$ in Eq. (2.85) are not completely clear. More detailed analysis of the wave number dependence of the diffusion coefficient leading to Eq. (2.85) was given by Abrahams and Lee [65] within the scaling approach. However, the complete solution of this problem is apparently still absent. In a recent paper [66] it was shown that Eq. (2.85) actually contradicts the general localization criterion of Berezinskii and Gorkov, from which it follows directly that at the localization transition the static diffusion coefficient $D(\omega = 0, \boldsymbol{q})$ vanishes for all q simultaneously. The detailed analysis performed in Ref. [66] demonstrates the absence of any significant spatial dispersion of the diffusion coefficient on the scale of $q \sim \xi^{-1}$, while its presence on the scale of $q \sim p_F$ is irrelevant for the critical behavior of the system close to the Anderson transition. In fact, in Ref. [66] it is claimed that the *exact* critical behavior at the mobility edge coincides with that predicted by the self-consistent theory of localization.

Finally, we would like to stress that the self-consistent theory of localization cannot be applied "deep" inside the localization region. Its derivation is based on a kind of extrapolation of "metallic" expressions and it does not take into account local discreteness of the energy spectrum in the region of localized states as discussed in the previous section. This is reflected in the form of the one-particle Green's function used in the self-consistent theory [53, 32, 29, 7]. It does not describe

the effects of local level repulsion, though it does not contradict it [67]. Thus the self-consistent theory of localization can be applied within the localized region only until local energy spacing given by Eq. (2.13) is much smaller than the other relevant energies of the problem under consideration. In fact, this always leads to a condition of sufficiently large localization length R_{loc}, i.e. the system must be in some sense close to the mobility edge.

2.3.2. Quasi-two-dimensional systems

Self-consistent theory of localization for quasi-two-dimensional systems was first analyzed by Prigodin and Firsov [68]. The electronic spectrum of a quasi-two-dimensional system can be modeled by nearly-free electrons within highly conducting planes and tight binding approximation for interplane electron transfer:

$$E(\boldsymbol{p}) - E_F = v_F(|\boldsymbol{p}_\parallel| - p_F) - w\varphi(p_\perp)\,. \tag{2.86}$$

Here w is the interplane transfer integral and $\varphi(p_\perp) = \cos p_\perp a_\perp$, where $-\pi/a_\perp \le p_\perp \le \pi/a_\perp$. Then the equations of self-consistent theory of localization for the anisotropic generalized diffusion coefficient take the following form [68]:

$$D_j(\omega) = D_j^0 - \frac{1}{\pi N(E_F)} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{D_j(\omega)}{-i\omega + D_\parallel(\omega)q_\parallel^2 + D_\perp(\omega)(1 - \varphi(q_\perp))}\,, \tag{2.87}$$

where $j = \parallel, \perp$, and $D_\parallel^0 = v_F^2\tau/2$, $D_\perp^0 = (wa_\perp)^2\tau$ are inplane and interplane bare Drude diffusion coefficients, τ is the mean free time due to elastic scattering (disorder). This approach is in complete correspondence with the analysis of Wolfle and Bhatt [69] who has shown that the effects of anisotropy can be completely absorbed into the anisotropic diffusion coefficient. It can be seen that the initial anisotropy of the diffusion coefficient does not change as disorder grows up to the Anderson transition and in fact we have only to find one unknown ratio

$$d(\omega) = D_j(\omega)/D_j^0 = \sigma_j(\omega)/\sigma_j^0 \tag{2.88}$$

which is determined by the algebraic equation following from Eq. (2.87):

$$d(\omega) = 1 - \frac{1}{2\pi E_F\tau} \ln \frac{2}{[-i\omega\tau/d(\omega)] + (w\tau)^2 + [(-i\omega\tau/d(\omega))(-i\omega/d(\omega) + 2w^2\tau^2)]^{1/2}}\,. \tag{2.89}$$

Due to a quasi-two-dimensional nature of the system there is no complete localization for any degree of disorder which is typical for a purely two-dimensional system. However, the tendency for a system to become localized at lower disorder than in the isotropic case is clearly seen. All states at the Fermi level become localized only for $w < w_c$, where

$$w_c = \sqrt{2}\tau^{-1}\exp(-\pi E_F\tau)\,. \tag{2.90}$$

Thus the condition for localization is actually more stringent than given by the simplest Ioffe–Regel type estimate as in Eq. (2.11). For fixed w the mobility edge appears at

$$E_F = E_c = (1/\pi\tau)\ln(\sqrt{2}/w\tau)\,. \tag{2.91}$$

Thus in the case of strong anisotropy when $w\tau \ll 1$ localization can in principle take place even in the case of $E_F \gg \tau^{-1}$, i.e. at relatively weak disorder. These estimates are in qualitative accordance with Eq. (2.11), which is valid in the case of relatively strong disorder $E_F\tau \sim 1$.

In the metallic phase close to the Anderson transition,

$$\sigma_j = \sigma_j^0(E_F - E_c)/E_c \,. \tag{2.92}$$

For $w \to 0$ we have $E_c \to \infty$ which reflects complete localization in two dimensions. We can also define in-plane Drude conductivity at $E_F = E_c$ as a kind of a "minimal metallic conductivity" in this case as a characteristic conductivity scale at the transition:

$$\sigma_\parallel^c = e^2 N(E_F) D_\parallel^0(E_F = E_c) = \frac{1}{\pi^2}\frac{e^2}{\hbar a_\perp}\ln\left(\frac{\sqrt{2}\hbar}{w\tau}\right) \approx \frac{1}{\pi^2}\frac{e^2}{\hbar a_\perp}\ln\left(\frac{E_F}{w}\right), \tag{2.93}$$

where we have used $N(E_F) = m/(\pi a_\perp \hbar^2)$, m is the in-plane effective mass, and the last equality is valid for $E_F\tau/\hbar \sim 1$, i.e. for a case of sufficiently strong disorder. For the time being we again use $\hbar$ explicitly. From these estimates it is clear that in-plane "minimal conductivity" is logarithmically enhanced in comparison with the usual estimates (cf. Eq. (2.5)). This logarithmic enhancement grows as the interplane overlap of the electronic wave functions diminishes. Accordingly, in case of small overlap ($w\tau/\hbar \ll 1$) this conductivity scale may be significantly larger than $(3–5)\times 10^2\,\Omega^{-1}\,\mathrm{cm}^{-1}$ which is characteristic for isotropic systems. Thus, in the quasi-two-dimensional case, Anderson transition may take place at relatively high values of inplane conductivity. For a typical estimate in a high-T_c system we can take something like $E_F/w > 10$ so that the value of $\sigma_\parallel^c$ may exceed $10^3\,\Omega^{-1}\,\mathrm{cm}^{-1}$. Obviously, these estimates are in qualitative accordance with elementary estimates based upon the Ioffe–Regel criterion of Eqs. (2.11) and (2.12). Similar conclusions can be deduced from the analysis presented in Ref. [70] where it was shown by a different method that in the case of the anisotropic Anderson model the growth of anisotropy leads to a significant drop of a critical disorder necessary to localize all the states in a conduction band.

Now let us quote some results for the frequency dependence of the generalized diffusion coefficient in the quasi-two-dimensional case which follow from the solution of Eq. (2.89) [68]. We shall limit ourselves only to the results valid close to the mobility edge in metallic phase:

$$d(\omega) \approx \begin{cases} \dfrac{E_F - E_c}{E_c}\,, & \omega \ll \omega_c\,, \\ (2\pi E_F w\tau^2)^{-2/3}(-i\omega\tau)^{1/3}\,, & \omega_c \ll \omega \ll \omega^2\tau\,, \\ 1 - \dfrac{1}{2\pi E_F\tau}\ln(1/-i\omega\tau)\,, & w^2\tau \ll \omega \ll \tau^{-1}\,, \end{cases} \tag{2.94}$$

where

$$\omega_c \approx [2\pi E_F w\tau^2]^2(1/\tau)|(E_F - E_c)/E_c|^3\,. \tag{2.95}$$

From these expressions we can see the crossover from the $\omega^{1/3}$-behavior typical for isotropic three-dimensional systems to a logarithmic dependence on frequency which is characteristic of two-dimensional systems.

2.3.3. Self-consistent theory of localization in a magnetic field

An early version of the self-consistent theory of localization as proposed by Vollhardt and Wolfle was essentially based upon the time-reversal invariance [53, 32]. This property is obviously absent in the presence of an external magnetic field. In this case in addition to Eq. (2.45) we have to consider the two-particle Green's function in the particle–particle (Cooper) channel:

$$\Psi_E^{RA}(\boldsymbol{q},\omega) = -\frac{1}{2\pi i}\sum_{p+p'_-} \langle G^R(\boldsymbol{p}_+,\boldsymbol{p}'_+,E+\omega)G^A(-\boldsymbol{p}'_-,-\boldsymbol{p}_-,E\rangle \tag{2.96}$$

which for small ω and $\boldsymbol{q}$ again has a diffusion-pole form like that of Eq. (2.46), but with a *different* diffusion coefficient. Appropriate generalization of the self-consistent theory of localization was proposed by Yoshioka et al. [71]. This theory is based on the following system of coupled equations for relaxation kernels $M_j(\boldsymbol{q},\omega)$, corresponding to the diffusion coefficients in the particle–hole and particle–particle channels:

$$M_1 = 2i\gamma\left\{1 - \frac{1}{\pi N(E)}\sum_{n=0}^{N_0}\frac{2}{\pi L_H}\int_0^{\sqrt{q_0^2-4m\omega_H(n+1/2)}}\frac{dq_z}{2\pi}\frac{1}{\omega-(D_0/\tau M_2)[q_z^2+4m\omega_H(n+1/2)]}\right\}, \tag{2.97}$$

$$M_2 = 2i\gamma\left\{1 - \frac{1}{\pi N(E)}\sum_{|q|<q_0}\frac{1}{\omega - D_0 q^2/(\tau M_1)}\right\}. \tag{2.98}$$

Here $\omega_H = eH/mc$ is the cyclotron frequency, $L_H = (c/eH)^{1/2}$ is the magnetic length and $N_0 = q_0^2/4m\omega_H$. These equations form the basis of the self-consistent theory of localization in the absence of time-reversal invariance and were extensively studied in Refs. [71–75]. Alternative formulations of self-consistent theory in a magnetic field were given in Refs. [76–80]. All these approaches lead to qualitatively similar results. Here we shall concentrate on formulations given in Ref. [75].

Let us introduce the dimensionless parameter $\lambda = \gamma/\pi E$ as a measure of disorder and the generalized diffusion coefficients in diffusion and Cooper channels D_1 and D_2 defined as in Eq. (2.47) with M replaced by M_1 and M_2, respectively. We shall use dimensionless $d_j = D_j/D_0$ ($j = 1, 2$) in the following.

We are mainly interested in the diffusion coefficient in the Cooper channel, which as we shall see defines the upper critical field of a superconductor. Both this coefficient as well as the usual one are determined by the following equations which follow from Eqs. (2.97) and (2.98) after the use of Poisson summation over Landau levels in the first equation which allows one to separate the usual diffusion coefficient independent of magnetic field and the field-dependent part:

$$d_1 = (1 + (3\lambda - \delta_2 - \Delta_2)/d_2)^{-1}, \qquad d_2 = (1 + (3\lambda - \delta_1)/d_1)^{-1}, \tag{2.99}$$

where

$$\delta_j = (3/2\pi\lambda)^{3/2}(-i\omega/E)^{1/2}d_j^{-1/2} \tag{2.100}$$

and

$$\Delta_2 = -3\lambda\sum_{p=1}^{\infty}(-1)^p\int_0^1 dx\, 2\int_0^{\sqrt{1-x}} dy\,\frac{\cos(2\pi p x_0^2/c^2)}{y^2 + x + 3/2\pi\lambda(-i\omega/E)/(d_2 x_0^2)}, \tag{2.101}$$

where $c = (2\omega_H/E)^{1/2}$. In the following we have to solve Eqs. (2.99) for the case of small δ_j and Δ_2. Limiting ourselves to terms linear in δ_1, δ_2 and Δ_2 we obtain

$$d_1/d_2 = 1 + \Delta_2/(1 + 3\lambda) . \tag{2.102}$$

Using Eq. (2.102) in Eqs. (2.99) we get an equation for the diffusion coefficient in the Cooper channel:

$$d_2 = 1 - 3\lambda + \delta_2 + [3\lambda/(1 + 3\lambda)]\Delta_2 . \tag{2.103}$$

Introducing Δ_1 which differs from Δ_2 by the replacement of d_2 by d_1 we can also write down the approximate equation for the usual diffusion coefficient:

$$d_1 = 1 - 3\lambda + \delta_1 + [1/(1 + 3\lambda)]\Delta_1 . \tag{2.104}$$

In the absence of the magnetic field ($\Delta_1 = \Delta_2 = 0$) Eqs. (2.103) and (2.104) are the same and lead to standard results of self-consistent theory quoted above. Eq. (2.103) can be written as

$$2mD_2 = \pm\left(\frac{\omega_c}{E}\right)^{1/3} + \left(-\frac{i\omega}{E}\right)^{1/2}(2mD_2)^{-1/2} + \frac{3\lambda}{1 + 3\lambda}\Delta_2 , \tag{2.105}$$

where + corresponds to metallic, and − to insulating phases, while the characteristic frequency

$$\omega_c = (|1 - 3\lambda|/(\tfrac{3}{2}\pi\lambda))^3 E \tag{2.106}$$

can be considered as a measure of disorder and separate regions with different frequency dependence of the diffusion coefficient.

Neglecting in Eq. (2.101) terms oscillating with the magnetic field (these oscillations are connected with the sharp cut-off in the momentum space used above and disappear for smooth cut-off) we get

$$\Delta_2 = -(2\omega_H/E)^{1/2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p^{1/2}} f(2\pi\rho\kappa) , \tag{2.107}$$

where

$$f(y) = \sqrt{2/\pi} \int_0^{\infty} \frac{\cos(t)\,dt}{\sqrt{t + y}} ; \qquad \kappa = \frac{-i\omega/E}{2\omega_H/E} \frac{1}{2mD_2} . \tag{2.108}$$

This gives

$$\Delta_2 = \begin{cases} W(2\omega_H/E)^{1/2} , & |\kappa| \ll 1 , \\ \dfrac{1}{48}\left((-i\omega/E)\dfrac{1}{2mD_2}\right)^{-3/2}(2\omega_H/E)^2 , & |\kappa| \gg 1 , \end{cases} \tag{2.109}$$

where $W = -\sum_{p=1}^{\infty}(-1)^p/p^{1/2} \approx 0.603$.

Solutions of Eq. (2.105) for different limiting cases can be found in Ref. [75]. Comparison of Eq. (2.104) and Eq. (2.103) shows that the usual diffusion coefficient D_1 is given by the same expressions as D_2 with the replacement of the coefficient $3\lambda/(1 + 3\lambda)$ before the field-dependent correction by

$1/(1+3\lambda)$. Here we only quote the results for D_2 in case of $\omega_c/E \ll (\omega_H/E)^{3/2}$, valid close to the transition in the absence of magnetic field:

$$D_2 = \frac{1}{2m}\left\{\pm(\omega_c/E)^{1/3} + \left[\frac{3\lambda}{1+3\lambda}\right] W(2\omega_H/E)^{1/2}\right\} \approx \frac{1}{4m} W(2\omega_H/E)^{1/2}\,, \quad \omega \ll \omega_c^*\,, \tag{2.110}$$

$$D_2 = \frac{1}{2m}\left\{\left(\frac{-i\omega}{E}\right)^{1/3} + \frac{2}{3}\left[\frac{3\lambda}{1+3\lambda}\right]\frac{1}{48}\frac{(2\omega_H/E)^2}{(-i\omega/E)}\right\}, \quad \omega \gg \omega_c^*\,, \tag{2.111}$$

where $\omega_c^* = (W/2)^3(2\omega_H/E)^{3/2}E$.

Note that for high frequencies larger than ω_c^* the correction term becomes quadratic in field which differs from the usual square root behavior at low frequencies.

It is easy to see that in the absence of the external magnetic field these equations reduce to the usual self-consistency equation as derived by Vollhardt and Wolfle with a single relaxation kernel.

Let us finally quote some results for the purely two-dimensional case [81]. Self-consistent equations for the diffusion coefficients take now the following form:

$$\begin{aligned}\frac{D_0}{D_2} &= 1 + \frac{1}{\pi N(E)}\sum_{|q|<q_0}\frac{1}{\omega + D_1 q^2}\,,\\ \frac{D_0}{D_1} &= 1 + \frac{1}{\pi N(E)}\sum_{|k|<q_0}\frac{1}{\omega + D_2 k^2}\,,\end{aligned} \tag{2.112}$$

where $k^2 = 4m\omega_H(n+\frac{1}{2})$, and we assume that ω here is the imaginary (Matsubara) frequency, which simplifies the analysis. Actually, only the dependence on the Matsubara's frequencies are important for further applications to superconductivity.

Introduce again the dimensionless diffusion coefficients $d_1 = D_1/D_0$, $d_2 = D_2/D_0$, so that Eqs. (2.112) are rewritten as

$$\begin{aligned}\frac{1}{d_2} &= 1 + \frac{\lambda}{d_1}\ln\left(1 + d_1\frac{1}{2\omega\tau}\right),\\ \frac{1}{d_1} &= 1 + \frac{\lambda}{d_2}\sum_{n=0}^{N_0}\frac{1}{n+\frac{1}{2}+(\omega/4m\omega_H D_0)(1/d_2)}\,,\end{aligned} \tag{2.113}$$

where $N_0 = 1/8m\omega_H D_0\tau$ is the number of Landau levels below the cut-off. We assume that the magnetic field is low enough, so that $N_0 \gg 1$, i.e.

$$H \ll \Phi_0/D_0\tau\,. \tag{2.114}$$

With accuracy sufficient for further use we can write down the following solution for the diffusion coefficient in the Cooper channel:

For weak magnetic field $\omega_H \ll \lambda e^{-1/\lambda}/\tau$

$$d_2 = \begin{cases} 1 & \text{for } \omega \gg e^{-1/\lambda}/2\tau\,,\\ 2\omega\tau e^{1/\lambda} & \text{for } \omega \ll e^{-1/\lambda}/2\tau\,,\end{cases} \tag{2.115}$$

and we can neglect the magnetic field influence upon diffusion.

For larger fields $\omega_H \gg \lambda e^{-1/\lambda}/\tau$

$$d_2 = \begin{cases} 1 & \text{for } \omega \gg e^{-1/\lambda}/2\tau\,, \\ 1/\lambda \ln(1/2\omega\tau) & \text{for } e^{-1/\lambda^2 \ln Q}/2\tau \ll \omega \ll e^{-1/\lambda}/2\tau\,, \\ 2\omega\tau\lambda \ln Q\, e^{1/\lambda^2 \ln Q} & \text{for } \omega \ll e^{-1/\lambda^2 \ln Q}/2\tau\,, \end{cases} \tag{2.116}$$

where $Q = \pi\gamma\lambda/\tau\omega_H$, $\gamma \approx 1.781$.

Here we neglect the magnetic field corrections small in comparison to the d_2 value in the absence of the magnetic field given by Eq. (2.115).

2.4. Phase transition analogy and scaling for correlators

Scaling description of a system close to the Anderson transition can be developed also on the basis of some analogies with usual phase transitions [4, 7, 6]. Most successful in this respect is an approach initially proposed by Wegner [82–84].

Let us consider Eqs. (2.67) and (2.83) which define basic electronic correlators (spectral densities) in a disordered system. For the metallic region we can write:

$$K_F(\boldsymbol{q}\omega) \equiv N(E)\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_q^F \sim \mathrm{Re}\,\frac{N(E)}{-i\omega + D_E q^2}\,, \tag{2.117}$$

$$K_H(\boldsymbol{q}\omega) \equiv N(E)\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_q^H \sim \mathrm{Re}\,\frac{D_E^{d/2}}{(-i\omega + D_E q^2)^{2-d/2}}\,. \tag{2.118}$$

Wegner has noted [83, 63] that these expressions are in some sense similar to analogous expressions for the transverse and longitudinal susceptibilities of a ferromagnet [47]:

$$\chi_\perp(\boldsymbol{q}) = M/(H + \rho_s q^2)\,, \tag{2.119}$$

$$\chi_\parallel(\boldsymbol{q}) \sim \frac{1}{(H + \rho_s q^2)^{2-d/2}}\,, \tag{2.120}$$

where M is the magnetization, H the external magnetic field and ρ_s is the spin-stiffness coefficient. Comparing Eqs. (2.117) with Eq. (2.119) and Eq. (2.118) with Eq. (2.120) we can write down a correspondence between electron diffusion in a random system and a ferromagnet as given in Table 1.

Now we can use the main ideas of the scaling approach in the theory of critical phenomena [45–47, 85] and formulate similar expressions for the electronic system close to the Anderson transition. As was noted above, scaling theory is based upon an assumption that a singular behavior of the physical parameters of a system close to a phase transition appears due to large scale (long wave-length) fluctuations of the order-parameter (e.g. magnetization) close to the critical temperature T_c. Scaling hypothesis claims that singular dependence on $T - T_c$ reflect the divergence of the correlation length of these fluctuations ξ and this length is the only relevant length-scale in the critical region. Scaling approach is based upon an idea of scale transformations

Table 1
Anderson transition and ferromagnet close to Curie point T_c

Localization	Ferromagnet
$E - E_c$	$T - T_c$
K_F	$\chi_\perp$
K_H	$\chi_\parallel$
$-i\omega$	H
$N(E)$	M
D_E	ρ_s
ξ_{loc}	ξ

and dimensional analysis. Under the scale transformation, the spatial interval Δx changes to $\Delta x'$, according to

$$\Delta x \to \Delta x' = s^{-1}\Delta x \,. \tag{2.121}$$

Accordingly, for the wave vector:

$$\boldsymbol{q} \to \boldsymbol{q}' = s\boldsymbol{q} \,. \tag{2.122}$$

Scaling dimension [85] of a physical quantity A is equal to λ if under scale transformations defined by Eqs. (2.121) and (2.122) we get

$$A \to A' = As^{\lambda} \,. \tag{2.123}$$

Scaling dimensions for the main characteristics of a ferromagnet are given in terms of standard critical exponents [85] in Table 2.

Correlation length of the theory of critical phenomena behaves like

$$\xi \sim |T - T_c|^{-\nu} \,. \tag{2.124}$$

The knowledge of scaling dimension of a given physical quantity allows to determine its dependence on ξ, i.e. on $T - T_c$. For example, magnetization M behaves according to Table 2 as

$$M \sim \xi^{-1/2(d-2+\eta)} \sim |T - T_c|^{\beta} \,, \tag{2.125}$$

where the critical exponent of magnetization equals

$$\beta = \tfrac{1}{2}\nu(d - 2 + \eta) \,. \tag{2.126}$$

Magnetic susceptibility is given by

$$\chi(\boldsymbol{q}, T - T_c) = \xi^{2-\eta} g(q\xi) \,, \tag{2.127}$$

where $g(x)$ is some universal function, such that $g(0) \sim$ const., $g(x \to \infty) \sim x^{-(2-\eta)}$. From Eq. (2.127) we get the standard results:

$$\chi(0, T - T_c) \approx \xi^{2-\eta} g(0) \sim |T - T_c|^{-\gamma} \,, \tag{2.128}$$

Table 2
Scaling dimensions in the theory of critical phenomena

ξ	q	M	H
-1	$+1$	$1/2(d-2+\eta)$	$1/2(d+2-\eta)$

where $\gamma = (2-\eta)\nu$ is the susceptibility exponent. Analogously,

$$\chi(\boldsymbol{q}, T = T_c) \sim q^{-2+\eta} . \tag{2.129}$$

Here η is sometimes called Fisher's exponent.

It is easy to see that Eq. (2.127) is equivalent to the scaling relation (H-dependence is taken from Table 2)

$$\chi(s\boldsymbol{q}, s^{-1}\xi, s^{1/2(d+2-\eta)}H) = s^{-(2-\eta)}\chi(\boldsymbol{q}, \xi, H) . \tag{2.130}$$

It is convenient to make the transformation $|T - T_c| \to b|T - T_c|$ so that $\xi \to b^{-\nu}\xi$ which is equivalent to the choice of $s = b^{\nu}$. Then Eq. (2.130) transforms to

$$\chi(b^{\nu}\boldsymbol{q}, b^{-\nu}\xi, b^{\nu(d+2-\eta)/2}H) = b^{-\gamma}\chi(\boldsymbol{q}, \xi, H) . \tag{2.131}$$

Finally, note that close to Curie point the spin-stiffness coefficient ρ_s satisfies the so-called Josephson relation [47]:

$$\rho_s \sim |T - T_c|^{(d-2)\nu} \tag{2.132}$$

and tends to zero as $T \to T_c$ from within the condensed phase.

Consider now the analogy formulated in Table 1. Density of states $N(E)$ is nonsingular at the mobility edge [27,7]. Then considering $N(E)$ as an analog of magnetization M we have to assume $\beta = 0$, i.e. at the localization transition

$$\eta = 2 - d \tag{2.133}$$

and the "order-parameter" $N(E)$ is nonsingular at the transition $E = E_c$. Accordingly, we have $\gamma = d\nu$. Josephson relation Eq. (2.132) now takes the form

$$D_E \sim |E - E_c|^{(d-2)\nu} , \tag{2.134}$$

i.e. it is in fact is equivalent to Wegner's relation for conductivity given by Eq. (2.31). Correlation length exponent ν remains unknown.

For electronic correlators of Eqs. (2.117) and (2.118) we obtain from Eq. (2.131) scaling relations [83,84]

$$K_{F,H}(b^{\nu}\boldsymbol{q}, b^{d\nu}\omega, b(E - E_c)) = b^{-d\nu}K_{F,H}(\boldsymbol{q}, \omega, E - E_c) . \tag{2.135}$$

Taking $\nu = 1/(d-2)$ from Eq. (2.29) for $d = 3$ and $E = E_c$ (i.e. at the mobility edge itself) we get from Eq. (2.135):

$$K_{F,H}(b\boldsymbol{q}, b^3\omega) = b^{-3}K_{F,H}(\boldsymbol{q}\omega) \tag{2.136}$$

which is equivalent to

$$K_{F,H}(\boldsymbol{q}\omega) = L_\omega^3 F_{F,H}(qL_\omega) \tag{2.137}$$

where $F_{F,H}(x)$ is some universal function and we introduced the characteristic length

$$L_\omega = [\omega N(E)]^{-1/3} . \tag{2.138}$$

Note that the same scaling dependence follows, e.g. for $K_H(\boldsymbol{q}\omega)$ from Eq. (2.83) or Eq. (2.84) after a simple replacement of D_0 by a diffusion coefficient given by

$$D_{E=E_c}(\boldsymbol{q}\omega) = L_\omega^{-1} f(qL_\omega) , \tag{2.139}$$

where $f(x \to 0) \to 1$ and $f(x \to \infty) \to x$. In particular, in the limit of $qL_\omega \to 0$ we get $F(x) = (1 + x^4)^{-1/4}$ and the replacement $D_0 \to D_0(\omega/\gamma)^{1/3}$ mentioned in connection with Eq. (2.84) is valid. On the other hand, from Eq. (2.133) it follows that at $\omega = 0$ we get from Eq. (2.129)

$$K(\boldsymbol{q}, \omega = 0, E = E_c) \sim q^{-d} \tag{2.140}$$

which is equivalent to Eq. (2.67) if we take $D_{E=E_c}(\omega = 0, \boldsymbol{q}) = \alpha q^{d-2}$ (cf. Eq. (2.85)).

Microscopic justification for this scaling hypothesis can be provided with one or other variant of the field-theory approach based upon nonlinear σ-model [82–84]. There exist several alternative schemes of "mapping" of the problem of an electron in a random field onto field-theoretic formalism of nonlinear σ-models [86, 88–92]. The main physical justification of this approach is to represent an effective Hamiltonian of an electronic system in a form similar to the analogous Hamiltonian of the Heisenberg ferromagnet below Curie point:

$$\mathscr{H} = \tfrac{1}{2}(\partial \boldsymbol{M}/\partial x_\alpha)^2 - \boldsymbol{HM} ; \quad \boldsymbol{M}^2 = \text{const} . \tag{2.141}$$

As a result, an effective Hamiltonian for an electron in a random field in terms of interacting modes responsible for the critical behavior close to mobility edge appears. Following Ref. [88] we can introduce an "order-parameter" as a $2n \times 2n$ matrix $\hat{Q}$ (n-integer). Every matrix element of $\hat{Q}$ can be represented as

$$Q_{ij} = \begin{pmatrix} D_{ij} & \Delta_{ij} \\ -\Delta_{ij}^* & D_{ij}^* \end{pmatrix}, \tag{2.142}$$

where $D_{ij} = D_{ji}^*$ and $\Delta_{ij} = -\Delta_{ji}^*$, i.e. they are elements of Hermitian and antisymmetric matrices, respectively. Analogously, $\boldsymbol{M}^2 = \text{const}$. in a ferromagnet. $\hat{Q}$-matrix must satisfy the condition:

$$\hat{Q}^2 = 1 ; \qquad \operatorname{Tr}\hat{Q} = 0 . \tag{2.143}$$

The effective Hamiltonian for diffusion modes takes the following form [82, 83]:

$$\mathscr{H} = D_0 \operatorname{Tr}(-i\nabla\hat{Q})^2 - i\omega \operatorname{Tr}\hat{\Lambda}\hat{Q} . \tag{2.144}$$

Here $\hat{\Lambda}$ is the diagonal matrix with the first n elements equal to 1 and the remaining n are -1. Correlation function of D-elements corresponds to diffusion, while that of Δ-elements to Cooperon. Parameter n should be put equal to zero at the end of calculations in the spirit of the famous "replica trick" in the theory of disordered systems [85, 4].

This formalism is useful also for the analysis of different kinds of external perturbations, such as external magnetic field, magnetic impurities, spin–orbital scattering etc. [88]. Standard methods of renormalization group using perturbation theory over $(p_F l)^{-1} \ll 1$ reproduces all the main results of elementary scaling theory of localization, including the qualitative form of the β-function as in Fig. 3. However, the formalism of the σ-model approach is quite complicated and practically does not allow to get explicit expressions for the physical characteristics of the system, especially in the localized phase.

Many problems of fundamental nature still remain unresolved. Most important are questions concerning the role of nonperturbative contributions close to the mobility edge [4, 7, 92–94]. Note, especially, the strong criticism about one-parameter scaling in Refs. [93–94]. Among the several results obtained within the σ-model approach we wish to mention an important paper by Lerner [95], where a distribution function for the local density of states in a system close to the Anderson transition was determined and shown to be essentially non-Gaussian.

For our future analysis it is important to stress that in most cases the results of the σ-model approach practically coincide with the predictions of the self-consistent theory of localization which also neglects all nonperturbative effects, except those determined by some infinite resummation of diagrams. It must be stressed that self-consistent theory is based upon some uncontrollable ad hoc assumptions and in this respect it is not as well justified as the σ-model approach. However, this simple theory as we have seen above allows practical calculation of any interesting characteristic of an electronic system close to the mobility edge including the localized phase.

2.5. Interaction effects and Anderson transition

The main unresolved problem of the theory of metal–insulator transition in disordered systems is the role of electron–electron interactions. The importance of interactions for this problem is known for a long time [2]. In recent years the decisive importance of interactions was revealed in the theory of "dirty metals" [38–40], as well as in the concept of the Coulomb gap at the Fermi level of strongly localized electrons [41–44]. We have already briefly discussed the Coulomb gap. It appears for strongly localized states. In case of "dirty metals", the diffusive nature of electronic transport leads to special interference effects between the Coulomb interaction and disorder scattering [38, 40]. Most important is an appearance of some kind of a precursor to the Coulomb gap already in the metallic state. It is connected with simple exchange correction to the electron self-energy (cf. Fig. 7) which leads to the following cusp-like correction to one-particle density of states in case of the screened Coulomb interaction in three-dimensional systems [38]:

$$\delta N(E) = \frac{|E - E_F|^{1/2}}{2\sqrt{2}\pi^2 D_0^{3/2}}, \tag{2.145}$$

where D_0 is the usual Drude diffusion coefficient. In the two-dimensional case this correction is logarithmic [96, 40]. General belief is that this cusp somehow transforms into the Coulomb gap as the system moves from metal to insulator. However, up to now there is no complete solution for this problem.

An early attempt to describe electron–electron interactions in Anderson insulators in a Fermi-liquid-like scheme is discussed in Ref. [97]. Simple generalization of the theory of "dirty metals"

[38–40] along the lines of self-consistent theory of localization was proposed in Refs. [98, 60, 7]. However, the most general approach to this problem was introduced by McMillan [99] who proposed to describe the metal–insulator transition in a disordered system by a scaling scheme similar in spirit to the elementary scaling theory of localization of noninteracting electrons discussed above. He formulated a simple system of coupled differential equations of the renormalization group for two effective "charges": dimensionless conductance g and single-particle density of states $N(E)$. Later it was realized that this simple scheme cannot be correct because it assumed for conductivity a relation like Eq. (2.59) with the density of states while the correct Einstein relation for the interacting system contains electron compressibility $dn/d\zeta$ (ζ is chemical potential) [100–102], which is not renormalized close to the metal–insulator transition as opposed to density of states. The most comprehensive approach to a scaling description of the metal–insulator transition in disordered systems was formulated by Finkelstein [102–105]. Unfortunately, more or less explicit solutions were only obtained neglecting the scattering and interaction processes in the Cooper channel which are mainly responsible, as we have seen above, for localization itself. Some attempts in this direction were undertaken only in Ref. [104]. This approach is still under very active discussion [107–115] and demonstrates the fundamental importance of interactions. However, the problem is still unresolved and most of these works consider only the metallic side of transition with no serious attempts to analyze the insulating state.

Below we consider only some qualitative results of this approach, following mainly Refs. [108, 109]. Fermi liquid theory survives the introduction of disorder [119], although with some important corrections [38, 40], and is actually valid up to the metal–insulator transition [102, 103, 108, 109].

In the absence of translation invariance there is no momentum conservation and we have to use some unknown exact eigenstate $\phi_\nu(\boldsymbol{r})$ representation for electrons in a random field to characterize quasi-particles with energies ε_ν (cf. Ref. [120]). The free energy as a functional of the quasi-particle distribution function $n_s(\varepsilon_\nu, \boldsymbol{r})$ (s-spin variable) is written as is usual in the Fermi liquid theory:

$$F\{n_s(\varepsilon_\nu, \boldsymbol{r})\} = \sum_{s,\nu} \int d\boldsymbol{r}\, n_s(\varepsilon_\nu \boldsymbol{r})(\varepsilon_\nu - \zeta) + \frac{1}{2}\sum_{ss'} \int d^d\boldsymbol{r}\, \delta N_s(\boldsymbol{r}) \delta N_{s'}(\boldsymbol{r}) f_{ss'} , \tag{2.146}$$

where $N_s = \sum_\nu n_s(\varepsilon_\nu \boldsymbol{r})$ is the total density per spin and $f_{ss'} = f^s + ss' f^a$ is the quasi-particle interaction function. The angular dependence of the f-function in the dirty case can be neglected, because $n_s(\varepsilon_\nu \boldsymbol{r})$ is assumed to describe electrons on distances larger than the mean free path where only s-wave scattering is important and the Fermi-liquid interaction becomes point-like. In an external spin dependent field V_s the quasi-particle distribution function obeys a kinetic equation:

$$\frac{\partial}{\partial t} n_s - D\nabla^2 n_s + (\partial n_s/\partial \varepsilon)(-D\nabla^2)\left[V_s + \sum_{s'} f_{ss'} N_{s'} \right] = 0 , \tag{2.147}$$

where D is the quasi-particle diffusion coefficient. Eq. (2.147) is obtained from the usual Fermi-liquid kinetic equation [120] by replacing $v_F \partial/\partial \boldsymbol{r}$ by $-D\nabla^2$ which reflects a crossover from ballistic to diffusive transport in disordered systems. Solving Eq. (2.147) for density–density and spin–spin response functions one gets [102, 103, 107]

$$\chi_\rho(\boldsymbol{q}\omega) = (dn/d\zeta) D_\rho q^2/(D_\rho q^2 - i\omega) , \tag{2.148}$$

$$\chi_s(\boldsymbol{q}\omega) = \chi D_s q^2/(D_s q^2 - i\omega) , \tag{2.149}$$

where $dn/d\zeta = N(E_F)/(1 + F_0^s)$, $\chi = N(E_F)\mu_B^2/(1 + F_0^a)$ (μ_B is Bohr's magneton) and

$$D_\rho = D(1 + F_0^s)\,, \tag{2.150}$$

$$D_s = D(1 + F_0^a)\,. \tag{2.151}$$

Landau parameters $F_0^{s,a}$ are defined by

$$N(E_F)f^s = F_0^s\,, \quad N(E_F)f^a = F_0^a\,, \tag{2.152}$$

where $N(E_F)$ is the quasi-particle density of states at the Fermi level (for both spin directions). If we neglect Fermi-liquid renormalization effects Eq. (2.148) reduces to Eq. (2.49). Conductivity is given now by $\sigma = e^2 D(dn/d\zeta)$.

As the system moves towards the metal–insulator transition Hubbard-like interaction of electrons close to a given impurity site becomes more and more important. It is known for a long time [2, 7] that this interaction leads to the appearance of a band of single-occupied states just below the Fermi level of a system on the dielectric side of the Anderson transition. These states actually simulate paramagnetic centers and lead to Curie-like contribution (diverging as temperature $T_c \to 0$) [2, 7]. Thus on the metallic side of transition static magnetic susceptibility χ is expected to diverge since it is infinite (at $T = 0$) on the insulating side. At the same time, $dn/d\zeta$ remains finite. Therefore, $D_s/D_\rho = (dn/d\zeta)/\chi$ goes to zero, i.e. *spin diffusion is much slower than charge diffusion* close to the metal–insulator transition. This fact was first noted in Ref. [104] where it was assumed that it leads to a possibility of local magnetic effects appearing in the metallic phase before a transition. It is interesting to note that the slowing down of spin diffusion due to interactions was actually discovered long before [106] it appeared in the context of the interaction picture of the metal–insulator transition. This idea was further elaborated in Refs. [112–114], where extensive discussion of this magnetic transition was given. There is an interesting problem why these localized moments are not quenched by the Kondo effect. This can apparently be explained by the local fluctuations of Kondo temperature due to fluctuations of local density of states induced by disorder [116]. The resulting distribution of Kondo temperatures is shown to be singular enough to induce diverging magnetic susceptibility as $T \to 0$.

The idea of paramagnetic moments appearing already in the metallic phase apparently can much simplify the analysis of the metal–insulator transition and allow its description by equations of elementary scaling theory of localization [117, 118, 40]. In the general case, electron interactions in the diffusion channel can be classified by the total spin of an electron and hole j [40]. It can be shown that all interaction corrections with $j = 0$ do not depend on the electron–electron coupling constant (charge) and are universal [40]. If paramagnetic scattering is operating in the system it dumps scattering processes in the Cooper (localization) channel [121] as well as interaction processes in the diffusion channel with $j = 1$ [40]. In this case, only interaction processes with $j = 0$ determine corrections to the classical (Drude) conductivity. Due to the universal nature of these corrections (independence of electronic charge) their structure actually coincides with that of localization corrections (Cooperon) [117, 118]. This means that renormalization group has only one effective "charge" – dimensionless conductance g. In this case, the differential equation for the conductance of a finite system is again given by Eq. (2.18) with the same asymptotic forms of $\beta_d(g)$. This approach is valid for systems with linear size $L < L_T = \sqrt{\hbar D/T}$. This length L_T replaces in the theory of interacting electrons the characteristic length of phase coherence L_φ of the noninteracting

theory. The appearance of this new length is due to the fact that the characteristic time of the interaction processes [40] is $\sim\hbar/T$. We must stress that these arguments are probably oversimplified as Refs. [102–104, 107] have demonstrated the relevance of interaction in the sense of appearance of additional coupling constants ("charges"). Also it is in no way clear that local moments appearing within this approach are acting just as the usual paramagnetic scatterers. However, the simple scheme following from Refs. [117, 118] seems to be too attractive on physical grounds just to be neglected.

As in the noninteracting case for $d = 3$, Eq. (2.18) again possess an unstable fixed point responsible for the existence of the mobility edge and absence of minimal metallic conductivity at the metal–insulator transition. However, in this case there are no special reasons to believe that the critical exponent ν of the localization correlation length $\xi_{\rm loc}$ will coincide with its value from the noninteracting theory. At finite temperatures, as in the usual scaling picture, conductivity for $d = 3$ is given by [117, 118, 40]

$$\sigma \approx (e^2/\hbar\xi_{\rm loc}) f(\xi_{\rm loc}/L_T) \,. \tag{2.153}$$

As the system approaches the insulating phase, $\xi_{\rm loc} \to \infty$. For $\xi_{\rm loc} \ll L_T$ we have $f(\xi_{\rm loc}/L_T) = A + B(\xi_{\rm loc}/L_T)$, where A and B are some numerical constants. Thus, in this region, conductivity corrections are proportional to $\sqrt{T}$ [38]. In case of $\xi_{\rm loc} \gg L_T$, i.e. very close to transition:

$$\sigma \approx C(e^2/\hbar L_T) = C(e^2/\hbar)\sqrt{T/D\hbar} \,, \tag{2.154}$$

where again $C \sim 1$. Using the Einstein relation [100] $\sigma = e^2 D(\mathrm{d}n/\mathrm{d}\zeta)$ we immediately obtain

$$D = (C^{2/3}/\hbar) T^{1/3} (\mathrm{d}n/\mathrm{d}\zeta)^{-2/3} \tag{2.155}$$

and

$$\sigma = C^{2/3}(e^2/\hbar)(T\,\mathrm{d}n/\mathrm{d}\zeta)^{1/3} \tag{2.156}$$

which is valid for $L_T < \xi_{\rm loc}$, where $L_T = [C/(T\,\mathrm{d}n/\mathrm{d}\zeta)]^{1/3}$.

