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I. M. KHALATNIKOV

Director, L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences, Moscow

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THEORY OF ELECTRON LOCALIZATION IN DISORDERED SYSTEMS

M. V. SADOVSKIĬ

Institute of Metal Physics, Ural Scientific Center, USSR Academy of Sciences, Sverdlovsk

Abstract

In this review article many aspects of the modern theory of electron localization in disordered systems are discussed and the general criterion for localization is formulated. This criterion is given as a requirement that a two-particle Green's function have a pole in terms of the frequency with a factorizable residue (in a momentum representation). A search for such a solution can be based on the use of a homogeneous Bethe-Salpeter equation, from which the point where the metallic phase is unstable (the mobility edge) can be determined but which does not describe the region of localized states. The self-consistent theory of localization of Vollhardt and Wölfle is extended to the space with a dimension $d > 2$ and the behavior of the principal physical quantities near the mobility edge is calculated. The mobility edge is situated in the "strong-coupling" region (which diverges in the limit $d \to 2$). This region is the exact analog of the "Ginzburg critical region" in the theory of critical phenomena, in which the perturbation theory breaks down. The analytic properties of the effective field theory, for an electron in a random field, are studied in the complex plane of the coupling constant. The role of finite-action nonlinear solutions (instantons) of the classical field equations in the formation of the "tail" in the density of states is demonstrated. A method of calculating the coefficient of the exponential function of the density of states is proposed. This method is based on the use of the dispersion relation over the coupling constant and on the correspondence with the standard theory of critical phenomena. It is demonstrated that a singular (pole) contribution, in terms of the frequency, to a two-particle Green's function with a factorizable residue, which corresponds to the proposed general criterion for localization, can be determined explicitly within the framework of the instanton approach. A unified approach for the search of instabilities in the system, giving rise to the localization, is formulated. This approach is based on the use of the effective-action formalism for composite fields. The Hartree–Fock corrections, resulting from interaction between the electrons, to the density of states and thermodynamic quantities near the mobility edge are examined. The localization corrections, which are linked directly to the probability for return of an electron, are found. It is shown that these corrections correspond to the formation of a band of singly occupied states below the Fermi level. A cusp in the state density at the Fermi level, which occurs in a "dirty" metal, is shown, within the framework of the self-consistent theory of localization, to smooth out in the insulator region. The correction to the density of states at the Fermi

level, however, diverges logarithmically in the entire region of localized states. The localization contribution to the polarization operator corresponding to a nonergodic behavior of the system, which accounts for the difference between the isothermal static response and the adiabatic static response, is analyzed. The isothermal static dielectric constant conserves the "metallic" behavior corresponding to the finite screening range even in the insulator phase: The "Fermi glass" screens the external static electric field.

Contents

1. Introduction

The concept of electron localization in disordered systems is central to the understanding of the modern theory of disordered systems. It is the foundation of the basic understanding of the energy spectrum and of the kinetic and other electronic properties of these systems [1, 2]. Formulated for the first time in a fundamental paper by Anderson [3], the concept of electron localization was developed qualitatively by Mott, who used it to formulate the foundations of electronic theory of disordered systems [1, 2].

The localization phenomenon has recently been discussed in many review articles and monographs [4–10] and its principal propositions are now well known. In spite of its importance, the problem of localization is, nonetheless, far from being solved satisfactorily. At issue here is principally our conceptual understanding of the behavior of electronic states near the so-called mobility edge and of the related question of the physical properties of the system, in which the Fermi level of electrons is situated near the mobility edge. The effect of electron–electron interaction, i.e., the relationship between the Mott and Anderson mechanisms for a metal-insulator transition in disordered systems, is yet to be studied extensively. The difficulty in understanding this relationship stems from the extreme mathematical complexity of this problem and from the fact that there is no direct experimental evidence for electron localization [1, 2]. In particular, it is very difficult to distinguish between disorder and electron–electron interaction in a real metal-insulator transition.

Let us summarize the principal propositions of the localization theory which is customarily based on the Anderson model [3]. In this model the electron which propagates in the regular lattice in a d-dimensional space is analyzed. Each lattice site has a random level E_j (j is the number of the lattice site in the lattice). It is assumed that there is a certain probability amplitude for the transition V_{ij} from the jth lattice site to the ith lattice site. This amplitude is usually assumed [3] to be nonvanishing and equal to a certain constant V for the transitions between the nearest neighbors. The energy levels E_j are assumed to be distributed independently at different sites and the energy distribution at a given site is usually assumed to be uniform

over a certain energy interval of width W. The qualitative results do not seem to depend too strongly on these assumptions. Another useful model for analyzing the localization is the model of free electrons in a field of point scatterers randomly distributed in a space with density ρ. Each of these point scatterers has the same scattering amplitude which we will denote by V [11].

If there is no disorder in the system ($W = 0$ in the Anderson model or $\rho = 0$ in the free-electron model), then the problem of the electronic spectrum can be solved in a straightforward manner. In the Anderson model the electronic states form a band of width $2ZV$, where Z is the number of nearest neighbors. An infinitely wide band of free electrons is formed in a similar manner. The introduction of disorder accounts for some important changes, giving rise to a strong dependence on the dimensionality of the space d. At $d = 1$ a disorder, however small, completely changes the nature of the electronic states, localizing all of them. In other words, the wave functions of these states begin to fall off exponentially in the coordinate space, while the static electrical conductivity of the system goes down to zero at $T = 0$ [12–15]. The two-dimensional ($d = 2$) systems are the limiting case ("lower critical dimensionality"). In these systems the electronic states presumably also become completely localized as a result of appearance of the slightest disorder. For $d > 2$ all the electronic states in the band become completely localized if the ratio W/V in the Anderson model is sufficiently large—larger than a certain critical ratio $(W/V)_c$, i.e., if the disorder is appreciable. If $W/V < (W/V)_c$, the electronic states become localized at the band edges but remain delocalized at the band center (Fig. 1a). This situation gives rise to critical values of the energy $\pm E_c$, which separates the regions of the localized states from those of the extended states, customarily called mobility edges. In a model of nearly free electrons a qualitative picture of the electronic states for $d > 2$ is also well known (Fig. 1b). If the Fermi level E_F is in the region of fairly high energies, the electronic states near it are just the plane waves that are slightly distorted by scattering. The importance of this scattering increases with decreasing Fermi energy toward the edge of the original band. A density of states "tail," which stems from the electron localization due to random-potential fluctuations of the scatterers [8], appears at