In the case of a system in an alternating electric field with frequency $\omega \gg T/\hbar$ the relevant length becomes $L_\omega = [D/\omega]^{1/2}$ as in Eq. (2.37). Accordingly, for $L_\omega \ll \xi_{\rm loc}$ instead of Eq. (2.156) we get

$$\sigma(\omega) \approx (e^2/\hbar)(\omega\,\mathrm{d}n/\mathrm{d}\zeta)^{1/3} \tag{2.157}$$

which is analogous to Eqs. (2.41) and (2.81). However, we must note that this result cannot be considered very reliable since the dynamical critical exponent in the general case is an independent one [103, 104].

The metal–insulator transition can be viewed as a gradual breakdown of the Fermi liquid state [109]. As we approach the transition, different Fermi-liquid parameters, such as D, $N(E_{\rm F})$, χ, etc. change continuously and at a critical point some of these may either diverge or go to zero. This behavior is related to the divergence of the correlation length $\xi_{\rm loc}$ characterized by a critical exponent ν. On the insulating side of the transition this length can also be interpreted as the scale inside which a Fermi liquid description of the system still holds.

At present, we are in need of some kind of new approach to the theory of interacting electrons in disordered systems which probably may be formulated along the lines of the self-consistent theory of localization. The attempt is to provide an effective formalism to calculate the basic physical

properties of the system in an interpolating scheme from metallic to insulating state. Below, we briefly describe an attempt to construct such a self-consistent approach [122].

The basic idea in equal footing (additive) is the treatment of both localization and interaction corrections to the current relaxation kernel defining the generalized diffusion coefficient in Eq. (2.47). As a zeroth approximation we take the Drude metal and consider the simplest localization and interaction corrections, so that the relaxation kernel takes the following form:

$$M(\omega) = M_0 + \delta M(\omega)\,, \tag{2.158}$$

where $\delta M(\omega) = \delta M_l(\omega) + \delta M_c(\omega) = -(M_0/D_0)(\delta D_l(\omega) + \delta D_c(\omega))$. Here the localization correction to the diffusion coefficient $D_l(\omega)$ is defined by the usual sum of "maximally crossed" diagrams which yields

$$\frac{\delta D(\omega)}{D_0} = -\frac{1}{\pi N_0(E_F)} \sum_{|q|<k_0} \frac{1}{-i\omega + D_0 q^2}\,, \tag{2.159}$$

while the Coulomb correction $D_c(\omega)$ is given by

$$\frac{\delta D_c(\omega)}{D_0} = \frac{\delta\sigma(\omega)}{2e^2 N_0(E_F) D_0}$$

$$= \frac{8i}{\pi d}\,\mu D_0 \frac{1}{\pi N_0(E_F)} \int_\omega^\infty d\Omega \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(-i(\Omega+\omega) + D(\Omega+\omega)q^2)(-i\Omega + D(\Omega)q^2)^2}\,, \tag{2.160}$$

where $\mu = N_0(E_F)v_0$ is the dimensionless point-like interaction with $N_0(E_F)$ now denoting the single-spin density of states at the Fermi level for the noninteracting case. The lowest order interaction corrections are shown in Fig. 7. Conductivity correction $\delta\sigma$ due to interactions was

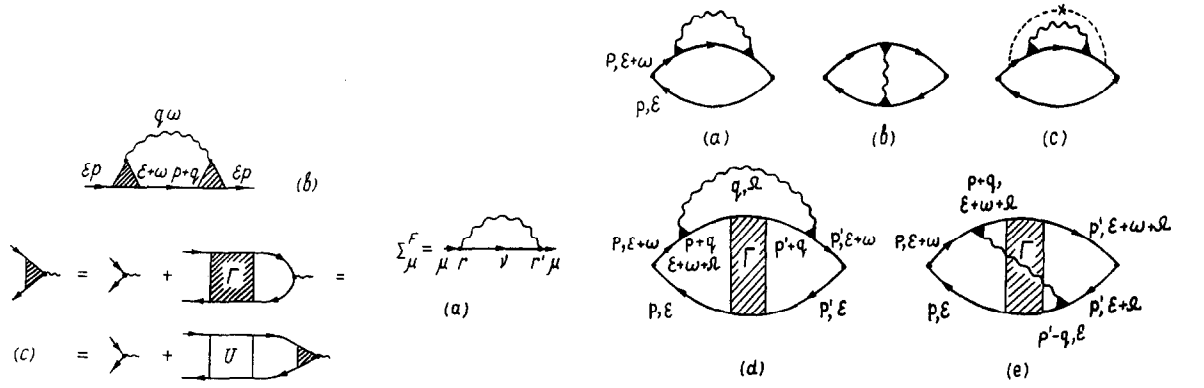

Fig. 7. Lowest order interaction corrections: (a) Simplest Fock correction for self-energy in exact eigenstate representation. (b) Equivalent diagram in momentum representation. (c) "Triangular" vertex defining diffusion renormalization. U – irreducible impurity scattering vertex, Γ – full impurity scattering vertex, and wavy line denotes interelectron interaction.

Fig. 8. Lowest order interaction corrections to conductivity.

defined by the lowest-order diagrams shown in Fig. 8 which were for the first time analyzed in Ref. [123], neglecting localization corrections. It was shown in Ref. [123] that the total contribution of diagrams (a)–(c) is actually zero and conductivity correction reduces to that determined by diagrams (d) and (e). Here we neglect also the so-called Hartree corrections to conductivity [40, 123], which is valid in the limit of $2k_F/\kappa_D \gg 1$, where κ_D is the inverse screening length. This inequality, strictly speaking, is valid for systems with low electronic density, which are most interesting for experimental studies of disorder induced metal–insulator transitions. Also, if we remember the divergence of the screening length at the metal–insulator transition, we can guess that this approximation becomes better as we approach the transition. The point-like interaction model used above has to be understood only in this sense.

Self-consistency procedure is reduced to the replacement of D_0 by the generalized diffusion coefficient in the denominators of all the diffusion forms. As a result, we obtain the following *integral* equation for the generalized diffusion equation:

$$\frac{D_0}{D(\omega)} = 1 + \frac{1}{\pi N_0(E_F)} \int \frac{d^d q}{(2\pi)^d} \frac{1}{-i\omega + D(\omega)q^2} - \frac{8i}{\pi d} \mu D_0 \frac{1}{\pi N_0(E_F)} \int_\omega^\infty d\Omega \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(-i(\Omega+\omega) + D(\Omega+\omega)q^2)(-i\Omega + D(\Omega)q^2)^2}. \tag{2.161}$$

This equation forms the basis of the proposed self-consistent approach. In the absence of interactions ($\mu = 0$) it obviously reduces to the usual self-consistent theory of localization. Let us transform it to dimensionless imaginary Matsubara frequencies which is the only case we need for further applications to the superconducting state: $-i\omega/D_0 k_0^2 \to \omega$, $-i\Omega/D_0 k_0^2 \to \Omega$, and also introduce the dimensionless diffusion coefficient $d(\omega) = D(\omega)/D_0$. In these notations integral equation (2.161) takes the following form:

$$\frac{1}{d(\omega)} = 1 + \frac{1}{d(\omega)} d\lambda\, x_0^{d-2} \int_0^1 \frac{dy\, y^{d-1}}{y^2 + (\omega/d(\omega))} + \frac{8}{\pi} \mu\lambda\, x_0^{d-2} \int_\omega^\infty \frac{d\Omega}{d(\omega+\Omega)d^2(\Omega)} \int_0^1 \frac{dy\, y^{d+1}}{(y^2 + (\omega+\Omega)/d(\omega+\Omega))(y^2 + \Omega/d(\Omega))^2}, \tag{2.162}$$

where $\lambda = \gamma/\pi E_F = 1/2\pi E_F \tau$ is the usual disorder parameter. In the following we shall limit ourselves only to the case of spatial dimension $d = 3$. The diffusion coefficient of the usual self-consistent theory of localization (2.78) in these notations reduces to

$$d(\omega) = \begin{cases} \alpha = 1 - 3\lambda x_0 \approx \dfrac{E_F - E_c}{E_c}, & \omega \ll \omega_c,\ \alpha > 0 \quad \text{(Metal)}, \\ \left(\dfrac{\pi}{2} 3\lambda x_0\right)^{2/3} \omega^{1/3}, & \omega \gg \omega_c \quad \text{(Metal and insulator)}, \\ \dfrac{((\pi/2)3\lambda x_0)^2}{\alpha^2}\omega = (\xi_{loc} k_0)^2 \omega, & \omega \ll \omega_c,\ \alpha < 0 \quad \text{(Insulator)}, \end{cases} \tag{2.163}$$

where $\omega_c = |\alpha|^3/((\pi/2)3\lambda x_0)^2$ and ξ_{loc} is the localization length and x_0 the dimensionless cutoff. Let us introduce $K(\omega) = \omega/d(\omega)$ and analyze Eq. (2.162) assuming that $K(\omega)$, $K(\Omega)$ and $K(\omega + \Omega) \ll 1$. Expanding the right-hand side of Eq. (2.162) over these small parameters we obtain

$$\frac{\alpha}{d(\omega)} = 1 - \frac{\pi}{2}\frac{3\lambda x_0}{d(\omega)} K^{1/2}(\omega) + 2\mu\lambda x_0 \int_\omega^\infty \frac{d\Omega}{d(\omega+\Omega)d^2(\Omega)} \frac{K^{1/2}(\Omega) + 2K^{1/2}(\omega+\Omega)}{(K^{1/2}(\Omega) + K^{1/2}(\omega+\Omega))^2} . \tag{2.164}$$

Consider the metallic phase and look for the solution for the diffusion coefficient $d(\omega)$ in the following form:

$$d(\omega) = \begin{cases} d , & \omega \ll \omega_c , \\ d(\omega/\omega_c)^{1/3} , & \omega \gg \omega_c . \end{cases} \tag{2.165}$$

Substituting (2.165) into Eq. (2.164) we find d and ω_c and for the diffusion coefficient we obtain

$$d(\omega) = \begin{cases} \alpha - \alpha^* , & \omega \ll \omega_c , \\ (\frac{1}{2}\pi 3\lambda x_0)^{2/3}\omega^{1/3} , & \omega \gg \omega_c , \end{cases} \tag{2.166}$$

where $\omega_c = |\alpha - \alpha^*|^3/(\frac{1}{2}\pi 3\lambda x_0)^2$, $\alpha^* = c\mu$, $c \approx 0.89$.

Thus for the metallic phase we come to a very simple qualitative conclusion: Anderson transition persists and the conductivity exponent remains as $\nu = 1$. The transition itself has shifted to the region of weaker disorder $\alpha = \alpha^* = c\mu$ – interaction facilitates transition to the insulating state. The frequency behavior of the diffusion coefficient in metallic phase is qualitatively similar to that in the usual self-consistent theory of localization (2.163). In the region of high frequencies $\omega \gg \omega_c$ the behavior of diffusion coefficient remains unchanged after the introduction of interelectron interactions.

Consider now the insulating phase. In the region of high frequencies $\omega \gg \omega_c$ the diffusion coefficient obviously possesses the frequency dependence like $d(\omega) \sim \omega^{1/3}$. Assume that for small frequencies it is also some power of the frequency:

$$d(\omega) = \begin{cases} d\left(\dfrac{\omega}{\omega_c}\right)^\delta , & \omega \ll \omega_c , \\ d\left(\dfrac{\omega}{\omega_c}\right)^{1/3} , & \omega \gg \omega_c , \end{cases} \tag{2.167}$$

where δ is some exponent to be determined.

Substituting (2.167) into (2.164) and considering the case of $\alpha < 0$ (insulating phase of the usual self-consistent theory of localization) and $|\alpha| \gg \alpha^*$, we get

$$d(\omega) = \begin{cases} \dfrac{(\frac{1}{2}\pi 3\lambda x_0)^2}{\alpha^2}\omega = (\xi_{loc}k_0)^2\omega , & \omega^* \ll \omega \ll \omega_c , \\ (\frac{1}{2}\pi 3\lambda x_0)^{2/3}\omega^{1/3} , & \omega \gg \omega_c , \end{cases} \tag{2.168}$$

where $\omega_c = |\alpha|^3/(\frac{1}{2}\pi 3\lambda x_0)^2$, while $\omega^* \approx 0.1\mu\alpha^2/(\frac{1}{2}\pi 3\lambda x_0)^2 = 0.1\,\mu/(\xi_{loc}k_0)^2$ is some new characteristic frequency defined by the interactions. Note that $\omega^* \to 0$ as we approach the transition point when $\xi_{loc} \to \infty$.

Thus, sufficiently deep inside the insulating phase when $\alpha < 0$ and $|\alpha| \gg \alpha^*$ and for the frequencies $\omega \gg \omega^*$, the diffusion coefficient remains the same as in the self-consistent theory of localization, i.e. at small frequencies it is linear over frequency, while for the higher frequencies it is $\sim\omega^{1/3}$.

The analysis of Eq. (2.164) shows that for the frequencies $\omega \ll \omega^*$ it is impossible to find the power-like dependence for $d(\omega)$, i.e. the diffusion coefficient in the insulating phase apparently cannot be represented in the form of $d(\omega) = d\frac{\omega^*}{\omega_c}(\frac{\omega}{\omega^*})^\delta$, where δ is some unknown exponent. Because of this we were unable to find any analytical treatment of Eq. (2.164) in the region of $\omega \ll \omega^*$ within the insulating phase.

Consider now the behavior of the system not very deep inside the insulating phase when $\alpha - \alpha^* < 0$ while $\alpha > 0$, that is when the system without interaction would be within the metallic phase. Let us assume that the frequency behavior of the diffusion coefficient for $\omega \ll \omega_c$ has the power-like form, i.e. the diffusion coefficient is defined by the expression (2.167). Substituting (2.167) into (2.164) we get $\delta = \frac{1}{3}$. As a result, for the diffusion coefficient we get

$$d(\omega) = \begin{cases} \left(4{,}2\,\dfrac{\mu\lambda x_0}{\alpha}\right)^{2/3}\omega^{1/3}\,, & \omega \ll \omega_c\,, \\ (\frac{1}{2}\pi 3\lambda x_0)^{2/3}\omega^{1/3}\,, & \omega \gg \omega_c\,, \end{cases} \tag{2.169}$$

where $\omega_c = |\alpha - \alpha^*|^3/(\frac{1}{2}\pi 3\lambda x_0)^2$. Naturally, the exact solution for the diffusion coefficient should show a continuous change of frequency around $\omega \sim \omega_c$.

Thus, within the insulating phase close to transition point, where the system without interactions should have been metallic, the diffusion coefficient behaves as $\sim\omega^{1/3}$ everywhere, though for the low frequency region the coefficient of $\omega^{1/3}$ differs from that of the usual self-consistent theory of localization and explicitly depends upon the interaction constant.

We have also performed the numerical analysis of the integral equation (2.162) for the wide region of frequencies, both for metallic (Fig. 9) and insulating phases (Fig. 10). Solution was achieved by a simple iteration procedure using the results of the usual self-consistent theory of localization as an initial approximation. Numerical data are in good correspondence with our analytical estimates. In the region of high frequencies, both for metallic and insulating phases, the frequency behavior of the diffusion coefficient is very close to that defined by the usual self-consistent theory of localization. In the region of small frequencies within the metallic phase diffusion coefficient $d(\omega)$ diminishes as interaction grows. Dependence of the static generalized diffusion coefficient on disorder for $\mu = 0.24$ is shown as an inset in Fig. 9, and is practically linear. Metal–insulator transition in this case is observed at $\alpha = \alpha^* = c\mu$, where $c \approx 0.5$, which is also in good correspondence with our qualitative analysis. Within the insulating phase for the region of small frequencies ($\omega \ll \omega^*$) we observe significant deviations from predictions of the usual self-consistent theory of localization. Diffusion coefficient is apparently nonanalytic in the frequency here and we clearly see the tendency to the formation of some kind of an effective gap for the frequencies $\omega \ll \omega^*$, with this "gap" closing as interactions are turned off.

Our numerical analysis was performed in the Matsubara frequency region, which was used in writing down Eq. (2.162). Analytical continuation of our numerical data to the real frequencies was

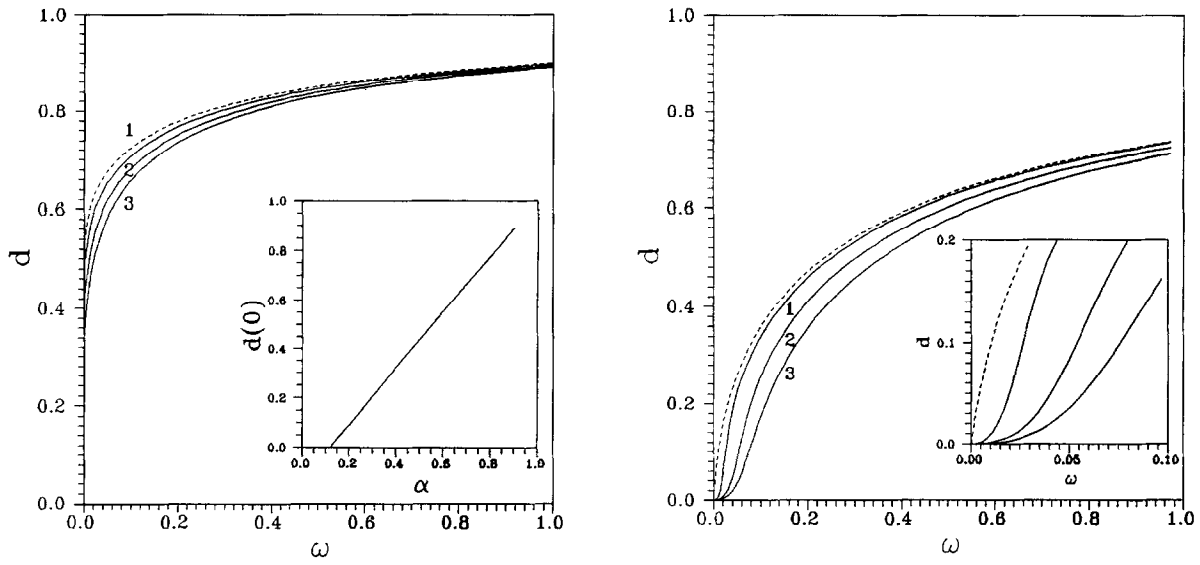

Fig. 9. Dependence of the dimensionless generalized diffusion coefficient on dimensionless Matsubara frequency in metallic phase ($\alpha = 0.5$), obtained by numerical solution for different values of μ: (1) 0.24; (2) 0.6; (3) 0.95; dashed line – the usual self-consistent theory of localization, $\mu = 0$. Inset: dependence of the static diffusion coefficient ($d = D(0)/D_0$) on disorder for $\mu = 0.24$.

Fig. 10. Dependence of dimensionless generalized diffusion coefficient on dimensionless Matsubara frequency in the dielectric phase ($\alpha = -0.5$), obtained by numerical solution for different values of μ: (1) 0.12; (2) 0.6; (3) 1.2; dashed line – the usual self-consistent theory of localization, $\mu = 0$.

not attempted, but as we stressed above the Matsubara frequency behavior is sufficient for our studies of the superconducting state discussed below.

In Ref. [122] we were also able to study the gradual evolution of the tunneling density of states from metallic to insulating region, demonstrating the continuous transformation of a cusp singularity of Eq. (2.145) in a metal into a kind of interaction induced pseudogap at the Fermi level in an insulator, which is in some respects similar to the Coulomb gap of Refs. [41–44].

For high-T_c superconductors, problems of interplay of localization and interactions become especially important because of the unusual nature of the normal state of these systems. In the absence of an accepted theory of this normal state we shall limit ourselves only to a few remarks on one specific model. The so-called "marginal" Fermi-liquid theory [124, 125] is a promising semi-phenomenological description of both normal and superconducting properties of these systems. We shall see that localization effects are apparently greatly enhanced in this case [126, 127].

Basically, the idea of "marginal" Fermi-liquid is expressed by the following form of one-particle Green's function [124].

$$G(\boldsymbol{Ep}) = Z_p/(\varepsilon - \xi_p - \mathrm{i}\gamma_p) + G_{\mathrm{incoh}}\,, \tag{2.170}$$

where ξ_p is the renormalized quasi-particle energy, $\gamma_p \sim \mathrm{Max}[\varepsilon, T]$ is the anomalous (linear) decay-rate for these quasiparticles which is quite different from quadratic in ε or T decay-rates of

the usual Fermi-liquid theory [120]. The concept of "marginality" arises due to the peculiar behavior of the quasi-particle residue:

$$Z_p^{-1} \approx \ln(\tilde{\omega}_c/|\xi_p|) \approx \ln(\tilde{\omega}_c/|\varepsilon|)\,, \tag{2.171}$$

where $\tilde{\omega}_c$ is the characteristic frequency scale of some kind of electronic excitations, which is the phenomenological parameter of the theory. From Eq. (2.171) it is clear that quasi-particle contribution to Green's function Eq. (2.170) vanishes precisely at the Fermi level, while it exists close to it though with logarithmically reduced weight. Note that in the case of the usual Fermi-liquid $Z_p \approx 1$ [120].

For the disordered system we can estimate the impurity contribution to the scattering rate of quasi-particles as [126]

$$\gamma = 2\rho V^2 Z_p \operatorname{Im} \sum_p \Lambda^2(\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}) G(\boldsymbol{p}+\boldsymbol{q}\varepsilon) \approx 2\pi\rho V^2 Z^2 \Lambda^2(\boldsymbol{q}\to 0) N(E_F) \approx Z\Lambda^2\gamma_0\,, \tag{2.172}$$

where Λ is the appropriate vertex-part renormalized by Fermi-liquid effects, ρ again is the impurity concentration, V the impurity potential and $N(E_F) = Z^{-1}N_0(E_F)$ is the renormalized density of states in the Fermi-liquid. Here $N_0(E_F)$ is the density of states for noninteracting electrons at the Fermi level, γ_0 is the scattering rate for the noninteracting case. To get the last relation in Eq. (2.172) a weak dependence of vertices and self-energy on momentum was assumed. Now we can use the Ward identity for $\Lambda(\boldsymbol{q}\to 0\omega = 0)$ vertex of the disordered Fermi-liquid theory [119, 108, 109]:

$$\Lambda(\boldsymbol{q}\to 0\omega = 0) = (1 + F_0^s)^{-1} Z^{-1}\,, \tag{2.173}$$

where F_0^s is Landau parameter introduced above. As a result, we can easily get a simple relation between the mean free paths of interacting and noninteracting quasi-particles [126, 127]:

$$l = (p_F/m^*)\gamma^{-1} = (p_F/m)\gamma_0^{-1}/\Lambda^2(\boldsymbol{q}\to 0) = l_0(1 + F_0^s)^2 Z^2\,. \tag{2.174}$$

Here $m^* = Z^{-1}m$ is the effective mass of the quasi-particle. Assuming $F_0^s \approx$ const. < 1 and using Eq. (2.171) we get at $T = 0$

$$l = l_0/[\ln(\tilde{\omega}_c/|\varepsilon|)]^2\,. \tag{2.175}$$

Then from the usual Ioffe–Regel criterion for localization $p_F l \approx 1$ we obtain that all the quasi-particle states within a region of the order of

$$|\varepsilon_c| \approx \tilde{\omega}_c \exp(-\sqrt{p_F l}) \tag{2.176}$$

around the Fermi-level in high-T_c oxides are localized even for the case of weak impurity scattering $p_F l \gg 1$. For realistic estimates of $\tilde{\omega}_c \approx 0.1$–$0.2$ eV [124] and $p_F l < 5$ the width of this localized band may easily be of the order of hundreds of degrees Kelvin, while for $p_F l \approx 10$ and $\tilde{\omega}_c \approx 1000$ K we get $|\varepsilon_c| \approx 40$ K. Obviously, this band grows with disorder as the mean free path l_0 drops. We can safely neglect this localization for $T \gg |\varepsilon_c|$, but for low enough temperatures localization effects become important and all states are localized in the ground state.

Of course, the formal divergence of the mean free path denominator in Eq. (2.175) is unphysical. Single-impurity scattering cannot overcome the so-called unitarity limit [126], so that we must always have

$$l \geq p_F^2/4\pi\rho \,. \tag{2.177}$$

In a typical metal with $p_F \sim a^{-1}$ this leads to $l \geq 1/4\pi\rho a^2$ and Ioffe–Regel criterion $l \leq a$ can be easily satisfied for large impurity concentrations $\rho \sim a^{-3}$. Thus the singularity in Eq. (2.175) does not mean that localization can appear for arbitrarily low concentration of impurities. We can safely speak only about the significant enhancement of localization effects in marginal Fermi liquids. These ideas are still at this elementary level and we may quote only one paper attempting to put them on a more sound basis of scaling theory of the metal–insulator transition of interacting electrons [128].

3. Superconductivity and localization: statistical mean-field approach

3.1. BCS model and Anderson theorem

We shall start our analysis of superconductivity in strongly disordered systems within the framework of the simple BCS-model [8,9] which assumes the existence of some kind of effective electron–electron attraction within the energy region of the order of $2\langle\omega\rangle$ around the Fermi level. In the usual superconductors, $\langle\omega\rangle \sim \omega_D$, where ω_D is the Debye frequency, because pairing is determined by electron–phonon mechanism; however, we shall use some effective $\langle\omega\rangle$ as an average frequency of some kind of Bose-like excitations responsible for pairing, e.g. in high-T_c superconductors. At the moment, we shall not discuss the microscopic nature of this attraction which in general case is determined by the balance of attraction due to Boson-exchange and Coulomb repulsion. Here we just assume (as always is done in simple BCS-approach) that this effective attraction is described by some interaction constant g, which is considered just as a parameter. More detailed microscopic approach will be given in later sections.

Nontrivial results concerning superconductivity in disordered systems were obtained very soon after the formulation of the BCS-theory [10–13]. The concept of a "dirty" superconductor described the experimentally very important case of the mean free path l short in comparison with the superconducting coherence length $\xi_0 \sim \hbar v_F/T_c$, i.e. the case when

$$\xi_0 \gg l \gg \hbar/p_F \,. \tag{3.1}$$

Already in this case of the not so strongly disordered (in the sense of closeness to metal–insulator transition) system, Cooper pairing takes place not between electrons with opposite momenta and spins as in regular case, but between time-reversed exact eigenstates of electrons in a disordered system [13,9].

$$(\boldsymbol{p}_\uparrow, -\boldsymbol{p}_\downarrow) \Rightarrow (\phi_\nu(\boldsymbol{r})_\uparrow, \phi_\nu^*(\boldsymbol{r})_\downarrow) \,. \tag{3.2}$$

In the following we consider only singlet isotropic (s-wave) pairing. Some aspects of anisotropic pairing are analyzed in Appendix C. The underlying physics is simple: in disordered systems such as e.g. an alloy the electron momentum is poorly determined due to the lack of translational

invariance. However, in a random potential field we can always define exact eigenstates $\phi_\nu(\boldsymbol{r})$, which are just solutions of the Schroedinger equation in this random field (for a given configuration of this field). We do not need to know the explicit form of these eigenstates at all, the pairing partner of $\phi_\nu(\boldsymbol{r})$ is being given by time-reversed $\phi_\nu^*(\boldsymbol{r})$. This leads to a relative stability of a superconducting state with respect to disordering in the absence of scattering mechanisms which break the time-reversal invariance such as e.g. of magnetic impurities.

Within the standard Green's function approach, the superconducting system is described by Gorkov equations [58, 129] which in the coordinate representation take the form:

$$\mathscr{G}_\uparrow(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = G_\uparrow(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) - \int \mathrm{d}\boldsymbol{r}''\, G_\uparrow(\boldsymbol{r}\boldsymbol{r}''\varepsilon_n)\Delta(\boldsymbol{r}'')\mathscr{F}(\boldsymbol{r}''\boldsymbol{r}'\varepsilon_n)\,, \tag{3.3}$$

$$\mathscr{F}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \int \mathrm{d}\boldsymbol{r}''\, G_\downarrow^*(\boldsymbol{r}\boldsymbol{r}''\varepsilon_n)\Delta^*(\boldsymbol{r}'')\mathscr{G}_\uparrow(\boldsymbol{r}''\boldsymbol{r}'\varepsilon_n)\,, \tag{3.4}$$

where $G(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ is an exact one-electron Matsubara Green's function of the normal state and the superconducting order-parameter (gap) $\Delta(\boldsymbol{r})$ is determined by the self-consistent gap equation:

$$\Delta(\boldsymbol{r}) = gT\sum_{\varepsilon_n} \mathscr{F}^*(\boldsymbol{r}\boldsymbol{r}\varepsilon_n)\,, \tag{3.5}$$

where $\mathscr{F}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ is (antisymmetric over spin variables) anomalous Gorkov Green's function, $\varepsilon_n = (2n+1)\pi T$.

If we consider temperatures close to the superconducting transition temperature T_c, when $\Delta(\boldsymbol{r})$ is small, $\mathscr{F}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ can be obtained from the linearized equation:

$$\mathscr{F}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \int \mathrm{d}\boldsymbol{r}''\, G_\downarrow^*(\boldsymbol{r}\boldsymbol{r}''\varepsilon_n)\Delta^*(\boldsymbol{r}'')G_\uparrow(\boldsymbol{r}''\boldsymbol{r}'\varepsilon_n)\,. \tag{3.6}$$

Then the linearized gap equation determining T_c takes the form

$$\Delta(\boldsymbol{r}) = gT\int \mathrm{d}\boldsymbol{r}'\sum_{\varepsilon_n} K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)\Delta(\boldsymbol{r}')\,, \tag{3.7}$$

where the kernel

$$K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = G_\uparrow(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)G_\downarrow^*(\boldsymbol{r}'\boldsymbol{r}\varepsilon_n) \tag{3.8}$$

is formed by exact one-electron Green's functions of a normal metal. Now we can use an exact eigenstate representation for an electron in a random field of a disordered system to write (cf. Eq. (A.13))

$$G_\uparrow(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \sum_\nu \frac{\phi_{\nu\uparrow}(\boldsymbol{r})\phi_{\nu\uparrow}^*(\boldsymbol{r}')}{\mathrm{i}\varepsilon_n - \varepsilon_\nu}\,, \tag{3.9}$$

where ε_ν are exact energy levels of an electron in the disordered system. Then

$$K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = Tg\sum_{\mu\nu} \frac{\phi_{\nu\uparrow}(\boldsymbol{r})\phi_{\nu\uparrow}^*(\boldsymbol{r}')\phi_{\mu\downarrow}^*(\boldsymbol{r}')\phi_{\mu\downarrow}(\boldsymbol{r})}{(\mathrm{i}\varepsilon_n - \varepsilon_\nu)(-\mathrm{i}\varepsilon_n + \varepsilon_\mu)}\,. \tag{3.10}$$

In the following for brevity we shall drop spin variables always assuming singlet pairing. In the case of a system with time-reversal invariance (i.e. in the absence of an external magnetic field, magnetic impurities, etc.) Eq. (3.10) can be rewritten as

$$K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = G(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)G(\boldsymbol{r}'\boldsymbol{r} - \varepsilon_n) = \sum_{\mu\nu} \frac{\phi_\nu(\boldsymbol{r})\phi_\nu^*(\boldsymbol{r}')\phi_\mu(\boldsymbol{r}')\phi_\mu^*(\boldsymbol{r})}{(\mathrm{i}\varepsilon_n - \varepsilon_\nu)(-\mathrm{i}\varepsilon_n - \varepsilon_\mu)} . \tag{3.11}$$

Averaging over disorder we get

$$\langle \Delta(\boldsymbol{r})\rangle = gT \int \mathrm{d}\boldsymbol{r}' \sum_{\varepsilon_n} \langle K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)\Delta(\boldsymbol{r}')\rangle . \tag{3.12}$$

Practically, in all the papers on superconductivity in disordered systems it is assumed that we can make the simplest decoupling in Eq. (3.12) to get the following linearized equation for the average order-parameter:

$$\langle \Delta(\boldsymbol{r})\rangle = gT \int \mathrm{d}\boldsymbol{r}' \sum_{\varepsilon_n} K(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n)\langle \Delta(\boldsymbol{r}')\rangle , \tag{3.13}$$

where the averaged kernel in the case of time-invariance is given by

$$\begin{aligned} K(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n) &= K^*(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n) = \langle K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)\rangle \\ &= \left\langle \sum_{\mu\nu} \frac{\phi_\nu(\boldsymbol{r})\phi_\mu^*(\boldsymbol{r})\phi_\mu(\boldsymbol{r}')\phi_\nu^*(\boldsymbol{r}')}{(\mathrm{i}\varepsilon_n - \varepsilon_\nu)(-\mathrm{i}\varepsilon_n - \varepsilon_\mu)} \right\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}E\, N(E) \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F}}}{(\mathrm{i}\varepsilon_n + E)(E + \omega - \mathrm{i}\varepsilon_n)} , \end{aligned} \tag{3.14}$$

where we have introduced Gorkov–Berezinskii spectral density [59] (cf. Eq. (A.2)):

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F}} = \frac{1}{N(E)} \left\langle \sum_{\mu\nu} \phi_\nu^*(\boldsymbol{r})\phi_\mu(\boldsymbol{r})\phi_\mu^*(\boldsymbol{r}')\phi_\nu(\boldsymbol{r}')\delta(E - \varepsilon_\nu)\delta(E + \omega - \varepsilon_{\nu'}) \right\rangle . \tag{3.15}$$

Here $N(E)$ is an exact electron density of states per *spin direction* as it always appears in superconductivity theory (above, while discussing localization we always used density of states for both spin directions).

Usually, the decoupling procedure used in Eq. (3.12) to reduce it to Eq. (3.13) is justified by the assumption that the averaging of $\Delta(\boldsymbol{r})$ and of Green's functions in Eq. (3.12) forming the kernel can be performed independently because of the essentially different spatial scales [12]. $\Delta(\boldsymbol{r})$ changes at a scale of the order of coherence length (Cooper pair size) ξ, while $G(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ are oscillating on the scale of interatomic distance $a \sim \hbar/p_{\mathrm{F}}$, and we always have $\xi \gg a$. Actually, it is clear that this decoupling is valid only if the order-parameter is *self-averaging* (i.e. in fact nonrandom) quantity: $\Delta(\boldsymbol{r}) = \langle \Delta(\boldsymbol{r})\rangle$, $\langle \Delta^2(\boldsymbol{r})\rangle = \langle \Delta(\boldsymbol{r})\rangle^2$. Below we shall see that for a system close to the mobility edge the property of self-averageness of $\Delta(\boldsymbol{r})$ is absent and situation is actually highly nontrivial. In this case, the so-called *statistical* fluctuations [62] leading to the inequality of $\langle \Delta^2(\boldsymbol{r})\rangle$ and $\langle \Delta(\boldsymbol{r})\rangle^2$ become quite important. However, we shall start with what we call the statistical mean-field approach which completely neglects these fluctuations and allows the simple analysis using

Eq. (3.13), as a necessary first step to understand superconductivity in strongly disordered systems, which will allow us to find most of the important deviations from the usual theory of "dirty" superconductors. The role of statistical fluctuations will be analyzed later.

If we look for the solution of Eq. (3.13) $\Delta(\boldsymbol{r}) = \text{const.}$ (homogeneous gap), we immediately obtain the following equation for the superconducting transition temperature T_c:

$$1 = gT_c \int d\boldsymbol{r} \sum_{\varepsilon_n} K(\boldsymbol{r} - \boldsymbol{r}' \varepsilon_n)$$

$$= gT_c \int d\boldsymbol{r} \sum_{\varepsilon_n} \int_{-\infty}^{\infty} dE\, N(E) \int_{-\infty}^{\infty} d\omega \frac{\langle\!\langle \rho_E(\boldsymbol{r}) \rho_{E+\omega}(\boldsymbol{r}') \rangle\!\rangle^{\mathrm{F}}}{(E + i\varepsilon_n)(E + \omega - i\varepsilon_n)} \tag{3.16}$$

Using the general sum-rule given in Eq. (A.5) [59]:

$$\int d\boldsymbol{r} \langle\!\langle \rho_E(\boldsymbol{r}) \rho_{E+\omega}(\boldsymbol{r}') \rangle\!\rangle^{\mathrm{F}} = \delta(\omega)\,, \tag{3.17}$$

we immediately reduce Eq. (3.16) to a standard BCS form

$$1 = gT_c \int_{-\infty}^{\infty} dE\, N(E) \sum_{\varepsilon_n} \frac{1}{E^2 + \varepsilon_n^2} = g \int_0^{\langle\omega\rangle} dE\, N(E) \frac{1}{E} \tanh(E/2T_c)\,, \tag{3.18}$$

where we introduced the usual cut off at $\varepsilon_n \sim 2\langle\omega\rangle$. Note that $N(E)$ here is an exact one-particle density of states (per spin direction) in a normal state of a disordered system. From Eq. (3.18) we get the usual result:

$$T_c = \frac{2\gamma}{\pi} \langle\omega\rangle \exp(-1/\lambda_p)\,, \tag{3.19}$$

where $\lambda_p = gN(E_F)$ is the dimensionless pairing constant and $\ln\gamma = C = 0.577\ldots$ the Euler constant. This is the notorious Anderson theorem: in the absence of scattering processes breaking time-reversal invariance, disorder influences T_c only through possible changes of the density of states $N(E_F)$ under disordering (which are usually relatively small).

Due to the sum-rule of Eq. (3.17), all the singularities of the Berezinskii–Gorkov spectral density, reflecting a possible localization transition, do not appear in the equation determining T_c: there is no explicit contribution from the $\delta(\omega)$ term of Eq. (A.8) and Eq. (3.18) has the same form in both metallic and localized phases (cf. Ref. [130]).

The only limitation here which appears on physical grounds is connected with the local discreteness of the electronic spectrum in the localized phase discussed above. It is clear that Cooper pairing is possible in the localized phase only between electrons with centers of localization within a distance of the order of $\sim R_{\mathrm{loc}}(E)$, because only in that case their wave functions overlap [21,22]. However, these states are split in the energy by δ_E defined in Eq. (2.13). Obviously, we have to demand that the superconducting gap Δ (at $T = 0$, $\Delta \sim T_c$) be much larger than this δ_E:

$$\Delta \sim T_c \gg \delta_E \sim 1/N(E) R_{\mathrm{loc}}^3(E)\,, \tag{3.20}$$

i.e. on the energy interval of the order of $\Delta \sim T_c$ there must be many discrete levels, with centers of localization within a distance $\sim R_{\mathrm{loc}}(E)$ from each other. In this case, the problem of Cooper pairs

formation within $\sim R_{loc}(E)$ is qualitatively the same as in the metallic state, e.g. we can replace the summation over discrete levels ε_ν by integration. An analogous problem was considered previously in the case of Cooper pairing of nucleons in finite nuclei [120] and also of Cooper pairing of electrons in small metallic particles (granular metals) [131, 132]. For strongly anisotropic high-T_c systems we must similarly have [16]

$$\Delta \sim T_c \gg [N(E)R^a_{loc}R^b_{loc}R^c_{loc}]^{-1}\,, \tag{3.21}$$

where we have introduced the appropriate values of localization lengths along the axes of an orthorhombic lattice.

Obviously, Eq. (3.20) is equivalent to a condition of large enough localization length:

$$R_{loc}(E) \gg [N(E)\Delta]^{-1/3} \sim (\xi_0/p_F^2)^{1/3} \sim (\xi_0 l^2)^{1/3}\,, \tag{3.22}$$

i.e. the system must be close enough to the mobility edge or just slightly localized. Here we used the usual estimate of mean free path close to the Anderson transition $l \sim p_F^{-1}$. Below we shall see that Eq. (3.22) is just a condition that Cooper pairs must be much smaller than the localization length, only in that case Cooper pairing is possible in the localized phase [21, 22].

3.2. T_c degradation

In the usual BCS model discussed above pairing interaction g is assumed to be a given constant in the vicinity of the Fermi level. In a more realistic approach this interaction is determined by the balance of interelectron attraction, due e.g. to electron–phonon coupling (as in traditional superconductors) or some other Boson-exchange mechanism (as is apparently the case in high-T_c superconductors), and Coulomb repulsion. It is clear that in a strongly disordered system all these interactions can, in principle, be strongly renormalized in comparison with the "pure" case. The aim of this section is to discuss these effects in the context of the metal–insulator transition induced by disorder.

Usually the Coulomb repulsion within a Cooper pair is strongly reduced in comparison with the electron–phonon attraction due to the retarded nature of the electron–phonon coupling [9]. The characteristic time of electron–phonon interaction is of the order of ω_D^{-1}, while for the Coulomb interaction in "pure" metal it is determined by $\sim\hbar/E_F$ – the time during which electrons "pass" each other in the pair. Due to metallic screening both interactions are more or less point-like. However, in a disordered metal, ballistic transport changes to diffusion and as disorder grows electron motion becomes slower effectively leading to the growth of Coulomb repulsion within the Cooper pair and the appropriate drop of T_c as was first claimed by Anderson et al. [20]. Actually, electron–phonon interaction can also change under disordering but a common belief is that these changes are less significant than in the case of Coulomb interaction [134, 135]. This problem is still under active discussion and some alternative points of view have been expressed [136–138]. However, the general agreement is that some kind of diffusion renormalization of the effective interaction of the electrons within a Cooper pair provides an effective mechanism of T_c degradation under disordering. Below we shall mainly use the approach of Ref. [22], with the main aim of finding the possibility of superconductivity surviving up to the Anderson transition.

Later in this section we shall also consider the possible mechanisms of T_c degradation under disordering due to magnetic fluctuations (or local moments) which appear close to the

metal–insulator transition. A possible relation of these mechanisms to enhanced Coulomb effects will also be discussed.

The general problem of T_c degradation under disordering becomes much more complicated in the case of high-temperature superconductors because of the unknown nature of pairing in these systems. However, we believe that the mechanism based upon the growth of Coulomb repulsion within the Cooper pair is also operational here, while of course it is difficult to say anything about disorder effects upon attractive interactions leading to Cooper pair formation in these systems.

If we assume a spin-independent Boson-exchange (phonons, excitons, etc.) model of the pairing interaction, T_c can be obtained from the generalized Eliashberg equations and thus be given by the famous Allen–Dynes expression [139]:

$$T_c = \frac{f_1 f_2}{1.20}\,\omega_{\log}\exp\left\{-\frac{1.04(1+\lambda)}{\lambda-\mu^*(1+0.62\lambda)}\right\}, \tag{3.23}$$

where

$$f_1 = [1+(\lambda/\lambda_1)^{3/2}]^{1/3}\,; \qquad f_2 = 1+\frac{[\langle\omega^2\rangle^{1/2}/\omega_{\log}-1]\lambda^2}{\lambda^2+\lambda_2^2}\,,$$

$$\lambda_1 = 2.46(1+3.8\mu^*)\,; \qquad \lambda_2 = 1.82(1+6.3\mu^*)\frac{\langle\omega^2\rangle^{1/2}}{\omega_{\log}}\,. \tag{3.24}$$

Here $\omega_{\log}$ is the mean logarithmic frequency and $\langle\omega\rangle^2$ is the mean square frequency of Bosons responsible for pairing (the averaging is over the spectrum of these Bosons), μ^* is the Coulomb pseudopotential, λ is the dimensionless pairing constant due to Boson-exchange. Strictly speaking, the Allen–Dynes formula has been derived for the electron–phonon model, with certain assumptions about the phonon spectrum. Its use for the general Boson-exchange model here serves only for illustrative purposes. At relatively weak coupling $\lambda \leq 1.5$, Allen–Dynes expression effectively reduces to McMillan formula [140]:

$$T_c = \frac{\omega_{\log}}{1.20}\exp\left\{-\frac{1.04(1+\lambda)}{\lambda-\mu^*(1+0.62\lambda)}\right\} \tag{3.25}$$

which in the weak coupling limit gives the usual BCS result $T_c \sim \langle\omega\rangle\exp(-1/\lambda-\mu^*)$. For a very strong pairing interaction, Eq. (3.23) gives the asymptotic behavior $T_c \approx 0.18\sqrt{\lambda\langle\omega^2\rangle}$. In most parts of this review we shall limit ourselves to the weak coupling approximation. The Coulomb pseudopotential μ^* in the "pure" system is given by

$$\mu^* = \frac{\mu}{1+\mu\ln(E_F/\langle\omega\rangle)}\,, \tag{3.26}$$

where μ is the dimensionless Coulomb constant. The mechanism of T_c degradation under disordering due to the growth of Coulomb repulsion is reflected in the appropriate growth of μ^* [20,22].

The singlet gap function with a simple s-wave symmetry which we have discussed above has a nonzero amplitude at zero separation of the two electrons in the pair. Thus it must pay the energy

price for the short-range repulsion due to a finite μ. In recent years, a number of new mechanisms of superconducting pairing were proposed which try to eliminate the effect of repulsion assuming a pair wave function which vanishes at zero separation. This is equivalent to the requirement that the sum over all momentum of the BCS gap function Δ must vanish [141]:

$$\Delta(\boldsymbol{r}) = \langle \psi\uparrow(\boldsymbol{r})\psi\downarrow(\boldsymbol{r})\rangle = \sum_{\boldsymbol{k}} \Delta(\boldsymbol{k}) = 0 \,. \tag{3.27}$$

A number of rather exotic schemes for this were proposed [141], but probably the simplest way of satisfying this requirement is by means of higher angular momentum pairing, e.g. d-wave which became rather popular as a possible explanation of high-T_c superconductivity within the spin-fluctuation exchange mechanism [142–145]. The sum in Eq. (3.27) is then zero because the gap changes sign as $\boldsymbol{k}$ goes around the Fermi surface. This leads, to a large extent, to the cancellation of Coulomb pseudopotential effects. However, this type of pairing is extremely sensitive to any kind of disordering (cf. Appendix C) and superconductivity is destroyed long before the localization transition. For these reasons we shall not discuss the disorder effects in such superconductors in this review. The same applies to more exotic pairing schemes such as the odd-gap pairing [146], where the usual scattering suppression of T_c is also very strong.

Among mechanisms discussed for high-T_c superconductors we should also mention different types of the so-called van-Hove scenarios [147–150], which are based upon the idea of T_c-enhancement due to some kind of the density of states singularity close to the Fermi level. For all such mechanisms, a rather strong T_c suppression may be due to the potential scattering smoothing out these singularities. Again we shall not discuss these mechanisms in our review as having nothing to do with localization effects.

3.2.1. Coulomb kernel

Let us use again the exact eigenstate $\phi_\nu(\boldsymbol{r})$ representation for an electron in a random system, with exact energy levels ε_ν. These functions and energies may correspond either to extended or to localized states. Consider the one-electron Green's function in this representation and take its diagonal element $G_{\nu\nu}(\varepsilon)$. The influence of interaction is described by the appropriate irreducible self-energy $\Sigma_\nu(\varepsilon)$ [120, 151]:

$$G_{\nu\nu}(\varepsilon) = 1/[\varepsilon - \varepsilon_\nu - \Sigma_\nu(\varepsilon)] \,. \tag{3.28}$$

Here energy zero is at the Fermi level. Let us introduce a "self-energy" $\Sigma_E(\varepsilon)$ averaged over some surface of constant energy $E = \varepsilon_\nu$ and over random field configurations [151]:

$$\Sigma_E(\varepsilon) = \frac{1}{N(E)} \left\langle \sum_\nu \delta(E - \varepsilon_\nu)\Sigma_\nu(\varepsilon) \right\rangle \,. \tag{3.29}$$

Consider a model with short-range static interelectron interaction $v(\boldsymbol{r} - \boldsymbol{r}')$. Then for the simplest Fock diagram shown in Fig. 7 we find

$$\Sigma_\mu^{\mathrm{F}} = -\int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}' v(\boldsymbol{r} - \boldsymbol{r}') \sum_\nu f_\nu \phi_\mu^*(\boldsymbol{r}')\phi_\nu^*(\boldsymbol{r})\phi_\mu(\boldsymbol{r})\phi_\nu(\boldsymbol{r}') \,, \tag{3.30}$$

where $f_\nu = f(\varepsilon_\nu)$ is the Fermi distribution function. Accordingly, from Eq. (3.29) we get [60]

$$\Sigma_E^{\mathrm{F}} = -\int_{-\infty}^{\infty} \mathrm{d}\omega\, f(E+\omega) \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}'\, v(\boldsymbol{r}-\boldsymbol{r}') \langle\!\langle \rho_E(\boldsymbol{r}) \rho_{E+\omega}(\boldsymbol{r}') \rangle\!\rangle^{\mathrm{F}} , \tag{3.31}$$

where we again introduced Berezinskii–Gorkov spectral density defined in Eqs. (A.2) and (3.15).

Let us define the Coulomb kernel by the following functional derivative:

$$K_{\mathrm{C}}(E-E') = -\delta\Sigma_E^{\mathrm{F}}/\delta f(E') \tag{3.32}$$

which characterizes the change of electron energy due to a variation of its distribution function. It is easy to see that

$$\begin{aligned} K_{\mathrm{C}}(\omega) &= \frac{1}{N(E)} \left\langle \sum_{\mu\nu} \langle \mu\nu | v(\boldsymbol{r}-\boldsymbol{r}') | \nu\mu \rangle \delta(E-\varepsilon_\nu) \delta(E+\omega-\varepsilon_\mu) \right\rangle \\ &= \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}'\, v(\boldsymbol{r}-\boldsymbol{r}') \langle\!\langle \rho_E(\boldsymbol{r}) \rho_{E+\omega}(\boldsymbol{r}') \rangle\!\rangle^{\mathrm{F}} \end{aligned} \tag{3.33}$$

is actually a Fock-type matrix element of interaction averaged over two surfaces of constant energy E and $E' = E + \omega$ and over disorder. We can use $K_{\mathrm{C}}(\omega)$ as a kernel in the linearized gap equation (cf. Appendix B) determining T_{c} which is a reasonable generalization of a Coulomb kernel used in the theory of ordered superconductors [152]. In the momentum representation,

$$K_{\mathrm{C}}(\omega) = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\, v(\boldsymbol{q}) \langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_{\boldsymbol{q}}^{\mathrm{F}} . \tag{3.34}$$

In the weak coupling approximation over pairing interaction it is the only relevant Coulomb contribution in the gap equation (cf. Appendix B), in case of strong coupling there are additional contributions, e.g. connected with diffusional renormalization of the density of states, Eq. (2.145) [136–138, 153, 154]. We refer to these papers for a detailed analysis of the density of states effects upon T_{c}.

In the following we assume point-like interaction: $v(\boldsymbol{q}) = v_0$. During our discussion of localization we have discovered that for small $\omega \ll \gamma$ and $q \ll l^{-1}$, the Gorkov–Berezinskii spectral density acquires a diffusional contribution:

$$\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_{\boldsymbol{q}}^{\mathrm{F_{diff}}} = \frac{1}{\pi N(E)} \operatorname{Im} \Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) , \tag{3.35}$$

where

$$\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) = -N(E)/(\omega + \mathrm{i}D_E(\omega)q^2) \tag{3.36}$$

and the generalized diffusion coefficient in the metallic phase is given by

$$D_E(\omega) \approx \begin{cases} D_E , & |\omega| \ll \omega_{\mathrm{c}} \approx 2\gamma(\sigma/\sigma_{\mathrm{c}})^3 , \\ D_0 \left(-\dfrac{\mathrm{i}\omega}{2\gamma} \right)^{1/3} , & |\omega| \gg \omega_{\mathrm{c}} . \end{cases} \tag{3.37}$$

In the absence of disorder this diffusional contribution disappears and the kernel $K_c(\omega)$ for $|\omega| < E_F$ reduces to the usual Coulomb potential $\mu = N(E)v_0$ [9, 152]. Accordingly, we can use the following approximation [22]:

$$K_c(\omega) \approx \mu\theta(E_F - |\omega|) + K_c^{diff}(\omega)\,, \tag{3.38}$$

where

$$K_c^{diff}(\omega) = \int \frac{d^3q}{(2\pi)^3}\, v_0 \langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_q^{F_{diff}}\,. \tag{3.39}$$

This form of the Coulomb kernel gives correct interpolation between the strong disorder limit and the "pure" case. Note that in the case of a disordered system, besides the diffusional contribution which contains singularities associated with the Anderson transition there also appear "regular" contributions to $K_c(\omega)$ which may be modelled by μ, making it different from its value in the "pure" system. The diffusional term in $K_c(\omega)$ is connected with diffusion renormalization of the electron–electron interaction vertex [38, 39, 151, 101, 98, 60]. Fig. 7 shows diagrams of standard perturbation theory responsible for this renormalization. In the approach based upon the self-consistent theory of localization "triangular" vertex defined by Fig. 7(c) is given by [98, 155]

$$\gamma^{RA}(\boldsymbol{q}\omega) \approx \frac{2\gamma}{-i\omega + D_E(\omega)q^2}\,, \quad \omega \ll \gamma,\ q \ll l^{-1}\,. \tag{3.40}$$

Singularity of Eq. (3.40) for small ω and q leads to a significant growth of the interaction in the disordered system. Actually, this expression is the same as in a "dirty" metal [38] but with the replacement of the Drude diffusion coefficient by the generalized one.

3.2.2. Electron–phonon interaction

The case of electron–phonon interaction is different. Diffusion renormalization of the electron–phonon vertex is unimportant because the relevant corrections compensate each other if we take into account the impurity vibrations [133–135]. Surely, the value of the electron–phonon contribution to the pairing interaction does change in a disordered system in comparison with the "pure" case [134]. However, these changes are relatively insignificant in the sense of absence of drastic changes at the Anderson transition. We shall demonstrate the absence of diffusion renormalization of the electron–phonon vertex using the lowest order diagrams of perturbation theory following the approach of Ref. [135].

Let us limit our analysis to a homogeneous continuous medium. The appearance of deformation $\boldsymbol{u}$ leads to the variation of density of the medium given by $\delta\rho = -\rho\,\mathrm{div}\,\boldsymbol{u}$. Accordingly, taking into account the electroneutrality condition we get the variation of electron density as $\delta n = -n\,\mathrm{div}\,\boldsymbol{u}$. This leads to the following change of the free electron Green's function:

$$\begin{aligned}\delta G^{-1}(E\boldsymbol{p}) &= -n\,\mathrm{div}\,\boldsymbol{u}\,\frac{d}{dn}\left[E - v_F(|\boldsymbol{p}| - p_F)\right] \\ &= -nv_F\,\mathrm{div}\,\boldsymbol{u}\,\frac{dp_F}{dn} = -\frac{1}{3}\,v_F p_F\,\mathrm{div}\,\boldsymbol{u}\,,\end{aligned} \tag{3.41}$$

where we have used $n = p_F^3/(3\pi^2)$. Let us define the electron–phonon vertex Λ by

$$\frac{\delta G}{\delta u} = G\Lambda G = -G(\delta G^{-1}/\delta u)G\,; \qquad \Lambda = -\delta G^{-1}/\delta u\,. \tag{3.42}$$

For $u(r,t) = u\exp(iqr - i\omega t)$ we get from Eq. (3.41)

$$\delta G^{-1}(Ep) = -\tfrac{1}{3}iqu\,v_F p_F \tag{3.43}$$

so that the "bare" electron–phonon vertex (i is vector index)

$$\Lambda_{1i}^{(0)} = \tfrac{1}{3}iq_i v_F p_F\,. \tag{3.44}$$

Consider the system with impurities randomly placed at points R_n which create the potential

$$U(r) = \sum_n V(r - R_n)\,. \tag{3.45}$$

Vibrations of the medium lead to vibrations of impurity atoms, so that $R_n \to R_{0n} + u_n(t)$ with $u_n(t) = u\exp(iqR_{0n} - i\omega t)$. The random field of static impurities leads to a simplest self-energy correction given by Fig. 11(a) [57, 58]. Impurity vibrations can be accounted for by the additional interaction term:

$$\delta V(r - R_n) = \frac{\partial V(r - R_{n0})}{\partial R_{n0}}\,u\exp(iqR_{n0} - i\omega t)$$

so that

$$\Lambda_{2i}u_i = \frac{\delta\Sigma}{\delta u_i}u_i = \left\langle \sum_n \left\{ \frac{\partial V(r - R_{n0})}{\partial R_{n0}^i}\,G(rt, r't')V(r' - R_{n0})u_{in} + V(r - R_{n0})G(rt, r't')\frac{\partial V(r' - R_{n0})}{\partial R_{n0}^i}\,u_{in} \right\} \right\rangle, \tag{3.46}$$

where the angular brackets define as usual the averaging over random impurity positions. In the momentum representation and for point-like impurities we get in the lowest order over ω/E_F and q/p_F

$$\Lambda_{2i}(p,q) = \rho V^2 \int \frac{d^3p'}{(2\pi)^3}\left[-i(p_i - p_i')G(Ep') + i(p_i' - p_i)G(Ep')\right]$$
$$= 2\rho V^2 \int \frac{d^3p'}{(2\pi)^3}\left[-i(p_i - p_i')G(Ep')\right] = 2\pi\rho V^2 N(E)p_i = 2\gamma p_i\,. \tag{3.47}$$

The relevant diagrams are shown in Fig. 11(b) [156]. A "bare" electron–phonon vertex is thus given by the sum of three diagrams shown in Fig. 11(b) and reduces to

$$\Lambda_i^{(0)} = \Lambda_{1i}^{(0)} + \Lambda_{2i}^{(0)} = \tfrac{1}{3}iq_i v_F p_F + 2\gamma p_i\,. \tag{3.48}$$

Diffusion renormalization of the electron–phonon vertex can appear due to impurity scattering ladder corrections as shown in Fig. 12(a). Similar diagrams shown in Fig. 7(c) lead to diffusion renormalization of the Coulomb vertex. However, in the case of electron–phonon interaction we

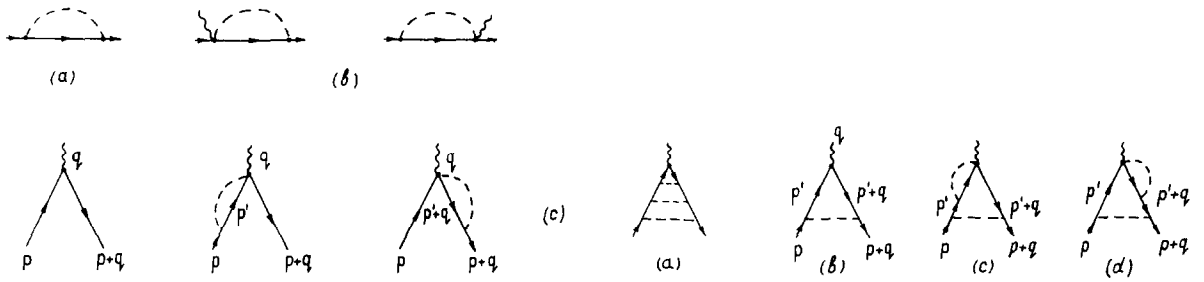

Fig. 11. Electron–phonon interaction and impurity scattering: (a) Self-energy due to impurity scattering, (b) Diagrams representing changes of (a) due to impurity vibrations, (c) diagrams for the "bare" electron–phonon vertex in the case of vibrating impurities.

Fig. 12. Electron–phonon vertex renormalization: (a) Impurity "ladder" (diffusion) renormalization. (b)–(d) Simplest corrections due to impurity vibrations.

have to make the same renormalization of the three diagrams of Fig. 7(c). Let us consider the simplest corrections shown in Fig. 12(b)–(d). For the contribution of graph of Fig. 12(b) we have

$$\Lambda_{1i}^{(1)} = \tfrac{1}{3}\rho V^2 \mathrm{i}q_i v_F p_F \int \frac{\mathrm{d}^3 \boldsymbol{p}'}{(2\pi)^3} G(E\boldsymbol{p}')G(E+\omega\boldsymbol{p}'+\boldsymbol{q})$$

$$\approx \tfrac{1}{3}\mathrm{i}q_i v_F p_F[1 + \mathrm{i}\omega/2\gamma - D_0 q^2/2\gamma] \approx \tfrac{1}{3}\mathrm{i}q_i v_F p_F\,, \quad \omega, q \to 0 \tag{3.49}$$

and for the sum of graphs of Fig. 12(c), (d):

$$\Lambda_{2i}^{(1)} = 2\rho V^2 \gamma \int \frac{\mathrm{d}^3 \boldsymbol{p}'}{(2\pi)^3} G(E\boldsymbol{p}')G(E+\omega\boldsymbol{p}'+\boldsymbol{q})p_i'$$

$$\approx 2\rho V^2 \gamma q_i \int \frac{\mathrm{d}^3 \boldsymbol{p}'}{(2\pi)^3} p_i' G(E\boldsymbol{p}') \frac{\partial}{\partial p_i} G(E+\omega\boldsymbol{p}')$$

$$\approx 2\gamma\rho V^2 q_i p_F \int \frac{\mathrm{d}^3 \boldsymbol{p}'}{(2\pi)^3} \frac{v_F}{3} G(E\boldsymbol{p}')G^2(E\boldsymbol{p}') = -\tfrac{1}{3}\mathrm{i}q_i v_F p_F\,. \tag{3.50}$$

Thus for $\omega \to 0$, $q \to 0$ we obtain

$$\Lambda_{1i}^{(1)} + \Lambda_{2i}^{(1)} = 0 \tag{3.51}$$

and we have total cancellation of initial diagrams contributing to the diffusion ladder. Apparently there is no diffusion renormalization of the electron–phonon vertex (for $\omega, q \to 0$): this cancellation is valid for any graph obtained from the simplest corrections by adding further impurity lines to the ladder. Similar cancellation takes place when corrections due to maximally crossed impurity lines (Cooper channel) are added to diagrams of Fig. 12(b)–(d). Thus there is no significant change of the electron–phonon vertex due to Cooperon and the only relevant contribution to the electron–phonon vertex in an impure system is defined by the sum of diagrams of Fig. 11(b) leading to Eq. (3.48) which does not contain diffusion type renormalization. Localization appears via a generalized diffusion coefficient which replaces the Drude one. Thus localization singularities does not

appear in the electron–phonon vertex, though surely this interaction is really changed by disorder scattering in comparison with the "pure" case. Of course, the question of whether localization effects contribute to a renormalization of the electron–phonon coupling is still open to discussion [136]. Probably, a more important aspect of this problem is reflected by the fact that superconductivity is actually determined not by the electron–phonon vertex itself, but by the famous integral expression over the phonon spectrum of the Eliashberg function $\alpha^2(\omega)F(\omega)$ which defines the pairing constant λ [139]. This integration will apparently smooth out all possible singularities.

In the following we shall model pairing interaction due to phonon exchange by some constant λ as in the BCS model. Of course, we must stress that this constant is different from that in a regular metal. It is constant in the sense that it does not contain singularities due to the metal–insulator transition. Electron–phonon kernel in the linearized gap equation (cf. Appendix B) can be taken in the simplest form

$$K_{\rm ph}(E,E') = \begin{cases} -\lambda & |E|,|E'| < \omega_{\rm D}\,, \\ 0 & |E|,|E'| > \omega_{\rm D}\,, \end{cases} \tag{3.52}$$

and consider λ as relatively weakly dependent on disordering. More detailed discussion of electron–phonon pairing in disordered systems can be found in Refs. [134, 136, 137].

As we mentioned above it is quite difficult to speculate on the disorder dependence of the pairing interaction in high-temperature superconductors. In the "marginal" Fermi-liquid approach [124, 125] pairing interaction can be modelled as in Eq. (3.52) with the replacement of Debye frequency $\omega_{\rm D}$ by some phenomenological *electronic* frequency $\tilde{\omega}_{\rm c}$ which we briefly mentioned before while discussing localization in a "marginal" Fermi-liquid. In the following we shall just assume that this pairing interaction is weakly dependent on disorder as in the case of the phonon mechanism of pairing.

3.2.3. *Metallic region*

In the metallic region we can use Eqs. (3.34)–(3.36) and (3.38) and find the diffusional contribution to the Coulomb kernel:

$$\begin{aligned} K_{\rm c}^{\rm diff}(\omega) &= -\int \frac{{\rm d}^3\boldsymbol{q}}{(2\pi)^3}\, v_0\, {\rm Im}\, \frac{1}{\omega + {\rm i}D_E q^2} \approx \frac{v_0}{2\pi^3}\left[\frac{1}{|D_E(\omega)l|} - \frac{|\omega|^{1/2}}{|D_E^{3/2}(\omega)|}\right] \\ &\approx \frac{v_0}{2\pi^3}\begin{cases} \dfrac{1}{D_E l} - \dfrac{|\omega|^{1/2}}{D_E^{3/2}}\,, & |\omega| \ll \omega_{\rm c}\,, \\ \dfrac{1}{D_0 l}\left(\dfrac{\omega}{2\gamma}\right)^{-1/3}\,, & |\omega| \gg \omega_{\rm c}\,. \end{cases} \end{aligned} \tag{3.53}$$

Accordingly, for the Coulomb kernel defined by Eq. (3.38) we get [22]

$$K_{\rm c}(\omega) = \mu\theta(E_{\rm F} - |\omega|) + \frac{\mu}{p_{\rm F} l}\begin{cases} \sigma_{\rm c}/\sigma & |\omega| < \omega_{\rm c} \\ (1/p_{\rm F} l)(\omega/2\gamma)^{-1/3} & \omega_{\rm c} < \omega < \gamma \sim E_{\rm F} \end{cases}. \tag{3.54}$$

Upper limit cut-off in the integral in Eq. (3.53) was taken $\sim l^{-1}$. Rough estimate of the contribution of higher momenta can be achieved introducing cut-off $\sim p_{\rm F}$ (cf. Ref. [153]). This will cancel $(p_{\rm F} l)^{-1}$ in Eq. (3.54). Close to the Anderson transition $l^{-1} \sim p_{\rm F}$ and this correction is irrelevant. We

shall assume that far from transition these higher momenta corrections can be included in the definition of μ. From Eq. (3.54) we can see that diffusion renormalization of the Coulomb kernel leads to substantial growth of the Coulomb repulsion close to Anderson transition (i.e. when conductivity drops below σ_c – "minimal metallic conductivity").

Superconducting transition temperature T_c is determined by the linearized gap equation [152] which in the weak coupling approximation can be written as (cf. Appendix B) [157,158]

$$\Delta(\omega) = \lambda\theta(\langle\omega\rangle - \omega)\int_0^{\langle\omega\rangle}\frac{d\omega'}{\omega'}\Delta(\omega')\tanh(\omega'/2T_c)$$
$$-\theta(E_F-\omega)\int_0^{E_F}\frac{d\omega'}{\omega'}K_c(\omega-\omega')\Delta(\omega')\tanh(\omega'/2T_c)\,. \tag{3.55}$$

Consider the metallic region and take $\omega_c \gg \langle\omega\rangle$ which in accordance with the ω_c estimate given in Eq. (3.37) roughly corresponds to $\sigma \geq \sigma_c$ for typical $E_F/\langle\omega\rangle \sim 10^2$, so that the system is not very close to the Anderson transition. The change of T_c due to the diffusion contribution in the Coulomb kernel (Eq. (3.54)) can be determined by perturbation theory over $K_c^{diff}(\omega)$ in the gap equation. First iteration of Eq. (3.55) gives

$$\frac{\delta T_c}{T_{co}} \approx \frac{\int_0^\infty (d\omega/\omega)\int_0^\infty (d\omega'/\omega')\Delta_0(\omega)\tanh(\omega/2T_{co})K_c^{diff}(\omega-\omega')\Delta_0(\omega')\tanh(\omega'/2T_{co})}{(1/2T_{co})\int_0^\infty d\omega\,[\Delta_0(\omega)]^2\,[\cosh(\omega/2T_{co})]^{-2}}\,, \tag{3.56}$$

where $\Delta_0(\omega)$ is the usual "two-step" solution of Eq. (3.55) [9,152] which is valid for the standard form of the Coulomb kernel $K_c(\omega) = \mu\theta(E_F - |\omega|)$,

$$T_{co} = 1.13\langle\omega\rangle\exp(-1/(\lambda-\mu_0^*)) \tag{3.57}$$

is a critical temperature in the regular superconductor when the Coulomb pseudopotential is given by

$$\mu_0^* = \frac{\mu}{1+\mu\ln(E_F/\langle\omega\rangle)}\,. \tag{3.58}$$

Using the first relation in Eq. (3.54) we get from Eq. (3.56)

$$\delta T_c/T_{c0} \approx -\frac{\mu}{(\lambda-\mu_0^*)^2}\frac{1}{p_F l}\frac{\sigma_c}{\sigma}\,. \tag{3.59}$$

This change of T_c is equivalent to the following change of Coulomb pseudopotential [22]

$$\delta\mu^* \approx \mu\sigma_c^2/[\sigma(\sigma+\sigma_c)]\,, \tag{3.60}$$

where we have used Eq. (2.75) and $p_F l \approx \sigma_0/\sigma_c = (\sigma+\sigma_c)/\sigma_c$ to replace $p_F l$ in Eq. (3.59). As we noted above this later factor disappears from Eq. (3.59) if we use cut-off at $q \sim p_F$ in Eq. (3.53). According to Eq. (3.60) Coulomb pseudopotential μ^* grows as σ drops and this dependence is stronger than a similar one obtained in Ref. [20], which is connected with our use of the results of the self-consistent theory of localization. The method of Ref. [20] is based upon the use of the

q-dependence of the diffusion coefficient as given by Eq. (2.85). Our expression for $\delta\mu^*$ leads to a significant growth of μ^* for conductivities $\sigma \leq 10^3\ \Omega^{-1}\,\mathrm{cm}^{-1}$. This growth can easily explain the typical T_c degradation in "very dirty" superconductors as their conductivity in normal state drops upon approaching the Ioffe–Regel limit [30]. At the same time expressions for μ^* proposed in Ref. [20] can explain experimental data only under the assumption that a characteristic conductivity scale determining μ^* is an order of magnitude larger than the Ioffe–Regel limit, for which we see no serious grounds. A more extensive discussion can be found in Ref. [136].

Let us consider now the situation at the mobility edge itself, when $\sigma = 0$ and $\omega_c = 0$ so that $K_c(\omega)$ is determined by the second expression in Eq. (3.54) for all the frequencies below $\gamma \sim E_F$. In this case we can show [22] that the influence of the Coulomb repulsion on T_c is again described by the effective pseudopotential μ^* which can be estimated as

$$\mu^* \sim \alpha\mu(\langle\omega\rangle/2\gamma)^{-1/3}, \quad \alpha \sim 1\,. \tag{3.61}$$

In this case T_c may remain finite at the mobility edge only under very strict conditions: both $E_F \sim \gamma$ and μ must be very small, while λ must be at least close to unity. As a crude estimate we can demand something like $\lambda \sim 1$, $\mu \leq 0.2$ and $E_F \leq 10^3\, T_{c0}$. Obviously, only some narrow band superconductors like Chevrel phases can satisfy these conditions among traditional systems. High-T_c superconductors are especially promising. The experimental situation will be discussed later.

Using Eq. (3.60) and Eq. (3.61) we can write down a simple interpolation formula for the conductivity dependence of μ^* [22]:

$$\mu^* \approx \mu_0^* + \frac{\alpha\mu(\langle\omega\rangle/2\gamma)^{-1/3} - \mu_0^*}{1 + (\langle\omega\rangle/2\gamma)^{-1/3}\sigma(\sigma + \sigma_c)/\sigma_c^2}\,. \tag{3.62}$$

To get an expression via observable parameters, take into account $\langle\omega\rangle/\gamma \approx (\langle\omega\rangle/E_F)(1 + \sigma/\sigma_c)$. These expressions describe continuous crossover from the region of weak localization corrections to the vicinity of Anderson transition where its influence upon T_c becomes very strong. This crossover takes place at $\omega_c \sim \langle\omega\rangle$.

3.2.4. *Localization region*

Let us now consider an Anderson insulator. According to Eqs. (3.34) and (A.9) the Coulomb kernel acquires in this case a $\delta(\omega)$-contribution:

$$K_c^{\mathrm{loc}}(\omega) = v_0 A_E \delta(\omega) = v_0 \frac{1}{N(E)} \left\langle \sum_\nu \delta(E - \varepsilon_\nu)|\phi(\boldsymbol{r})|^2 |\phi(\boldsymbol{r})|^2 \right\rangle, \tag{3.63}$$

$$A_E = A_E(\boldsymbol{r} - \boldsymbol{r}')|_{r=r'} \sim R_{\mathrm{loc}}^{-3} \tag{3.64}$$

which is actually connected with the "Hubbard-like" repulsion of electrons in a single quantum state becoming nonzero in the localization region [159,98,7]. This mechanism contributes in addition to the diffusion contributions in the Coulomb pseudopotential μ^* considered above, which are due to the "regular" part of the Gorkov–Berezinskii spectral density. Using Eq. (3.64) as a full Coulomb vertex in the linearized gap equation (3.55) we can solve it exactly [22] and find

$$\Delta(\omega) = \frac{\theta(\langle\omega\rangle - |\omega|)\Delta_1}{1 + (\mu A_E/2N(E))(1/\omega)\tanh(\omega/2T_c)}\,, \tag{3.65}$$

where

$$\Delta_1 = \lambda \int_0^{\langle\omega\rangle} d\omega\, \Delta(\omega)(1/\omega)\tanh(\omega/2T_c) \tag{3.66}$$

and the equation for T_c takes the form

$$1 = \lambda \int_0^{\langle\omega\rangle} d\omega \frac{\tanh(\omega/2T_c)}{\omega + (\mu A_E/2N(E))\tanh(\omega/2T_c)} . \tag{3.67}$$

To account for the "regular" diffusion contributions to μ^* we can just replace here $\lambda \to \lambda^* = \lambda - \mu^*$, where μ^* is given by Eq. (3.61). Then our equation for T_c can be approximately represented by [22]

$$\ln\frac{T^*}{T_c} \approx \psi\left(\frac{1}{2} + \frac{\mu A_E}{4T_c N(E)}\right) - \psi\left(\frac{1}{2}\right), \tag{3.68}$$

where $\psi(x)$ is digamma function, and T^* is taken to be equal to T_c of the system at the mobility edge which is given by Eq. (3.57) with μ_0^* replaced by μ^* from Eq. (3.61). Here we slightly overestimate the role of the Coulomb repulsion in the localization region. We can see that this additional "Hubbard-like" repulsion acts upon T_c as magnetic impurities [9, 152] with an effective spin-flip scattering rate:

$$\frac{1}{\tau_{sf}} = \pi \mu A_E/N(E) \sim \mu/(N(E)R_{loc}^3(E)) . \tag{3.69}$$

Obviously, this result is connected with the appearance of singly occupied electron states of width [159, 98, 7, 160] $v_0 R_{loc}^{-3}$ below the mobility edge of the "band". Superconductivity persists until $\tau_{sf}^{-1} < 0.57T_c^*$, i.e. until

$$R_{loc}(E) > [\mu/N(E)T_c^*]^{1/3} \sim (\xi_0 p_F^{-2})^{1/3} \sim (\xi_0 l^2)^{1/3} , \tag{3.70}$$

where the last estimates are valid for typical values of parameters and correspond to the simple estimate of Eq. (3.20). Thus the Coulomb repulsion in a single (localized) quantum state leads to a sharp reduction of T_c below the mobility edge even if superconductivity survived up to the Anderson transition. Another interpretation of this effect is the influence of "free" spins of Mott's band of singly occupied states below the Fermi level of an Anderson insulator.

Coulomb gap [41–44] effects can be neglected here [22] because according to the estimates given in Eqs. (2.14) and (2.15) the Coulomb gap width

$$\Delta_c \sim [N(E)R_{loc}^3(E)]^{-1} \ll T_c \sim \Delta , \tag{3.71}$$

i.e. is small in comparison to the superconducting gap Δ (or T_c) under the conditions given by Eq. (3.20) which is necessary for the observation of superconductivity in the localization region.

3.2.5. Spin fluctuations

As we mentioned during our discussion of interaction effects upon the Anderson transition the role of magnetic fluctuations (spin effects) in general becomes stronger as we approach the metal–insulator transition. A band of single-occupied states is being formed below the Fermi

level of the Anderson insulator, which is equivalent to the appearance of localized moments [159,7,160]. These effects actually may become important even before the metal–insulator transition [104,112–114,117,118,40] and lead to the additional mechanism of T_c degradation. Unfortunately, there is no complete theoretical understanding of these effects and accordingly only few estimates can be done concerning superconductivity. Here we shall mention only some of these crude estimates following Refs. [161–163].

In the framework of the Hubbard model with weak disorder it can be shown [161] that the spin susceptibility is represented by

$$\chi_s = \frac{\chi_0}{1 - UN(E) + \gamma_0 - \gamma'} = \frac{\chi_0}{\eta_0 - \gamma'} \equiv \frac{\chi_0}{\eta}, \tag{3.72}$$

where χ_0 is the spin susceptibility of free electrons, $\eta_0 = 1 - UN(E) + \gamma_0$ is the enhancement factor for the ordered case (U is the Hubbard interaction, γ_0 the correlation correction to the RPA approximation), γ' is the correction due to the interference of Hubbard interaction and disorder scattering:

$$\gamma' = B\lambda^2, \quad B = 6\sqrt{3}\pi^2[N(E)U]^2\{1 - \tfrac{1}{2}UN(E)\}. \tag{3.73}$$

Here $\lambda = 1/(2\pi E\tau) = 1/(p_F l)$ is the usual perturbation theory parameter for disorder scattering. As $\gamma' > 0$ we can see from Eq. (3.72) that disordering leads to a diminishing denominator $\eta = \eta_0 - \gamma'$. If we reach a critical disorder defined by

$$\lambda_c = \sqrt{\frac{\eta_0}{B}}, \quad p_F l|_c = 2\sqrt{\frac{B}{\eta_0}}, \tag{3.74}$$

we get $\chi_s \to \infty$. It should be stressed that this divergence of χ_s in a disordered system must not be identified with any kind of ferromagnetic instability but may signify something like the appearance of a spin-glass state or just of localized moments. In any case, it means the growth of spin dependent effects under disordering.

If the initial enhancement of spin susceptibility is strong enough (e.g. due to a large U), i.e. $\eta_0 \ll 1$, the critical disorder defined by Eq. (3.74) may be lower than the critical disorder for Anderson localization, appearing at $p_F l \sim 1$. Then these spin dependent effects may become important well before Anderson transition. In the opposite case, these effects will appear only very close to the metal–insulator transition. In the general case, the relation between these two transitions depends on parameters.

If spin fluctuations are strong enough ($\eta \ll 1$) a strong mechanism for T_c degradation in the superconducting state appears [162] analogous to the similar effect due to magnetic impurities [9,152]

$$\ln\frac{T_{c0}}{T_c} = \psi\left(\frac{1}{2} + \rho\right) - \psi\left(\frac{1}{2}\right), \tag{3.75}$$

where [162]

$$\rho = \frac{9\sqrt{3}\pi}{2}\lambda^2\frac{UN(E)}{\eta} = \frac{9\sqrt{3}}{2}\left[\frac{UN(E)}{B}\right]\frac{\lambda^2}{\lambda_c^2 - \lambda^2}. \tag{3.76}$$

As ρ from Eq. (3.76) diverges as $(\lambda_c - \lambda)^{-1}$ for $\lambda \to \lambda_c$ superconducting transition, temperature T_c drops to zero.

If $\lambda_c \ll 1$, which is possible for $\eta_0 \ll 1$, superconductivity will be destroyed long before the metal–insulator transition. In the opposite case this mechanism may lead to its destruction on either side of the metal–insulator transition depending on the parameters of the system, such as U. In general, we need a more accurate analysis which must include the mutual interplay of magnetic fluctuations and disorder scattering leading to a metal–insulator transition. In any case, magnetic mechanisms of T_c degradation close to the metal–insulator transition may be as important as the Coulomb effects considered above.

3.3. Ginzburg–Landau theory and Anderson transition

3.3.1. General analysis

The main result of the previous analysis may be formulated as follows. Despite many mechanisms leading to T_c degradation and destruction of superconductivity in strongly disordered systems there seems to be no general rule prohibiting the possibility of a superconducting state in the Anderson insulator. Of course we must meet very rigid conditions if we hope to observe this rather exotic state. There is almost no chance to observe it in traditional superconductors but high-T_c systems seem promising. The following analysis will be based on the general assumption that T_c survives in a strongly disordered system or even in an Anderson insulator, i.e. that these strict conditions are met. Our aim is to study superconducting properties of such a strongly disordered system to determine specific characteristics which will make this case different from the usual case of "dirty" superconductors. We shall see that even before the Anderson transition there are significant deviations from the predictions of standard theory which make a strongly disordered system different. So, on the practical side, our aim is simply to generalize the usual theory of "dirty" superconductors for the case of a strong disorder in the sense of the mean free path of the order of the interatomic spacing or $l \sim p_F^{-1}$.

To claim that superconductivity is possible close to a disorder-induced metal–insulator transition it is not sufficient just to demonstrate the finite values of T_c. Even more important is to show the existence of the superconducting response to an external electromagnetic potential $\boldsymbol{A}$. In the general case, the analysis of response functions of a superconductor with strong disorder seems to be a difficult task. However, close to T_c, significant simplifications take place and actually we only have to show that the free-energy density of the system can be expressed in the standard Ginzburg–Landau form [164, 12, 9]:

$$F = F_n + A|\Delta|^2 + \tfrac{1}{2}B|\Delta|^4 + C|(\nabla - 2ie\boldsymbol{A}/\hbar c)\Delta|^2 , \tag{3.77}$$

where F_n is the free energy density of the normal state. Our problem is thus reduced to a microscopic deviation of expressions for the coefficients A, B, and C of the Ginzburg–Landau expansion Eq. (3.77) taking into account the possibility of electron localization. This will be the generalization of the famous Gorkov's derivation [12] of similar expressions for the case of "dirty" superconductors. Such an analysis was first done by Bulaevskii and Sadovskii [21, 22] and later by Kotliar and Kapitulnik [23, 24]. Recently, the same results were obtained by Kravtsov [166].

Within the BCS model, coefficients A and B which determine the transition temperature and the equilibrium value of the order-parameter Δ do not change in comparison with their values found in

the theory of "dirty" superconductors, even if the system is close to the Anderson transition. This corresponds to the main statement of the Anderson theorem. Less trivial is the behavior of the coefficient C, which in fact defines the superconducting response. In the usual theory of "dirty" superconductors [12] this coefficient is proportional to the diffusion coefficient of electrons, i.e. to conductivity (at $T = 0$). As the Fermi level approaches the mobility edge, conductivity drops to zero. However, we shall see that the coefficient C remains finite in the vicinity of the Anderson transition, even in the region of localized states.