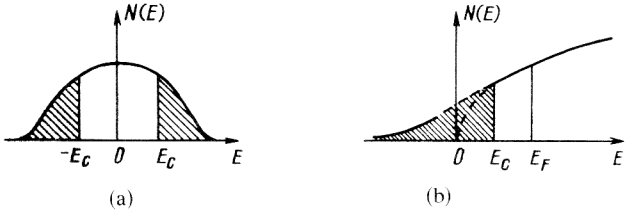

Figure 1. The density of states and location of the mobility edges in the Anderson model (a) and in the model of nearly free electrons (b). The hatched regions represent the localized states.

the band-edge and an energy E_c, which separates the delocalized states from the localized states (the mobility edge), appears near the former band edge. The term "mobility edge" is used because the localized states do not contribute to the static conductivity at zero temperature $T = 0$. If the Fermi level E_F of the many-electron system is at $T = 0$ in the energy region corresponding to the localized states, then the system is an insulator: conductivity appears only at $T \neq 0$ or when the electrons are excited by an alternating electromagnetic field. The conductivity in this case is realized by means of the hopping mechanism [1, 6]. If, however, the Fermi level is situated in the region of delocalized states, then the conductivity is metallic in nature. Such a metal-insulator transition, which occurs when the Fermi level crosses the mobility edge, is customarily called the Anderson transition.

One of the central questions raised in the theory is how the metallic state conductivity changes when the Fermi level E_F crosses the mobility edge E_c (at $T = 0$). Some possible alternatives are illustrated in Fig. 2 [1, 2, 16]. The conductivity may go down to zero discontinuously, after reaching a certain minimum value σ_{mm}—the minimum metallic conductivity. Such a behavior was suggested by Mott [1, 2] on the basis of a qualitative analysis of the conductivity in the Anderson model and on the basis of extensive experimental data. Theoretically, the conductivity can also decrease continuously down to zero, while the value σ_{mm} determines the conductivity scale for this continuous transition. This idea was initially suggested by Cohen

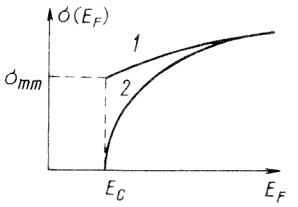

Figure 2. Variation of the metallic conductivity during the passage of the Fermi level through the mobility edge. 1—According to Mott [1, 2]; 2—according to Cohen [16].

[16], who basically used the analogy of the localization with percolation [6]. Clearly, either alternative leads (when the conductivity decreases sharply but continuously) to roughly the same experimental effects, at least at reasonably high temperatures. Highly accurate measurements at extremely low temperatures must, however, be performed before this problem can be solved experimentally. The experiments [17–19] in the temperature region of 10^{-3} K, which have recently been performed, tend to suggest that the conductivity goes continuously down to zero at the mobility edge. The current theoretical picture based on the scaling localization theory also supports this viewpoint [7].

A qualitative estimate of the typical conductivity scale near the mobility edge, as well as the estimate of its location in the band, can easily be obtained using the arguments advanced by Ioffe and Regel [20]. Their argument essentially reduces to the fact that the metallic conductivity remains in effect so long as the mean free path of an electron exceeds its wavelength. With increasing disorder, the corresponding inequality is disrupted and the nature of the conductivity changes considerably, principally because of the localization. In the case of the Anderson model with a half-filled band (one electron per site), we run across this situation when the mean free path l is of the order of the interatomic distance. Using the standard Drude equation $\sigma = ne^2\tau/m$ (n is the electron density, τ is the mean free time, and m is the electron mass) and setting $n \sim a^{-d}$ and $\tau \sim ma^2$, we thus estimate σ_{mm} to be (we will use everywhere the system of units $\hbar = 1$)

$$\sigma_{mm} \sim \text{const} \frac{e^2}{a^{d-2}}.$$

According to Mott [1, 2], the constant in this equation, which is specified by the critical ratio $(W/V)_c$, lies within the range 0.025–0.06 when $d = 3$, giving $\sigma_{mm} \sim 1\text{–}5 \times 10^2$ mho/cm, typically for $a \sim 2\text{–}3$ Å.

In a model of nearly free electrons the Ioffe–Regel criterion indicates that the Fermi energy E_F is of the order of the reciprocal mean free time, τ^{-1}. Using the standard Born expression to estimate this time $\tau^{-1}(E) \sim \rho V^2 N(E)$, where $N(E) \sim m^{d/2} E^{(d/2)-1}$ is the density of states of the free electrons, we estimate from the condition $E_c \sim \tau^{-1}(E_c)$ the mobility edge to be at

$$E_c \sim m^{d/(4-d)} (\rho V^2)^{2/(4-d)}$$

near the band edge. This characteristic energy will have an important role in our discussion below. This energy defines the energy region [7] at the edge of the band, in which the strong scattering by disorder becomes important. Reckoning the energy from the edge of the original band, we find

$$E_c \sim V \left(\frac{W}{V} \right)^{4/(4-d)}$$

for the Anderson model by using the same procedure. In fact, we see from these estimates the prominent role of the dimensionalities of space $d = 2$ and $d = 4$ in the localization problem.

The localization theory has recently been developed extensively. The development of this theory is attributable primarily to the formulation of the basic concepts and methods of the scaling theory of localization, which are based on the use of the present-day theory of critical phenomena [7]. In the theory of critical phenomena [21–23] the scaling description of the fluctuation region of the phase transitions is generally motivated by the growth of correlation length of the fluctuations of the order parameter near the critical point. An analog of this effect in the localization theory is the increase of the localization length of the wave functions as the mobility edge is approached from the side of the localized states [24]. We should emphasize, however, that there is no analog of such diverging length

in the region of extended states, just as there is no obvious order parameter associated with the Anderson transition. The various suggestions proposed in many papers (see review article [7]) are, from our viewpoint, largely not satisfactory. Immediately after the scaling model of the Anderson transition was initially suggested [24, 25], a considerable progress was, nonetheless, made in the understanding of this region. This progress was particularly rapid after Abrahams *et al.* [26] published a landmark paper, in which the scaling equations for the conductivity of a finite system, based principally on certain ideas advanced earlier by Thouless *et al.* [27–30], were formulated. This progress was particularly impressive in the analysis of two-dimensional systems [7, 9, 10], where it was possible to formulate not only totally new concepts but also to perform important experiments which have confirmed the theoretical predictions. The restrictions on the analogy with the phase transitions indicated above show, however, that the localization problem, on the whole, is much more complex than the problem of critical phenomena. Serious difficulties must yet be dealt with before this problem is finally solved.