To derive Ginzburg–Landau coefficients we must know the two-electron Green's function in the normal state [12]. Let us introduce the following two-particle Matsubara Green's functions in momentum representation [22]:

$$\Psi_E(\boldsymbol{q}, \omega_m, \varepsilon_n) = -\frac{1}{2\pi i} \sum_{p_+ p'_-} \langle G(\boldsymbol{p}_+, \boldsymbol{p}'_+, -\varepsilon_n + \omega_m) G(-\boldsymbol{p}'_-, -\boldsymbol{p}_-, -\varepsilon_n) \rangle \tag{3.78}$$

$$\Phi_E(\boldsymbol{q}, \omega_m, \varepsilon_n) = -\frac{1}{2\pi i} \sum_{p_+ p'_-} \langle G(\boldsymbol{p}_+, \boldsymbol{p}'_+, -\varepsilon_n + \omega_m) G(\boldsymbol{p}'_-, \boldsymbol{p}_-, -\varepsilon_n) \rangle \tag{3.79}$$

where $\boldsymbol{p}_{+-} = \frac{1}{2}(\boldsymbol{p} \pm \boldsymbol{q})$ and $\omega_m = 2\pi m T$. Graphically, these functions are represented in Fig. 13. Then Ginzburg–Landau coefficients are defined by [12, 165]

$$A = (1/g) + 2\pi i T \sum_{\varepsilon_n} \Psi_E(\boldsymbol{q} = 0 \omega_m = 2\varepsilon_n) \tag{3.80}$$

$$C = i\pi T \sum_{\varepsilon_n} \frac{\partial^2}{\partial q^2} \Psi_E(\boldsymbol{q} \omega_m = 2\varepsilon_n)|_{q=0} \,. \tag{3.81}$$

Thus the superconducting properties are determined by the Green's function Ψ_E describing the propagation of the electron (Cooper) pair. At the same time we have seen that the Green's function Φ_E determines transport properties of a normal metal and Anderson transition. In the case of time-invariance (i.e. in the absence of the external magnetic field or magnetic

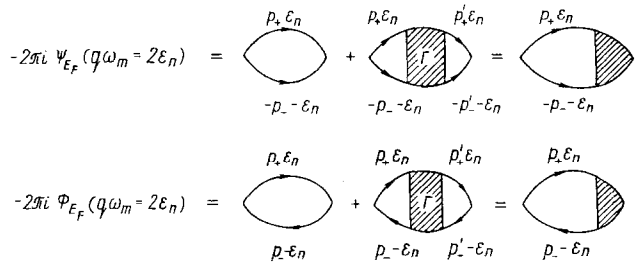

Fig. 13. Graphical representation of the two-particle Green's functions $\Psi_E(\boldsymbol{q}\omega_m)$ and $\Phi_E(\boldsymbol{q}\omega_m)$ (for $\omega_m = 2\varepsilon_n$). There is no summation over ε_n in the loops.

impurities) we have [71]

$$\Psi_E(\boldsymbol{q}\omega_m\varepsilon_n) = \Phi_E(\boldsymbol{q}\omega_m\varepsilon_n) \tag{3.82}$$

and it is sufficient to know only $\Phi_E(\boldsymbol{q}\omega_m = 2\varepsilon_n)$ to determine the Ginzburg–Landau coefficients.

As a one-electron model of Anderson transition we can take the self-consistent theory of localization which will allow us to perform all calculations explicitly. We only have to formulate the main equations of this theory in the Matsubara formalism (finite T) [22]. For small q and ω_m, analogous to Eq. (2.46), we have

$$\Phi_E(\boldsymbol{q}\omega_m) = -\frac{N(E)}{\mathrm{i}|\omega_m| + D_E(|\omega_m|)q^2}\,, \quad \omega_m = 2\pi mT\,, \tag{3.83}$$

where the generalized diffusion coefficient $D_E(\omega_m)$ is determined by the self-consistency equation analogous to Eq. (2.57):

$$\frac{D_0}{D_E(\omega_m)} = 1 - \frac{\mathrm{i}}{\pi N^2(E)} \sum_{|q|<k_0} \Phi_E(\boldsymbol{q}\omega_m)\,. \tag{3.84}$$

In the three-dimensional case, Eq. (3.84) reduces to (cf. Eq. (2.77))

$$\frac{D_E(\omega_m)}{D_0} = 1 - \frac{\lambda}{\lambda_c} + \frac{\pi}{2}\frac{\lambda}{\lambda_c}\left[\frac{D_0}{D_E(\omega_m)}\frac{\omega_m}{2\gamma}\right]^{1/2}, \tag{3.85}$$

where we have used the same notations as in our discussion of the self-consistent theory of localization. Analogous to Eq. (2.78) and with accuracy sufficient for our purpose we can write down the solution of Eq. (3.85) as

$$D_E(\omega_m) \approx \mathrm{Max}\left\{D_E\frac{\omega_m}{\omega_m + 3D_E\omega_0^2(E)/v_F^2};\ D_0\left(\frac{\omega_m}{2\gamma}\right)^{1/3}\right\}, \tag{3.86}$$

where D_E is the renormalized diffusion coefficient defined in Eq. (2.80) and ω_0 is the fundamental frequency defined by Eq. (2.64), which signals a transition to insulator.

As we have already noted Ginzburg–Landau coefficients A and B are given by the usual expressions valid also for "dirty" superconductors [12, 22]:

$$A = N(E_F)\ln\frac{T}{T_c} \approx N(E_F)\frac{T - T_c}{T_c}\,, \tag{3.87}$$

where T_c is given by the usual BCS relation of Eq. (3.19), and

$$B = (7\zeta(3)/8\pi^2 T_c^2)N(E_F)\,, \tag{3.88}$$

where $\zeta(x)$ is the Riemann zeta-function ($\zeta(3) = 1.202\ldots$). These coefficients depend on disorder only through the appropriate disorder dependence of $N(E_F)$ and are valid even in the localized phase. This is equivalent to the main statement of Anderson theorem.

Significant changes appear in the gradient term coefficient C. Using Eqs. (3.81)–(3.83) with Eq. (3.86) we can find that in different limiting cases this coefficient can be expressed as [21,22]:

$$C \equiv N(E_F)\xi^2 \approx N(E_F)\begin{cases} \dfrac{\pi}{8T_c} D_{E_F}, & \xi_{loc}(E_F) < (\xi_0 l^2)^{1/3};\ E_F > E_c, \\ \left(\dfrac{D_0 l}{T_c}\right)^{2/3} \approx (\xi_0 l^2)^{2/3}, & \xi_{loc}(E_F) > (\xi_0 l^2)^{1/3};\ E_F \sim E_c, \\ R_{loc}^2(E_F) \ln \dfrac{1.78 D_{E_F}}{\pi T_c R_{loc}^2(E_F)}, & R_{loc}(E_F) < (\xi_0 l^2)^{1/3};\ E_F < E_c, \end{cases} \tag{3.89}$$

where we have defined the coherence length ξ, and $\xi_0 = 0.18 v_F/T_c$ is the BCS coherence length, l as usual is the mean free path. Practically, the same results were obtained in Refs. [23,24] using the approach based upon elementary scaling theory of localization, which is as we already noted is equivalent to our use of the self-consistent theory of localization. In Ref. [166] the same results were confirmed using the σ-model approach to localization.

In the metallic state, as the Fermi level E_F moves towards the mobility edge E_c localization correlation length ξ_{loc} grows and the coefficient C initially drops as the generalized diffusion coefficient D_{E_F}, i.e. as conductivity of a system in the normal state. However, in the vicinity of Anderson transition, while $\sigma \to 0$ the drop in C saturates and it remains finite even for $E_F < E_c$, i.e. in Anderson insulator. With further lowering of E_F into the localization region (or E_c growth with disorder) the C coefficient is determined by the localization radius R_{loc} which diminishes as E_F moves deep into the insulating state. However, remembering Eqs. (3.20) and (3.22) we recognize that our analysis is valid only for large enough values of localization length, which satisfy Eq. (3.22). In this sense the last expression in Eq. (3.89) is actually outside these limits of applicability.

The finite value of the coefficient C in the Ginzburg–Landau expansion in the vicinity of Anderson transition signifies the existence of a superconducting (Meissner) response to an external magnetic field. Accordingly, for $T < T_c$ the system can undergo a transition from Anderson insulator to superconductor. The physical meaning of this result can be understood from the following qualitative picture (cf. Ref. [131] where similar estimates were used for the granular metal). In an Anderson insulator all electrons with energies E close to the Fermi level are localized in spatial regions of the size of $\sim R_{loc}(E)$. Nearby regions are connected by a tunneling amplitude $\mathscr{V}$ which determines the probability of electron transition between such regions as

$$P_T \approx 2\pi |\mathscr{V}|^2 N(E) R_{loc}^3(E). \tag{3.90}$$

However, Anderson localization means that

$$|\mathscr{V}| < 1/(N(E)R_{loc}^3(E)) \tag{3.91}$$

and coherent tunneling between states localized in these regions is impossible, and we have $P_T < 2\pi N^{-1}(E) R_{loc}^{-3}$. At the same time if conditions given by Eqs. (3.20) or (3.22) are satisfied inside each region $\sim R_{loc}$, Cooper pairs may form and a superconducting gap Δ appears in the spectrum. Then a kind of "Josephson" coupling appears between regions of localized states which determines the possibility of *pairs* tunneling:

$$E_J \approx \pi^2 [N(E) R_{loc}^3(E)]^2 |\mathscr{V}|^2 \Delta. \tag{3.92}$$

It is easy to see that for

$$\Delta > (2/\pi)\, 1/(N(E) R_{\mathrm{loc}}^3(E)) \tag{3.93}$$

we have $E_J > P_T$, so that if Eq. (3.20) is satisfied we can get $E_J \gg N^{-1}(E) R^{-3}(E)$ irrespective of Eq. (3.91) and tunneling of pairs between nearby regions of localized states is possible, even in the absence of single-particle tunneling.

It is convenient to rewrite Eq. (3.89) using the relation between a generalized diffusion coefficient and conductivity similar to Eq. (2.59) as well as Eqs. (2.73), (2.75). Then using the Ginzburg–Landau expansion and the expressions for its coefficients we can easily find the temperature dependent coherence length $\xi(T)$ [9,21,22]:

$$\xi^2(T) = \frac{T_c}{T_c - T} \begin{cases} \xi_0 l \dfrac{\sigma}{\sigma + \sigma_c}, & \sigma > \sigma^* \quad (E_F > E_c), \\ (\xi_0 l^2)^{2/3}, & \sigma < \sigma^* \quad (E_F \sim E_c), \end{cases} \tag{3.94}$$

where $\sigma_c = e^2 p_F/(\pi^3 \hbar^2)$ and characteristic conductivity scale σ^* is given by

$$\sigma^* \approx \sigma_c (p_F \xi_0)^{-1/3} \approx \sigma_c (T_c/E_F)^{1/3}. \tag{3.95}$$

Thus in the region of very small conductivities $\sigma < \sigma^*$ the scale of $\xi(T)$ is defined not by $\xi \sim \sqrt{\xi_0 l}$ as in the usual theory of "dirty" superconductors [12,9] but by the new length $\xi \sim (\xi_0 l^2)^{1/3} \sim (\xi_0/p_F^2)^{1/3}$, which is now the characteristic size of Cooper pair close to Anderson transition.

If the $\omega^{1/3}$-law for a diffusion coefficient at the mobility edge is invalid and we have ω^{δ}-behavior, with some unknown critical exponent δ (which is possible because the modern theory actually cannot guarantee precise values of critical exponents at the Anderson transition [51,7]) we can easily show in a similar way that for conductivities $\sigma < \sigma^* \approx \sigma_c (p_F \xi_0)^{-\delta}$ the coherence length is defined by $\xi \sim \xi_0^{(1-\delta)/2} l^{(1+\delta)/2}$. Qualitatively, this leads to the same type of behavior as above.

From Eq. (3.94) we can see that $\xi^2(T)$ initially diminishes as we approach the metal–insulator transition proportionally to σ as in the case of a "dirty" superconductor. However, as in the metallic region for $\sigma < \sigma^*$ it diminishes more slowly remaining finite both at the transition itself and below.

The superconducting electron density n_s can be defined as [9]

$$n_s(T) = 8mC\Delta^2(T) = 8mC(-A)/B. \tag{3.96}$$

Close to the Anderson transition we can estimate

$$n_s \sim mN(E_F)\xi^2\Delta^2 \sim mp_F(\xi_0/p_F^2)^{2/3}\Delta^2 \sim n(T_c^{1/2}/E_F^2)^{2/3}(T_c - T), \tag{3.97}$$

where $n \sim p_F^3$ is the total electron density. If we take here $T \sim 0.5T_c$, i.e. more or less low temperatures, we get a simpler estimate:

$$n_s \sim n(T_c/E_F)^{4/3} \tag{3.98}$$

which is actually valid up to $T = 0$, as we shall see below. From these estimates we can see that only a small fraction of electrons are superconducting in a strongly disordered case. However, this confirms the possibility of a superconducting response of the Anderson insulator.

Characteristic conductivity σ^* defined in Eq. (3.95) gives an important conductivity scale at which significant influence of localization effects upon superconducting properties appear [22]. While σ_c is of the order of Mott's "minimal metallic conductivity" [2, 3], σ^* is in general even lower. However, for small enough Cooper pairs (i.e. small ξ_0 which is characteristic of strong coupling and high-T_c superconductors) it is more or less of the order of σ_c. Experimentally, it can be defined as a conductivity scale at which significant deviations from predictions of the standard theory of "dirty" superconductors appear under disordering.

We must stress that these results show the possibility of Cooper pairs being delocalized in an Anderson insulator, while single-particle excitations of such a superconductor are apparently localized, which may lead to some peculiar transport properties of "normal" electrons for $T < T_c$. First attempts to explore this peculiar situation were undertaken in Refs. [167–170].

These results are easily generalized for the case of strongly anisotropic quasi-two-dimensional systems such as high-T_c superconducting oxides. Using the analysis of such systems within the self-consistent theory of localization [68] we can write down the following Matsubara generalization of Eq. (2.94):

$$\frac{D_j(\omega_m)}{D_j^0} \approx \begin{cases} \mathrm{Max}\left[\dfrac{E_F - E_c}{E_c};\ (2\pi E_F w\tau^2)^{-2/3}(\omega_m\tau)^{1/3}\right], & \omega_m \ll w^2\tau\ , \\ 1 - \dfrac{1}{2\pi E_F\tau}\ln\left(\dfrac{1}{\omega_m\tau}\right), & \omega_m \gg w^2\tau\ , \end{cases} \tag{3.99}$$

where $j = \|, \perp$. Now carrying out calculations similar to that of Ref. [22] we obtain for the coefficients of gradient terms in the Ginzburg–Landau expansion [16, 171]:

$$C_{\|,\perp} = N(E_F)\xi_{\|,\perp}^2\ , \tag{3.100}$$

where for the coherence lengths $\xi_{\|,\perp}$ we obtain a number of different expressions, depending on the value of the ratio $w^2\tau/2\pi T_c\hbar$ which determines as we shall see the "degree of two-dimensionality" of the problem under study. For the case of $w^2\tau/2\pi T_c\hbar \gg 1$, corresponding to an anisotropic but three-dimensional system, we have

$$\xi_{\|,\perp}^2 = \frac{\pi}{8T_c} D_{\|,\perp}^0\left(\frac{E_F - E_c}{E_c}\right) \approx \xi_{\|,\perp}^0 l_{\|,\perp}\left(\frac{E_F - E_c}{E_c}\right), \tag{3.101}$$

where $\xi_\|^0 \sim \hbar v_F/T_c$, $\xi_\perp^0 \sim wa_\perp/T_c$, $l_\| = v_F\tau$ and $l_\perp = wa_\perp\tau/\hbar$ are the longitudinal and transverse BCS coherence lengths and mean free paths. The above expressions are valid in the conductivity region $\sigma_\| > \sigma^*$, where

$$\sigma^* \sim \sigma_\|^c(\xi_\|^0/l_\|)(T_c^2/E_F w)^{2/3}\ . \tag{3.102}$$

where $\sigma_\|^c$ is defined as in Eq. (2.93). The condition of $w^2\tau/2\pi T_c\hbar \gg 1$ is equivalent to the requirement:

$$\xi_\perp \sim \sqrt{\xi_\perp^0 l_\perp} \gg a_\perp \tag{3.103}$$

which clarifies its physical meaning: the transverse size of a Cooper pair must be much greater than the interplane lattice spacing. In this case we have just anisotropic three-dimensional superconductivity.

In the immediate vicinity of the Anderson transition, for $\sigma_\parallel < \sigma^*$ we have

$$\xi^2_{\parallel,\perp} \approx (1 - 2^{-5/3})(16\pi^4)^{-1/3}\zeta(5/3)\frac{D_{\parallel,\perp}}{(E_F T_c w)^{2/3}\tau} \approx (\xi^0_{\parallel,\perp})^2\left(\frac{T_c^2}{E_F w}\right)^{2/3}. \tag{3.104}$$

It is easy to see that for $w \sim E_F$ all these expressions naturally go over to those derived above for the three-dimensional case.

For $w^2\tau/2\pi T_c\hbar < 1$ which corresponds to the "almost two-dimensional" case of

$$\xi_\perp \sim \sqrt{\xi^0_\perp l_\perp} \leq a_\perp , \tag{3.105}$$

i.e. of transverse size of Cooper pairs smaller than the interplane spacing, we have

$$\xi^2_{\parallel,\perp} \approx \left\{\begin{array}{ll} \dfrac{D^0_{\parallel,\perp}}{\pi T_c}\dfrac{E_F - E_c}{E_c}, & (\sigma_\parallel > \sigma^*) \\ \dfrac{D^0_{\parallel,\perp}}{(4\pi^2 E_F T_c w)^{2/3}\tau}, & (\sigma_\parallel < \sigma^*) \end{array}\right\} + (\pi^2/8 - 1)\frac{D^0_{\parallel,\perp}}{\pi T_c}\left(1 - \frac{1}{2\pi E_F\tau}\ln\frac{1}{2\pi T_c\tau}\right). \tag{3.106}$$

Essential difference from just the anisotropic case of Eqs. (3.101) and (3.104) is the appearance here of a second term of "two-dimensional" type. In a purely two-dimensional problem ($w = 0$) we have [165]:

$$\xi^2_\parallel = \frac{\pi D^0_\parallel}{8T_c}\left(1 - \frac{1}{2\pi E_F\tau}\ln\frac{1}{2\pi T_c\tau}\right). \tag{3.107}$$

For high-T_c oxides it is reasonable to estimate $\xi^0_\parallel \sim l_\parallel$, $T_c \sim w$, $T_c \sim 0.1E_F$, so that $\sigma^* \sim \sigma^c_\parallel$, i.e. these systems are always more or less close to the Anderson transition. For $T_c \sim w$ and $\hbar/\tau \sim E_F$ which is characteristic of the rather strongly disordered case, we have $w^2\tau/2\pi T_c\hbar < 1$, so that for these systems we can realize almost two-dimensional behavior, though in general high-T_c oxides are apparently an intermediate case between the strongly anisotropic three-dimensional and nearly two-dimensional superconductors.

The significant change of Ginzburg–Landau coefficients and the new scale of coherence length close to the Anderson transition lead to an increased width of the critical region of thermodynamic fluctuations near T_c [23,24]. These are well known to be important for any second-order phase transition. The width of the critical region is defined by the so-called Ginzburg criterion [45,47] which may be expressed via the coefficients of Landau expansion. Mean-field approximation for the order parameter in Landau theory is valid (for $d = 3$) for [45,47]

$$1 \gg \left|\frac{T - T_c}{T_c}\right| \gg \frac{B^2 T_c^2}{\alpha C^3} \equiv \tau_G , \tag{3.108}$$

where α is defined by $A = \alpha(T - T_c)/T_c$. In the case of a superconducting transition we have: $\alpha = N(E_F)$, $B \sim N(E_F)/T_c^2$ and $C = N(E_F)\xi^2$. Accordingly, from Eq. (3.108) we get the following estimate for the critical region:

$$\tau_G \sim \frac{1}{N^2(E_F)\xi^6 T_c^2} \sim \left(\frac{E_F}{T_c}\right)^2\frac{1}{\xi^6 p_F^6} . \tag{3.109}$$

In the "pure" limit $\xi = \xi_0 \sim v_F/T_c$ and we get $\tau_G \sim (T_c/E_F)^4$, so that the critical region is practically unobservable. In a "dirty" superconductor $\xi \sim \sqrt{\xi_0 l}$ and

$$\tau_G \sim (T_c/E_F)[1/(p_F l)^3] \tag{3.110}$$

and again we have $\tau_G \ll 1$. However, for a superconductor close to the mobility edge $\xi \sim (\xi_0/p_F^2)^{1/3}$ and from Eq. (3.109) we get [23,24]

$$\tau_G \sim 1 . \tag{3.111}$$

Note that in fact τ_G may still be small because of numerical constants which we have dropped in our estimates. Anyhow, the critical region in this case becomes unusually wide and the superconducting transition becomes similar in this respect to the superfluid transition in Helium. Fluctuation effects may thus become observable even in a bulk three-dimensional superconductor. Note that in localized phase $\xi \sim R_{loc}$ and $\tau_G \sim [N^2(E_F)R_{loc}^6 T_c^2]^{-1} > 1$ if the condition given by Eq. (3.20) is violated.

Finally we should like to mention that thermodynamic fluctuations lead [23,24] to an additional mechanism of T_c degradation for a system which is close to Anderson transition. This follows from the general result on the reduction of mean-field transition temperature due to critical fluctuations. If these fluctuations are small (and we can use the so-called one-loop approximation) for a three-dimensional system it can be shown that [23,24]:

$$T_c = T_{c0} - \frac{7\zeta(3)}{16\pi^4 \xi^3 N(E_F)} , \tag{3.112}$$

where T_{c0} is the mean-field transition temperature. If we use here our expressions for ξ valid close to the metal–insulator transition we easily find for $\sigma > \sigma^*$ [24]:

$$T_c \approx T_{c0}[1 - 0.5(\sigma_c/\sigma)^{3/2}(T_{c0}/E_F)^{1/2}] . \tag{3.113}$$

For $\sigma < \sigma^*$ this fluctuation correction saturates as the further drop of coherence length stops there. Obviously, higher-order corrections are important here, but unfortunately little is known on the importance of this mechanism of T_c degradation outside the limits of one loop approximation.

3.3.2. *Upper critical field*

Direct information on the value of $\xi^2(T)$ can be obtained from the measurements of the upper critical field H_{c2} [9]:

$$H_{c2} = \phi_0/2\pi\xi^2(T) , \tag{3.114}$$

where $\phi_0 = \pi c\hbar/e$ is the superconducting magnetic flux quantum. Using Eq. (3.94) we obtain the following relation between normal state conductivity σ, the slope of the upper critical field at $T = T_c$ given by $(dH_{c2}/dT)_{T_c}$ and the value of the electronic density of states at the Fermi level (per one spin direction) $N(E_F)$ [21,22]:

$$-\frac{\sigma}{N(E_F)}\left(\frac{dH_{c2}}{dT}\right)_{T_c} \approx \begin{cases} \dfrac{8e^2}{\pi^2\hbar}\phi_0 , & \sigma > \sigma^* , \\ \phi_0 \dfrac{\sigma}{N(E_F)(\xi_0 l^2)^{2/3} T_c} \approx \phi_0 \dfrac{\sigma}{[N(E_F)T_c]^{1/3}}, & \sigma < \sigma^* . \end{cases} \tag{3.115}$$

For $\sigma > \sigma^*$ the r.h.s. of Eq. (3.115) contains only the fundamental constants. This so-called Gorkov's relation [12] is often used to interpret experimental data in "dirty" superconductors. Using it we may find $N(E_F)$ for different degrees of disorder from measurements of $(dH_{c2}/dT)_{T_c}$ and conductivity σ. On the other hand, $N(E_F)$ can in principle be determined from independent measurements e.g. of the electronic contribution to specific heat. However, our expression for $\sigma < \sigma^*$ which is valid close to the metal–insulator transition shows that in this region Gorkov's relation becomes invalid and its use can "simulate" the drop of $N(E_F)$ with the growth of resistivity (disorder). Roughly speaking, Eq. (3.115) shows that under the assumption of relatively smooth change of $N(E_F)$ and T_c with disorder the usual growth of $(dH_{c2}/dT)_{T_c}$ with disorder saturates in the conductivity region of $\sigma < \sigma^*$ close to the Anderson transition and the slope of the upper critical field becomes independent of resistivity. This stresses the importance of independent measurements of $N(E_F)$.

Note that the qualitative behavior given by Eq. (3.115) is retained also when the dependence of the diffusion coefficient at the mobility edge is ω^δ (with some arbitrary critical exponent δ); only the expression for σ^* is changed as noted above. Thus this behavior is not related to any specific approximations of the self-consistent theory of localization, except the general concept of continuous transition.

For an anisotropic (quasi-two-dimensional) system we have similar relations:

$$\left(\frac{dH_{c2}^{\perp}}{dT}\right)_{T_c} = -\frac{\phi_0}{2\pi\xi_{\parallel}^2 T_c}, \tag{3.116}$$

$$\left(\frac{dH_{c2}^{\parallel}}{dT}\right)_{T_c} = -\frac{\phi_0}{2\pi\xi_{\parallel}\xi_{\perp} T_c} \tag{3.117}$$

with $\xi_{\parallel,\perp}$ given in our discussion above after Eq. (3.100). This leads to relations and qualitative behavior similar to Eq. (3.115). However, we should like to note an especially interesting relation for the anisotropy of the slopes of the upper critical field [16]:

$$\frac{(dH_{c2}^{\parallel}/dT)_{T_c}}{(dH_{c2}^{\perp}/dT)_{T_c}} = \frac{\xi_{\parallel}}{\xi_{\perp}} = \frac{v_F}{wa/h}. \tag{3.118}$$

We see that the anisotropy of $(dH_{c2}/dT)_{T_c}$ is actually determined by the anisotropy of the Fermi velocity irrespective of the superconductivity regime: from the "pure" limit, through the usual "dirty" case, up to the vicinity of the Anderson transition.

The above derivation of C coefficient of Ginzburg-Landau expansion explicitly used the time-reversal invariance expressed by Eq. (3.82). This is valid in the absence of the external magnetic field and magnetic impurities. Accordingly the previous results for the upper critical field are formally valid in the limit of infinitesimal external field and this is sufficient for the demonstration of superconducting (Meissner) response and for the determination of $(dH_{c2}/dT)_{T_c}$, because $H_{c2} \to 0$ as $T \to T_c$. In a finite external field we must take into account its influence upon localization. The appropriate analysis was performed in Refs. [75, 172] and with a slightly different method in Ref. [80]. The results are essentially similar and below we shall follow Ref. [75]. The standard scheme for the analysis of the superconducting transition in an external magnetic field

[12,9,173,174] gives the following equation determining the temperature dependence of $H_{c2}(T)$:

$$\ln\frac{T}{T_c}=2\pi T\sum_{\varepsilon_n}\left\{\frac{1}{2|\varepsilon_n|+2\pi D_2(2|\varepsilon_n|)H/\phi_0}-\frac{1}{2|\varepsilon_n|}\right\}, \tag{3.119}$$

where $D_2(2|\varepsilon_n|)$ is the generalized diffusion coefficient in the Cooper channel as defined after Eqs. (2.97) and (2.98). Eq. (3.119) is valid [9] for

$$R_H=mcv_F/eH\gg\xi\,, \tag{3.120}$$

where R_H is the Larmor radius of an electron in a magnetic field and ξ the coherence length. Note that Eq. (3.119) describes only the orbital motion contribution to H_{c2}. In fact H_{c2} is also limited by the paramagnetic limit [9,174]:

$$\tfrac{1}{2}g_0\mu_B H<\Lambda\,, \tag{3.121}$$

where g_0 is the usual g-factor of an electron and μ_B the Bohr magneton.

Standard approach of the theory of "dirty" superconductors is based upon the replacement of $D_2(2|\varepsilon_n|)$ in Eq. (3.119) by the Drude diffusion coefficient D_0 which is valid for a metal with $l\gg p_F^{-1}$. For a system which is close to the Anderson transition we must take into account both the frequency dependence of the diffusion coefficient and the fact that in a magnetic field D_2 is not equal to the usual diffusion coefficient determining electronic transport, D_1. Actually, we shall see that the external magnetic field's influence upon localization leads to rather small corrections to $H_{c2}(T)$ practically everywhere except in the region of localized states [75]. Thus we may really neglect this influence as a first approximation as was done in Refs. [21,22] and start with the replacement of D_2 in Eq. (3.119) by $D_1=D_E$, where D_E is the frequency dependent generalized diffusion coefficient in the absence of magnetic field. Detailed analysis of Eq. (3.119) can be found in Ref. [75].

Summation over Matsubara frequencies in Eq. (3.119) must be cutoff at some frequency of the order of $\langle\omega\rangle$ the characteristic frequency of Bose excitations responsible for the pairing interaction. It is convenient here to measure the distance from the Anderson transition (degree of disorder) via frequency ω_c defined in Eqs. (2.42), (2.79) or Eq. (2.106). If a system is far from Anderson transition, so that $\omega_c\gg\langle\omega\rangle$ we can completely neglect the frequency dependence of the diffusion coefficient and find the usual results of the theory of "dirty" superconductors:

$$H_{c2}(T)=\frac{4}{\pi^2}\frac{\phi_0 T_c}{D_0}\ln\frac{T_c}{T}\,,\quad T\sim T_c\,, \tag{3.122}$$

$$H_{c2}=\frac{1}{2\gamma}\frac{\phi_0 T_c}{D_0}\left[1-\frac{1}{24}\left(\frac{4\gamma T}{T_c}\right)^2\right],\quad T\ll T_c\,, \tag{3.123}$$

where $\gamma=1.781]\ldots$. For the H_{c2} derivative at $T=T_c$ we find from here the first relation of Eq. (3.115), and $H_{c2}(T=0)$ is conveniently expressed as [12,173]

$$-H_{c2}(0)/[T_c(\mathrm{d}H_{c2}/\mathrm{d}T)_{T_c}]=\pi^2/8\gamma\approx 0.69\,. \tag{3.124}$$

In this case, the $H_{c2}(T)$ curve is convex at all temperatures below T_c [12,173,9,174]. Very close to the Anderson transition, when $\omega_c\ll 2\pi T$, only $\omega^{1/3}$ behavior of the diffusion coefficient is

important in Eq. (3.119) and it takes the following form [75]:

$$\ln\frac{T}{T_c}=\sum_{n=0}^{\infty}\{[(n+\tfrac{1}{2})+(n+\tfrac{1}{2})^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)]^{-1}-[n+\tfrac{1}{2}]^{-1}\}\,, \tag{3.125}$$

where $\omega_H = eH/mc$. From the above we get

$$H_{c2}(T)=m(\phi_0/\pi)[(4\pi)^{2/3}c_1]\,T^{2/3}E^{1/3}\ln(T_c/T)\,,\quad T\sim T_c\,, \tag{3.126}$$

$$H_{c2}(T)=m(\phi_0/\pi)(\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}[1-\tfrac{2}{3}c_2(4\gamma T/T_c)^{2/3}]\,,\quad T\ll T_c\,, \tag{3.127}$$

where $c_1=\sum_{n=0}^{\infty}(n+\frac{1}{2})^{-5/3}\approx 4.615$ and $c_2\approx 0.259$. From these expressions we get

$$-\frac{1}{N(E)}\left(\frac{dH_{c2}}{dT}\right)_{T_c}=\frac{(4\pi)^{2/3}}{\pi c_1}m\phi_0(E/T_c)^{1/3}=\frac{2\pi}{c_1}\frac{\phi_0}{[N(E)T_c]^{1/3}} \tag{3.128}$$

which makes precise the second relation in Eq. (3.115), while for $H_{c2}(T=0)$ we obtain

$$-H_{c2}(0)/(T_c(dH_{c2}/dT)_{T_c})=c_1/(4\gamma)^{2/3}\approx 1.24\,. \tag{3.129}$$

As was first noted in Refs. [21,22] this ratio for the system at the mobility edge is significantly larger than its classical value 0.69. In this case $H_{c2}(T)$ curve is concave for all temperatures below T_c [22]. Detailed expressions for the intermediate disorder when $2\pi T \ll \omega_c \ll \langle\omega\rangle$ can be found in Ref. [75].

In Fig. 14 we present the results of the numerical solution of Eq. (3.119) for different values of the characteristic frequency ω_c, i.e. for different disorders. A smooth crossover from the classical behavior of the theory of "dirty" superconductors [173, 9, 174] to anomalous temperature dependence close to the Anderson transition [22] is clearly seen.

Below the mobility edge (i.e. in the Anderson insulator) and for $\omega_c = 1/(2\pi^2 N(E)R_{loc}^3) \ll 2\pi T$, i.e. very close to the mobility edge we can again use the $\omega^{1/3}$ behavior of the diffusion coefficient and find the same temperature dependence of H_{c2} as at the mobility edge itself or just above it. For $2\pi T \ll \omega_c \ll 2\pi T_c$ Eq. (3.119) takes the form [75]

$$\ln\frac{T}{T_c}=\sum_{n=0}^{n_0-1}\{(n+\tfrac{1}{2})[1+(E/\omega_c)^{2/3}(\omega_H/E)]\}^{-1}$$
$$+\sum_{n=n_0}^{\infty}\{(n+\tfrac{1}{2})+(n+\tfrac{1}{2})^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)\}^{-1}-\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{-1}\,, \tag{3.130}$$

where $n_0=\omega_c/4\pi T_c$ corresponds to a change of frequency behavior of the diffusion coefficient. Defining $x=\omega_H/\omega_c^{2/3}E^{1/3}$ we can reduce Eq. (3.130) to

$$\ln(T/T_c)=x\ln(\gamma\omega_c/\pi T_c)+\tfrac{3}{2}(1+x)\ln(1+x) \tag{3.131}$$

which implicitly defines $H_{c2}(T)$ and shows [75] that now $H_{c2}(T)\to\infty$ for $T\to 0$ (logarithmic divergence). Numerical solution of Eq. (3.130) is shown in the inset in Fig. 14. Below we shall see, however, that this divergence of H_{c2} is lifted by the inverse influence of magnetic field upon diffusion.

Let us now turn to the problem of the influence of magnetic field upon diffusion and its consequences for H_{c2} temperature behavior. If we are far from the Anderson transition, magnetic

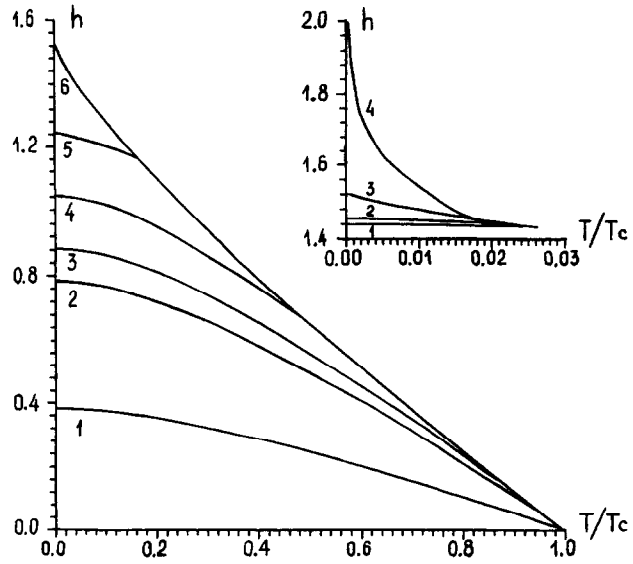

Fig. 14. Temperature dependence of the upper critical field H_{c2}. Numerical solution for the dependence of $h = \omega_H/T_c^{2/3}E_{1/3}$ on T/T_c for different values of $\theta = \omega_c/T_c$: (1) $\theta = 100$; (2) $\theta = 10$; (3) $\theta = 2\pi$; (4) $\theta = 3$; (5) $\theta = 1$; (6) $\theta = 0$ (mobility edge). Metallic state, no magnetic field influence on diffusion. At the inset: Low temperature dependence of h on T/T_c close to the Anderson transition; mobility edge ($\theta = 0$) with magnetic field influence on diffusion; metallic phase ($\theta = 0.1$), no magnetic field influence; mobility edge ($\theta = 0$), no magnetic field influence; and insulating phase ($\theta = 0.1$), no magnetic field influence. Numerical cut-off was taken at $\langle\omega\rangle = 100T_c$.

field influence is small on parameter $\sim\sqrt{\omega_H/E}$ and its influence upon H_{c2} is insignificant. Close to the transition, magnetic field correction may overcome the value of $D(H = 0)$ and we have to consider its influence in detail [75]. Accordingly, we shall limit ourselves to the case of $\omega_c/E \ll (\omega_H/E)^{2/3}$ for which we have already discussed the magnetic field behavior of the generalized diffusion coefficient in the Cooper channel. This was given in Eqs. (2.110) and (2.111). In this case, we have seen that the characteristic frequency ω_c is replaced by

$$\omega_c' = (\varphi\omega_H/E)^{3/2}E\,, \tag{3.132}$$

where $\varphi = \frac{1}{2}W^2 \approx 0.18$. ($W$ was defined during our discussion of localization in a magnetic field.) For $T \sim T_c$ there is no change in the slope of H_{c2} given by Eq. (3.128) as was noted already in Ref. [22]. Here we shall consider the case of $T \ll T_c$.

For $2\pi T > \omega_c'$ in all sums over Matsubara frequencies we can take $D(\omega) \sim \omega^{1/3}$ and actually we can neglect the magnetic field's influence upon diffusion. In this case, $H_{c2}(T)$ behaves as in Eq. (3.127) i.e. as at the mobility edge in the absence of magnetic field effects. For $2\pi T < \omega_c'$ the equation for $H_{c2}(T)$ takes the form [75]

$$\ln\frac{T}{T_c} = \sum_{n=0}^{n_0-1}\left[(n+\tfrac{1}{2}) + (\omega_c/E)^{1/3}(\omega_H/4\pi T)\right]^{-1} + \sum_{n=n_0}^{\infty}\left\{(n+\tfrac{1}{2}) + (n+\tfrac{1}{2})^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)\right\}^{-1} - \sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{-1} \tag{3.133}$$

where $n_0 = \omega_c'/4\pi T$. In this case we find

$$H_{c2}(T) = m\frac{\phi_0}{\pi}(1+\varphi)^{-1/3}(\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}\left[1 - \frac{4\gamma}{3\varphi^{1/3}(1+\varphi)}\frac{T}{T_c}\right]. \tag{3.134}$$

Accordingly, we have

$$-\frac{H_{c2}(0)}{T_c(dH_{c2}/dT)_{T_c}} = (1+\varphi)^{-1/3}\frac{c_1}{(4\gamma)^{2/3}} \approx 1.18 \tag{3.135}$$

and the change when compared with Eq. (3.129) is actually small. However, for $2\pi T < \omega_c'$ the $H_{c2}(T)$ curve becomes convex. The inflexion point can be estimated as $T^* = \omega_c'/2\pi \approx 0.02T_c$. This behavior is shown in the inset of Fig. 14.

Consider now the insulating region. We shall see that the magnetic field effects on diffusion lead to the effective cut-off of the weak divergence of H_{c2} as $T \to 0$ noted above. Generalized diffusion coefficient D_2 in the insulating phase and at low enough frequencies is determined by the following equation [75]:

$$2mD_2 = -(\omega_c/E)^{1/3} + (-i\omega/E)^{1/2}(2mD_2)^{-1/2} + \tfrac{1}{2}W(2\omega_H/E)^{1/2}. \tag{3.136}$$

Now we can see that the external field defined by

$$\frac{W}{2}\sqrt{2\omega_H/E} > (\omega_c/E)^{1/3} \tag{3.137}$$

transfers the system from the insulating to the metallic state. If the system remains close to the mobility edge we can estimate the upper critical field as above by $\omega_H \approx (\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}$ and Eq. (3.137) reduces to

$$\omega_c \approx \frac{1}{2\pi^2 N(E)R_{loc}^3} < \frac{\pi}{\gamma}(W/\sqrt{2})^3 T_c \approx 0.14T_c \tag{3.138}$$

and practically in the entire interval of localization lengths, where according to our main criterion of Eq. (3.20) we can have superconductivity in an Anderson insulator, *the upper critical field in fact destroys localization* and the system becomes metallic. Accordingly there is no way to observe the divergence of the upper critical field as $T \to 0$ and the $H_{c2}(T)$ curves in the "insulating" phase all belong to the region between the curves of $H_{c2}(T)$ at the mobility edge defined in the absence of magnetic field (curve 3 in the inset on Fig. 14) and at the mobility edge defined in the magnetic field (curve 1 in the inset). This result actually shows that it may be difficult to confirm the insulating ground state of a strongly disordered superconducting system just by applying a strong enough magnetic field to destroy superconductivity and perform the usual transport measurements at low temperatures.

Note that another mechanism for the change of $H_{c2}(T)$ at low temperatures was proposed by Coffey et al. [175]. They have found the enhancement of H_{c2} at low temperatures due to the magnetic field dependence of the Coulomb pseudopotential μ^* which appears via the magnetic field dependence of the diffusion coefficient. Magnetic field suppression of localization effects leads to the reduction of Coulomb pseudopotential enhancement due to these effects [20]. Accordingly,

we get the enhancement of H_{c2} at low temperatures. Unfortunately, the apparently more important effects of the frequency dependence of the generalized diffusion coefficient were dropped.

Returning to the general criteria for the validity of Eq. (3.119) we note that the condition of $R_H \gg \xi$ is reduced to $\omega_H \ll T_c^{1/3} E_F^{2/3}$ which is obviously satisfied in any practical case. Note, however, that our estimates for H_{c2} at low temperatures lead to $\omega_H = \Delta_0 (E_F/\Delta_0)^{1/3} > \Delta_0$ which can easily overcome the paramagnetic limit. In this case, the experimentally observed H_{c2} of course will be determined by the paramagnetic limit and the anomalous behavior due to localization will be unobservable at low temperatures. At the same time in case of H_{c2} being determined by the paramagnetic limit it may become possible to obtain the insulating ground state of the system by applying a strong enough magnetic field. Note that the effective masses entering to cyclotron frequency and the paramagnetic splitting may actually be very different and there may be realistic cases when the orbital critical field may dominate at low T. For $T \sim T_c$, H_{c2} is always determined by the orbital contribution.

Similar analysis can be performed for the two-dimensional and quasi-two-dimensional cases [81], which are important mainly due to the quasi-two-dimensional nature of high-temperature superconductors. We shall limit ourselves only to the case of a magnetic field perpendicular to the highly conducting planes, when the temperature dependence of $H_{c2}(T)$ is again determined by Eq. (3.119) with $D_2(\omega)$ having the meaning of a diffusion coefficient in the Cooper channel along the plane.

If we neglect the magnetic field influence upon diffusion the frequency dependence of the diffusion coefficient in the purely two-dimensional case is determined by Eq. (2.115). It is easy to see that the possible anomalies in the temperature behavior of the upper critical field due to the frequency dependence of the diffusion coefficient will appear only at temperatures $T \ll e^{-1/\lambda}/\tau$. At higher temperatures we obtain the usual dependence of the "dirty" limit. Accordingly, from Eq. (3.119) we obtain two different types of behavior of $H_{c2}(T)$:

1. For $T_c \gg e^{-1/\lambda}/\tau$

$$H_{c2}(T) = \frac{4}{\pi^2} \frac{\phi_0}{D_0} T \ln\left(\frac{T_c}{T}\right) \quad \text{for } T \sim T_c\,, \tag{3.139}$$

$$H_{c2}(T) = \frac{1}{2\gamma} \frac{\phi_0 T_c}{D_0} \left(1 - 2.12 \left(\frac{T}{T_c}\right)^2\right) \quad \text{for } \frac{e^{-1/\lambda}}{\tau} \ll T \ll T_c\,. \tag{3.140}$$

For $T \ll e^{-1/\lambda}/\tau$ the upper critical field is defined by the equation:

$$\ln\left(\frac{\gamma}{2\pi} \frac{e^{-1/\lambda}}{\tau T}\right) = \left(1 + 4\pi \frac{D_0}{\phi_0} \frac{\tau H_{c2}}{e^{-1/\lambda}}\right) \ln\left(\frac{\gamma}{2\pi} \frac{e^{-1/\lambda}}{\tau T_c} \left(1 + 4\pi \frac{D_0}{\phi_0} \frac{\tau H_{c2}}{e^{-1/\lambda}}\right)\right) \tag{3.141}$$

from which we can explicitly obtain the dependence of $T(H_{c2})$.

Thus, up to very low temperatures of the order of $\sim e^{-1/\lambda}/\tau$, the upper critical field is determined by the Drude diffusion coefficient and we obtain the standard $H_{c2}(T)$ dependence of a "dirty" superconductor. The ratio $-H_{c2}(T)/[T_c(dH_{c2}/dT)]_{T_c}$ for $e^{-1/\lambda}/\tau \ll T \ll T_c$ is equal to the usual value of 0.69. For low temperatures $T \ll e^{-1/\lambda}/\tau$ we obtain significant deviations from the predictions of the usual theory of "dirty" superconductors. $H_{c2}(T)$ dependence acquires positive

curvature and the upper critical field diverges as $T \to 0$. The behavior of the upper critical field for the case of $T_c \gg e^{-1/\lambda}/\tau$ is shown in Fig. 15, curve 1.

2. For $T_c \ll e^{-1/\lambda}/\tau$ the upper critical field behavior for any temperature is defined by Eq. (3.141). $H_{c2}(T)$-dependence acquires positive curvature and H_{c2} diverges for $T \to 0$. For small fields $H_{c2} \ll (\phi_0/D_0)\, e^{-1/\lambda}/\tau$, i.e. for $T \sim T_c$, Eq. (3.141), gives the explicit expression for H_{c2}:

$$H_{c2} = \frac{1}{4\pi}\frac{\phi_0}{D_0}\frac{e^{-1/\lambda}}{\tau}\frac{\ln\left(\dfrac{T_c}{T}\right)}{\ln\left(\dfrac{\gamma}{2\pi}\dfrac{e^{-1/\lambda}}{\tau T}\right)}. \tag{3.142}$$

The slope of $H_{c2}(T)$ at the superconducting transition is determined by

$$-\frac{\sigma_0}{N(E)}\left(\frac{dH_{c2}}{dT}\right)_{T_c} = \frac{e^2}{2\pi}\phi_0\frac{e^{-1/\lambda}}{\tau T_c \ln\left(\dfrac{\gamma}{2\pi}\dfrac{e^{-1/\lambda}}{\tau T_c}\right)}. \tag{3.143}$$

The behavior of the upper critical field for the case of $T_c \ll e^{-1/\lambda}/\tau$ is shown in Fig. 16, curve 1.

It is clearly seen from Eqs. (2.115) and (2.116) that the magnetic field's influence upon diffusion becomes relevant only for high enough magnetic fields $H_{c2} \gg (\phi_0/D_0)(e^{-1/\lambda}/\tau)$, i.e. for very low

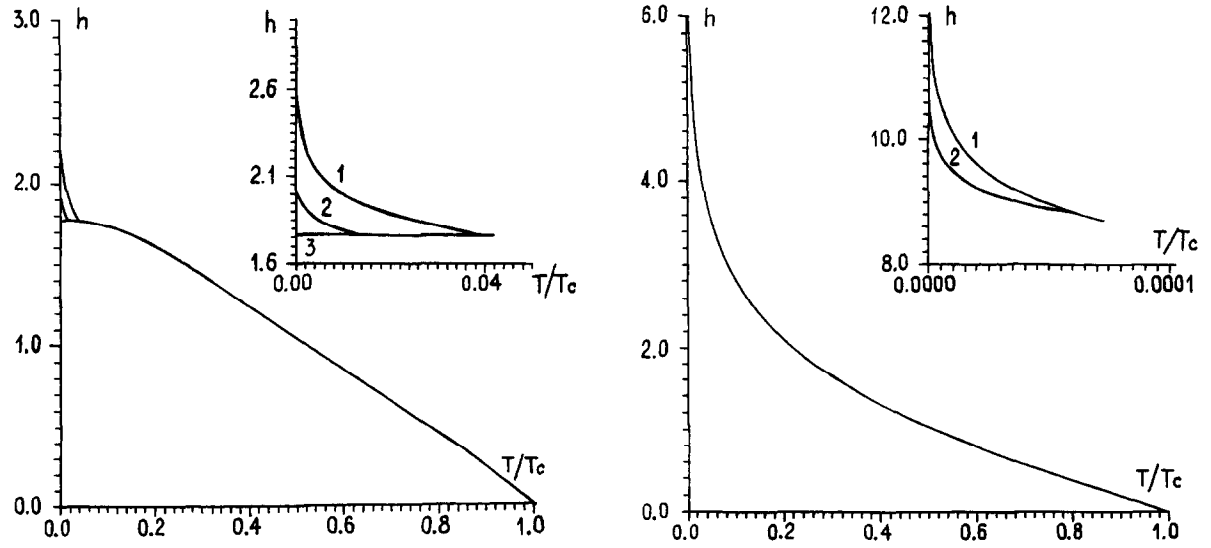

Fig. 15. Temperature dependence of the upper critical field for the two-dimensional superconductor $(e^{-1/\lambda}/\tau T_c) = 0.4$, $\lambda = 0.1$, $h = \omega_H/\pi\lambda T_c$): (1) no magnetic field influence upon diffusion, (2) with magnetic field influence upon diffusion and (3) standard theory of "dirty" superconductors.

Fig. 16. Temperature dependence of the upper critical field for the two-dimensional superconductor $(e^{-1/\lambda}/\tau T_c = 4$, $\lambda = 0.126$, $h = \omega_H/\pi\lambda T_c$): (1) no magnetic field influence upon diffusion and (2) with magnetic field influence upon diffusion.

temperatures $T \ll T_c$. If we use Eq. (2.116) in the main equation (3.119), we obtain the following results:

1. The case of $e^{-1/\lambda}/\tau \ll T_c \ll 1/\tau$: For high enough temperatures $T \gg e^{-1/\lambda}/\tau$ the diffusion coefficient entering Eq. (3.119) coincides with Drude's D_0 and the upper critical field is determined by Eqs. (3.139) and (3.140).

For $\exp[-1/\lambda^2 \ln(\gamma^2/\pi)(1/\tau T_c)]/\tau \ll T \ll e^{-1/\lambda}/\tau$ we obtain:

$$H_{c2}(T) = (1/2\gamma)(\phi_0 T_c/D_0)(1 - 3.56(T/T_c)) . \tag{3.144}$$

Eq. (3.144) differs from Eq. (3.140) only by the temperature dependent corrections and we can say that the magnetic field's influence upon diffusion in this case leads to the widening of the temperature region where we can formally apply the usual theory of "dirty" superconductors.

For $T \ll \exp[-1/\lambda^2 \ln(\gamma^2/\pi)(1/\tau T_c)]/\tau \ll T \ll e^{-1/\lambda}/\tau$ the upper critical field is defined by

$$\ln\left(\frac{\gamma}{2\pi e}\frac{e^{-1/\lambda^2 \ln Q}}{\tau T}\right) = \frac{2\gamma}{Q}\frac{\lambda \ln Q}{e^{-1/\lambda^2 \ln Q}} \ln\left(\frac{\gamma^2}{\pi Q}\frac{1}{\tau T_c}\right), \tag{3.145}$$

where $Q = (\gamma/2\pi)(\phi_0/D_0 H_{c2})(1/\tau)$. From Eq. (3.145) we can obtain the explicit dependence $T(H_{c2})$. The upper critical field in this case is slightly concave as in Eq. (3.141) where we have neglected the magnetic field influence upon diffusion. However, now we have no divergence of H_{c2} for $T \to 0$ and

$$H_{c2}(T = 0) = (\gamma/2\pi)(\phi_0/D_0)(1/\tau) . \tag{3.146}$$

In fact, the value of $H_{c2}(T = 0)$ will be even smaller, because for these values of the field the number of Landau levels below the cutoff will be of the order of unity and we are now outside the limits of applicability of Eqs. (2.113). However, the order of magnitude of $H_{c2}(T = 0)$ given by Eq. (3.146) is correct. $H_{c2}(T)$ behavior with the influence of magnetic field upon diffusion is shown in Fig. 15, curve 2.

2. The case of $T_c \ll e^{-1/\lambda}/\tau$: For small fields $H_{c2} \ll (\phi_0/D_0)(e^{-1/\lambda}/\tau)$, i.e. for $T \sim T_c$, the magnetic field's influence upon diffusion is irrelevant and the upper critical field is determined by Eq. (3.142). For low temperatures, $H_{c2}(T)$ is determined by Eq. (3.145), i.e. the magnetic field's influence upon diffusion liquidates the divergence of the upper critical field as $T \to 0$. The behavior of $H_{c2}(T)$ for $T_c \ll e^{-1/\lambda}/\tau$ is shown in Fig. 16, curve 2.

It should be noted that the case of $T_c \ll e^{-1/\lambda}/\tau$ is possible only for a sufficiently strong disorder. For typical $T_c \sim 10^{-4} E_F$, this case can occur only for $\lambda > 0.2$. Superconducting pairing can exist only when a condition similar to Eq. (3.20) is satisfied. In the two-dimensional case, this condition leads to the inequality $T_c \gg \lambda e^{-1/\lambda}/\tau$ which makes the region under discussion rather narrow.

The quasi-two-dimensional case was extensively discussed in Ref. [81]. The situation here is in many respects similar to that of two-dimensions, e.g. the anomalies in the upper critical field behavior due to the frequency dependence of the diffusion coefficient appear only for temperatures $T \ll e^{-1/\lambda}/\tau$, while at higher temperatures $H_{c2}(T)$ is well described by the usual theory of "dirty" superconductors. As the interplane transfer integral w grows, the smooth transition from the purely two-dimensional behavior to that of a three-dimensional isotropic system can be demonstrated. When $T_c \gg e^{-1/\lambda}/\tau$, deviations from the usual temperature behavior of H_{c2} is observed only for very low temperatures $T \ll e^{-1/\lambda}/\tau$, while close to T_c there are no significant changes from the standard dependence of $H_{c2}(T)$. For $T_c \ll e^{-1/\lambda}/\tau$, as the interplane transfer integral w grows, the

temperature dependence of $H_{c2}(T)$ changes from a purely two-dimensional concave behavior for all temperatures to a convex three-dimensional like dependence. In Fig. 17 we show the typical transformations of the $H_{c2}(T)$ behavior, as the transfer integral w changes, driving the system through the metal–insulator transition [81]. This clearly demonstrates the sharp anomalies in the H_{c2} behavior which can appear due to localization effects.

3.4. Fluctuation conductivity near the Anderson transition

Fluctuation conductivity of Cooper pairs (above T_c) is especially interesting in strongly disordered systems because the usual single-particle contribution to conductivity drops to zero as the system moves towards the Anderson transition. We shall use the standard approach [176] which takes into account fluctuational Cooper pairs formation above T_c. We assume that it is possible to neglect the so-called Maki–Thompson correction which describes the increased one-particle contribution to conductivity due to superconducting fluctuations [177]. We expect that these estimates [178] will enable us to find a correct scale of fluctuation conductivity close to the mobility edge.

Consider first the averaged fluctuation propagator:

$$L^{-1}(\boldsymbol{q}, \Omega_k) = \lambda^{-1} - \Pi(\boldsymbol{q}, \Omega_k)\,, \tag{3.147}$$

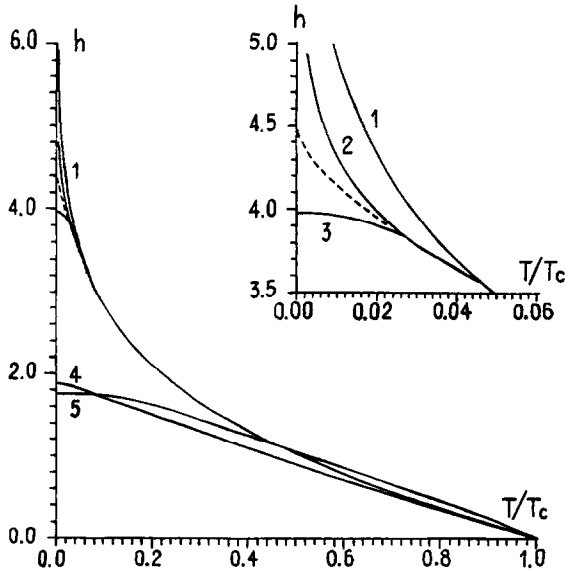

Fig. 17. Temperature dependence of the upper critical field for a quasi-two-dimensional superconductor ($e^{-1/\lambda}/\tau T_c = 4$, $\lambda = 0.126$, $h = \omega_H/\pi\lambda T_c$) for different values of the interplane transfer integral around the critical value of w_c corresponding to Anderson transition at a given disorder: (1) purely two-dimensional behavior ($w = 0$), (2) dielectric side close to Anderson transition ($L = |2\ln(w/w_c)| = 0.7$), (3) metallic side close to Anderson transition ($L = 2\ln(w/w_c) = 0.7$), and (4) metallic state far from the Anderson transition ($L = 3$). Dashed line represents the behavior at the Anderson transition ($L = 0$).

where the polarization operator

$$\Pi(\boldsymbol{q},\Omega_k) = \sum_{\varepsilon_n}\sum_{pp'} \langle G(\boldsymbol{p}_+\boldsymbol{p}'_+\varepsilon_n + \Omega_k)G(\boldsymbol{p}_-\boldsymbol{p}'_- - \varepsilon_n)\rangle$$
$$= -2i\pi T \sum_{\varepsilon_n} \Phi_E(\boldsymbol{q},\omega_m = -2\varepsilon_n + \Omega_k)\,, \quad \omega_m = 2\pi m T\,. \tag{3.148}$$

During our analysis of Ginzburg–Landau coefficients we were interested in $\omega_m = 2\varepsilon_n$, so that one of the Green's functions in Φ_E was automatically retarded, while the other was advanced. Now we need a more general expression of Eq. (3.148) with $\omega_m = 2\varepsilon_n + \Omega_k$. Accordingly, instead of Eq. (3.83) we must use the following expression with an additional θ-function:

$$\Phi_E(\boldsymbol{q},\omega_m = 2\varepsilon_n + \Omega_k) = -\frac{N(E)\theta[\varepsilon_n(\varepsilon_n + \Omega_k)]}{i|2\varepsilon_n + \Omega_k| + iD_E(|2\varepsilon_n + \Omega_k|)q^2}\,, \tag{3.149}$$

where the generalized diffusion coefficient is again determined by Eqs. (3.85) and (3.86). From Eqs. (3.147)–(3.149), performing summation over ε_n we get the following form of the fluctuation propagator for small $\boldsymbol{q}$ ($D_E q^2 < T$):

$$L^{-1}(\boldsymbol{q},\Omega_k) = -N(E)\left\{\ln\frac{T}{T_c} + \psi\left(\frac{1}{2} + \frac{|\Omega_k|}{4\pi T}\right) - \psi\left(\frac{1}{2}\right) + \eta(|\Omega_k|)q^2\right\}, \tag{3.150}$$

where

$$\eta(|\Omega_k|) = 4\pi T \sum_{n=0}^{\infty} \frac{D_E(2\varepsilon_n + |\Omega_k|)}{(2\varepsilon_n + |\Omega_k|)^2}$$
$$= \begin{cases} \dfrac{D_E}{4\pi T}\psi'\left(\dfrac{1}{2} + \dfrac{|\Omega_k|}{4\pi T}\right), & \xi_{\mathrm{loc}} < (\xi_0 l^2)^{1/3}\,,\ E > E_c\,, \\ \dfrac{D_0}{(4\pi T)^{2/3}(2\gamma)^{1/3}}\zeta\left(\dfrac{5}{3};\dfrac{1}{2} + \dfrac{|\Omega_k|}{4\pi T}\right), & \xi_{\mathrm{loc}} > (\xi_0 l^2)^{1/3}\,. \end{cases} \tag{3.151}$$

It is also useful to know the form of the fluctuation propagator for $|\Omega_k| \gg T$. In this case, close to the Anderson transition, we may replace the sum over ε_n in Eq. (3.148) by an integral, while far from the transition it can be calculated exactly. As a result, we get

$$L^{-1}(\boldsymbol{q},\Omega_k) =$$
$$-N(E)\begin{cases} \ln\dfrac{T}{T_c} + \psi\left(\dfrac{1}{2} + \dfrac{|\Omega_k|}{4\pi T} + \dfrac{D_E q^2}{4\pi T}\right) - \psi\left(\dfrac{1}{2}\right), & \xi_{\mathrm{loc}} < (\xi_0 l^2)^{1/3}\,,\ E > E_c\,, \\ \ln\dfrac{T}{T_c} + \dfrac{3}{2}\ln\left[\left(\dfrac{|\Omega|}{4\pi T}\right)^{2/3} + \dfrac{D_0 q^2}{(4\pi T)^{2/3}(2\gamma)^{1/3}}\right], & \xi_{\mathrm{loc}} > (\xi_0 l^2)^{1/3}\,. \end{cases} \tag{3.152}$$

Diagrams determining fluctuation conductivity are shown in Fig. 18. Contributions of graphs, Fig. 18(a) and (b), are nonsingular close to T_c because at least one of the fluctuation propagators transfers a large momentum of the order of p_F. Thus we have to consider independent contributions $\boldsymbol{B}$ formed by three Green's functions. We can calculate these contributions using the usual approximations of the self-consistent theory of localization taking into account the renormalization of triangular vertices by maximally-crossed graphs [98, 155] (cf. Eq. (3.40)) as in Fig. 18(c).

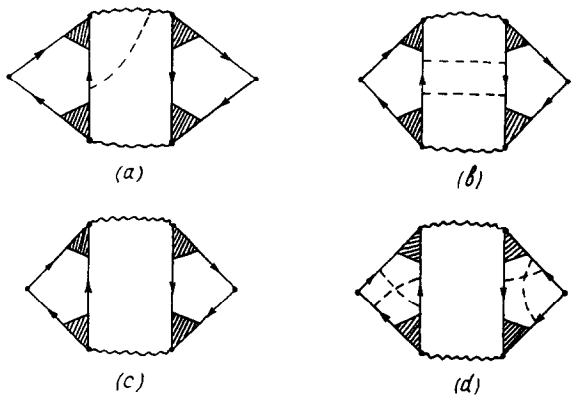

Fig. 18. Diagrams for fluctuation conductivity. Wavy lines denote the fluctuation propagator, and dashed lines represent disorder scattering.

We shall neglect graphs like the ones in Fig. 18(d) where the topology of the disorder scattering lines is not reduced to the renormalization of triangular vertices. We assume that these approximations are sufficient at least for a qualitative inclusion of localization effects. Note that it is sufficient to calculate the contribution of three Green's functions $\boldsymbol{B}(\boldsymbol{q}, \Omega_k, \omega_m)$ for small $\boldsymbol{q}$ and zero external frequency $\omega_m = 0$. It can be easily found by differentiating the polarization operator of Eq. (3.148)

$$\boldsymbol{B}(\boldsymbol{q}, \Omega_k, 0) = \boldsymbol{q}C = -\frac{\partial}{\partial \boldsymbol{q}} \Pi(\boldsymbol{q}, \Omega_k) \,. \tag{3.153}$$

The contribution of diagram of Fig. 18(c) to the operator of electromagnetic response [58] is determined by the following expression:

$$Q_{\alpha\beta} = -\frac{4e^2 T}{m^2} \sum_{\Omega_k} \int \frac{\mathrm{d}^3 \boldsymbol{q}}{(2\pi)^3} (Cq_\alpha)(Cq_\beta) L(\boldsymbol{q}, \Omega_k) L(\boldsymbol{q}, \Omega_k + \omega_m) \,. \tag{3.154}$$

Close to T_c we can also neglect the dependence of C on Ω_k. Then C reduces to Eq. (3.89) and we have $C = N(E)\xi^2$. Fluctuation propagator analytically continued to the upper halfplane of complex ω takes the usual form

$$L(\boldsymbol{q}, \omega) = -\frac{1}{N(E)[(T - T_c)/T_c] - (\mathrm{i}\pi\omega/8T_c) + \xi^2 q^2} \,. \tag{3.155}$$

Further calculations can be performed in a standard way and for fluctuation conductivity for $(T - T_c)/T_c \ll 1$ we get the usual result [176]:

$$\sigma_{AL} = (e^2/32\xi\hbar)(T_c/(T - T_c))^{1/2} \tag{3.156}$$

but with the coherence length ξ being defined as (cf. Eq. (3.89)):

$$\xi = \begin{cases} \left(\dfrac{\xi_0 l}{p_F \xi_{loc}}\right)^{1/2}, & \xi_{loc} < (\xi_0 l^2)^{1/3}, \quad E > E_c \,, \\ (\xi_0 l^2)^{1/3} \sim (\xi_0 / p_F^2)^{1/3}, & \xi_{loc} > (\xi_0 l^2)^{1/3}, \quad E \sim E_c \,. \end{cases} \tag{3.157}$$

From these estimates we can see that as the system approaches the Anderson transition a temperature interval, where the fluctuation contribution to conductivity is important, widens. Fluctuation Cooper pair conductivity becomes comparable to that for a single particle for $\sigma < \sigma^* \approx \sigma_c(p_F\xi_0)^{-1/3} \approx \sigma_c(T_c/E_F)^{1/3}$, i.e. close enough to the mobility edge. In fact this confirms the above picture of Cooper pairs remaining delocalized while single-particle excitations localize as the system undergoes the metal–insulator transition.

It is not difficult to find also the fluctuation contribution to diamagnetic susceptibility [178]. Close to T_c it is determined by a standard expression:

$$\chi_{fl} = \frac{e^2 T_c}{6\pi c^2}\,\xi\left(\frac{T_c}{T - T_c}\right)^{1/2}, \tag{3.158}$$

where the coherence length is again defined as in Eq. (3.157).

Thus our expressions for fluctuation effects follow more or less obviously from our general picture of the Ginzburg–Landau expansion: for systems close to the Anderson transition we have to only replace the usual coherence length $\sqrt{\xi_0 l}$ of a "dirty" superconductor by $\xi \sim (\xi_0 l^2)^{1/3} \sim (\xi_0/p_F)^{1/3}$.

3.5. *Superconductivity in an Anderson insulator at $T = 0$*

We have already considered the superconducting response of a system which is close to the Anderson transition within the Ginzburg–Landau approximation, i.e. for temperatures $T \sim T_c$. In fact it is not difficult to obtain similar results also for $T = 0$ [25].

Superconducting current density at $T = 0$ is given by [9]:

$$\boldsymbol{j}_s = -\frac{n_s e^2}{mc}\boldsymbol{A}, \tag{3.159}$$

where n_s is density of the superconducting electrons and $\boldsymbol{A}$ is the vector potential of an external magnetic field. On the other hand, using exact eigenstates representation DeGennes has obtained the following beautiful relation between superconducting response at $T = 0$ and the conductivity of a system in the normal state [9,25]:

$$\boldsymbol{j}_s = \left\{\frac{1}{2\pi c}\int d\xi \int d\xi'\, L(\xi,\xi')\,\mathrm{Re}\,\sigma(\xi - \xi') - \frac{ne^2}{mc}\right\}\boldsymbol{A}. \tag{3.160}$$

All characteristics of a superconducting state are contained here in the kernel:

$$L(\xi,\xi') = \frac{1}{2}\,\frac{EE' - \xi\xi' - \Delta_0^2}{EE'(E + E')}, \tag{3.161}$$

where $E = \sqrt{\xi^2 + \Delta_0^2}$ and Δ_0 is the superconducting gap at $T = 0$. Note that in the normal state $\boldsymbol{j}_s = 0$ and we can rewrite Eq. (3.160) as

$$\boldsymbol{j}_s = \frac{1}{2\pi c}\int d\xi \int d\xi' [L(\xi,\xi')|_{\Delta=\Delta_0} - L(\xi,\xi')|_{\Delta=0}]\,\mathrm{Re}\,\sigma(\xi - \xi')\,\boldsymbol{A}, \tag{3.162}$$

Taking into account that for large $|\xi - \xi'|$ $L(\xi,\xi')|_{\Delta=\Delta_0} - L(\xi,\xi')|_{\Delta=0}$ drops as $|\xi - \xi'|^{-3}$ it is sufficient to know only the low-frequency response of a system in normal state. In particular, for a "pure" system (with no scattering) we have $\mathrm{Re}\,\sigma(\omega) = (ne^2/m)\pi^{-1}\delta(\omega)$ and comparing Eq. (3.159) with Eq. (3.162) it is immediately clear that at $T = 0$ we have $n_s = n$, i.e. in an ideal system all electrons are superconducting.

Close to the Anderson transition we can use the results of elementary scaling theory of localization, e.g. Eqs. (2.31) and (2.33) to write

$$\sigma(\omega) \approx \begin{cases} A\,\dfrac{g_c}{\xi_{loc}}\,, & \omega < \omega_c\,, \\ A\,\dfrac{g_c}{\xi_{loc}}\left(\dfrac{\omega}{\omega_c}\right)^{1/3}, & \omega > \omega_c\,, \end{cases} \tag{3.163}$$

where $\omega_c \sim [N(E)\xi_{loc}^3]^-$ is defined in Eq. (2.42), g_c is the critical conductance of scaling theory ($g_c \sim 1$) $A \sim 1$. From Eqs. (3.161) and (3.162) it is clear that the main contribution to the integral in Eq. (3.162) comes from $|\xi - \xi'| \sim \Delta_0$, so that the value of n_s depends on the relation between Δ_0 and ω_c. For $\Delta_0 < \omega_c$ we have $\sigma(\Delta_0) = Ag_c/\xi_{loc}$ and

$$n_s = A(m/e^2)\Delta_0(g_c/\xi_{loc})\,. \tag{3.164}$$

For $\Delta_0 > \omega_c$ we have $\sigma(\Delta_0) = Ag_c[N(E)\Delta_0]^{1/3}$ and it becomes independent on further growth of ξ_{loc} in the region of $\xi_{loc} > [N(E)\Delta_0\,]^{1/3}$. Accordingly, n_s does not vanish at the mobility edge but saturates at

$$n_s = A(m/e^2)g_c[N(E)\Delta_0]^{1/3}\,. \tag{3.165}$$

In the localization region we can write instead of Eq. (3.163)

$$\sigma(\omega) \approx \begin{cases} 0\,, & \omega < \omega_c \\ Ag_c[N(E)\omega]^{1/3}\,, & \omega > \omega_c \end{cases} \tag{3.166}$$

which again leads to $\sigma(\Delta_0) \approx Ag_c[N(E)\Delta_0]^{-1/3}$ and Eq. (3.165) remains valid until $R_{loc} > [N(E)\Delta_0]^{-1/3}$. Thus the density of superconducting electrons n_s remains finite close to the Anderson transition both in the metallic and insulating states.

However, from Eq. (3.165) it is easy to see that close to the Anderson transition

$$n_s/n \sim (\Delta_0/E_F)^{4/3}\,. \tag{3.167}$$

This coincides with an estimate of Eq. (3.98) based upon the Ginzburg–Landau expansion. For typical Δ_0 and E_F only a small part ($\sim 10^{-4}$ in traditional superconductors) of conduction electrons form Cooper pairs. The condition of $R_{loc} > [N(E)\Delta_0]^{-1/3} \sim a(E_F/\Delta_0)^{1/3}$ as discussed above defines the size of the possible superconducting region in an Anderson insulator. This region is of course quite small, e.g. if metal–insulator transition takes place with a change of some external parameter x (impurity concentration, pressure, fluence of fast neutrons, etc.), so that $R_{loc} \sim a|(x - x_c)/x_c|^{-\nu}$, then for $\nu \approx 1$ and typical $E_F/\Delta_0 \sim 10^4$ we get $|x - x_c| < 0.1x_c$.

These estimates are in complete accordance with the results of our discussion of Ginzburg–Landau approximation [21,22] and we can obtain the qualitative picture of superconductivity in an Anderson insulator both for $T \sim T_c$ and $T \to 0$, i.e. in the ground state.

4. Statistical fluctuations of the superconducting order parameter

The previous discussion of superconductivity in a strongly disordered system is based upon the important assumption of the existence of a self-averaging superconducting order-parameter Δ. This assumption was first used in the theory of "dirty" superconductors [10,11,13,9] and also in all early papers on the interplay of localization and superconductivity. It was expected that spatial fluctuations of this order parameter $\Delta(\boldsymbol{r})$ are actually small and we can always use some disorder averaged parameter $\langle\Delta(\boldsymbol{r})\rangle$. It seems natural for $\sigma \gg \sigma_e$ and it really can be justified in this region as we shall see below. However, close to the mobility edge there are no special reasons to believe in the correctness of this assumption. In this case electronic characteristics of the system become strongly fluctuating and we shall see that these lead to the strong spatial (*statistical*) fluctuations of the superconducting order parameter, or even to the regime of inhomogeneous superconductivity. At the same time, we must stress that these fluctuations are in some sense similar to the usual thermodynamic critical fluctuations of the order parameter and become important in some new critical region (we call it the statistical critical region) close to T_c. In this sense, all the previous analysis is just a kind of statistical mean-field approximation and of course it is a necessary step for further studies taking into account the statistical fluctuations. The importance of these fluctuations is stressed by the fact that the statistical critical region widens (similarly to the usual critical region) as the system goes to the Anderson transition and apparently the role of fluctuations becomes decisive for the physics of the interplay of localization and superconductivity.

4.1. Statistical critical region

Here we shall start by a demonstration of the appearance of the new type of fluctuations which are at least of the same importance as the usual critical fluctuations of the superconducting order-parameter. We call them statistical fluctuations [62] and their nature is closely connected to the problem of self-averaging properties of this order parameter (i.e. with a possibility of decoupling which transforms Eq. (3.12) into Eq. (3.13)). We shall more or less follow Ref. [62], equivalent results were recently obtained in Ref. [179].

Let us return to the Eq. (3.7) and analyze the situation in more details. We shall use a simple iteration procedure assuming that fluctuations of the kernel $K(\boldsymbol{r}\boldsymbol{r}')$ due to disorder are small. Similar approach was first used in Ref. [180]. In this case we can represent $K(\boldsymbol{r}\boldsymbol{r}')$ and $\Delta(\boldsymbol{r})$ as

$$K(\boldsymbol{r}\boldsymbol{r}') = K_0(\boldsymbol{r}-\boldsymbol{r}') + K_1(\boldsymbol{r}\boldsymbol{r}'); \quad K_0(\boldsymbol{r}-\boldsymbol{r}') = \langle K(\boldsymbol{r}\boldsymbol{r}')\rangle\,, \quad \Delta(\boldsymbol{r}) = \langle\Delta\rangle + \Delta_1(\boldsymbol{r})\,, \tag{4.1}$$

where $\langle\Delta\rangle$ is the solution of the linearized gap equation with averaged kernel $K_0(\boldsymbol{r}-\boldsymbol{r}')$ while $\Delta_1(\boldsymbol{r})$ is the first order correction for the perturbation defined by $K_1(\boldsymbol{r}\boldsymbol{r}')$. We have seen that the linearized gap equation (Eq. (3.13)) with the averaged kernel $K_0(\boldsymbol{r}-\boldsymbol{r}')$ determines the standard transition temperature of BCS theory given by Eq. (3.19) which we shall now denote as T_{c0}. In the first order over K_1 there is no correction to T_{c0}: $\langle K_1\rangle = 0$. In the second order of this perturbation theory we obtain the following change of transition temperature, defined as the temperature of appearance of the homogeneous order-parameter:

$$\frac{T_c - T_{c0}}{T_{c0}} = \frac{1}{\lambda_p}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\,\frac{K_1(\boldsymbol{q}_0)K_1(0\boldsymbol{q})}{1-K_0(\boldsymbol{q},T_c)}\,, \qquad K_0 = \int d\boldsymbol{r}\, e^{i\boldsymbol{q}\boldsymbol{r}} K(\boldsymbol{r},T_c)\,, \tag{4.2}$$

where

$$K_1(0q) = K_1(-q0) = \int dr \int dr' e^{iqr}[K(rr') - K_0(r-r')]$$

$$= \lambda_p \int_0^{\langle\omega\rangle} \frac{dE}{E} \tanh\left(\frac{E}{2T_c}\right) \int dr\, e^{iqr}\left[\frac{1}{N(E)} \sum_\mu |\phi_\mu(r)|^2 \delta(E-\varepsilon_\mu) - 1\right]. \tag{4.3}$$

Here $\lambda_p = gN(E_F)$ and we have used the completeness and orthonormality of the exact eigenfunctions $\phi_\mu(r)$. It is obvious that correction to T_{c0} given by Eq. (4.2) is always positive. After averaging Eq. (4.2) over disorder we get the relative change of the transition temperature due to fluctuations as

$$\frac{\delta T_c}{T_{c0}} = \left\langle \frac{T_c - T_{c0}}{T_{c0}} \right\rangle = \lambda_p \int \frac{d^3q}{(2\pi)^3} \frac{\varphi(q)}{1 - K_0(q, T_c)}, \quad \varphi(q) = \int dr\, e^{iqr} \varphi(r), \tag{4.4}$$

where

$$\varphi(r) = \int_0^{\langle\omega\rangle} \frac{dE}{E} \tanh\left(\frac{E}{2T_c}\right) \int_0^{\langle\omega\rangle} \frac{dE'}{E'} \tanh\left(\frac{E'}{2T_c}\right) \left\{\frac{1}{N(E)} \langle\!\langle \rho_E(r)\rho_{E'}(0)\rangle\!\rangle^H - 1\right\} \tag{4.5}$$

and we have introduced the spectral density of Eq. (A.3):

$$\langle\!\langle \rho_E(r)\rho_{E'}(r')\rangle\!\rangle^H = \frac{1}{N(E)} \left\langle \sum_{\mu\nu} |\phi_\mu(r)|^2 |\phi_\nu(r')|^2 \delta(E-\varepsilon_\mu)\delta(E'-\varepsilon_\nu) \right\rangle \tag{4.6}$$

which is actually a correlation function of the local densities of states.

Remember now that in a "dirty" system [174]:

$$1 - K_0(q, T) = 1 - 2\pi T \lambda_p \sum_{\varepsilon_n} \frac{1}{2|\varepsilon_n| + D_E(2|\varepsilon_n|)q^2}$$

$$\approx \lambda_p \left[\frac{T - T_{c0}}{T_{c0}} + \xi^2 q^2\right], \quad \varepsilon_n = (2n+1)\pi T, \tag{4.7}$$

where ξ is the coherence length defined previously, e.g. in Eq. (3.89). The approximate equality here is valid for $|T - T_{c0}|/T_c \ll 1$, $\xi^2 q^2 \ll 1$. From Eqs. (4.4) and (4.7) we get the change of transition temperature in the following form:

$$\frac{\delta T_c}{T_{c0}} = \int \frac{d^3q}{(2\pi)^3} \frac{\varphi(q)}{\xi^2 q^2}. \tag{4.8}$$

Here we must cut off integration at $q \sim \xi^{-1}$ in accordance with the limits of applicability of the last expression in Eq. (4.7). However, the contribution of short-wave fluctuations here may be also important.

The Ginzburg–Landau functional expressed via the non-averaged order parameter $\Delta(r)$ has the following form [9]:

$$F\{\Delta\} = \int dr \left\{\frac{N(E_F)}{\lambda_p} |\Delta(r)|^2 - N(E_F) \int dr'\, K(rr')\Delta(r')\Delta(r) + \tfrac{1}{2} B |\Delta(r)|^4\right\}, \tag{4.9}$$

where we have neglected the fluctuations of the pairing interaction λ_p and of the coefficient B, which is defined by the standard expression given in Eq. (3.88). Using Eqs. (4.1)–(4.3) we can find the Ginzburg–Landau equations which describe the slow changes of $\Delta(\boldsymbol{r})$:

$$\left\{N(E_F)\frac{T_{c0}-T}{T_{c0}} + \delta A(\boldsymbol{r}) - B|\Delta(\boldsymbol{r})|^2 + C\frac{\partial^2}{\partial r^2}\right\}\Delta(\boldsymbol{r}) = 0\,, \tag{4.10}$$

where

$$\delta A(\boldsymbol{r}) = N(E_F)\int_0^{\langle\omega\rangle}\frac{dE}{E}\tanh\left(\frac{E}{2T_{c0}}\right)\left\{\frac{1}{N(E_F)}\sum_\nu |\phi_\nu(\boldsymbol{r})|^2\delta(E-\varepsilon_\nu) - 1\right\} \tag{4.11}$$

describes the fluctuations of the coefficient A of the Ginzburg–Landau expansion and we have neglected the fluctuations of the C coefficient.

Ginzburg–Landau equations with fluctuating coefficients were analyzed for the first time by Larkin and Ovchinnikov [181]. It was shown that $\delta A(\boldsymbol{r})$-fluctuations lead to a shift of transition temperature given by Eq. (4.8) and the solution of Eq. (4.10) for the order parameter in the first order over fluctuations has the form of Eq. (4.1) with

$$\Delta_1(\boldsymbol{r}) = \int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\Delta_1(\boldsymbol{q})e^{i\boldsymbol{q}\boldsymbol{r}}\,, \qquad \Delta_1(\boldsymbol{q}) = -\frac{\langle\Delta\rangle}{N(E_F)}\frac{\delta A(\boldsymbol{q})}{\xi^2q^2+2\tau}\,, \tag{4.12}$$

where $\tau = (T_c - T)/T_c$ is the temperature measured relative to the new transition temperature. The mean-square fluctuation of the order-parameter itself is determined from Eq. (4.12) by

$$\frac{\langle\Delta^2\rangle}{\langle\Delta\rangle^2} - 1 = \int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{\varphi(\boldsymbol{q})}{[\xi^2q^2+2\tau]^2}\,, \tag{4.13}$$

where $\varphi(\boldsymbol{q})$ was introduced in Eqs. (4.4), (4.5). It is important to note that fluctuations of $\Delta(\boldsymbol{r})$ as opposed to the T_c-shift are determined by the small $\boldsymbol{q}$ behavior of $\varphi(\boldsymbol{q})$.

We can see now that all the physics of statistical fluctuations is described by the correlation function of local densities of states (or spectral density of Eq. (4.6)). This function was determined above in Eqs. (2.83), (2.84) within the self-consistent theory of localization or by Eqs. (2.137), (2.139) which follow from the scaling approach close to the mobility edge.

Using Eq. (2.84) for the metallic state not very close to the mobility edge we can get from Eq. (4.5)

$$\varphi(\boldsymbol{q}=0) \sim \frac{\xi}{N^2(E_F)D_0^2}\,, \tag{4.14}$$

where $\xi = \sqrt{\xi_0 l}$ and D_0 is the Drude diffusion coefficient. Estimating the T_c-shift from Eq. (4.8) we get

$$\frac{\delta T_c}{T_{c0}} \sim \frac{1}{N^2(E_F)D_0^2\xi^2} \sim \frac{T_c}{E_F}\frac{1}{(p_F l)^3} \sim \tau_G\,, \tag{4.15}$$

where τ_G is the size of the Ginzburg critical region defined by Eq. (3.108). We have seen that in the usual "dirty" superconductor $\tau_G \ll 1$. For the order-parameter fluctuations from Eq. (4.13) we obtain

$$\frac{\langle\Delta^2\rangle}{\langle\Delta\rangle^2} - 1 \approx \frac{1}{8\pi}\frac{\varphi(\boldsymbol{q}=0)}{\xi^3\sqrt{2|\tau|}} \approx \left(\frac{\tau_D}{|\tau|}\right)^{1/2}. \tag{4.16}$$

From here we can see that the width of the temperature region where statistical fluctuations are important is given by

$$\tau_D \sim \frac{\varphi^2(0)}{\xi^6} \sim \frac{1}{N^4(E_F)D_0^4\xi^4} \sim \left(\frac{T_c}{E_F}\right)^2 \frac{1}{(p_F l)^6} \sim \tau_G^2 . \tag{4.17}$$

It is obvious that in a "dirty" superconductor we have $\tau_D \ll \tau_G \ll 1$ and statistical fluctuations are absolutely unimportant.

Situation changes for a system which is close to the mobility edge. Using Eq. (2.84) with D_0 replaced by $D_0(\omega/\gamma)^{1/3}$ or Eqs. (2.137)–(2.139) we obtain

$$\varphi(\boldsymbol{q}) \approx \frac{\gamma^{1/2}}{N^2(E_F)D_0^3 T_c} \int_0^{T_c} \frac{d\omega}{\omega^{1/2}} [\omega^2 + D_0\gamma^{-2/3}\omega^{2/3}q^4]^{-1/4} \sim \xi^2 \ln \frac{1}{\xi_q} , \tag{4.18}$$

where $\xi \sim (\xi_0 p_F^{-2})^{1/3}$. Similarly, we get

$$\frac{\langle \Delta^2 \rangle}{\langle \Delta \rangle^2} - 1 \approx \int_0^{\xi^{-1}} \frac{\xi^3 q^2\, dq}{(\xi^2 q^2 + 2\tau)^2} \ln \frac{1}{q\xi} \sim \frac{1}{\sqrt{|\tau|}} \ln \frac{1}{|\tau|} . \tag{4.19}$$

From Eq. (4.19) it follows that close to the mobility edge statistical fluctuations become important and even overcome thermodynamic fluctuations due to the logarithmic factor in $\varphi(\boldsymbol{q})$. Thus in this region we have $\tau_D > \tau_G \sim 1$.