In this review article we discuss in detail the theoretical results recently obtained in this field by the author. The main attention is focused on the discussion of the particular difficulties of the theory. The earlier theoretical studies in this field were elucidated in a review article by Sadovskiĭ [7], who also included in this study most of the experimental results, which we basically overlook in this article. The theoretical studies of other authors will be used only as the need arises. The literature cited in this article is by no means complete. An outline of this article is clear from the Contents.

2. The Electron in a Field of Random Scatterers and the Localization Theory

2.1. The General Criterion for Localization. Localization Viewed in Terms of the Bethe–Salpeter Equation

Two alternative methods can be used in a rigorous analysis of the localization phenomenon. The Anderson method, which was

described in a basic paper [3] and which was used until recently to get all the principal assertions of the theory, is distinguished by its nonconventional features. The main point of this method is an analysis of the Green's functions of an electron which have not been averaged over the random configurations. In a certain sense, the *most probable single-electron Green's function* is analyzed in this method, because the *average* single-particle Green's function, as is well known [3–5], contains no information on electron localization. This problem can, nonethelesss, be studied by analyzing the convergence of the stochastic series of the perturbation theory for an unaveraged single-particle Green's function [3, 31]. In particular, the condition of convergence (convergence in probability) of this series determines the location of the mobility edge. The Anderson method can be used to solve particular problems, ranging from the construction of the scaling theory of localization [24] to the analysis of new, specific models of disorder and effects of external fields [32]. A serious drawback of this method, however, is that it allows virtually no calculation of the observable physical quantities which are defined by the correlation functions (the Green's functions) that are averaged over the ensemble of random configurations of the system. These quantities can be calculated by using the well-developed formalism and a diagram technique [11, 33]. Unfortunately, the question of how the localization manifests itself in the principal quantities, such as the average Green's functions, which are used in the standard theory, has been studied inadequately until recently. The lack of sufficiently clear analysis of this problem leads to the difficulties in the solution of the localization problem and establishing a connection between this phenomenon and the observed characteristics of the system. Clearly, the problem of localization, for example, generally differs from that of the behavior of the electrical conductivity near the mobility edge, whose solution may be considerably more difficult.

In this section we will examine, according to Refs. 34 and 35, the general condition for localization in terms of the standard formalism, keeping in mind the traditional problem encountered in the Edwards method [11]. This problem involves the noninteracting electrons which move in a field of the scatterers which are randomly distributed in space.

The Hamiltonian of the system is

$$H = \int d^d\mathbf{r}\psi^+(\mathbf{r})\left\{-\frac{\nabla^2}{2m} + \sum_i V(\mathbf{r}-\mathbf{R}_i)\right\}\psi(\mathbf{r}), \tag{2.1}$$

where $V(\mathbf{r}-\mathbf{R}_i)$ is the potential of the scatterer (an impurity in the crystal, for example) which is situated at the point $\mathbf{R}_i$, and $\psi^+(\mathbf{r})$ and $\psi(\mathbf{r})$ are the electron creation and annihilation operators. For generality, we will examine a d-dimensional space. Let us introduce a complete orthonormal set of exact wave functions $\varphi_\nu(\mathbf{r})$ of the Hamiltonian (2.1)

$$H\varphi_\nu(\mathbf{r}) = \epsilon_\nu\varphi_\nu(\mathbf{r}), \tag{2.2}$$

where ϵ_ν are the exact eigenvalues of the energy of an electron in a field of random scatterers. Obviously each wave function $\varphi_\nu(\mathbf{r})$ and each eigenvalue of the electron energy ϵ_ν are also functionals of the location of the scatterers $\mathbf{R}_i$ in a given random system. Given that such a dependence exists, we will not write it out explicitly in the arguments of these functions. In the absence of an external magnetic field, we can assume, without loss of generality, that the functions $\phi_\nu(\mathbf{r})$ are real [36]. However, we will write out the complex conjugation explicitly. According to Berezinskiĭ and Gor'kov [34], define the two-particle spectral density is as

$$\langle\langle\rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle = \frac{1}{N(E)}\left\langle\sum_{\nu\nu'}\varphi^*_\nu(\mathbf{r})\varphi_{\nu'}(\mathbf{r})\varphi^*_{\nu'}(\mathbf{r}')\varphi_\nu(\mathbf{r}')\right.$$
$$\left.\times\,\delta(E-\epsilon_\nu)\delta(E+\omega-\epsilon_{\nu'})\right\rangle. \tag{2.3}$$

In this equation the angular brackets denote averaging over the random configurations of the scatterers, and

$$N(E) = \left\langle\sum_\nu \varphi_\nu(\mathbf{r})\varphi^*_\nu(\mathbf{r})\delta(E-\epsilon_\nu)\right\rangle \tag{2.4}$$

is the single-electron (average) density of states. The spectral density (2.3) has the following general properties [34], which can easily be checked by invoking the conditions under which the

function $\varphi_\nu(\mathbf{r})$ are complete and orthonormal:

$$\int d^d\mathbf{r}\langle\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle = \delta(\omega)$$
$$\int d\omega\langle\langle \rho_E(\mathbf{r})\rho_{E+\omega'}(\mathbf{r}')\rangle\rangle = \delta(\mathbf{r}-\mathbf{r}'). \tag{2.5}$$

For a spatial Fourier transform of (2.3) we find $\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_\mathbf{q}$ in an analogous manner,

$$\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{q}=0} = \delta(\omega)$$
$$\int d\omega\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_\mathbf{q} = 1; \qquad \langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_\mathbf{q} \geq 0. \tag{2.6}$$

In general, expression (2.3) has terms which correspond to the contribution of $\epsilon_\nu = \epsilon_{\nu'}$. If, however, the corresponding states of the system are extended states, then their wave functions are normalized to the total volume of the system Ω. For this reason, this contribution in (2.3), which is proportional to Ω^{-1}, vanishes in the limit of $\Omega \to \infty$. The situation changes dramatically if the states are localized. The wave functions in this case are normalized to the volume $\sim R_{\text{loc}}^d$, where R_{loc} is the localization length, and a simple estimate gives the value $\sim R_{\text{loc}}^{-d}$ for the contribution of $\epsilon_\nu = \epsilon_{\nu'}$ in (2.3). Thus a δ-function contribution in terms of the frequency ω appears in the spectral density (2.3)

$$\langle\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle = A_E(\mathbf{r}-\mathbf{r}')\delta(\omega) + \rho_E(\mathbf{r}-\mathbf{r}', \omega) \tag{2.7}$$

or in the momentum representation

$$\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_\mathbf{q} = A_E(\mathbf{q})\delta(\omega) + \rho_E(\mathbf{q}\omega), \tag{2.8}$$

where the second terms are regular terms in ω. The incorporation of such a singular contribution into (2.3) as a general localization criterion was suggested by Berezinskiĭ and Gor'kov [34]. It is easy to see that

$$A_E(\mathbf{r}-\mathbf{r}') = \frac{1}{N(E)}\left\langle \sum_\nu \delta(E-\epsilon_\nu)|\varphi_\nu(\mathbf{r})|^2|\varphi_\nu(\mathbf{r}')|^2 \right\rangle \tag{2.9}$$

is a generalized "inverse participation ratio." The nonvanishing of this ratio was frequently regarded as the localization criterion, alternative to the well-known Economou–Cohen criterion [4, 31, 37]. Roughly speaking, this ratio is inversely proportional to the number of atomic orbitals which contribute effectively to the superposition corresponding to the state ν.