The crossover from the regime of weak statistical fluctuations ($\tau_D \ll \tau_G$) to the strong fluctuation regime occurs at the conductivity scale $\sigma \sim \sigma^* \approx \sigma_c(p_F\xi_0)^{-1/3}$ which was extensively discussed above. Thus close to the mobility edge the superconducting order-parameter is no more a self-averaging quantity. Here the mean-field theory approach becomes formally invalid due to thermodynamic and also because of statistical fluctuations. Below we shall analyze this situation in more details.

Finally, we shall briefly discuss the region of localized states. The appearance here of a singular $\delta(\omega)$-contribution to the correlator of local densities of states given by Eqs. (A.8)–(A.10) leads to the additional contribution to $\varphi(\boldsymbol{q})$:

$$\varphi(\boldsymbol{q}) = \int_0^{\langle\omega\rangle} \frac{dE}{E^2} (\tanh(E/2T_c)) \frac{A_E(\boldsymbol{q})}{N(E_F)} + \cdots \sim \frac{A_{E_F}}{N(E_F)T_{c0}} + \cdots$$

$$= \frac{1}{N(E_F)T_c(1 + R_{loc}^2 q^2)} + \cdots . \tag{4.20}$$

Accordingly, a new contribution to $\Delta(\boldsymbol{r})$ fluctuations is given by

$$\frac{\langle \Delta^2 \rangle}{\langle \Delta \rangle^2} - 1 \approx \frac{1}{N(E_F)T_c} \int_0^{R_{loc}^{-1}} \frac{q^2\, dq}{(\xi^2 q^2 + 2|\tau|)^2(1 + R_{loc}^2 q^2)} \sim \frac{1}{N(E_F)T_c R_{loc}^3 \tau^2} \tag{4.21}$$

and it grows fast as the localization length R_{loc} diminishes. Using our main criterion of superconductivity in localized phase given by Eq. (3.20) we can see that in all regions of possible

superconductivity statistical fluctuations of $\Delta(\boldsymbol{r})$ remain of the order of unity and are important in a rather wide temperature interval around T_c.

4.2. Superconducting transition at strong disorder

We consider now superconductivity in systems with strong statistical fluctuations of the "local transition temperature" $T_c(\boldsymbol{r})$ as described by Eqs. (4.10) and (4.11). In this analysis we shall follow Refs. [182, 183]. For simplicity we assume a Gaussian nature for these fluctuations. Note, however, that close to the mobility edge the fluctuations of local density of states become strongly non-Gaussian [95] and this can complicate the situation. Unfortunately, the importance of this non-Gaussian behavior for superconductivity has not been studied up to now. We shall see that in our model, depending on the degree of disorder, which we shall measure by the ratio τ_D/τ_G, two types of superconducting transitions are possible. For τ_D smaller than some critical value τ_D^* the superconducting transition is the usual second-order phase transition at $T = T_c$. The superconducting order-parameter is in this case equal to zero for $T > T_c$ and is spatially homogeneous over scales exceeding the correlation length $\xi(T)$ below T_c. Statistical fluctuations lead only to a change of critical exponents at the transition [184, 185].

At $\tau_D > \tau_D^*$ the superconducting state appears in an inhomogeneous fashion even if the correlation length of disorder induced fluctuations of $T_c(\boldsymbol{r})$ is small compared with the superconducting correlation length ξ (microscopic disorder). This case was first analyzed by Ioffe and Larkin [189]. Investigating the case of extremely strong disorder they have shown that as the temperature is lowered the normal phase acquires localized superconducting regions (drops) with characteristic size determined by $\xi(T)$. Far from T_c their density is low, but with further cooling the density and dimensions of the drops increase and they begin to overlap leading to a kind of percolative superconducting transition.

According to our previous estimates, if we take into account only the fluctuations of local density of states, the parameter τ_D/τ_G increases from very small values to a value greater than unity as the system moves towards the mobility edge. The onset of an inhomogeneous superconducting regime is therefore to be expected as the localization transition is approached.

Our treatment of superconductors with large statistical fluctuations will be based on the Ginzburg–Landau functional:

$$F\{\boldsymbol{A}(\boldsymbol{r}), \Delta(\boldsymbol{r})\} = \int d\boldsymbol{r} \left\{ \frac{\boldsymbol{B}^2(\boldsymbol{r})}{8\pi} + N(E_F) \left[(\tau + t(\boldsymbol{r}))|\Delta(\boldsymbol{r})|^2 + \xi^2 \left| \left(\nabla - \frac{2ie}{\hbar c} \boldsymbol{A}(\boldsymbol{r}) \right) \Delta(\boldsymbol{r}) \right|^2 + \frac{1}{2} \lambda |\Delta(\boldsymbol{r})|^4 \right] \right\}, \tag{4.22}$$

where $\boldsymbol{B} = \mathrm{rot}\boldsymbol{A}$ is the magnetic field and we have redefined the coefficient of the quartic term as $B = N(E_F)\lambda$. Here $t(\boldsymbol{r})$ is defined by Eq. (4.11) as $\delta A(\boldsymbol{r}) = N(E_F)t(\boldsymbol{r})$ and plays the role of the fluctuation of local "critical temperature", which appears due to fluctuations of the local density of states. In the general case, it can also have contributions from the local fluctuations of the pairing interaction or other types of microscopic inhomogeneities. As noted above, we assume Gaussian statistics of these fluctuations, though real situation close to the mobility edge may be more

complicated [95]. Given the distribution of $t(\boldsymbol{r})$, the free energy of the system and the order-parameter correlator are equal to

$$\mathscr{F}\{t(\boldsymbol{r})\} = -T\ln Z\,, \quad Z = \int D\{\boldsymbol{A},\Delta\}\exp[-F\{\boldsymbol{A}(\boldsymbol{r}),\Delta(\boldsymbol{r})\}/T]\,, \tag{4.23}$$

$$\langle\Delta(\boldsymbol{r})\Delta(\boldsymbol{r}')\rangle = Z^{-1}\int D\{\boldsymbol{A},\Delta\}\Delta(\boldsymbol{r})\Delta(\boldsymbol{r}')\exp[-F\{\boldsymbol{A}(\boldsymbol{r}),\Delta(\boldsymbol{r})\}/T]\,, \tag{4.24}$$

and must be averaged over the Gaussian distribution of $t(\boldsymbol{r})$. From our definition of $t(\boldsymbol{r})$ and using the approach of the previous section, assuming the short-range of fluctuations of local density of states (on the scale of ξ), it is easy to estimate the correlator of $t(\boldsymbol{r})$ as

$$\langle t(\boldsymbol{r})t(\boldsymbol{r}')\rangle = \gamma\delta(\boldsymbol{r}-\boldsymbol{r}')\,, \quad \gamma \approx \tau_{\mathrm{D}}^{1/2}\xi^3\,. \tag{4.25}$$

Then the probability of a configuration with a given $t(\boldsymbol{r})$ is given by

$$\mathscr{P}\{t(\boldsymbol{r})\} = \exp\left[-\frac{1}{2\gamma}\int \mathrm{d}\boldsymbol{r}\, t^2(\boldsymbol{r})\right]. \tag{4.26}$$

The problem reduces thus to the calculation of the functions $\mathscr{F}\{t(\boldsymbol{r})\}$ and $\langle\Delta(\boldsymbol{r})\Delta(\boldsymbol{r}')\rangle$ and their subsequent averaging over $\mathscr{P}\{t(\boldsymbol{r})\}$.

We shall limit ourselves to the consideration of noninteracting drops and no vortices. Then we can consider the phase of the order-parameter $\Delta(\boldsymbol{r})$ as nonsingular. After the gauge transformation

$$\boldsymbol{A}(\boldsymbol{r}) \to \boldsymbol{A}(\boldsymbol{r}) + (c\hbar/2e)\nabla\phi(\boldsymbol{r})\,, \qquad \Delta(\boldsymbol{r}) \to \Delta(\boldsymbol{r})\exp[-\mathrm{i}\phi(\boldsymbol{r})]\,, \tag{4.27}$$

where $\phi(\boldsymbol{r})$ is the *phase of the order parameter*; we can use real $\Delta(\boldsymbol{r})$ and the Ginzburg–Landau functional of Eq. (4.22) becomes

$$F\{\boldsymbol{A}(\boldsymbol{r}),\Delta(\boldsymbol{r})\} = \int \mathrm{d}\boldsymbol{r}\left\{\frac{\boldsymbol{B}^2(\boldsymbol{r})}{8\pi} + N(E_{\mathrm{F}})\left[(\tau + t(\boldsymbol{r}))\Delta^2(\boldsymbol{r}) + \frac{4e^2\xi^2}{c^2\hbar^2}\boldsymbol{A}^2(\boldsymbol{r})\Delta^2(\boldsymbol{r}) + \xi^2(\nabla\Delta(\boldsymbol{r}))^2 + \frac{1}{2}\lambda\Delta^4(\boldsymbol{r})\right]\right\}. \tag{4.28}$$

Integration over phase in Eq. (4.23) gives an inessential constant factor to the partition function which we disregard.

To average the logarithm of the partition function, Eq. (4.23), over $t(\boldsymbol{r})$ we can use the replica trick [186] which permits the averaging to be carried out in explicit form. We express the average free energy, Eq. (4.23), of the system in the form

$$\langle\mathscr{F}\rangle = -T\lim_{n\to 0}\frac{1}{n}[\langle Z^n\rangle - 1]\,. \tag{4.29}$$

To calculate $\langle Z^n\rangle$ in accordance with the idea of the replica method, we first assume n to be an arbitrary integer. Expressing Z^n in terms of an n-fold functional integral over the fields of the

replicas A_α, $\Delta_\alpha(r)$, $\alpha = 1, \ldots, n$ and carrying out exact Gaussian averaging over $t(r)$, we get

$$\langle Z^n \rangle = \int D\{A, \Delta\} \exp[-S_n\{A_\alpha, \Delta_\alpha\}] , \tag{4.30}$$

$$S\{A_\alpha, \Delta_\alpha\} = \int dr \left\{ \sum_\alpha^n \frac{B^2(r)}{8\pi T} + \frac{N(E_F)}{T} \sum_\alpha^n \left[\tau \Delta_\alpha(r)^2 + \frac{4e^2\xi^2}{e^2\hbar^2} A_\alpha^2(r)\Delta_\alpha^2(r) + \xi^2 (\nabla\Delta_\alpha(r))^2 + \frac{1}{2}\lambda\Delta_\alpha^4(r) \right] - \frac{1}{2}\frac{N(E_F)}{T}\gamma \left[\sum_{\alpha=1}^n \Delta_\alpha^2(r) \right]^2 \right\} .$$

The last expression here represents the "effective action" and $\tilde{\gamma} = \gamma N(E_F)/T_c \approx \tau_D^{1/2} N(E_F)/T_c$ grows with disorder. Note that the random quantities $t(r)$ have already dropped out of these expressions, and that the action $S\{A_\alpha, \Delta_\alpha\}$ is translationally invariant. For the correlator of Eq. (4.24) we obtain

$$\langle \Delta(r)\Delta(r') \rangle = \lim_{n\to 0} \frac{1}{n} \int D\{A, \Delta\} \exp[-S_n\{A_\alpha, \Delta_\alpha\}] \sum_{\alpha=1}^n \Delta_\alpha(r)\Delta_\alpha(r') , \tag{4.31}$$

where we have symmetrized over the replica indices.

Far from the region of strong fluctuations of the order parameter $|\tau| \gg \tau_D$, τ_G the functional integrals in Eqs. (4.30) and (4.24) can be calculated by the saddle-point method. The extrema of the action are determined by classical equations:

$$\left[\tau - \xi^2 \nabla^2 + \lambda\Delta_\alpha^2 - \tilde{\gamma} \sum_{\beta=1}^n \Delta_\beta^2(r) \right] \Delta_\alpha(r) = 0 , \quad A_\alpha = 0 . \tag{4.32}$$

The nontrivial conclusion is that these equations for $\Delta_\alpha(r)$, besides having spatially homogeneous solutions do have localized solutions with finite action (*instantons*). These correspond at $\tau > 0$ to the superconducting drops. We shall limit ourselves to a picture of noninteracting drops and consider only instanton solutions above T_c (at $\tau > 0$). We shall be interested only in those solutions that admit analytic continuation as $n \to 0$. We designate them $\Delta_\alpha^{(i)}(r)$, where the superscript i labels the type of solution. To find their contribution we must expand the action of Eq. (4.30) up to the terms quadratic in deviations $\varphi_\alpha(r) = \Delta_\alpha(r) - \Delta_\alpha^{(i)}(r)$. It can be shown that fluctuations of the fields $A_\alpha(r)$ can be neglected if we consider noninteracting drops [182, 183].

For $\tau > 0$ and for $\tilde{\gamma} > \lambda$, Eq. (4.32) possesses (besides the trivial solution $\Delta_\alpha = 0$) the following nontrivial solution with finite action (instanton) (cf. Refs. [187, 188, 7]):

$$\Delta_\alpha^{(i)}(r) = \Delta_0(r)\delta_{\alpha i} , \quad i = 1, \ldots, n ,$$

$$\Delta_0(r) = \sqrt{\frac{\tau}{\tilde{\gamma} - \lambda}}\, \chi\left[\frac{\tau}{\xi(T)}\right] , \quad \xi(T) = \frac{\xi}{\sqrt{\tau}} , \tag{4.33}$$

where the dimensionless function $\chi(x)$ satisfies the condition $d\chi(x)/dx|_{x=0} = 0$ and its asymptotic form: $\chi(x) \sim x^{-1}\exp(-x)$ for $x \gg 1$ (for spatial dimension $d = 3$). The qualitative form of this solution is shown in Fig. 19.

From Eq. (4.33) it is seen that instantons are oriented along axes of replica space (there are n types of instanton solutions) which is due to the "cubic anisotropy" term $\lambda\Delta_\alpha^4$ in the effective

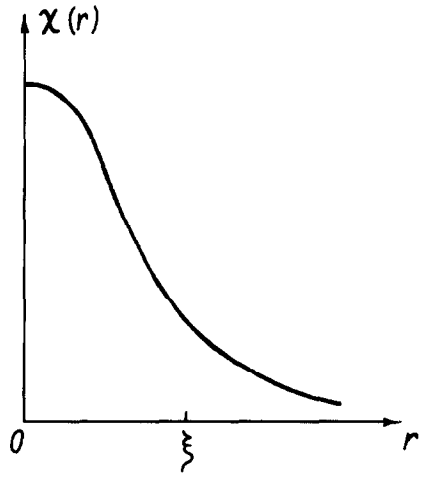

Fig. 19. Qualitative form of instanton solution.

action of Eq. (4.30). Index i characterizes the direction in replica space along which the symmetry breaking takes place. For $\lambda \to 0$ the action becomes O(n)-symmetric and instantons take the form

$$\Delta_\alpha(\boldsymbol{r}) = \Delta_0(\boldsymbol{r})e_\alpha\,, \quad \sum_{\alpha=1}^{n} e_\alpha^2 = 1\,, \tag{4.34}$$

i.e. are oriented along the arbitrary unit vector $\boldsymbol{e}$ in replica space. Such instantons earlier were studied in the theory of localization [187, 188, 7].

The quadratic expansion of the effective action near the instanton solution takes the form (cf. analogous treatment in Refs. [187, 188, 7]):

$$S\{\Delta_\alpha\} = S\{\Delta_\alpha^{(i)}\} + \frac{1}{2}\int d\boldsymbol{r} \sum_{\alpha,\beta} (\varphi_\alpha \hat{M}_{\alpha\beta}^{(i)} \varphi_\beta)\,, \tag{4.35}$$

where the operator $M_{\alpha\beta}^{(i)}$ on instanton solutions is equal to

$$\hat{M}_{\alpha\beta}^{(i)} = [\hat{M}_L \delta_{\alpha i} + \hat{M}_T(1 - \delta_{\alpha i})]\delta_{\alpha\beta} \tag{4.36}$$

with

$$M_{L,T} = (2N(E_F)/T)[-\xi^2 \nabla^2 + \tau U_{L,T}(\boldsymbol{r})] \tag{4.37}$$

where

$$U_L(\boldsymbol{r}) = 1 - 3\chi^2[r/\xi(T)]\,, \qquad U_T(\boldsymbol{r}) = 1 - (1 - \lambda/\tilde{\gamma})^{-1}\chi^2[r/\xi(T)]\,. \tag{4.38}$$

The value of the Gaussian functional integer is determined by the spectra of eigenstates of operators M_L and M_T. Detailed analysis can be found in Refs. [182, 183]. The qualitative form of these spectra is shown in Fig. 20. Operator M_L always possess an eigenvalue $\varepsilon_1^L = 0$ the so-called translation zero-mode, connected with translation symmetry: instanton center may be placed anywhere in space, the action does not change. However, this is not a lowest eigenvalue of M_L,

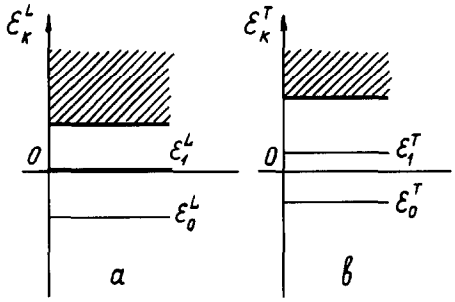

Fig. 20. Qualitative structure of eigenvalues of M_L (a) and M_T (b) operators. $\varepsilon_1^L = 0$: zero translation mode; $\varepsilon_0^T \to 0$ for $\lambda \to 0$ transforms to zero "rotation" mode. The continuous part of the spectrum is shaded.

there is always a negative eigenvalue $\varepsilon_0^L < \varepsilon_1^L = 0$. It can be shown rigorously that it is the only negative eigenstate of M_L [190]. Operator M_T possess also a single negative eigenvalue $\varepsilon_0^T < 0$ [182,183]; however, this eigenvalue tends to zero for $\lambda \to 0$ becoming the "rotation" zero-mode, reflecting the arbitrary "direction" of instanton in replica space in the absence of cubic anisotropy in the action [187,188,7]. For $\lambda = \lambda^* = 2/3\tilde{\gamma}$ we have $M_L = M_T$ and the spectra of both operators coincide.

Including the contributions of instantons oriented along all the axes in replica space we obtain the following one-instanton contribution to the partition function entering Eq. (4.29) [182,183]:

$$\langle Z^n \rangle = n\Omega \left(\frac{J_L}{2\pi}\right)^{d/2} [\mathrm{Det}' M_L]^{-1/2} [\mathrm{Det}\, M_T]^{/1-n)/2} \exp\{-S_0(\tau)\}\,, \tag{4.39}$$

where Ω is the system volume,

$$J_L = \frac{1}{d}\int \mathrm{d}r \left(\frac{\partial \Delta_0}{\partial r}\right)^2 = \frac{T}{2N(E_F)}\frac{S_0(\tau)}{\xi^2} \tag{4.40}$$

and the action at the instanton is given by

$$S_0(\tau) = \mathscr{A}\frac{\xi^3 \tau^{1/2}}{\gamma - \lambda T/N(E_F)}\,, \tag{4.41}$$

where $\mathscr{A} \approx 37.8$ is a numerical constant [191]. The prime on Det M_L means that we must exclude the zero-eigenvalue $\varepsilon_1^L = 0$ from the product of eigenvalues determining this determinant. The condition of applicability of the saddle-point approximation looks like $S_0(\tau) \gg 1$, and in fact all our analyses are valid outside the critical regions both for thermodynamic and statistical fluctuations.

In the limit of $n \to 0$ the total cancellation of imaginary contributions appearing due to negative eigenvalues takes place in Eq. (4.39) and using Eq. (4.29) we get for $\tilde{\gamma} > 3/2\lambda$ the following *real* contribution to the free energy:

$$\mathscr{F} = -\rho_s(\tau) T \Omega \tag{4.42}$$

where the density of superconducting "drops"

$$\rho_s(\tau) = \left[\frac{T}{4\pi N(E_F)} S_0(\tau)\right]^{3/2} \xi^{-3} \left[\frac{\mathrm{Det}\, M_T}{\mathrm{Det}' M_L}\right]^{1/2} \exp\{-S_0(\tau)\}\,. \tag{4.43}$$

Thus for $\tilde{\gamma} > 3/2\lambda$ even for $T > T_c$ the superconducting "drops" (instantons) appear in the system which directly contribute to the equilibrium free energy. This contribution given by Eqs. (4.42) and (4.43) exists along the usual thermodynamic fluctuations. The condition of $\tilde{\gamma} > 3/2\lambda$ defines the critical disorder $\tau_D > \tau_D^* > \tau_G$, and this inhomogeneous picture of superconducting transition appears only for the case of sufficiently strong statistical fluctuations. The knowledge of the qualitative structure of spectra of eigenvalues of M_L and M_T allows to analyze different asymptotics of Eq. (4.42) [182, 183]. For $\tilde{\gamma} S_0(\tau) \ll \lambda \ll \lambda^*$ we get

$$\rho_s(\tau) \approx \xi^{-3}(T)(\lambda/\tilde{\gamma})^{1/2} S_0^{3/2}(\tau) \exp[-S_0(\tau)] . \tag{4.44}$$

For $\lambda \to \lambda^*$, we obtain

$$\rho_s(\tau) \approx \xi^{-3}(T)((\lambda^*/\lambda) - 1)^{3/2} S_0^{3/2}(\tau) \exp[-S_0(\tau)] . \tag{4.45}$$

Thus the density of superconducting "drops" $\rho_s(\tau)$ vanishes as $\lambda \to \lambda^*$, i.e. they are destroyed by thermodynamic fluctuations.

For the order-parameter correlator of Eq. (4.24) we get the following result:

$$\langle \Delta(\boldsymbol{r})\Delta(\boldsymbol{r}')\rangle \approx \rho_s(\tau) \int d\boldsymbol{R}_0 \, \Delta_0(\boldsymbol{r} + \boldsymbol{R}_0)\Delta_0(\boldsymbol{r}' + \boldsymbol{R}_0) . \tag{4.46}$$

The integration over instanton center $\boldsymbol{R}_0$ here means in fact averaging over different positions of "drops". Note that over large distances this correlator decreases like $\exp[-|\boldsymbol{r} - \boldsymbol{r}'|/\xi(T)]$ and does not contain the usual Ornstein–Zernike factor $|\boldsymbol{r} - \boldsymbol{r}'|^{-1}$.

We have found the free-energy of the inhomogeneous superconducting state in the temperature region $\tau \gg \tau_D$, where the "drop" concentration is exponentially small and the picture of non-interacting "drops" is valid. They give exponentially small contribution to the specific heat and diamagnetic susceptibility. The characteristic size of "drops" is determined by $\xi(T)$ and as $T \to T_c$ the "drops" grow and begin to overlap leading to a percolative superconducting transition. Thus for $\tau_D > \tau_D^* > \tau_G$ superconductivity first appears in isolated "drops". This is similar to the picture of decay of a metastable state in the case of first-order phase transitions [192]. However, in this latter case instantons give imaginary contribution to the free energy determining the decay rate of a "false" equilibrium state (critical bubble formation). Here instanton contributions lead as was noted above to real free energy and "drops" appear in the true equilibrium state.

It is more or less obvious that between isolated "drops" a kind of Josephson coupling may appear and lead to rather complicated phase diagram of the system in an external magnetic field, e.g. including the "superconducting glass" phase [193, 194]. The existence of the inhomogeneous regime of superconductivity will obviously lead to the rounding of BCS-like singularities of the density of states and superconductivity may become gapless. Note that diffusion-enhanced Coulomb interactions can also lead to the gaplessness of strongly disordered superconductors via Coulomb-induced inelastic scattering [195]. Fluctuation conductivity in a similar inhomogeneous superconducting state was studied in Ref. [196]. Note the closely related problem of strongly disordered superfluids [197, 198]. Some results here may be quite useful for the case of strongly disordered superconductors, though the limitations of this analogy are also important.

A major unsolved problem here is the possible influence of statistical fluctuations of the coefficient of the gradient term in the Ginzburg–Landau expansion which has been neglected above, or the equivalent problem (cf. Eq. (3.96)) for superconducting electron density n_s. This

problem was briefly considered for the case of weak disorder in Ref. [199]. It was shown that

$$\langle(\delta n_s/n_s)^2\rangle \sim (\xi_0 p_F^2 l)^{-1} \sim e^4/g(\xi)\,, \tag{4.47}$$

where $g(\xi) = \sigma\xi$ is the conductance of the metallic sample with the size of the order of the superconducting coherence length $\xi = \sqrt{\xi_0 l}$. Extrapolating this estimate up to the Anderson transition using $\xi = (\xi_0/p_F^2)^{1/3}$ we get

$$\langle(\delta n_s/n_s)^2\rangle \sim \frac{e^4}{\sigma^2(\xi_0/p_F^2)^{2/3}}\,. \tag{4.48}$$

Obviously, we get $\langle(\delta n_s/n_s)^2\rangle \geq 1$ for $\sigma \leq \sigma^*$ so that statistical fluctuations of n_s become important close to the Anderson transition in the same region we have discussed above. This further complicates the picture of the superconducting transition and can also be very important for the possible anomalous behavior of H_{c2} which was recently studied on similar lines in Ref. [200]. Some qualitative conjectures for the case of $\langle(\delta n_s/n_s)^2\rangle \geq 1$ were formulated in Ref. [201], where it was argued that in this case there will occur regions in the sample with locally *negative* values of superfluid density. This is equivalent to a negative sign of the Josephson coupling between the "drops". In this sense, the disordered superconductor is unlike a Bose liquid. This leads to an important prediction that in a small superconducting ring, if there is a segment with negative n_s, the ground state of the ring will spontaneously break the time-reversal invariance. The ground state will have nonzero supercurrent and magnetic flux (or rather random, trapped fluxes in the ground state) and will be two-fold degenerate. At longer times, the symmetry will be restored due to the thermal activation of the macroscopic quantum tunneling between the two states, but according to Ref. [201] it can be expected that for dirty metal rings with conductance of the order of $e^2/\hbar$ there will be "roughly 50% chance that the ground state will break time-reversal symmetry". By the way, this means that in the presence of disorder there may be no way to distinguish between an anyon superconductor [202] and a conventional superconductor. Of course we must stress that these speculations are entirely based upon a simple extrapolation of Eq. (4.47) to the vicinity of metal–insulator transition and there is no complete theory of statistical fluctuations of the gradient term in this region at the moment.

5. Superconductivity in strongly disordered metals: experiment

Our review of experiments on strongly disordered superconductors will be in no sense exhaustive. This is mainly a theoretical review and the author is in no way an expert on experiments. However, we shall try to illustrate the situation with the interplay of Anderson localization and superconductivity in *bulk* (three-dimensional) superconductors, both traditional and high-temperature. Again we must stress that we exclude any discussion of the numerous data on thin films which are to be described by two-dimensional theories. In this case we just refer to existing reviews [17–19]. Here we shall confine ourselves to a limited number of experiments, which we consider most interesting from the point of view of illustration of some of the ideas expressed above, just to convince the reader, that previous discussion, while purely theoretic, has something to do with the real life. More than anywhere else in this review our choice of material is based on personal

interests of the author, or our direct involvement in the discussion of experiments. We shall not deal with the general problem of the influence of disorder upon superconductivity, but shall consider only the systems which remain superconducting close to the disorder-induced metal–insulator transition.

5.1. Traditional superconductors

There exists a number of strongly disordered systems which remain superconducting close to the metal–insulator transition induced by disorder.

The drop of T_c with the decrease of conductivity from a value of the order of $10^4\,\Omega^{-1}\,\mathrm{cm}^{-1}$ was observed in amorphous alloys of GeA [203], SiAu [204] and MoRe [205], in Chevrel phase superconductors disordered by fast neutron irradiation, such as $Pb_{1-x}U_xMo_6S_8$ [207], Mo_6Se_8 [208], in amorphous InO_x [209], in $BaPb_{1-x}Bi_xO_3$ in the concentration interval $0.25 < x < 0.30$ [210] and in metallic glass $Zr_{0.7}Ir_{0.3}$ [212]. In all these systems, superconducting transition is observed apparently not very far from the metal–insulator transition. For many of these systems, such as $Pb_{1-x}U_xMo_6S_8$, $SnMo_6S_8$, Mo_6Se_8, $Zr_{0.7}Ir_{0.3}$ and $BaPb_{0.75}Bi_{0.25}O_3$ [210] and some others a characteristic strongly negative temperature resistivity coefficient has been observed. Note, however, that this fact alone in no way indicates that a specimen is on one side or the other of the metal–insulator transition. The drop of T_c close to the mobility edge apparently was also observed in As_2Te_3 [213]. However, in all of these systems T_c apparently vanishes before the metal–insulator transition. Below we present some of the data on these and other similar systems.

In Fig. 21 we show the dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in $SnMo_5S_6$ (Chevrel phase superconductor) on the fluence of fast neutron irradiation (the number of neutrons which passed through a cross-section of a sample during irradiation) [207]. In the region of large fluences (large disorder), when the system becomes amorphous, characteristic values of conductivity in the normal state are of the order of $\sim 10^3\,\Omega^{-1}\,\mathrm{cm}^{-1}$, which is not far from the values of "minimal metallic conductivity" $\sigma_c \sim 5\times 10^2\,\Omega^{-1}\,\mathrm{cm}^{-1}$, which define the conductivity scale of a disorder induced metal–insulator transition. A negative temperature coefficient of resistivity was observed in this conductivity range. The experimental data on T_c-decrease with the growth of resistivity in this system were rather well fitted in Ref. [22] using the μ^* dependence on resistivity given by Eq. (3.62). A clear tendency for $|dH_{c2}/dT|_{T_c}$ saturation with disorder is also observed. Analogous dependence of T_c and $|dH_{c2}/dT|_{T_c}$ on the resistivity in the normal state for Mo_6Se_8 disordered by fast neutrons is shown in Fig. 22 [208]. Here superconductivity exists up to conductivities $\sigma \sim 250\,\Omega^{-1}\,\mathrm{cm}^{-1}$. Further disordering (irradiation) leads to the destruction of the superconducting state and metal–insulator transition (an unlimited growth of resistivity with decrease of T, with variable-range hopping conduction [2, 3] is observed). The slope of the upper critical field $|dH_{c2}/dT|_{T_c}$ also has a tendency to saturate with the growth of resistivity. Standard interpretation of such behavior of $|dH_{c2}/dT|_{T_c}$ was based upon the use of Gorkov's relation (cf. first relation in Eq. (3.115)) and lead to the conclusion that $N(E_F)$ decreases under disordering. In fact, we have seen that no such conclusion can be reached for systems with conductivities $\sigma < 10^3\,\Omega^{-1}\,\mathrm{cm}^{-1}$, because such saturation behavior may be a natural manifestation of the approaching metal–insulator transition. Similar dependences were observed in other Chevrel phase superconductors [206, 214, 215].

In Fig. 23 we show the dependence of conductivity and T_c on the parameter $p_F l/\hbar$ in amorphous InO_x alloy [209]. In Fig. 24, the data on the temperature dependence of H_{c2} in amorphous In/InO_x

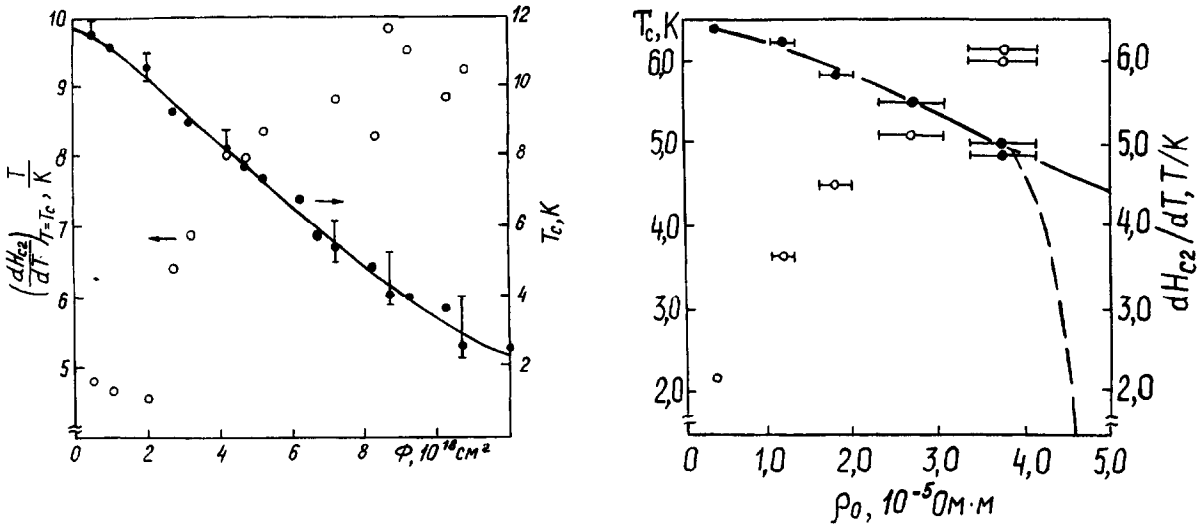

Fig. 21. Fluence dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in $SnMo_5S_6$.

Fig. 22. Resistivity dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in Mo_6Se_8.

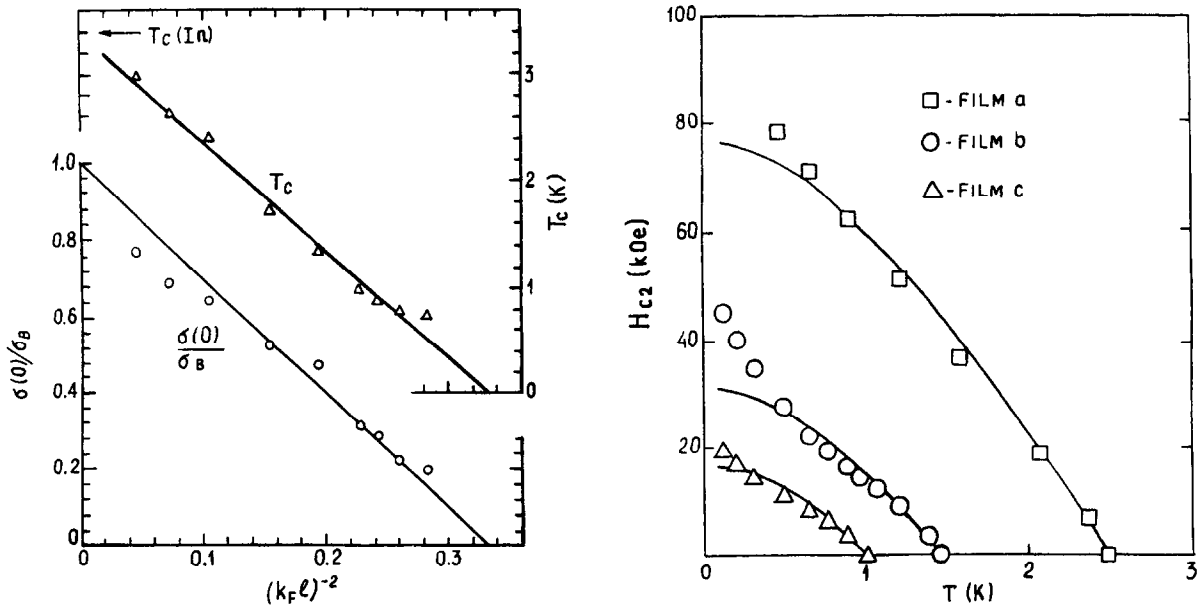

Fig. 23. Conductivity σ and T_c dependence on the parameter $p_F l/\hbar$ in amorphous InO_x; σ_B is the estimated Drude conductivity.

Fig. 24. $H_{c2}(T)$ in amorphous films of In/InO_x. Lines show standard theoretical dependence.

(bulk) films from Ref. [216] are presented for different degrees of disorder. We can see that in the low temperature region $H_{c2}(T)$ deviates from the standard temperature dependence, but apparently confirms the qualitative form predicted above for systems which are close to the Anderson transition. The same system was also studied in Ref. [217]. In Fig. 25 we show the dependence of two characteristic energies on disorder which in the opinion of the authors of Ref. [217] demonstrate the narrow region of coexistence of superconductivity and insulating state. In Fig. 26 we show the dependence of localization length and superconducting coherence length on disorder according to Ref. [217]. It demonstrates qualitative agreement with our general criterion of coexistence of superconductivity and localization; localization length must be larger or at least of the order of the size of the Cooper pair.

Very impressive are the data for amorphous $Si_{1-x}Au_x$ alloy [203, 204, 218]. In Fig. 27 [204] the data on T_c and conductivity dependence on the Au concentration x are shown. In Fig. 28, $H_{c2}(T)$ dependence for this system is shown for different alloy compositions [204]. From these data it is

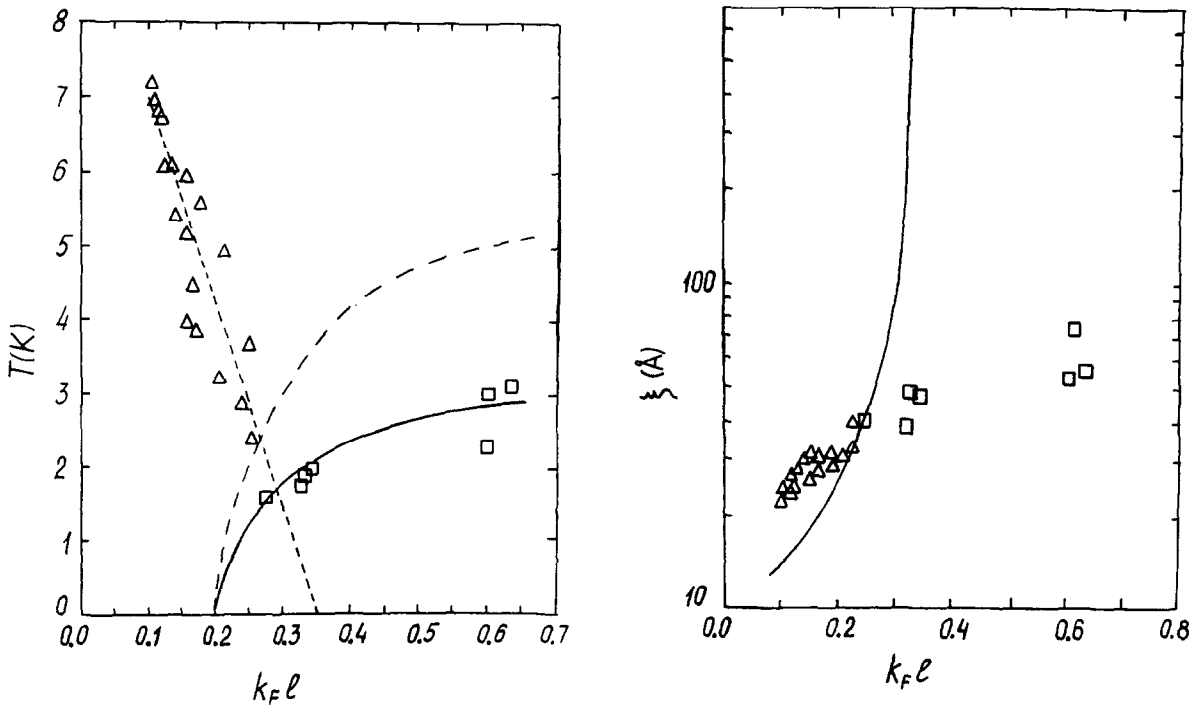

Fig. 25. The dependence of activation energy of hopping conductivity (triangles) and superconducting transition temperature T_c (squares) in amorphous films of In/InO_x on disorder parameter $p_F l/\hbar$ as determined from room-temperature conductivity and Hall measurements. Long-dashed line represents $\Delta = 1.76T_c$ following the BCS gap formula. The short-dashed line best fits the insulating data points with $(p_F l/\hbar)_c \approx 0.35$ – the critical disorder of metal–insulator transition. A narrow region of superconductivity within the insulating phase can be inferred from these data.

Fig. 26. Disorder dependence of localization length (full curve) and superconducting coherence length in amorphous In/InO_x films. Squares represent superconducting ξ for metallic films while triangles refer to insulating samples.

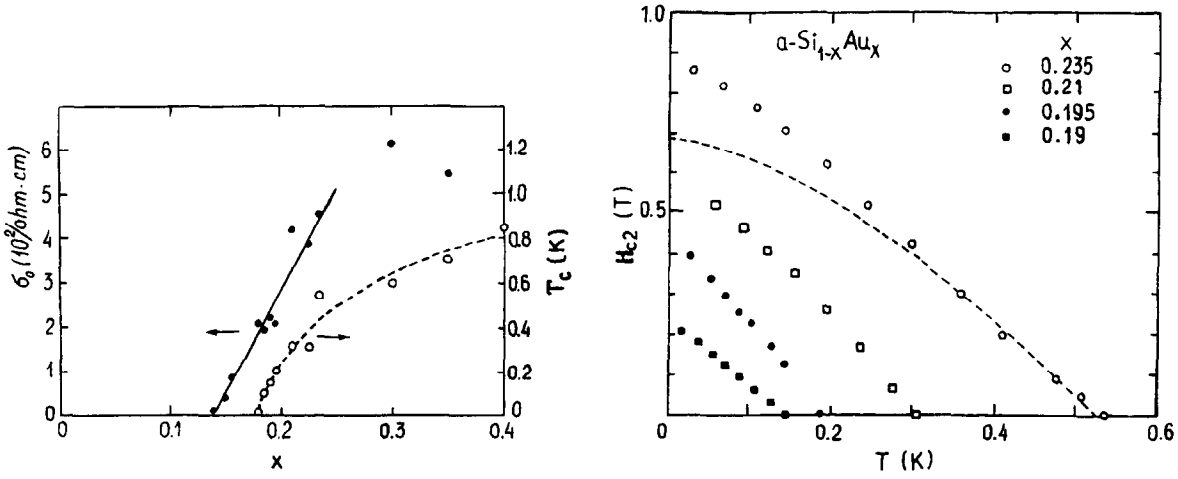

Fig. 27. Conductivity σ and T_c dependence on gold concentration in amorphous $Si_{1-x}Au_x$ alloy.

Fig. 28. $H_{c2}(T)$ in amorphous $Si_{1-x}Au_x$ alloy.

clearly seen that T_c vanishes *before* the metal–insulator transition. The metal–insulator transition itself is *continuous*, conductivity vanishes linearly with the decrease of gold concentration and the values of conductivity significantly less than the estimated "minimal metallic conductivity" are definitely observed. The system remains superconducting even for such low conductivity values. The slope of $H_{c2}(T)$ at $T = T_c$ is practically constant, irrespective of the change of conductivity (disorder) in a rather wide range. This behavior apparently cannot be explained only by the appearance of the correlation pseudogap in the density of states observed in Ref. [218], which becomes significant only very close to the metal–insulator transition. Low temperature deviation from standard convex dependence on T is also clearly seen. In Fig. 29 from Ref. [218] we show the temperature dependence of resistivity and the superconducting energy gap (determined by tunneling) of a sample with $x = 0.21$. It nicely demonstrates the superconducting transition in a system which is very close to a disorder-induced metal–insulator transition. Note that according to Ref. [218] the superconducting energy gap in this sample is substantially broadened which may indicate the growth of the statistical gap fluctuations due to the same fluctuations of the local density of states. These data are in obvious qualitative correspondence with the general theoretical picture described throughout this review.

These data show that in systems which are superconducting close to the disorder-induced metal–insulator (Anderson) transition, T_c decreases rather quickly and practically in all reliable cases vanishes before transition to the insulating state. At the same time the temperature dependence of H_{c2} is not described by the standard theory of "dirty" superconductors both with respect to the $(dH_{c2}/dT)_{T_c}$ behavior and at low temperatures, where the upward deviations from the standard dependence are readily observed. This confirms most of our theoretical conclusions.

Some indications of a possible superconducting state in the insulating phase of granular Al and Al–Ge were observed in Refs. [219, 220]. Obviously, the granular systems are more or less outside the scope of our review. However, we should like to mention that the strong smearing of BCS-like

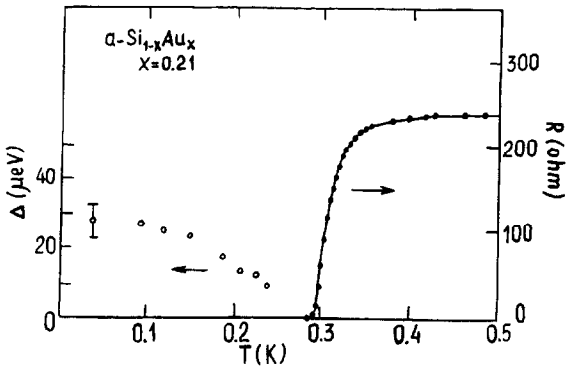

Fig. 29. Temperature dependences of the superconducting energy gap Δ and of the resistance R for amorphous $Si_{0.79}Au_{0.21}$.

density of states and the gapless regime of superconductivity was observed (via tunneling measurements) in Refs. [221, 222] close to the metal–insulator transition in these systems. This may confirm our picture of statistical fluctuation smearing of the density of states. Note that a more recent work on granular Al [223] apparently excludes the possibility of superconductivity in the insulating phase. In this work a small amount of Bi was added to granular Al in order to enhance spin–orbit scattering, which leads to an *antilocalization* effect [39]. This shifts both metal–insulator and to the *same extent* the superconducting transition, with the preservation of a narrow range of concentration on the metallic side where the material is not fully superconducting. The fact that the superconducting transition shifts with the metal–insulator transition demonstrates that its position is determined by its vicinity to the metal–insulator transition, and that it is the impending transition to the insulating state which inhibits superconductivity. Similar conclusions on superconductivity vanishing at the point of metal–insulator transition were reached for amorphous Al_xGe_{1-x} [224] and amorphous Ga–Ar mixtures [225]. This later case is particularly interesting because it has been shown that the conductivity exponent at the metal–insulator transition here is $\nu \approx 0.5$ which places this system in a different universality class than those discussed above and similar to that observed in some doped uncompensated semiconductors like Si:P [226]. Usual interpretation of this difference is based upon the importance of interaction effects in these systems [108]. Starting with the value of T_c of amorphous Ga ($T_c = 7.6$ K), T_c decreases rather slowly with decreasing Ga volume fraction v, until one enters the critical region near $v_c \approx 0.145$. Further approach to v_c leads to a rapid decrease of T_c. Taking McMillan formula Eq. (3.25) for T_c (with $\omega_{\log}/1.20 = 320$ K and $\lambda = 0.45$) and assuming negligible Coulomb repulsion μ^* for pure amorphous Ga the increase of μ^* on the approach of metal–insulator transition can be determined from the experimental data for T_c. This increase is approximately given by $\mu^* \sim (v - v_c)^{-0.33}$. From this it is easy to see that $T_c \to 0$ for $v \to v_c$, so that these data do not indicate the survival of superconductivity beyond the metal–insulator transition. These results are not surprising since we have seen the existence of strong mechanisms of T_c degradation close to disorder-induced metal–insulator transition.

The interesting new high-pressure metastable metallic phase of an amorphous alloy $Cd_{43}Sb_{57}$ exhibiting the gradual metal–insulator transition during the slow decay at room temperature and atmospheric pressure has been studied in Refs. [227, 228]. Authors claim that during this decay the

system remains homogeneous while going from metallic to insulating phase. At the same time the metallic phase is superconducting with $T_c \approx 5$ K and remains so up to the metal–insulator transition. Close to it the superconducting transition becomes smeared, while incomplete transition persists even in the insulating state. While these data are reminiscent of data on quench-condensed films of Sn and Ga [229], which were interpreted as reentrant superconductivity due to sample inhomogeneities, it is stressed in Refs. [227,228] that, in this new system situation is different and we are dealing with an intrinsically inhomogeneous state of superconductors discussed in Refs. [62,182,183]. From our point of view, further studies of this system are necessary in order to show unambiguously the absence of structural inhomogeneities. Also, a rather peculiar characteristic of this system is the almost complete independence of the onset temperature of the superconducting transition on disorder.

The general conclusion is that, in most cases of traditional superconducting systems, we cannot find an unambiguous demonstration of the possibility of superconductivity in an insulating state induced by disorder. At the same time, we can see a rather rich variety of data on superconductivity close to the metal–insulator transition which stimulate further studies. Some of the anomalies of the superconducting behavior discussed above can be successfully explained by theories presented in this review, while the others require further theoretical investigations.

5.2. *High-T_c superconductors*

Very soon after the discovery of high-temperature oxide superconductors [14,15] it was recognized that localization effects have an important role to play in these systems. There are many sources of disorder in these systems and the low level of conductivity indicate from the very beginning their closeness to Anderson transition. In the field where there are hundreds of papers published on the subject it is impossible to review or even to quote all of them. A more or less complete impression about the status of high-T_c research can be obtained from Conference Proceedings [230]. Here we shall concentrate almost only on papers which deal with disordering by fast neutron irradiation which we consider probably the "purest" method to introduce disorder into the system (allowing to neglect the complicated problems associated with chemical substitutions). Also, historically, it is apparently the earliest method used to study the disorder effect in high-T_c superconductors in a controllable way [231,232].

There are several reasons for localization effects to be important in high-T_c oxides:

- *Two-dimensionality.* All the known high-T_c systems (with $T_c > 30$ K) are strongly anisotropic or quasi-two-dimensional conductors. We have seen above that for such systems it is natural to expect the strong enhancement of localization effects due to the special role of spatial dimensionality $d = 2$: in the purely two-dimensional case, localization appears for infinitely small disorder [31,4,6,7]. The inplane conductivity scale for the metal–insulator transition in such systems as given by Eq. (2.12) or Eq. (2.93) is larger than in the isotropic case. Reasonable estimates show that the values of inplane "minimal metallic conductivity" may exceed $10^3\,\Omega^{-1}\,\mathrm{cm}^{-1}$. While due to the continuous nature of the Anderson transition there is no rigorous meaning of minimal metallic conductivity, these estimates actually define the scale of conductivity near the metal–insulator transition caused by disorder. Then it is clear that most of the real samples of high-T_c superconductors are quite close to the Anderson transition and even the very slight disordering is sufficient to transform them into Anderson insulators [171].

- *"Marginal" Fermi liquid.* During our discussion of interaction effects we have seen that there are serious reasons to believe that importance of localization effects in high-T_c oxides may be actually due to more fundamental reasons connected with the anomalous electronic structure and interactions in these materials. The concept of a "marginal" Fermi liquid [124] leads to extreme sensitivity of such a system to disordering and the appearance of localized states around the Fermi level at a rather weak disorder [126, 127].

On the other hand, high-T_c systems are especially promising from the point of view of the search for superconductivity in the Anderson insulator:

- High transition temperature T_c itself may guarantee the survival of superconductivity at relatively high disorder.
- Due to the small size of Cooper pairs high-T_c systems in combination with high-T_c (large gap!) we can easily satisfy the main criterion for superconductivity in the localized phase as given by Eq. (3.20).
- Being narrow band systems, as most of the conducting oxides, high T_c systems are promising due to low values of the Fermi energy E_F which leads to less effective T_c degradation due to localization enhancement of the Coulomb pseudopotential μ^* (cf. Eq. (3.61)).

Anomalous transport properties of high-T_c oxides are well known [233]. Experimentally, there are two types of resistivity behavior of good single-crystals of these systems. In the highly conducting *ab* plane of $YBa_2Cu_3O_{7-\delta}$ and other oxides, resistivity of a high-quality single-crystal always shows the notorious linear-T behavior (by "good" we mean the samples with resistivity $\rho_{ab} < 10^3\ \Omega$cm). However, along the orthogonal c direction the situation is rather curious: most samples produce semiconductor-like behavior $\rho_c \sim 1/T$, though some relatively rare samples (apparently more pure) show metallic-like $\rho_c \sim T$ (with strong anisotropy $\rho_c/\rho_{ab} \approx 10^2$ remaining) [233, 234]. Metallic behavior in the c direction was apparently observed only in the best samples of $YBa_2Cu_3O_{7-\delta}$ and almost in no other high-T_c oxide. In Fig. 30 taken from Ref. [235] we show the temperature dependence of ρ_c in a number of high-T_c systems. It is seen that $\rho_c(T)$ changes between metallic and semiconducting behavior depending on whether the resistivity is below or above the Ioffe–Regel limit defined for the quasi-two-dimensional case by Eq. (2.12). Rather strange is the absence of any obvious correlation between the behavior of ρ_c and T_c.

This unusual behavior leads us to the idea that most of the samples of high-T_c systems which are studied in the experiment are actually already in a localized phase due to internal disorder which is always present. Surely, we realize that such a drastic assumption contradicts the usual expectations and propose it just as an alternative view open for further discussion. The attempted justification of this idea may be based upon the quasi-two-dimensional nature of these systems or on marginal Fermi liquid effects. In this case a simple conjecture on the temperature behavior of resistivity of single-crystals can be made which qualitatively explains the observations [236, 237]. In case of localized states at the Fermi level and for finite temperatures it is important to compare the localization length R_{loc} with the diffusion length due to inelastic scattering $L_\varphi \approx \sqrt{D\tau_\varphi}$, where D is the diffusion coefficient due to elastic scattering on disorder, while τ_φ is the phase coherence time determined by inelastic processes. For $T > 0$ this length L_φ effectively replaces the sample size L in all expressions of scaling theory of localization when $L \gg L_\varphi$, because on distances larger than L_φ all information on the nature of wave functions (e.g. whether they are localized or extended) is smeared out. Taking into account the usual low-temperature dependence like $\tau_\varphi \sim T^{-p}$ (where p is some integer, depending on the mechanism of inelastic scattering) this can lead to a nontrivial

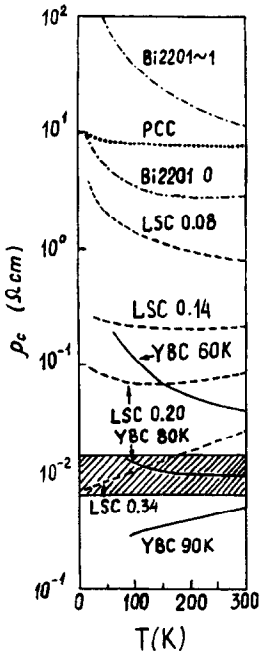

Fig. 30. Temperature dependence of ρ_c for different high-T_c cuprates. The dashed region indicates the resistivity range corresponding to Ioffe–Regel limit.

temperature dependence of conductivity, in particular to the possibility of a negative temperature coefficient of resistivity [33]. Similar expressions determine the temperature dependence of conductivity also for the localized phase unit $L_\varphi < R_{\rm loc}$. In this case, electrons do not "feel" being localized and conductivity at high enough T will show metallic like behavior. For localization to be important we must go to low enough temperatures, so that L_φ becomes greater than $R_{\rm loc}$. If disordered high-T_c superconductors are in fact Anderson insulators with a very anisotropic localization length, $R^{ab}_{\rm loc} \gg R^{c}_{\rm loc}$ and both localization lengths diminish as disorder grows, L_φ is also anisotropic and we can have three different types of temperature behavior of resistivity [236]:

1. Low T or strong disorder, when we have

$$L^{ab}_\varphi \approx \sqrt{D_{ab}\tau_\varphi} \gg R^{ab}_{\rm loc}\,, \quad L^{c}_\varphi \approx \sqrt{D_{c}\tau_\varphi} \gg R^{c}_{\rm loc}\,. \tag{5.1}$$

This gives semiconductor-like behavior for both directions.

2. Medium T or medium disorder, when

$$L^{ab}_\varphi < R^{ab}_{\rm loc}\,, \quad L^{c}_\varphi > R^{c}_{\rm loc}\,, \tag{5.2}$$

and metallic behavior is observed in the *ab* plane, while semiconducting temperature dependence of resistivity is observed along the *c*-axis.

3. High T or low disorder, when

$$L^{ab}_\varphi < R^{ab}_{\rm loc}\,, \quad L^{c}_\varphi < R^{c}_{\rm loc} \tag{5.3}$$

and metallic behavior is observed in both directions.

Here we do not speculate on the inelastic scattering mechanisms leading to the concrete temperature behavior in high-T_c oxides, in particular on linear T behavior in the *ab* plane or $1/T$ behavior in the *c* direction. Unfortunately too little is known on these mechanisms [233] to be able to make quantitative estimates on the different types of behavior predicted above. Of course detailed studies of such mechanisms are necessary to prove the proposed idea and to explain the temperature dependence of resistivity in high-T_c systems on its basis. However, most of the experimental data as we shall see below at least do not contradict the idea of the possibility of Anderson localization in disordered high-T_c cuprates.

Now let us consider the experiments on controllable disordering of high-temperature superconductors. Already the first experiments on low temperature ($T = 80$ K) fast neutron irradiation of ceramic samples of high-T_c systems [238–242, 244] have shown that the growth of structural disorder leads to a number of drastic changes in their physical properties:

- continuous metal–insulator transition at very slight disordering,
- rapid degradation of T_c,
- apparent coexistence of hopping conductivity and superconductivity at intermediate disorder,
- approximate independence of the slope of H_{c2} at $T \sim T_c$ on the degree of disorder,
- anomalous *exponential* growth of resistivity with defect concentration.

These anomalies were later confirmed on single-crystals and epitaxial films [243, 245–247], and were interpreted [171, 16] using the ideas of possible coexistence of Anderson localization and superconductivity.

In Fig. 31 we show data [171] on the dependence of the superconducting transition temperature and resistivity (at $T = 100$ K, i.e. just before the superconducting transition) on fast neutron fluence for $YBa_2Cu_3O_{6.95}$. In all high-T_c compounds, introduction of defects leads to a strong broadening of the superconducting transition. The derivative $(dH_{c2}/dT)_{T_c}$ in ceramic samples measured at the midpoint of the superconducting transition does not change as ρ_{100K} grows by an order magnitude. In Fig. 32 [171] we show the temperature dependence of resistivity for samples of $YBa_2Cu_3O_{6.95}$ and $La_{1.83}Sr_{0.17}CuO_4$ for different degrees of disorder. In all these materials the $\rho(T)$ curves vary in the same way. In the fluence range $\Phi > 10^{19}$ cm^{-2}, where superconductivity is absent, $\rho(T)$ follows a dependence which is characteristic of conductivity via localized states [2, 3]:

$$\rho(T) = \rho_0 \exp(Q/T^{1/4}), \quad Q = 2.1[N(E_F)R_{loc}^3]^{-1/4} \tag{5.4}$$

as shown in Fig. 33 (Mott's variable-range hopping conduction).

The most striking anomaly of resistivity behavior of all high-T_c systems under disordering is nonlinear, practically *exponential* growth of resistivity at fixed temperature (e.g. $\rho(T = 100\text{ K})$) with fluence, starting from the low fluences $\Phi < 7 \times 10^{18}$ cm^{-2}, including superconducting samples [238, 171, 239–242]. These data are shown in Fig. 34 [171] for the dependence of $\rho(T = 80\text{ K})$ on Φ obtained from measurements made directly during the process of irradiation. For comparison, the similar data for $SnMo_6S_8$ are shown which do not demonstrate such an anomalous behavior, its resistivity is just proportional to Φ and saturates at large fluences. We relate this exponential growth of ρ with the increase of Φ (i.e. of defect concentration) in all high-T_c systems to localization, which already appears for very small degrees of disorder in samples with high values of T_c. As we have seen in samples with much reduced or vanishing T_c localization is observed directly via Mott's hopping in the temperature behavior of resistivity given by Eq. (5.4).

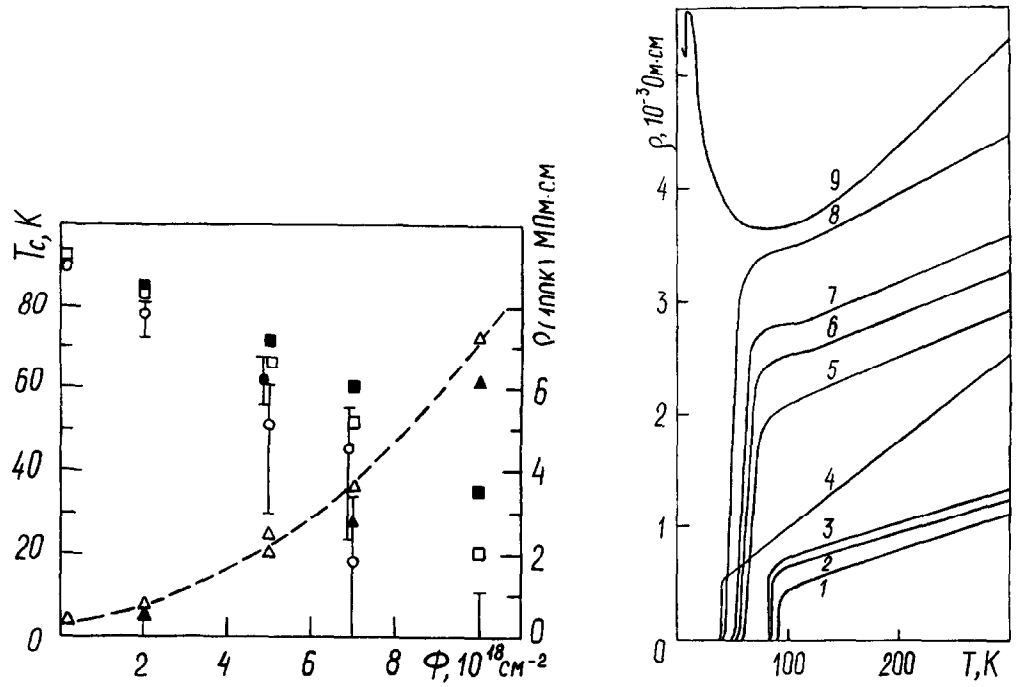

Fig. 31. Dependence of the superconducting transition temperature and resistivity (at $T = 100$ K) on neutron fluence for ceramic $YBa_2Cu_3O_{6.95}$. Different notations correspond to different methods of measurement and also evolution after annealing at 300 K.

Fig. 32. Temperature dependence of resistivity ρ for ceramic samples of $YBa_2Cu_3O_{6.95}$ (curves 1–3 and 5–8) and $La_{1.83}Sr_{0.17}CuO_4$ (curves 4, 9) irradiated at $T = 80$ K with different fluences: 1, $\Phi = 0$; 3, 6, 8, $\Phi = 2.5$ and 7×10^{18} cm^{-2} plus annealing for 2 h at 300 K; 2, 5, 7, irradiated with $\Phi = 2.5$ and 7×10^{18} cm^{-2} plus annealing for two weeks at 300 K; 4, $\Phi = 0$; 9, $\Phi = 5 \times 10^{18}$ cm^{-2} plus annealing for 2 h at 300 K.

From these results it follows that the electronic system of high-T_c superconductors is very close to the Anderson transition. The observed variation of ρ as a function of both fluence and of temperature can be described by the following empirical formula [238]:

$$\rho(T, \Phi) = (a + cT)\exp(b\Phi/T^{1/4}) . \tag{5.5}$$

Identifying the exponential factors in Eqs. (5.4) and (5.5) it is possible to obtain a fluence dependence of localization length (cf. Ref. [171] and below).

Detailed neutron diffraction studies of structural changes in irradiated samples were also performed [238, 171, 248]. These investigations have shown definitely that there is no oxygen loss in $YBa_2Cu_3O_{6.95}$ during low temperature irradiation. Only some partial rearrangement of oxygens between positions O(4) and O(5) in the elementary cell occur as radiation-induced defects are introduced. In addition, in all high-T_c compounds the Debye–Waller factors grow and the lattice parameters a, b, c increase slightly [171, 248]. The growth of Debye–Waller factors reflect significant atomic shifts, both static and dynamic, from their regular positions, which induce a random potential. This disorder is pretty small from the structural point of view, the lattice is only

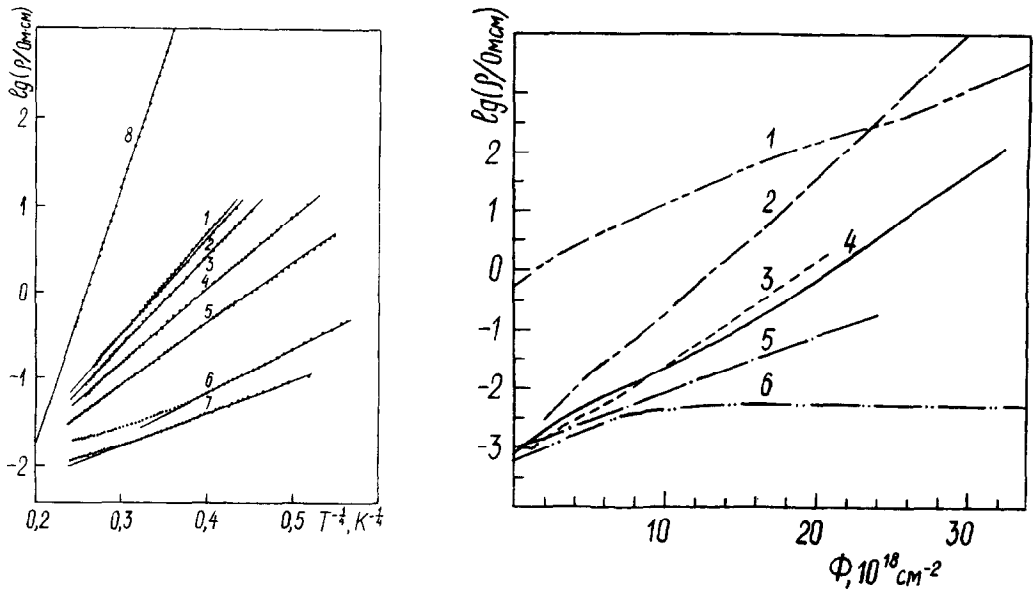

Fig. 33. Dependence of $\ln\rho$ on $T^{-1/4}$ for $YBa_2Cu_3O_{6.95}$ irradiated with a fluence of $\Phi = 1.2\times10^{19}\,cm^{-2}$ at $T = 80$ K (curve 1), and after 20 min annealing at $T = 150$ K (2); 200 K (3); 250 K (4); 300 K (5) and two weeks annealing at $T = 300$ K (7). Similar dependence for $La_{1.83}Sr_{0.17}CuO_4$ for $\Phi = 2\times10^{19}\,cm^{-2}$ annealed for 2 h at 300 K (6) and for La_2CuO_4 for $\Phi = 2\times10^{19}\,cm^{-2}$ annealed for 2 h at 300 K (8).

Fig. 34. Dependence of $\ln\rho$ on fluence Φ during irradiation at $T = 80$ K: 1 – La_2CuO_4; 2 – $YBa_2Cu_3O_{6.95}$; 3 – single crystalline ρ_{ab} in $YBa_2Cu_3O_{6.95}$; 4 – $La_{1.83}Sr_{0.17}CuO_4$; 5 – Bi–Sr–Ca–Cu–O; 6 – $SnMo_6Se_8$.

slightly distorted. However, we have seen that this small disorder is sufficient to induce a metal–insulator transition and complete degradation of superconductivity. The absence of oxygen loss implies that there is no significant change in concentration of carriers and we really have a disorder-induced metal–insulator transition. This is also confirmed by other methods [244,249]. In Fig. 35 we show the data [244,245] regarding temperature dependence of the Hall concentration of ceramic samples of irradiated and oxygen deficient $YBa_2Cu_3O_{7-\delta}$. It is seen that disordering weakens the anomalous temperature dependence of Hall effect, but Hall concentration n_H at low T practically does not show significant difference with the data on oxygen deficient samples, where n_H drops by several orders. This also confirms the picture of a disorder-induced metal–insulator transition in radiation disordering experiments. Similar Hall data were obtained on epitaxial films [246] and single-crystals [247].

Qualitatively identical resistivity behavior was also obtained in the experiments on radiation disordering of single-crystals [243, 245] and epitaxial films [246]. Electrical resistivities of $YBa_2Cu_3O_{7-\delta}$ single crystals were measured at $T = 80$ K directly during irradiation by fast neutrons. The data are shown in Fig. 36. We can see that ρ_{ab} increases exponentially with Φ (defect concentration) starting from the smallest doses, while ρ_c grows more slowly and only for $\Phi > 10^{19}\,cm^{-2}$ they grow at the same rate. At large fluences, both ρ_{ab} and ρ_c demonstrate [250] Mott's hopping $\ln\rho_{ab,c} \sim T^{-1/4}$. Similar data of Ref. [246] show $\ln\rho \sim T^{-1/2}$ characteristic of the

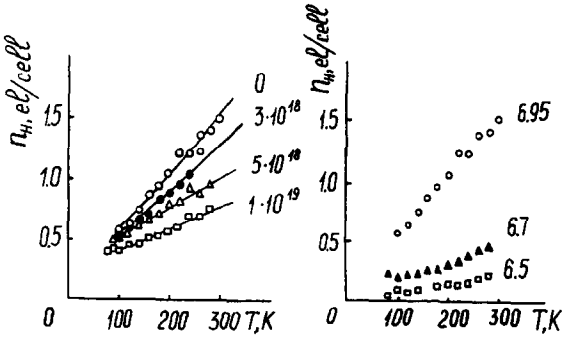

Fig. 35. Temperature dependence of Hall concentration for the irradiated (left) and oxygen deficient (right) ceramic samples of $YBa_2Cu_3O_{7-\delta}$.

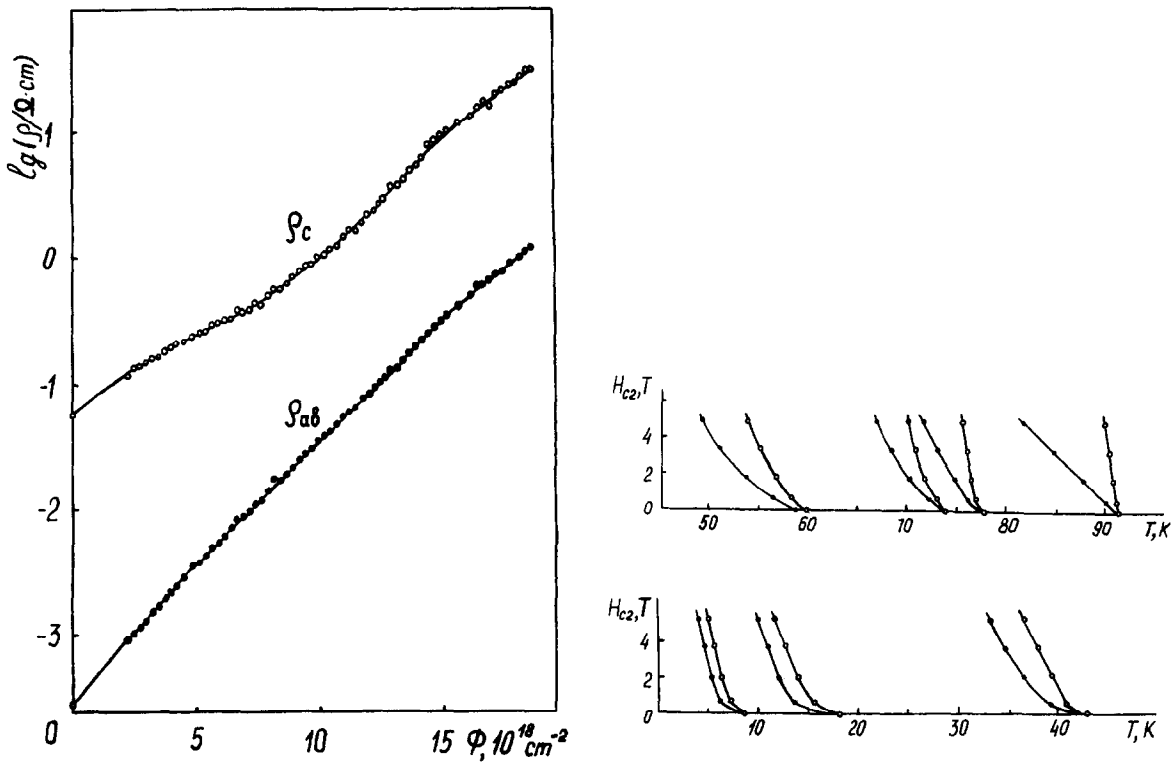

Fig. 36. Fluence dependence of ρ_{ab} and ρ_c at $T = 80$ K during fast neutron irradiation.

Fig. 37. Temperature dependence of $H_{c2}^{\parallel}$ (upper curves) and $H_{c2}^{\perp}$ (lower curves) for the single-crystals of $YBa_2Cu_3O_{7-\delta}$ with different degrees of disorder.

Coulomb gap. We do not know the reasons for this discrepancy between single-crystalline and epitaxial films data (note that another method of disordering by 1 MeV Ne^+ ions was used in Ref. [246]). Anisotropy ρ_c/ρ_{ab} at $T = 80$ K drops rapidly (to the values ~ 30 for $\Phi = 10^{19}$ cm^{-2}) and then practically remains unchanged and the "residual" anisotropy is of the order of the room-temperature value as in initial samples. This means that the temperature dependence of anisotropy weakens in the disordered samples. Note that, unfortunately, only the single-crystals with "semiconducting" temperature dependence of resistivity along the c axis were investigated up to now.

The upper critical fields of $YBa_2Cu_3O_{7-\delta}$ single-crystals (determined from standard resistivity measurements) for different degrees of disorder are shown in Fig. 37 [245]. Temperature dependence of H_{c2} in disordered samples is essentially nonlinear, especially for samples with low T_c. The temperature derivative of $H_{c2}^{\perp}$ (field along the c axis) estimated from high-field regions increases with disorder. However, a similar derivative of $H_{c2}^{\parallel}$ (field along ab plane) drops in the beginning and then does not change. Anisotropy of H_{c2} decreases with disorder and in samples with $T_c \sim 10$ K the ratio of $(H_{c2}^{\parallel})'/(H_{c2}^{\perp})'$ is close to unity. According to Eq. (3.118) this means the complete isotropisation of the Cooper pairs. This is illustrated by Fig. 38 [251]. The remaining anisotropy of resistivity may be connected with some kind of planar defects in the system.

In a recent paper [252] Osofsky et al. presented the unique data on the temperature dependence of the upper critical field of high-temperature superconductor $Bi_2Sr_2CuO_y$ in a wide temperature interval from $T_c \approx 19$ K to $T \approx 0.005 T_c$, which has shown a rather anomalous dependence with positive curvature at any temperature. The authors of Ref. [252] have noted that this type of behavior is difficult to explain within any known theory. It is sharply different from the standard behaviour the BCS-model. It was demonstrated in Refs. [253, 254] that the observed dependence of $H_{c2}(T)$ can be satisfactorily explained by localization effects in a two-dimensional (quasi-two-dimensional) model in the limit of sufficiently strong disorder. Measurements of H_{c2} in Ref. [252] were performed on epitaxially grown films of $Bi_2Sr_2CuO_y$: however it is quite possible that the films were still disordered enough, which can be guessed from the rather wide (~ 7 K) superconducting transition. Unfortunately, the relevant data, in particular, regarding the conductivity of the films studied are absent. This gives us some ground to try to interpret the data obtained in Ref. [252] in the framework of a very strong disorder, the effects of which are obviously enhanced by the quasi-two-dimensional nature of high-temperature superconductors.

The general discussion of the temperature dependence of the upper critical field in the two-dimensional and quasi-two-dimensional cases with strong localization effects was presented above in Section 3.3.1. Note that we mainly analyzed there the case of a magnetic field perpendicular to the highly conducting planes, which is precisely the case of Ref. [252]. We have seen [81] that the anomalies of the upper critical field due to the frequency dependence of the diffusion coefficient appear only for temperatures $T \ll e^{-1/\lambda}/\tau$. For higher temperatures we obtained the usual behavior of "dirty" superconductors. Also we have noted [81] that superconductivity survives in a system with finite localization length if $T_c \gg \lambda\, e^{-1/\lambda}/\tau$, which is equivalent to our criteria for the smallness of Cooper pair size compared with localization length. This latter length is exponentially large in two-dimensional systems with small disorder ($\lambda \ll 1$). The most interesting (for our aims) limit of a relatively strong disorder is defined by $T_c \ll e^{-1/\lambda}/\tau$, so that in fact we are dealing with a pretty narrow region of λs when $\lambda\, e^{-1/\lambda}/\tau \ll T_c \ll e^{-1/\lambda}/\tau$. In this case we have seen that the upper critical

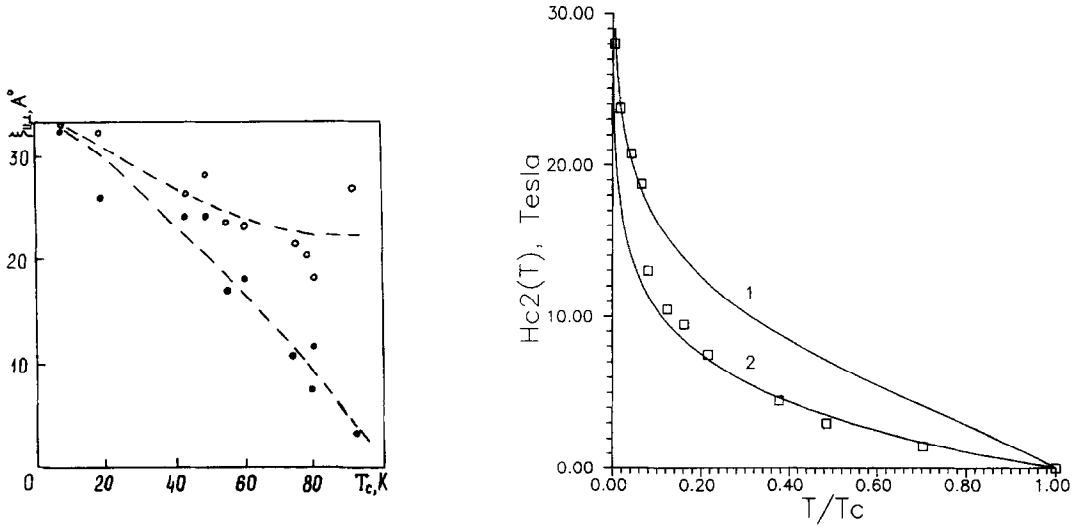

Fig. 38. The dependence of coherence lengths determined from H_{c2} behavior under disordering on the critical temperature T_c: $\xi_\parallel$ – open circles; $\xi_\perp$ – filled circles.

Fig. 39. Temperature dependence of the upper critical field: theoretical curve (1) is given for the case of $e^{-1/\lambda}/T_c\tau = 2$, $\lambda = 0.18$, while curve (2) is for $e^{-1/\lambda}/T_c\tau = 20$, $\lambda = 0.032$. Squares represent the experimental data for $Bi_2Sr_2CuO_y$.

field is practically defined by Eq. (3.141):

$$\ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T}\right) = \left(1 + 4\pi\frac{D_0}{\phi_0}\frac{\tau H_{c2}}{e^{-1/\lambda}}\right)\ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T_c}\left(1 + 4\pi\frac{D_0}{\phi_0}\frac{\tau H_{c2}}{e^{-1/\lambda}}\right)\right) \tag{5.6}$$

($\gamma = 1.781$) from which we can directly obtain the $T(H_{c2})$-dependence. The appropriate behavior of the upper critical field for two sets of parameters is shown in Fig. 39. The curve of $H_{c2}(T)$ demonstrates positive curvature and H_{c2} diverges for $T \to 0$. We have seen that this weak (logarithmic) divergence is connected with our neglect of the magnetic field influence upon diffusion. Taking this influence into account we can suppress this divergence of H_{c2} as $T \to 0$. This is the main effect of broken time invariance and it is clear that it is important only for extremely low temperatures [81]. In the following we neglect it. For the quasi-two-dimensional case on the dielectric side of Anderson's transition, but not very close to it, the behavior of diffusion coefficient is quite close to that of purely two-dimensional case, so that the upper critical field can be analyzed within the two-dimensional approach. Close to the transition (e.g. over interplane transfer integral) both for metallic and insulating sides and for parameters satisfying the inequality $\lambda e^{-1/\lambda}/\tau \ll T_c \ll e^{-1/\lambda}/\tau$, the temperature dependence of H_{c2} is in fact again very close to those in the purely two-dimensional case considered above [81]. Some deviations appear only in a very narrow region of very low temperatures [81].

In Fig. 39 we also show the experimental data for H_{c2} from Ref. [252]. Theoretical curve (1) is given for the parameters which lead to rather good agreement with experiment in the low temperature region. The curve (2) corresponds to parameters giving good agreement in a wide temperature region except the lowest temperatures. The cyclotron mass m was always assumed to

be equal that of the free electron. In general we observe satisfactory agreement between theory and experiment. Unfortunately, the values of the ratio $e^{-1/\lambda}/T_c\tau$ for the second curve, while corresponding to quite reasonable values of λ, lead to nonrealistic (too small) values of $T_c\tau$, which are rather doubtful for a system with relatively high T_c. For the first curve, the situation is much better though the electron damping on the scale of T_c is still very large which corresponds to strong disorder. Note however, that the detailed discussion of these parameters is actually impossible without the knowledge of additional characteristics of the films studied in Ref. [252]. In particular, it is quite interesting to have an independent estimate of λ. We also want to stress the relatively approximate nature of these parameters due to our two-dimensional idealization. More serious comparison should be done using the expressions of Ref. [81] for the quasi-two-dimensional case, which again requires the additional information on the system, in particular, the data on the anisotropy of electronic properties.

In our opinion, the relatively good agreement of experimental data of Ref. [252] with theoretical dependences obtained for the two-dimensional (quasi-two-dimensional) case of a disordered system with Anderson localization illustrates the importance of localization effects for the physics of high-temperature superconductors. However, we must note that similar anomalies of the temperature dependence of the upper critical field were also observed in Ref. [255] for the single crystals of the overdoped $Tl_2Ba_2CuO_{6+\delta}$ which authors claim to be extremely clean, so that apparently no explanation based upon strong localization effects can be used. Similar data were recently obtained for thin films of underdoped $YBa_2(Cu_{0.97}Zu_{0.03})_3O_{7-\delta}$ with pretty low transition temperatures [256]. These films again seem to be disordered enough to call localization effects as a possible explanation of the unusual positive curvature of $H_{c2}(T)$ dependence for all temperatures.

Under irradiation, localized moment contribution appears in the magnetic susceptibility of high-T_c oxides [238, 171]. In the temperature range from T_c to 300 K, $\chi(T)$ is satisfactorily described by a Curie–Weiss type dependence: $\chi(T) = \chi_0 + C/(T - \Theta)$. The value of χ_0 and the Curie constant C as a function of fluence Φ are given in Fig. 40. The value of C is proportional to the fluence. Note that the threefold larger slope of $C(\Phi)$ in $YBa_2Cu_3O_{6.95}$ as compared with $La_{1.83}Sr_{0.17}CuO_4$ is an evidence that this Curie-law temperature dependence is associated with localized moments forming on Cu (there are three times more copper in an elementary cell of Y compound than in an La compound).

The data presented above show that electronic properties of high-T_c systems are quite different under disordering from that of traditional superconductors [214, 215] or even some closely related metallic oxides [247, 257]. We associate these anomalies with the closeness of the Anderson transition and believe that real samples of high-T_c systems which always possess some noticeable disorder may well be already in the state of the Anderson insulator. However, we must stress that it is quite difficult to decide from the experiments described above the precise position of the Anderson transition on disorder scale. Some additional information on this problem may be obtained from experiments on NMR relaxation in disordered state, using the approach proposed rather long ago by Warren [258] and later quantified theoretically in Refs. [259, 260]. The study of NMR relaxation rate on [89] Y nuclei in radiationally disordered $YBa_2Cu_3O_{6.95}$ (which is opposite to Cu nuclei demonstrate Korringa behavior) [261, 262] has shown the anomalies (a *maximum* in the so-called Warren's enhancement factor) which according to Ref. [260] may indicate the Anderson transition somewhere in the fluence interval $\Phi = (1\text{–}2) \times 10^{19}\ \text{cm}^{-2}$. Unfortunately, the number of samples in these experiments was too limited to place the transition point

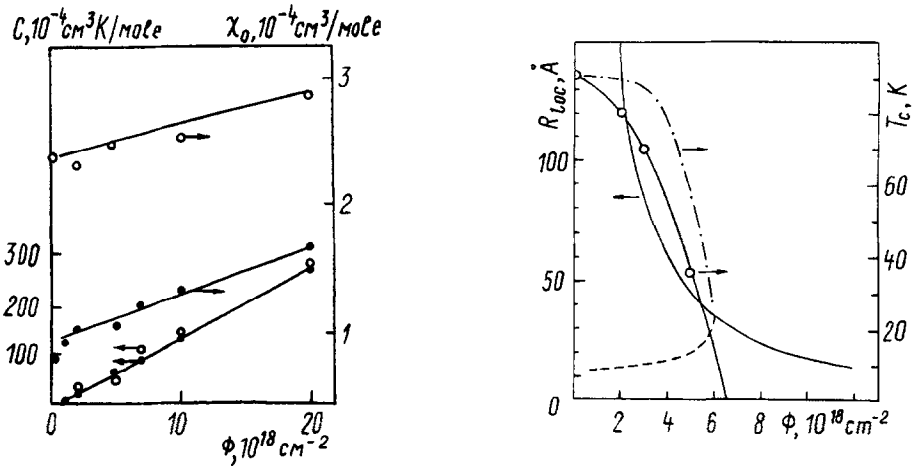

Fig. 40. Dependence of the Curie constant C and the temperature-independent part χ_0 of magnetic susceptibility on neutron fluence Φ for $La_{1.83}Sr_{0.17}CuO_4$ (filled circles) and $YBa_2Cu_3O_{6.95}$ (open circles).