The $\delta(\omega)$ singularity in the spectral density of (2.7) and (2.8), which is a manifestation of *nonergodicity* of the system [38], corresponds to the appearance of *time-independent* contributions in the corresponding correlation functions. Such contributions in this instance appear in the retarded density–density Green's function, for example [34]. It is important, however, that the $\delta(\omega)$ anomaly of the spectral density does not contribute to the *commutative* Green's functions, which determine the linear response in the Kubo theory [38, 39]. These anomalies account for the difference between the adiabatic static response and the isothermal static response of the system [39, 40]. The connection between the localization and the nonergodic behavior was, in fact, pointed out by Anderson in his first paper [3]. Later we will return to the discussion of the physical manifestations of this important effect.

From the general properties of (2.5) and (2.6) we find in the limit $\mathbf{q} \to 0$ (Ref. 34)

$$\langle\langle \rho_E \rho_{E+\omega} \rangle\rangle_{\mathbf{q}} \approx \{1 - R_{\text{loc}}^2(E)\mathbf{q}^2\}\delta(\omega) + \mathbf{q}^2 \rho_E(\omega), \tag{2.10}$$

where

$$R_{\text{loc}}^2(E) = \int_{-\infty}^{\infty} d\omega \rho_E(\omega)$$

$$= \frac{1}{2dN(E)} \int d^d r r^2 \left\langle \sum_{\nu} \delta(E - \epsilon_\nu) |\varphi_\nu(\mathbf{r})|^2 |\varphi_\nu(0)|^2 \right\rangle. \tag{2.11}$$

We can assume that this expression defines the localization length.

A delocalization corresponds to a spreading out of the δ-function singularity in (2.8) for the finite values of $\mathbf{q}$. The simplest example of such spreading, which accounts for the diffusional behavior, is

$$\langle\langle \rho_E \rho_{E+\omega} \rangle\rangle_{\mathbf{q}} = \frac{1}{\pi} \frac{D_E \mathbf{q}^2}{\omega^2 + (D_E \mathbf{q}^2)^2}. \tag{2.12}$$

As was emphasized in Ref. 34, this example is only a special case among the different possible versions of the mathematical behavior of $\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_{\mathbf{q}}$ in the limit $\mathbf{q} \to 0$.

It is easy to see [34] that the conductivity of the system is

$$\begin{aligned} \sigma(\omega) &= \lim_{\mathbf{q}\to 0} \left(-\frac{i\omega}{\mathbf{q}^2} \right) e^2 \chi^R(\mathbf{q}\omega) \\ &= \lim_{\mathbf{q}\to 0} \frac{i\omega}{\mathbf{q}^2} e^2 \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\omega' N(E) \frac{f(E) - f(E+\omega')}{\omega - \omega' + i\delta} \langle\!\langle \rho_E \rho_{E+\omega'} \rangle\!\rangle_{\mathbf{q}}, \end{aligned} \tag{2.13}$$

where $\chi^R(\mathbf{q}\omega)$ is the retarded response function of the density–density type, which is determined by the corresponding commutative Green's function [34], and e is the electronic charge. Using (2.8), we see that $A_E(\mathbf{q})$ does not contribute to the conductivity, consistent with the general property mentioned above. Using (2.10), we find

$$\begin{aligned} \sigma(\omega) &\simeq -i\omega e^2 \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\omega' N(E) \omega' \left(-\frac{\partial f(E)}{\partial E} \right) \frac{\rho_E(\omega')}{\omega - \omega' + i\delta} \\ \operatorname{Re} \sigma(\omega) &\simeq \pi e^2 \omega^2 N(E_F) \rho_{E_l}(\omega), \end{aligned} \tag{2.14}$$

so that at low frequencies the behavior of $\operatorname{Re} \sigma(\omega)$ is determined entirely by the $\rho_E(\omega)$ function or by the "regular" part of the spectral density (2.8). Thus, in such a general analysis, strictly speaking, only an indirect correlation [through the sum rules such as (2.5) and (2.6), for example] can be established between the "localization signal" [the quantity $A_E(\mathbf{q})$] and the frequency dependence of the conductivity. In the diffusion (metallic) regime, the use of (2.12) yields

$$\begin{aligned} \sigma(\omega) &\simeq \frac{e^2}{i\omega} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\omega' N(E) \left(-\frac{\partial f(E)}{\partial E} \right) \frac{1}{\pi} \frac{D_E \omega'}{\omega - \omega' + i\delta} \to \\ &\xrightarrow[\omega\to 0]{} e^2 N(E_F) D_{E_l}, \end{aligned} \tag{2.15}$$

i.e., we obtain the standard result (the Einstein relation) for the metallic conductivity of the noninteracting electrons, and the quantity D_E is the diffusion coefficient of the electron with energy E.

Since $A_E(\mathbf{q})$ in (2.8) "signals" the appearance of localized states with energy E, it would be useful [35] to determine it from the standard formalism (Green's functions). Using the standard unaveraged retarded (R) and advanced (A) single-electron Green's functions

$$G^{R,A}(\mathbf{r}\mathbf{r}'E) = \sum_{\nu} \frac{\varphi_\nu(\mathbf{r})\varphi_\nu^*(\mathbf{r}')}{E - \epsilon_\nu \pm i\delta} \tag{2.16}$$

and the definition (2.3), we immediately obtain

$$\begin{aligned} \langle\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle &= \frac{1}{\pi^2 N(E)} \langle \operatorname{Im} G^{R,A}(\mathbf{r}\mathbf{r}'E+\omega) \operatorname{Im} G^{R,A}(\mathbf{r}'\mathbf{r}E)\rangle \\ &= \frac{1}{2\pi^2 N(E)} \operatorname{Re}\{\langle G^R(\mathbf{r}\mathbf{r}'E+\omega) G^A(\mathbf{r}'\mathbf{r}E)\rangle \\ &\quad - \langle G^{R,A}(\mathbf{r}\mathbf{r}'E+\omega) G^{R,A}(\mathbf{r}'\mathbf{r}E)\rangle\} \end{aligned} \tag{2.17}$$

or we obtain the following expression in the momentum representation:

$$\langle\langle \rho_E \rho_{E+\omega}\rangle\rangle_{\mathbf{q}} = \frac{1}{\pi N(E)} \operatorname{Im}\{\phi_E^{RA}(\omega\mathbf{q}) - \phi_E^{RR}(\omega\mathbf{q})\}. \tag{2.18}$$