Fig. 41. Dependence of T_c on fluence for $YBa_2Cu_3O_{6.95}$ (circles). The solid curve is the localization length calculated from hopping conductivity. Dashed curve defines the minimum localization length at which superconductivity can exist at given T_c. Dashed–dotted curve is the theoretical fit using expressions described in the text.

more precisely, while superconductivity disappears exactly in this interval. In this sense we still have no direct proof of coexistence of superconductivity and localization in disordered high-T_c oxides. However, the method used in Refs. [261, 262] seems to be very promising. Note that Knight shift data of Refs. [261, 262] strongly indicate Coulomb gap opening at the Fermi level of strongly disordered oxides. Independently, this conclusion was reached in tunneling experiments of Ref. [263] on a number of oxides disordered by doping.

Using the experimental data on electrical resistivity of disordered samples of $YBa_2Cu_3O_{6.95}$ and the relations given by Eq. (5.4) and Eq. (5.5) (assuming that exponentials there are identical) we can calculate the change of localization length R_{loc} as a function of fluence [171, 239–242]. This dependence is shown on Fig. 41 along with the fluence dependence of T_c. It is clearly seen that superconductivity is destroyed when localization length R_{loc} becomes smaller than ~ 30 Å, i.e. it becomes of the order or smaller than a typical size of the Cooper pair in this system (cf. Fig. 38) in complete accordance with our basic criterion of Eq. (3.20). We can estimate the minimal value of R_{loc} for which superconductivity can still exist in a system of localized electrons via Eq. (3.20) [171] taking the free-electron value of $N(E_F) \approx 5 \times 10^{33}$ (erg cm^3)$^{-1}$ (for carrier concentration of $\sim 6 \times 10^{21}$ cm^{-3}) and the gap value $\Delta \sim 5T_c$, corresponding to very strong coupling [232]. We obtain the result shown in Fig. 41. In any case we can see that criterion of Eq. (3.20) ceases to be fulfilled for $\Phi \sim (5\text{–}7) \times 10^{18}$ cm^{-2} in remarkably good agreement with the experiment.

In the absence of accepted pairing mechanism for high temperature superconductors it is very difficult to speculate on the reasons for T_c degradation in these systems. If we assume that the main mechanism of T_c degradation is connected with the growth of Coulomb effects during disordering, as discussed above in this review, we can try to use appropriate expressions to describe the experimental data. Assuming superconductivity in the localized phase we can use Eq. (3.68),

estimating R_{loc} as above from empirical relation (5.5) and (5.4) (or directly expressing the parameters entering Eq. (3.68) via experimental dependence of resistivity on fluence as described by Eq. (5.5) [171]). The results of such a fit (with the assumption of $\mu \approx 1$) are also shown in Fig. 41. The agreement is also rather satisfactory, the more rapid degradation of T_c for small degrees of disorder can be related to additional contributions to Coulomb repulsion within Cooper pairs neglected in the derivation of Eq. (3.68). Surely, we do not claim that this is a real explanation of T_c degradation in disordered high temperature superconductors. However, note its relation to localized moment formation under disordering which leads to the usual Abrikosov–Gorkov mechanism of depairing due to spin-flip scattering on magnetic impurities. According to Mott [159] (cf. also Refs. [160, 7]) the appearance of localized moments may be related to the presence of localized states (single occupied states below the Fermi level as briefly discussed above). We can then estimate the value of the effective magnetic moment (in Bohr magnetons) in a unit cell as [171].

$$\mu R_{loc}^{-3} \Omega_0 = p_{theor}^2 , \tag{5.7}$$

where Ω_0 is the volume of a unit cell. For large degrees of disorder ($\Phi = 2 \times 10^{19}\,cm^{-2}$) and $R_{loc} \approx 8$ Å with $\mu \approx 1$ we obtain $p_{theor}^2 = 0.66$ for $YBa_2Cu_3O_{6.95}$ in full agreement with experiment. However, for smaller fluences p_{theor} is considerably smaller than the experimental value. Note, though, that the estimate of Eq. (5.7) is valid only for small enough values of R_{loc}, i.e. when the Fermi level is well inside the localized region. On the other hand, the accuracy with which the Curie constant is determined in weakly disordered samples is considerably less than that in the strongly disordered case. Of course, the other mechanisms of local moment formation, which were discussed above and can become operational even before the metal–insulator transition can be important here.

Of course, plenty of work on localization effects in high-T_c oxides use disorder induced by different types of chemical substitutions in these systems. Of these we shall rather arbitrarily quote Refs. [264–268], which provide data quite similar, though not necessarily identical, to those described above on different types of systems and obtained by different experimental methods. We note that the effects of chemical disorder are always complicated by the inevitable changes of carrier concentration due to doping effects. Still all these data indicate that superconductivity in high-T_c systems is realized close to disorder induced metal–insulator transition, so that these systems provide us with plenty of possibilities to study experimentally the general problems discussed in our review. More details can be found in the extensive review paper [269].

Special attention should be paid to a recent study of angle resolved photoemission in Co doped single-crystals of $Bi_2Sr_2CaCu_2O_{8+y}$ [270]. Doping $Bi_2Sr_2CaCu_2O_{8+y}$ with Co causes superconducting–insulator transition, Co doping decreases T_c and causes increase in residual resistivity. The changes in the temperature behavior of resistivity from metallic to insulating phase correlate with the disappearance of the dispersing band-like states in angle-resolved photoemission. Authors believe that Anderson localization caused by the impurity potential of the doped Co atoms provides a consistent explanation of all experimental features and T_c reduction is not caused by magnetic impurity pairbreaking effects but by spatial localization of carriers with the superconducting ground state being formed out of spatially localized carriers. Similar data were also obtained for some exceptional (apparently strongly disordered) samples of undoped $Bi_2Sr_2CaCu_2O_{8+y}$ [271]. Of course, it will be very interesting to perform similar type of experiments on neutron irradiated samples where we deal with pure disorder.

Finally, we must stress that in our opinion these data on rather strongly disordered samples of high-temperature superconductors more or less definitely exclude the possibility of d-wave pairing in these systems. As is well known (and also can be deduced from our discussion in Appendix C), d-wave pairing is much more sensitive to disordering and is completely suppressed roughly speaking at the disorders measured by the energy scale $1/\tau \sim T_{c0}$, which is at least an order of magnitude smaller than the disorder necessary to induce the metal–insulator transition which can be estimated as $1/\tau \sim E_F$. This apparently excludes the possibility to observe any manifestations of localization effects in d-wave superconductors, though these are clearly observed in high-T_c systems. Of course, these qualitative conclusions deserve further studies within the specific models of microscopic mechanisms of high-temperature superconductivity.

We shall limit ourselves to the discussion of localization effects in high temperature superconductors. Our conclusion is that these effects are extremely important in these systems and some of the anomalies can be successfully described by theoretical ideas formulated in this review. We must stress that much additional work is needed both theoretical and experimental to clarify the general picture of disorder effects in high-T_c superconductors and we can expect that future progress, especially with the quality of samples, may provide some new and exciting results.

6. Conclusions

We conclude our review by trying to recapitulate the basic unsolved problems. From the theoretical point of view probably the main problem is to formulate the theory of superconducting pairing in strongly disordered system along the lines of the general theory of interacting Fermi systems. This problem is obviously connected with the general theory of metal–insulator transition in such an approach, which as we mentioned during our brief discussion above is rather far from its final form. Nevertheless, there were several attempts to analyze the superconducting transition within this framework [272–275, 115]. In all cases, the authors limited themselves to certain universality classes within the general renormalization group approach of the interaction theory of the metal–insulator transition. Ref. [272] dealt only with the two-dimensional problem, while Refs. [273–275, 115] also considered the bulk case. These papers have demonstrated a large variety of possible behavior of superconductivity under disordering, from disorder-induced (triplet) superconductivity [274] to a complete destruction of it close to [272, 273] or even long before the metal–insulator transition [275]. Our point of view is that at the moment it is rather difficult to make any *general* conclusions from the results of these approaches. In particular, we do not believe that the present status of these theories is sufficient to prove or disprove the general possibility of superconductivity in Anderson insulators. However, it is obvious that further theoretical progress in the problem of T_c behavior under disordering will be largely possible only within this general approach. In this sense, our simplified discussion of Coulomb effects and other mechanisms of T_c degradation in this review is only of qualitative nature. Still, more general approaches apparently do not change our qualitative conclusions. These problems become even more complicated if we address ourselves to the case of high temperature superconductors, where we do not know precisely the nature of the pairing interaction in a regular system.

Concerning the semiphenomenological approach to the theory of superconductivity close to the Anderson transition we must stress the necessity of further investigation of the region of strong

statistical fluctuations with the aim of a more detailed study of their influence upon different physical properties, like e.g. the upper critical field, density of states, nuclear relaxation, etc. Obviously, all of them may be significantly changed in comparison with predictions of what we called the statistical mean-field theory. Especially important are further studies of rather exotic predictions of random fluxes in the ground state [201].

Despite our explicit limitation to a discussion of superconductivity in *bulk* disordered superconductors we have to mention the extremely interesting problem of universal conductivity at the superconductor–insulator transition at $T = 0$ in two-dimensional systems which attracted much attention recently [276–278, 19]. It is argued that the transition between the insulating and superconducting phases of a disordered two-dimensional system at zero temperature is of continuous quantum nature, but the system behaves like a normal metal right at the transition, i.e. the conductivity has a finite, nonzero value. This value is *universal* and, apparently, equal to $(2e)^2/h$ (with $2e$ being the Cooper pair charge). There is strong experimental evidence [216, 279–282, 19] that a variety of systems (metallic films, high-T_c films, etc.) show the onset of superconductivity to occur when their sheet resistance falls below a value close to $h/4e^2 \approx 6.45\,\mathrm{k\Omega}$. The theoretical analysis here is based upon boson (Cooper pairs) approach to superconductivity and the main conclusion is that in contrast to the case of localization of fermions in two dimensions, bosons exhibit a superconductor to insulator transition (as disorder grows) with the value of conductivity at the critical point being independent of microscopic details. A major theoretical problem arises to describe a crossover to such behavior e.g. in the quasi-two-dimensional case of BCS superconductivity as interplane coupling goes to zero.

So we are not short of theoretical problems in this important field of research. As to the experiments, certainly much is still to be done for unambiguous demonstration of the exotic possibility of the superconductivity of Anderson insulators.

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Appendix A. Spectral densities and criterion for localization

A convenient formalism to consider general properties of a disordered system is based upon exact eigenstate representation for an electron in a random field created by disorder. These eigenstates

$\phi_\nu(\boldsymbol{r})$ are formally defined by the Schroedinger equation

$$H\phi_\nu(\boldsymbol{r}) = \varepsilon_\nu\phi_\nu(\boldsymbol{r}) , \tag{A.1}$$

where H is the one-particle Hamiltonian of the disordered system under consideration, ε_ν are exact eigenvalues of electron energy in a random potential. Obviously, $\phi_\nu(\boldsymbol{r})$ and ε_ν are dependent on locations of scatterers $\boldsymbol{R}_n$ for a given realization of random field.

Let us define two-particle spectral densities [59, 7]:

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F}} = \frac{1}{N(E)}\left\langle \sum_{\nu\nu'} \phi^*_\nu(\boldsymbol{r})\phi_{\nu'}(\boldsymbol{r})\phi^*_{\nu'}(\boldsymbol{r}')\phi_\nu(\boldsymbol{r}')\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'})\right\rangle , \tag{A.2}$$

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle^{H} = \frac{1}{N(E)}\left\langle \sum_{\nu\nu'} |\phi_\nu(\boldsymbol{r})|^2|\phi_{\nu'}(\boldsymbol{r}')|^2\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'})\right\rangle , \tag{A.3}$$

where angular brackets denote averaging over disorder and

$$N(E) = \left\langle \sum_\nu |\phi_\nu(\boldsymbol{r})|^2\delta(E-\varepsilon_\nu)\right\rangle \tag{A.4}$$

is one-electron (average) density of states. Obviously Eq. (A.3) is just a correlation function of *local* densities of states in a disordered system. Spectral density given by Eq. (A.2) determines electronic transport [59]. The following general properties are easily verified using the completeness and orthonormality of functions $\phi(\boldsymbol{r})$:

$$\int d\boldsymbol{r}\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F}} = \delta(\omega) , \qquad \int d\omega\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F}} = \delta(\boldsymbol{r}-\boldsymbol{r}') \tag{A.5}$$

or for the Fourier-components:

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_{q=0} = \delta(\omega) , \qquad \int d\omega\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_q = 1 \tag{A.6}$$

and $\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_q \geq 0$. From general definitions given in Eqs. (A.2) and (A.3) it is clear that

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r})\rangle\!\rangle^{\mathrm{F}} = \langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r})\rangle\!\rangle^{H} , \tag{A.7}$$

i.e. these spectral densities coincide for $\boldsymbol{r} = \boldsymbol{r}'$.

Terms with $\varepsilon_\nu = \varepsilon_{\nu'}$ are in general present in Eqs. (A.2) and (A.3). However, if these states are extended the appropriate wave-functions $\phi_\nu(\boldsymbol{r})$ are normalized on the total volume Ω of the system and these contributions to Eqs. (A.2) and (A.3) are proportional to Ω^{-1} and vanish as $\Omega \to \infty$. Things change if states are localized. In this case states are normalized on a finite volume of the order of $\sim R^d_{\mathrm{loc}}$. This leads to the appearance of a $\delta(\omega)$-contribution to spectral densities:

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F},H} = A_E(\boldsymbol{r}-\boldsymbol{r}')\delta(\omega) + \rho_E^{\mathrm{F},H}(\boldsymbol{r}-\boldsymbol{r}'\omega) \tag{A.8}$$

or in momentum representation:

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_q^{\mathrm{F},H} = A_E(\boldsymbol{q})\delta(\omega) + \rho_E^{\mathrm{F},H}(\boldsymbol{q}\omega) , \tag{A.9}$$

where the second term is regular in ω. This singular behavior was proposed as a general criterion for localization [59]. It is easy to show that

$$A_E(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{N(E)}\left\langle \sum_\nu \delta(E-\varepsilon_\nu)|\phi_\nu(\boldsymbol{r})|^2|\phi_\nu(\boldsymbol{r}')|^2 \right\rangle \tag{A.10}$$

$$A_E = A_E(\boldsymbol{r}-\boldsymbol{r}')|_{r=r'} \sim R_{\mathrm{loc}}^{-d} .$$

$A_E(\boldsymbol{r}-\boldsymbol{r}')$ represents the so-called inverse participation ratio [27, 87]. Roughly speaking, its value at $\boldsymbol{r}=\boldsymbol{r}'$ is inversely proportional to the number of atomic orbitals which effectively form the quantum state ν.

These $\delta(\omega)$-singularities in spectral densities signal nonergodic behavior of the system in a localized state. This leads to a difference between the so-called adiabatic and isothermal response functions [100, 60, 7]. The intimate connection between localization and nonergodic behavior was already noted in the first paper by Anderson [1].

From general properties given by Eqs. (A.5) and (A.6) for $\boldsymbol{q}\to\boldsymbol{0}$ in the localization region we have [59]:

$$\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_{\boldsymbol{q}}^{\mathrm{F}} \approx [1 - R_{\mathrm{loc}}^2 q^2]\delta(\omega) + \cdots , \tag{A.11}$$

where

$$R_{\mathrm{loc}}^2 = \frac{1}{2dN(E)}\int \mathrm{d}^d r\, r^2 \left\langle \sum_\nu \delta(E-\varepsilon_\nu)|\phi_\nu(\boldsymbol{r})|^2|\phi_\nu(0)|^2 \right\rangle \tag{A.12}$$

defines the localization length. Delocalization leads to a smearing of the $\delta(\omega)$-singularity for finite q.

Spectral densities of Eqs. (A.2) and (A.3) can be expressed via two-particle Green's functions [7]. Using nonaveraged retarded and advanced Green's functions

$$G^{\mathrm{R}}(\boldsymbol{r}\boldsymbol{r}'E) = G^{\mathrm{A}*}(\boldsymbol{r}\boldsymbol{r}'E) = \sum_\nu \frac{\phi_\nu(\boldsymbol{r})\phi_\nu^*(\boldsymbol{r}')}{E-\varepsilon_\nu+\mathrm{i}\delta} , \tag{A.13}$$

we immediately get from Eqs. (A.2) and (A.3)

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E'}(\boldsymbol{r}') \rangle\!\rangle^{\mathrm{F}} = \frac{1}{2\pi^2 N(E)}\,\mathrm{Re}\{\langle G^{\mathrm{R}}(\boldsymbol{r}\boldsymbol{r}'E')G^{\mathrm{A}}(\boldsymbol{r}'\boldsymbol{r}E)\rangle - \langle G^{\mathrm{R,A}}(\boldsymbol{r}\boldsymbol{r}'E')G^{\mathrm{R,A}}(\boldsymbol{r}'\boldsymbol{r}E)\rangle\} , \tag{A.14}$$

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E'}(\boldsymbol{r}') \rangle\!\rangle^{II} = \frac{1}{2\pi^2 N(E)}\,\mathrm{Re}\{\langle G^{\mathrm{R}}(\boldsymbol{r}\boldsymbol{r}E')G^{\mathrm{A}}(\boldsymbol{r}'\boldsymbol{r}'E)\rangle - \langle G^{\mathrm{R,A}}(\boldsymbol{r}\boldsymbol{r}E')G^{\mathrm{R,A}}(\boldsymbol{r}'\boldsymbol{r}'E)\rangle\} . \tag{A.15}$$

In the momentum representation, Eq. (A.14) is equivalent to

$$\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_{\boldsymbol{q}}^{\mathrm{F}} = \frac{1}{\pi N(E)}\,\mathrm{Im}\{\Phi_E^{\mathrm{RA}}(\omega\boldsymbol{q}) - \Phi_E^{\mathrm{RR}}(\omega\boldsymbol{q})\} , \tag{A.16}$$

where

$$\Phi_E^{\mathrm{RA(R)}}(\boldsymbol{q}\omega) = -\frac{1}{2\pi\mathrm{i}}\sum_{\boldsymbol{p}\boldsymbol{p}'} \langle G^{\mathrm{R}}(\boldsymbol{p}+\boldsymbol{p}'_+E+\omega)G^{\mathrm{A(R)}}(\boldsymbol{p}'_-\boldsymbol{p}_-E)\rangle \tag{A.17}$$

and $p_{+-} = p \pm 1/2q$. It can be shown [53, 54] that $\Phi_E^{RR(AA)}(\omega q)$ are nonsingular for small ω and q. Accordingly, $\delta(\omega)$-singularity signalling localization can appear only from the first term in Eq. (A.16).

Appendix B. Linearized gap equation in disordered system

Let us consider the derivative of the linearized gap equation Eq. (3.55) used to determine T_c [157, 158]. Equation for Gorkov's anomalous Green's function in an inhomogeneous disordered system (before any averaging procedure) at $T = T_c$ takes the following form:

$$(\varepsilon_n^2 + \hat{\varepsilon}_r^2) F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = -T_c \sum_m V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m) F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_m) , \tag{B.1}$$

where $\varepsilon_n = (2n+1)\pi T_c$ and $V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m)$ is an effective interelectron potential, $\hat{\varepsilon}_r$ is a one electron energy operator (energy zero is at the Fermi energy). Define

$$\Delta(\boldsymbol{r}\boldsymbol{r}') = -2\hat{\varepsilon}_r \coth \frac{\hat{\varepsilon}_r}{2T_c} T_c \sum_n F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) \tag{B.2}$$

and assume the following relation between $\Delta(\boldsymbol{r}\boldsymbol{r}')$ and $F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$:

$$F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_r^2} T_c \sum_m V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_r^2} \hat{Q}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_m) \Delta(\boldsymbol{r}\boldsymbol{r}') , \tag{B.3}$$

where $\hat{Q}$ is some unknown operator. Then after substitution of Eq. (B.3) into Eq. (B.2) we get a BCS-like equation for T_c:

$$\Delta(\boldsymbol{r}\boldsymbol{r}') = -\hat{U}(\boldsymbol{r}\boldsymbol{r}') \frac{\tanh(\hat{\varepsilon}_r)/2T_c}{2\hat{\varepsilon}_r} \Delta(\boldsymbol{r}\boldsymbol{r}') , \tag{B.4}$$

where the operator of "effective" interaction is defined by

$$\hat{U}(\boldsymbol{r}\boldsymbol{r}') = 2\hat{\varepsilon}_r \coth\left(\frac{\hat{\varepsilon}_r}{2T_c}\right) T_c \sum_n \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_r^2} T_c \sum_m V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m)$$

$$\times \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_r^2} \hat{Q}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_m) 2\hat{\varepsilon}_r \coth(\hat{\varepsilon}_r/2T_c) . \tag{B.5}$$

From Eqs. (B.1)–(B.3) we obtain the following equation for $\hat{Q}$ (we drop $\boldsymbol{r}\boldsymbol{r}'$ for brevity):

$$\hat{Q}(\varepsilon_n) = 1 - T_c \sum_m V(\varepsilon_n - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}^2} \hat{Q}(\varepsilon_m)$$

$$+ 2\hat{\varepsilon} \coth\left(\frac{\hat{\varepsilon}}{2T_c}\right) T_c \sum_{n'} \frac{1}{\varepsilon_{n'}^2 + \hat{\varepsilon}^2} T_c \sum_m V(\varepsilon_{n'} - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}^2} \hat{Q}(\varepsilon_m) . \tag{B.6}$$

In case of weak coupling in the lowest order over interaction in Eq. (B.6) we can leave only the first term $\hat{Q}(\varepsilon_n) = 1$. Then Eq. (B.5) reduces to

$$\hat{U}(\boldsymbol{r}\boldsymbol{r}') = 2\hat{\varepsilon}_r \coth\left(\frac{\hat{\varepsilon}_r}{2T_c}\right) T_c \sum_n \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_r^2} T_c \sum_m V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_r^2} 2\hat{\varepsilon}_r \coth\left(\frac{\hat{\varepsilon}_r}{2T_c}\right) \tag{B.7}$$

and Eq. (B.4) completely determines T_c.

Using the usual definition of the superconducting gap

$$\Delta(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = T_c \sum_m V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m) F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_m) = -(\varepsilon_n^2 + \hat{\varepsilon}_r^2) F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) \tag{B.8}$$

it is easy to get

$$\Delta(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \hat{Q}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)\Delta(\boldsymbol{r}\boldsymbol{r}') \tag{B.9}$$

so that $\Delta(\boldsymbol{r}\boldsymbol{r}')$ represents the energy gap in the absence of frequency dispersion, while $\hat{Q}$ describes the frequency dependence of the energy gap.

Cooper pairing takes place in states which are time-reversed, thus in the exact eigenstate representation of an electron in disordered system we have

$$\Delta(\boldsymbol{r}\boldsymbol{r}') = \sum_\nu \Delta_\nu \phi_\nu^*(\boldsymbol{r}')\phi_\nu(\boldsymbol{r}) \tag{B.10}$$

and Eq. (B.4) gives

$$\Delta_\nu = -\sum_{\nu\nu'} \frac{1}{2\varepsilon_{\nu'}} \tanh\frac{\varepsilon_{\nu'}}{2T_c} U_{\nu\nu'}\Delta_{\nu'}\,, \tag{B.11}$$

where the kernel

$$U_{\nu\nu'} = \int d\boldsymbol{r} \int d\boldsymbol{r}'\, \phi_\nu^*(\boldsymbol{r})\phi_{\nu'}^*(\boldsymbol{r}')\hat{U}(\boldsymbol{r}\boldsymbol{r}')\phi_\nu(\boldsymbol{r}')\phi_\nu(\boldsymbol{r}) \tag{B.12}$$

has the form of a "Fock" matrix element of an effective interaction. From Eq. (B.7) we have

$$U_{\nu\nu'} = T_c^2 \frac{2\varepsilon_\nu\varepsilon_{\nu'}}{\tanh(\varepsilon_\nu/2T_c)\tanh(\varepsilon_{\nu'}/2T_c)} \sum_n \sum_m \frac{1}{\varepsilon_n^2 + \varepsilon_\nu^2} \frac{1}{\varepsilon_m^2 + \varepsilon_{\nu'}^2}$$
$$\times \int d\boldsymbol{r} \int d\boldsymbol{r}'\, \phi_\nu^*(\boldsymbol{r})\phi_{\nu'}^*(\boldsymbol{r}') V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m)\phi_\nu(\boldsymbol{r}')\phi_{\nu'}(\boldsymbol{r})\,. \tag{B.13}$$

It is convenient to rewrite Eq. (B.11) introducing summation over states belonging to some surface of constant energy with subsequent integration over energies:

$$\Delta_\nu = -\int_{-\infty}^{\infty} dE' \frac{1}{2E'} \tanh\left(\frac{E'}{2T_c}\right) \sum_{\nu' \in E'} \mathcal{N}(E') U_{\nu\nu'(E)}\Delta_{\nu'(E')}\,, \tag{B.14}$$

where $\mathcal{N}(E) = \sum_\nu \delta(E - \varepsilon_\nu)$.

Consider now averaging of the gap equation. Define

$$\Delta(E) = \frac{1}{N(E)} \left\langle \sum_{\nu} \Delta_{\nu} \delta(E - \varepsilon_{\nu}) \right\rangle , \tag{B.15}$$

i.e. the gap averaged over disorder and a surface of constant energy $E = \varepsilon_{\nu}$. Here as usual we denote $N(E) = \langle \mathcal{N}(E) \rangle$. Suppose now that $\Delta_{\nu} = \Delta(\varepsilon_{\nu}) = \Delta(E = \varepsilon_{\nu})$, i.e. that Δ_{ν} depends only on energy $E = \varepsilon_{\nu}$, but not on the quantum numbers ν. This is similar to the usual assumption of $\Delta(\boldsymbol{p})$ depending only on $|\boldsymbol{p}|$ in a homogeneous and isotropic system [152].

After the usual decoupling used e.g. in transforming Eq. (3.12) into (3.13), i.e. assuming the self-averaging of the gap, we obtain the following linearized gap equation determining T_c:

$$\Delta(E) = -\int_{-\infty}^{\infty} dE' \, K(E, E') \frac{1}{2E'} \tanh\left(\frac{E'}{2T_c}\right) \Delta(E') , \tag{B.16}$$

where

$$\begin{aligned} K(E, E') &= \frac{1}{N(E)} \left\langle \sum_{\nu\nu'} U_{\nu\nu'} \delta(E - \varepsilon_{\nu}) \delta(E' - \varepsilon_{\nu'}) \right\rangle \\ &= T_c^2 \sum_n \sum_m \left[\frac{2E}{\tanh(E/2T_c)} \frac{1}{\varepsilon_n^2 + E^2} \right] \left[\frac{2E'}{\tanh(E'/2T_c)} \frac{1}{\varepsilon_m^2 + E'^2} \right] \\ &\quad \times \int d\boldsymbol{r} \int d\boldsymbol{r}' \, V(\boldsymbol{r} - \boldsymbol{r}' \varepsilon_n - \varepsilon_m) \, \langle\!\langle \rho_E(\boldsymbol{r}) \rho_{E'}(\boldsymbol{r}') \rangle\!\rangle^{\mathrm{F}} , \end{aligned} \tag{B.17}$$

where we have again introduced the Gorkov–Berezinskii spectral density defined in Eq. (A.2). Effective interaction can be written as

$$V(\boldsymbol{r} - \boldsymbol{r}' \varepsilon_n - \varepsilon_m) = V_{\mathrm{p}}(\boldsymbol{r} - \boldsymbol{r}' \varepsilon_n - \varepsilon_m) + V_{\mathrm{C}}(\boldsymbol{r} - \boldsymbol{r}' \varepsilon_n - \varepsilon_m) , \tag{B.18}$$

i.e. as the sum of some kind of Boson-exchange attractive interaction V_{p} and Coulomb repulsion V_{C}, which leads to

$$K(E, E') = K_{\mathrm{p}}(E, E') + K_{\mathrm{C}}(E, E') . \tag{B.19}$$

Assuming $V_{\mathrm{C}}(\boldsymbol{r} - \boldsymbol{r}' \varepsilon_n - \varepsilon_m) = v(\boldsymbol{r} - \boldsymbol{r}')$, i.e. static approximation for Coulomb repulsion, we obtain

$$K_{\mathrm{C}}(E, E') = \int d\boldsymbol{r} \int d\boldsymbol{r}' \, v(\boldsymbol{r} - \boldsymbol{r}') \, \langle\!\langle \rho_E(\boldsymbol{r}) \rho_{E'}(\boldsymbol{r}') \rangle\!\rangle^{\mathrm{F}} \tag{B.20}$$

which coincides with Eq. (3.33) used above in our analysis of Coulomb repulsion within Cooper pairs in disordered systems. Above we have used the approximation of Eq. (3.52) to model K_{p} due to electron–phonon pairing mechanism (or similar model for some kind of excitonic pairing). In this case Eq. (B.16) reduces to Eq. (3.55).

Note that $V_{\mathrm{C}}(\boldsymbol{r} - \boldsymbol{r}' \varepsilon_n - \varepsilon_m)$ may be taken also as a dynamically screened Coulomb interaction. Then we must use the appropriate expressions for the dielectric function $\varepsilon(\boldsymbol{q}\omega_m)$ which may be found using the self-consistent theory of localization [98, 60]. Then after some tedious calculations we can get the expressions for $K_{\mathrm{C}}(E, E')$ which for small $|E - E'|$ practically coincide with those used by us above for the case of static short-range interactions [158].

Appendix C. Localization and *d*-wave pairing

There is a growing body of experimental evidence in high-T_c superconductors that indicate that the pairing state is of $d_{x^2-y^2}$ symmetry [283, 284]. In superconductors with an anisotropic order parameter, both magnetic and non-magnetic impurities are pair breaking. For d-wave symmetry, the effect of non-magnetic impurities is equivalent to magnetic impurities in s-wave superconductors [285, 286]. Effectively, this means that superconductivity in such systems cannot persist until disorder becomes high enough to transform the system into an Anderson insulator. The situation is different for the so-called extended s wave symmetry. This corresponds to an order parameter with uniform sign which could, in particular, vanish at certain directions in momentum space [287]. Point impurities are not pair breaking in this case, but they are "pair-weakening": for small impurity concentration, T_c decreases linearly with disorder, but the critical impurity concentration is formally infinite, i.e. Anderson's theorem works after essential isotropisation of the gap [288].

We shall present now some of the relevant equations along the lines of our discussion of the Anderson theorem in the main body of the review. Here, we partly follow Ref. [289]. We shall consider d-wave pairing on a two-dimensional lattice induced by the following interaction Hamiltonian:

$$H_{\text{int}} = -g\sum_{r}\hat{\Delta}_r^{\dagger}\hat{\Delta}_r \tag{C.1}$$

where $\boldsymbol{r}$ denotes lattice sites. This Hamiltonian corresponds to an instantaneous anisotropic attractive interaction with an implicit cutoff at a characteristic energy $\langle\omega\rangle$. In order to model $d_{x^2-y^2}$ symmetry we choose $\hat{\Delta}_r^{\dagger}$ in the following form:

$$\hat{\Delta}_r^{\dagger} = \frac{1}{\sqrt{2}}\sum_{\delta}\varepsilon_{\delta}(c_{r\uparrow}^{\dagger}c_{r+\delta\downarrow}^{\dagger} - c_{r\downarrow}^{\dagger}c_{r+\delta\uparrow}^{\dagger}) \tag{C.2}$$

with $\delta = \pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2$ being the lattice vectors, and $\varepsilon_{\pm e_1} = -\varepsilon_{\pm e_2} = 1$.

Next we can perform the analysis similar to that used in deriving Eqs. (3.7)–(3.16) and find that now we again have Eq. (3.16) determining the critical temperature T_c with the kernel $K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ in the exact eigenstates representation taking the following form:

$$\begin{aligned} K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) &= gT\left\langle\sum_{\mu\nu\delta\delta'}\varepsilon_{\delta}\varepsilon_{\delta'}\frac{\phi_{\mu}^{*}(\boldsymbol{r})\phi_{\nu}^{*}(\boldsymbol{r}+\boldsymbol{\delta})\phi_{\nu}(\boldsymbol{r}')\phi_{\mu}(\boldsymbol{r}'+\boldsymbol{\delta})}{(\varepsilon_{\nu}-\mathrm{i}\varepsilon_n)(\varepsilon_{\mu}+\mathrm{i}\varepsilon_n)}\right\rangle \\ &= \int_{-\infty}^{\infty}\mathrm{d}E\,N(E)\int_{-\infty}^{\infty}\mathrm{d}\omega\frac{\langle\!\langle d_E(\boldsymbol{r})d_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle}{(\mathrm{i}\varepsilon_n+E)(E+\omega-\mathrm{i}\varepsilon_n)}, \end{aligned} \tag{C.3}$$

where we have introduced the spectral density:

$$\langle\!\langle d_E(\boldsymbol{r})d_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle = \frac{1}{N(E)}\left\langle\sum_{\mu\nu}\sum_{\delta\delta'}\varepsilon_{\delta}\varepsilon_{\delta'}\phi_{\nu}^{*}(\boldsymbol{r}+\boldsymbol{\delta})\phi_{\mu}(\boldsymbol{r})\phi_{\mu}^{*}(\boldsymbol{r}')\phi_{\nu}(\boldsymbol{r}'+\boldsymbol{\delta}')\delta(E-\varepsilon_{\nu})\delta(E+\omega-\varepsilon_{\nu'})\right\rangle. \tag{C.4}$$

Now we can rewrite Eq. (3.16) for T_c as

$$1 = gT_c \int_{-\infty}^{\infty} dE\, N(E) \int_{-\infty}^{\infty} d\omega \sum_{\varepsilon_n} \frac{g(\omega)}{(E + i\varepsilon_n)(E + \omega - i\varepsilon_n)}, \tag{C.5}$$

where

$$g(\omega) = \int dr' \langle\langle d_E(r) d_{E+\omega}(r') \rangle\rangle = \langle\langle d_E d_{E+\omega} \rangle\rangle_{q=0} . \tag{C.6}$$

No sum rules similar to that given by Eqs. (A.5), (A.6) exist for the spectral density of Eq. (C.4). However, it can be easily expressed via the Green's functions and we obtain the following relations similar to those obtained in Appendix A:

$$\langle\langle d_E d_{E+\omega} \rangle\rangle^q = \frac{1}{\pi N(E)} \mathrm{Im}\{\Phi_{Ed}^{RA}(\omega q) - \Phi_{Ed}^{RR}(\omega q)\} , \tag{C.7}$$

where

$$\Phi_{Ed}^{RA(R)}(q\omega) = -\frac{1}{2\pi i} \sum_{pp'} \gamma_p^d \langle G^R(p_+ p'_+ E + \omega) G^{A(R)}(p'_- p_- E) \rangle \gamma_{p'}^d \tag{C.8}$$

with the vertices $\gamma_p^d = \cos p_x - \cos p_y$ for d-wave. If from now on we ignore the lattice effects then $\gamma_p^d = \cos 2\theta_p$, which corresponds to a gap function $\Delta(k) = \Delta(T) \cos 2\theta_p$, where θ_p is the polar angle in the plane [287]. Similar expressions will determine T_c for the case of anisotropic s-wave pairing with the vertices γ_p^d replaced by appropriate angle-dependent expressions [287].

Now we can write as usual:

$$g(\omega) = \frac{1}{\pi N(E)} \mathrm{Im}\{\Phi_{Ed}^{RA}(\omega q = 0)\} = \frac{1}{\pi N(E)} \mathrm{Im}\left\{ -\frac{1}{2\pi i} \sum_{pp'} \cos 2\theta_p \, \Phi_{pp'}^{RA}(E\omega q = 0) \cos 2\theta_{p'} \right\} . \tag{C.9}$$

Here $\Phi_{pp'}^{RA}(E\omega q = 0)$ obeys the $q = 0$ limit of the Bethe–Salpeter equation (Eq. (2.44)) which is easily transformed to the following kinetic equation [53]:

$$\left(\omega - \frac{i}{\tau}\right) \Phi_{pp'}^{RA}(E\omega) = -\Delta G_p \left[\delta(p - p') + \sum_{p''} U_{pp''}^{E}(\omega) \Phi_{p''p'}^{RA}(E\omega) \right] \tag{C.10}$$

with $\Delta G_p \equiv G^R(pE + \omega) - G^A(pE)$. If we replace in (C.10) the irreducible vertex by the bare vertex $U_0 = \rho V^2$, we obtain finally

$$g(\omega) = \frac{1}{4\pi} \frac{\tau}{1 + (\omega\tau)^2} \tag{C.11}$$

with the usual scattering rate $1/\tau = 2\pi\rho V^2 N(E)$. Inserting (C.11) in (C.5) and following the standard analysis [9] we obtain the well-known expression for the critical temperature variation [286] $\ln(T_{c0}/T_c) = \Psi(1/2 + 1/4\pi\tau T_c) - \Psi(1/2)$ which is similar to the case of magnetic impurity scattering in superconductors. However, here the normal potential scattering rate is operational leading to very fast degradation of T_c – superconducting state is completely destroyed for

$1/\tau > 1.76T_{c0}$. Actually, this result does not depend on the spatial dimensionality of the system, i.e. the same dependence works in three dimensions.

Effectively, this makes it impossible to reach the Anderson transition before superconductivity is destroyed: critical disorder for metal–insulator transition is determined by $1/\tau \sim E_F \gg T_c$. The only hope seems to be to analyze the quasi-two-dimensional case, where this critical disorder can be reduced due to a small enough interplane transfer integral w as in Eqs. (2.90), (2.91). Localization appears for $w < w_c = \sqrt{2}/\tau \exp(-\pi E_F \tau)$ and take as an estimate some $1/\tau \approx T_{c0}$, so that superconductivity is still possible, we can arrive at the following criterion of coexistence of localization and superconductivity:

$$w < T_{c0} \exp(-\pi E_F/T_{c0}) . \tag{C.12}$$

In a typical situation even for high-temperature superconductors we have $T_{c0} < 0.1E_F$ and the inequality in Eq. (C.12) can be satisfied only for extremely anisotropic systems with $w \ll T_{c0}$. Most known superconductors apparently fail in this respect. This probably makes d-wave pairing irrelevant for the main body of our review. It is then quite difficult to reconcile the existing data on the closeness of e.g. radiationally disordered high-T_c systems to the disorder-induced metal–insulator transition and all the evidence for d-wave pairing in these systems. However, this reasoning does not apply to the case of anisotropic s-wave pairing, where Anderson theorem effectively works for large degrees of disorder [287]. In this respect the experiments on disordering in high-T_c systems can become crucial in solving the problem of the nature of pairing (and thus of its microscopic mechanisms) in these systems.

Still, even in the case of d-wave pairing localization effects may become important and interesting, but for a quite different problem – that of localization of BCS-quasi-particles within the superconducting gap at relatively small disorder [290–293]. It is known that while in the pure d-wave superconductor, density of states close to the Fermi level is linear in energy $N(E) \sim E$ due to the gap nodes at the Fermi surface, the impurity scattering makes it finite at $E = 0$ [285]. In this sense the system becomes similar to the *normal* metal and we can calculate [290] the low lying quasi-particle contribution to conductivity $\sigma(\omega \to 0)$. This conductivity equals

$$\sigma \approx \frac{e^2}{2\pi\hbar}\frac{\xi_0}{a} , \tag{C.13}$$

where $\xi_0 = v_F/\pi\Delta_0$ is the superconducting coherence length and a is the lattice spacing (we assume $T = 0$). The surprising thing is that σ is *independent of the scattering rate* $1/\tau$, i.e. of disorder. For the two-dimensional case (applicable probably for high-T_c systems) we know that all states are localized with localization controlled by dimensionless conductance which now is equal to $g = \sigma/(e^2/2\pi\hbar) = \xi_0/a$. The value of g may be small enough in high temperature superconductors due to the small values of ξ_0, which are typically only slightly larger than the lattice constant. This can make localization effects important with BCS-quasi-particles forming a mobility gap in the vicinity of the Fermi level, leading to anomalies in the low temperature behavior of microwave conductivity and the penetration depth of a d-wave superconductor [290].

These results were first obtained [290] for point-like impurity scattering, later it was shown in Ref. [291] that the finite range of the impurity potential can lead to the nonuniversal disorder-dependent behavior of conductivity which becomes proportional to the normal state scattering

rate. Situation was further complicated by the claim made in Refs. [292, 293] that the more rigorous analysis leads to the density of states of the impure d-wave superconductor behaving as $N(E) \sim |E|^{\alpha}$ with $\alpha > 0$, but dependent on the type of disorder. The renormalization group for the conductivity then apparently leads to some kind of a fixed point of intermediate nature, suggesting the finite conductivity in two-dimensions. All these aspects of disorder and localization for d-wave superconductors deserve further intensive studies.

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