In this expression we have introduced, for brevity, the following notation for the two-particle Green's function [41]:

$$\phi_E^{RA(R)}(\mathbf{q}\omega) = -\frac{1}{2\pi i} \sum_{\mathbf{p}\mathbf{p}'} \langle G^R(\mathbf{p}_+\mathbf{p}'_+E+\omega) G^{A(R)}(\mathbf{p}'_-\mathbf{p}_-E)\rangle, \tag{2.19}$$

where $\mathbf{p}_\pm = \mathbf{p} \pm \frac{1}{2}\mathbf{q}$. For small values of $\mathbf{q}$ and ω the functions $\phi_E^{RR}(\mathbf{q}\omega)$ and $\phi_E^{AA}(\mathbf{q}\omega)$ are non singular functions [40, 41]. It is clear, therefore, that the δ-function component in (2.7) and (2.8), which corresponds to the appearance of localized states, may arise

only from the first term in (2.18). Using in (2.8) the formal transition from ω to $\omega + i\delta$ ($\delta \to 0^+$), i.e., "smearing out" the δ-function and comparing it with (2.18), we immediately obtain an expression for the amplitude of $A_E(\mathbf{q})$ in front of the δ-function in (2.8), which is written in terms of the two-particle Green's function

$$A_E(\mathbf{q}) = \lim_{\delta \to 0} \frac{1}{N(E)} \delta \operatorname{Im} \phi_E^{RA}(\omega + i\delta \mathbf{q})\Big|_{\omega=0}$$

$$= \frac{1}{2\pi N(E)} \lim_{\delta \to 0} \delta \sum_{\mathbf{pp}'} \operatorname{Re}\langle G^R(\mathbf{p}_+\mathbf{p}'_+ E + i\delta) G^A(\mathbf{p}'_-\mathbf{p}_- E - i\delta)\rangle. \tag{2.20}$$

Switching to the coordinate representation, we find

$$A_E(\mathbf{r} - \mathbf{r}') = \int \frac{d^d\mathbf{q}}{(2\pi)^d} e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')} A_E(\mathbf{q})$$

$$= \frac{1}{2\pi N(E)} \lim_{\delta \to 0} \delta\langle |G(\mathbf{r}\mathbf{r}' E + i\delta)|^2\rangle. \tag{2.21}$$

It would be useful to introduce the quantity

$$A_E = A_E(\mathbf{r} - \mathbf{r}')\Big|_{\mathbf{r}=\mathbf{r}'} = \int \frac{d^d\mathbf{q}}{(2\pi)^d} A_E(\mathbf{q})$$

$$= \frac{1}{2\pi N(E)} \lim_{\delta \to 0} \delta\langle |G(\mathbf{rr} E + i\delta)|^2\rangle$$

$$= \frac{1}{2\pi N(E)} \lim_{\delta \to 0} \sum_{\mathbf{pp}'\mathbf{q}} \operatorname{Re}\langle G^R(\mathbf{p}_+\mathbf{p}'_+ E + i\delta) G^A(\mathbf{p}'_-\mathbf{p}_- E - i\delta)\rangle. \tag{2.22}$$

In the region of localized states ($E < E_c$ "below" the mobility edge) we have $A_E > 0$. On the other hand, if we assume that the Green's function corresponds physically to the probability amplitude of the transition, we find that (2.22) is proportional to the average *probability of return* of the electron with a given energy E to the starting point in an infinite time [31]. The condition under which (2.22) is nonvanishing is therefore none other than the averaged version of

Economou–Cohen localization criterion [31]. The Economou–Cohen localization criterion in this sense is equivalent to the Berezinskiĭ–Gor'kov localization criterion [35].

As we have seen in (2.10), at $\mathbf{q}=0$ we have the general property $A_E(\mathbf{q}=0)=1$. We easily see that this property is equivalent to the condition

$$\phi_E^{RA}(0\omega)=-\frac{N(E)}{\omega}. \tag{2.23}$$

Such a divergence in the limit $\omega\to 0$ is a consequence of the conservation of the number of particles (of the equation of continuity) [41, 42].

To determine the value of $\Phi_E^{RA}(\mathbf{q}\omega)$ (2.19) we must be able to calculate the two-particle Green's function:

$$\phi_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega)=-\frac{1}{2\pi i}\langle G^R(\mathbf{p}_+\mathbf{p}'_+E+\omega)G^A(\mathbf{p}'_-\mathbf{p}_-E)\rangle. \tag{2.24}$$

Within the framework of Edwards' diagram technique, this function, as we know [11, 33, 41, 42], is determined by the Bethe–Salpeter integral equation (see Fig. 3):

$$\phi_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega)=G^R(E+\omega\mathbf{p}_+)G^A(E\mathbf{p}_-)\left\{-\frac{1}{2\pi i}\,\delta(\mathbf{p}-\mathbf{p}')+\sum_{\mathbf{p}''}U_{\mathbf{pp}''}^{E}(\mathbf{q}\omega)\phi_{\mathbf{p}''\mathbf{p}'}^{RA}(E\mathbf{q}\omega)\right\}, \tag{2.25}$$

where $G^{R,A}(E\mathbf{p})$ is the dressed, average, retarded (advanced), single-electron Green's function, and the irreducible vertex $U_{\mathbf{pp}'}^{E}(\mathbf{q}\omega)$ is determined by the sum of all the diagrams that cannot be cut along two lines (the advanced and the retarded line).

Let us see [35] whether a solution of Eq. (2.25) leads to a two-particle Green's function that contains some features corresponding to a localization. We assume that $\Phi_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega)$ has a pole contribution in the energy region in which the system possesses localized states:

$$\phi_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega)=-\frac{\psi_{\mathbf{p}}^{\mathbf{q}}(E)\psi_{\mathbf{p}'}^{-\mathbf{q}}(E)}{\omega+i\delta}+\tilde{\phi}_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega), \tag{2.26}$$

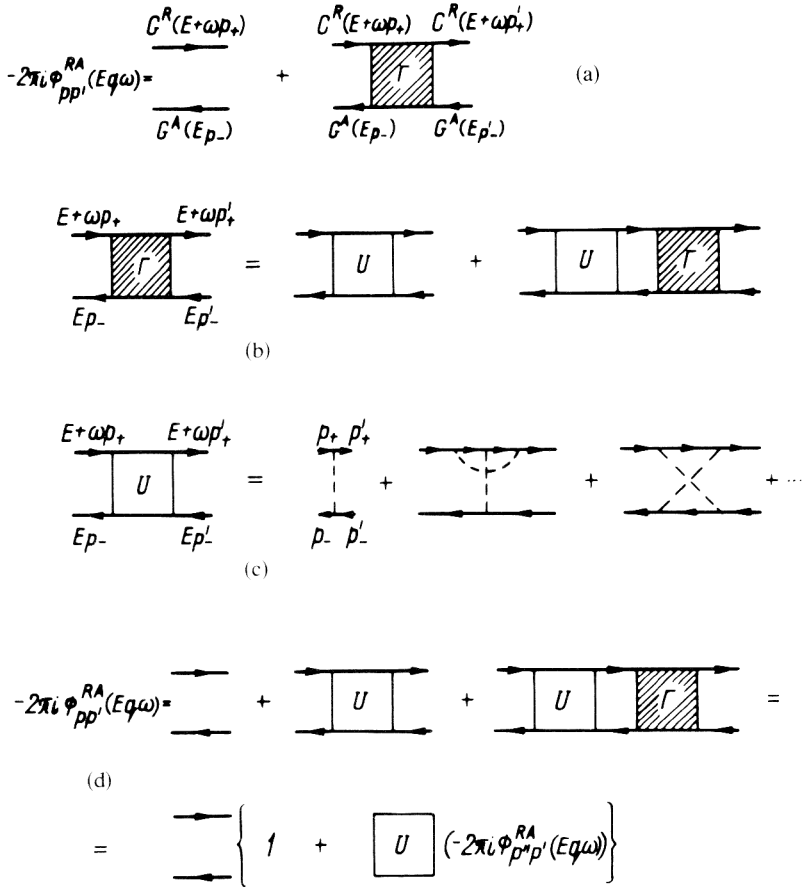

Figure 3. (a) Graphic definition of two-particle Green's function (2.24). (b) Equation for a complete vertex $\Gamma^{E}_{\mathbf{pp}'}(\mathbf{q}\omega)$. (c) Typical diagrams for an irreducible vertex $U^{E}_{\mathbf{pp}'}(\mathbf{q}\omega)$. (d) Bethe–Salpeter equation (2.25). The dashed curve denotes the "interaction" $U_0(\mathbf{p}-\mathbf{p}') = \rho|V(\mathbf{p}-\mathbf{p}')|^2$, where ρ is the density of the scatterers, and $V(\mathbf{p}-\mathbf{p}')$ is the Fourier transform of the potential of a single scattering center.

where $\tilde{\Phi}^{RA}_{\mathbf{pp}'}(E\mathbf{q}\omega)$ is the regular part, and the factorization of the residue at the pole (in the momentum representation) is assumed in analogy with the problem of the bound state. We will see later that this assumption may in a certain sense be justified.

From (2.19) and (2.26) we find

$$\phi_E^{RA}(\mathbf{q}\omega) = -\frac{\chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E)}{\omega + i\delta} + \sum_{\mathbf{pp}'} \tilde{\phi}_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega). \tag{2.27}$$

In this equation

$$\chi_{\mathbf{q}}(E) = \sum_{\mathbf{p}} \psi_{\mathbf{p}}^{\mathbf{q}}(E). \tag{2.28}$$

It then follows from (2.20) that

$$A_E(\mathbf{q}) = \frac{1}{N(E)} \chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E). \tag{2.29}$$

It is easy to see that $\chi_{\mathbf{q}}^*(E) = \chi_{-\mathbf{q}}(E)$, and $A_E(\mathbf{q}) > 0$. The general property of $A_E(\mathbf{q} = 0) = 1$ reveals the normalization condition,

$$\chi_0(E) = \sqrt{N(E)} \tag{2.30}$$

and from the asymptotic behavior of (2.10) we find

$$\chi_{\pm\mathbf{q}}(E) \approx \sqrt{N(E)}\{1 \pm iR_{\text{loc}}(E)\mathbf{q}\}^{-1}; \quad \mathbf{q} \to 0. \tag{2.31}$$

For the "probability of return" (2.22) it follows from these equations that

$$A_E = \frac{1}{N(E)} \sum_{\mathbf{q}} \chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E). \tag{2.32}$$

After the substitution of (2.26) into (2.25) the pole term is dominant (in the limit $\omega \to 0$) and we obtain a *homogeneous* Bethe–Salpeter integral equation for $\psi_{\mathbf{p}}^{\mathbf{q}}(E)$,

$$\psi_{\mathbf{p}}^{\mathbf{q}}(E) = G^R(E\mathbf{p}_+)G^A(E\mathbf{p}_-) \sum_{\mathbf{p}'} U_{\mathbf{pp}'}^E(\mathbf{q}\omega = 0)\psi_{\mathbf{p}'}^{\mathbf{q}}(E). \tag{2.33}$$

The localization would correspond to a nontrivial solution $\psi_{\mathbf{p}}^{\mathbf{q}}(E) \neq 0$ of Eq. (2.33), which would remain nonvanishing in the entire energy region $E \leqslant E_c$, where E_c is the mobility edge. It may turn out,

however, that we can use Eq. (2.33) only to determine the location of the mobility edge E_c as a point on the energy axis at which we find the first nontrivial solution of (2.33). To describe the region $E < E_c$, however, may require going beyond the scope of the perturbation theory, which defines the structure of the Bethe–Salpeter equation. Equation (2.33) obviously cannot be analyzed in its general form. Some approximations must therefore be used for the functions $G^{R,A}(E\mathbf{p})$ and $U^{E}_{\mathbf{pp}'}(\mathbf{q}\omega = 0)$. We will use the simplest approximation for $G^{R,A}(E\mathbf{p})$, which is valid in the region in which the perturbation theory is applicable [11, 33, 42]:

$$G^{R,A}(E\mathbf{p}) = \frac{1}{E - \dfrac{\mathbf{p}^2}{2m} \pm i\gamma(E)}, \tag{2.34}$$

where

$$\gamma(E) = \pi\rho V^2 N(E) \tag{2.35}$$

is the "Born" frequency of the collisions of the electrons with the impurities which we will assume, for simplicity, to be point like. This result, as we know, is obtained by summing those diagrams in Fig. 4a

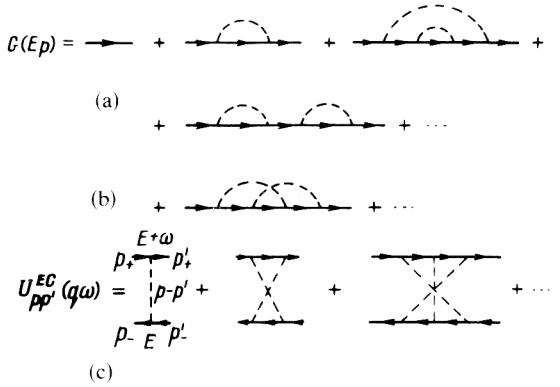

Figure 4. (a) Diagrams which determine the single-electron Green's function. (b). Langer–Neal diagrams for the irreducible vertex of the Bethe–Salpeter equation (cooperon).

which do not have any interesting "interaction" lines, i.e., diagrams similar to those shown in Fig. 4b. This approximation can be used if the inequality $E \gg \gamma(E)$ holds. As was noted by Gor'kov *et al.* [43] for the first time, the localization is linked, at least in the limit $d \to 2$, with a specific sequence of diagrams for $U^{E}_{\mathbf{pp}'}(\mathbf{q}\omega)$ that have a "maximum number of intersections". These diagrams are illustrated in Fig. 4b. Langer and Neal [44] studied this diagram sequence more than a decade earlier. This sequence can easily be summed by using Eq. (2.34). We thus find a "cooperon" propagator for small values of ω and $\mathbf{p}+\mathbf{p}'$ (Refs. 41 and 43),

$$U^{EC}_{\mathbf{pp}'}(\mathbf{q}\omega) = \frac{2\gamma(E)\rho V^2}{D_0^E(\mathbf{p}+\mathbf{p}')^2 - i\omega}, \tag{2.36}$$

where

$$D_0^E = \frac{E}{md\gamma(E)} \tag{2.37}$$

is a classical diffusion coefficient. In addition to the aforementioned condition under which the perturbation theory is applicable, we see that the condition for the applicability of the diffusion approximation $|\mathbf{p}+\mathbf{p}'| \ll l^{-1}$, where $l \sim \sqrt{E/m}\gamma^{-1}(E)$ is the mean free path, must also be satisfied in order for expression (2.36) to be valid. This condition is equivalent to $|\mathbf{p}+\mathbf{p}'| \ll \sqrt{2mE} \simeq \mathbf{p}_F$ ($\mathbf{p}_F$ is the Fermi momentum) near the mobility edge, where, as we will see below (see also the Introduction), $E \sim \gamma(E)$.

Using (2.34) and (2.36), we can finally write Eq. (2.33) as follows:

$$\left\{E - \frac{1}{2m}\left(\mathbf{p}+\frac{1}{2}\mathbf{q}\right)^2 + i\gamma(E)\right\}\left\{E - \frac{1}{2m}\left(\mathbf{p}-\frac{1}{2}\mathbf{q}\right)^2 - i\gamma(E)\right\}\psi^{\mathbf{q}}_{\mathbf{p}}(E)$$

$$= \lambda(E)\int \frac{d^d\mathbf{p}'}{(2\pi)^d}\frac{\psi^{\mathbf{q}}_{\mathbf{p}'}(E)}{(\mathbf{p}+\mathbf{p}')^2}, \tag{2.38}$$

where $\lambda(E) = 2dm\gamma^2(E)(\rho V^2/E)$. Switching to dimensionless variables $\mathbf{p} \to \mathbf{p}/\sqrt{2mE}$, we can write this equation as follows:

$$\psi^{\mathbf{q}}_{\mathbf{p}}(E) = \frac{\lambda_E}{\left\{1 - \left(\mathbf{p}-\frac{1}{2}\mathbf{q}\right)^2 + \frac{i\gamma}{E}\right\}\left\{1 - \left(\mathbf{p}+\frac{1}{2}\mathbf{q}\right)^2 - \frac{i\gamma}{E}\right\}}\int d^d\mathbf{p}'\,\frac{\psi^{\mathbf{q}}_{\mathbf{p}'}(E)}{|\mathbf{p}-\mathbf{p}'|^2}, \tag{2.39}$$

where $\lambda_E = 4m^2/(2\pi)^d (2mE)^{(d/2)-3} \lambda(E)$. The integration in (2.39) is over d-dimensional sphere of radius $\sim p_0/\sqrt{2mE}$, where $p_0 \sim a^{-1}$, and a is the "minimum length" in the system.

The homogeneous integral equation in (2.39) resembles a similar equation in the Wick–Cutkosky relativistic model [45–47], which describes the bound states of the scalar particles produced as a result of massless quantum exchange. We can easily transform to a symmetrized equation

$$\tilde{\psi}^{\mathbf{q}}_{-\mathbf{p}}(E) = \lambda_E \int d^d\mathbf{p}' K^E_{\mathbf{q}}(\mathbf{p}, \mathbf{p}') \tilde{\psi}^{\mathbf{q}}_{\mathbf{p}'}(E), \tag{2.40}$$

where

$$\begin{aligned} &\tilde{\psi}^{\mathbf{q}}_{-\mathbf{p}}(E) = \sqrt{R^{-1}_{\mathbf{q}}(\mathbf{p})}\, \psi^{\mathbf{q}}_{-\mathbf{p}}(E) \\ &R_{\mathbf{q}}(\mathbf{p}) = \left\{1 - \left(\mathbf{p} + \frac{1}{2}\mathbf{q}\right)^2 + \frac{i\gamma}{E}\right\}^{-1} \left\{1 - \left(\mathbf{p} - \frac{1}{2}\mathbf{q}\right)^2 - \frac{i\gamma}{E}\right\}^{-1} \\ &R_{\mathbf{q}}(\mathbf{p}) = R^*_{\mathbf{q}}(-\mathbf{p}); \qquad R_{\mathbf{q}}(\mathbf{p}) = R^*_{-\mathbf{q}}(\mathbf{p}). \end{aligned} \tag{2.41}$$

Here

$$K^E_{\mathbf{q}}(\mathbf{p}, \mathbf{p}') = R^{1/2}_{\mathbf{q}}(\mathbf{p}) R^{1/2}_{\mathbf{q}}(-\mathbf{p}') \frac{1}{|\mathbf{p} - \mathbf{p}'|^2} \tag{2.42}$$

is a symmetrical (Hermitian) positive kernel [48]:

$$\begin{gathered} K^E_{\mathbf{q}}(\mathbf{p}, \mathbf{p}') = K^{*E}_{\mathbf{q}}(\mathbf{p}', \mathbf{p}); \qquad K^E_{-\mathbf{q}}(\mathbf{p}, \mathbf{p}') = K^{*E}_{\mathbf{q}}(\mathbf{p}, \mathbf{p}') \\ K^E_{\mathbf{o}}(\mathbf{p}, \mathbf{p}') > 0. \end{gathered} \tag{2.43}$$

It is clear that

$$|K^E_{\mathbf{q}}(\mathbf{p}, \mathbf{p}')| < \frac{E^2}{\gamma^2} \frac{1}{|\mathbf{p} - \mathbf{p}'|^2}. \tag{2.44}$$

We see that at $2 < d < 4$ this equation is an equation with a weak singularity [48] and at $d \geq 4$ we have the Fredholm equation. The integral kernel which we are considering, being a fully continuous

operator, has a finite or denumerable eigenvalue spectrum which lies on the segment of the real axis. The length of this segment is determined by the norm of the integral operator $\|K\|$ which in this case is given by

$$\|K\psi\|^2 \equiv \left\{ \int d^d\mathbf{p}' K_{\mathbf{q}}^E(\mathbf{p}, \mathbf{p}')\psi_{\mathbf{p}}^{\mathbf{q}}(E) \right\}^2 \leqslant \|K\|^2 \left\{ \int d^d\mathbf{p}' \psi_{\mathbf{p}}^{\mathbf{q}}(E) \right\}^2. \quad (2.45)$$

This inequality is valid for any quadratically integrable function $\psi_{\mathbf{p}}^{\mathbf{q}}(E)$. The integrability of $\psi_{\mathbf{p}}^{\mathbf{q}}(E)$ is necessary for the solution of (2.39) to define $\chi_{\mathbf{q}}(E)$ from (2.28). It has also been established elsewhere [48] that the first eigenvalue of the kernel under consideration is a simple, positive eigenvalue, while the corresponding eigenfunction is a positive definite eigenfunction. We see from (2.44) that the norm of the operator under consideration is

$$\|K\| < \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{E^2}{\gamma^2} \frac{D_\Omega^{d-2}}{d-2}, \quad (2.46)$$

where D_Ω is the diameter of the integration range. It may seem that $D_\Omega \sim p_0/\sqrt{2mE}$. We easily see, however, that a factor such as $\{(1-\mathbf{p}'^2)^2 + \gamma^2/E^2\}^{-1/2}$, which at $\gamma \ll E$ has a sharp peak when $\mathbf{p}' \simeq 1$, whose width is $\sim \gamma/E$. The cutoff therefore occurs when $\mathbf{p}' \sim 1$ (i.e., $\mathbf{p}' \sim \sqrt{2mE}$ in dimensional variables), so that $D_\Omega \sim 1$. We can infer from this fact that the equation we are considering *does not have* any nontrivial solutions for

$$\lambda_E < \left\{ \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{d-2} \frac{E^2}{\gamma^2} \right\}^{-1}. \quad (2.47)$$

We see that there are no nontrivial solutions in the energy region given by

$$E > \left(\frac{A_d}{d-2} \right)^{2/(4-d)} E_{sc}. \quad (2.48)$$

In this inequality we have introduced the characteristic energy (compare with the Introduction) (Refs. 7 and 35):

$$E_{sc} = m^{d/(4-d)}(\rho V^2)^{2/(4-d)}, \tag{2.49}$$

where $A_d = 2^{1-d/2}\pi^{-d/2}d/\Gamma(d/2)$. As we have already seen in the Introduction, the condition $E \sim E_{sc}$, which is equivalent to $\gamma(E) \sim E$, define the limit of applicability of the perturbation theory. Since this choice of diagrams is not valid in the "strong-coupling" region $E \leqslant E_{sc}$, we must sum all the perturbation-theory diagrams. We see from (2.48) that the threshold energy E_c, where the first nontrivial solution of Eq. (2.38) appears, at $d = 3$ falls directly into the strong-coupling region $E < E_{sc} = m^3(\rho V^2)^2$. The energy region, where there is clearly no solution, goes to infinity in the limit $d \to 2$. This means that the first eigenvalue of Eq. (2.38), which is identified with the mobility edge, also goes to infinity in the limit $d \to 2$: $E_c \to \infty$, consistent with the proposition that a complete localization occurs at $d = 2$ [26]. Since $E_c \gg E_{sc}$, the mobility edge may seem, in the limit $d \to 2$, to be in the region where the perturbation theory is valid. We will see below that this is not true, since the condition under which the perturbation theory is valid in the limit $d \to 2$ must be defined more accurately. In fact, it coincides with inequality (2.48).

This analysis shows qualitatively that the homogeneous Bethe–Salpeter equation of the type in (2.33), which is based on the perturbation theory, cannot correctly describe the energy region corresponding to the localized states, if only because of the discrete nature of its spectrum in the region $E > E_c$. This discrete nature of the spectrum will probably be found even in more complex approximations. It is clear, however, that the location of the mobility edge, the "instability" point of the metallic phase, where appears the first nontrivial solution $\psi_{\mathbf{p}}^{\mathbf{q}}(E) \neq 0$ can be determined from this equation by some approximation for $G^{R,A}(E\mathbf{p})$ and $U_{\mathbf{pp}'}^{E}(\mathbf{q}\omega)$. We will give later a complete justification of this viewpoint.

2.2. *Self-Consistent Theory of Localization*

An important step in the evolution of the localization theory, which made it possible to calculate many physical quantities within the framework of the standard formalism of the average Green's func-

tions, was, in our view, the development of the so-called self-consistent theory of localization. The original version of this theory was proposed by Götze [49–51]. A significant contribution to the development of this theory is attributed to Vollhardt and Wölfe [41, 52, 53] who were responsible for the diagram formulation of this theory, based on a self-consistent generalization of the formalism of the Bethe–Salpeter equation. The main advantage of this method is its simplicity and a certain automatism allowing further generalization.

Using the following relation from (2.25)

$$G^R(E+\omega\mathbf{p}_+)G^A(E\mathbf{p}_-) = -\frac{\Delta G_{\mathbf{p}}}{\omega - \frac{1}{m}\mathbf{p}\cdot\mathbf{q} - \Sigma^R(E+\omega\mathbf{p}_+) + \Sigma^A(E\mathbf{p}_-)}, \tag{2.50}$$

where

$$\Delta G_{\mathbf{p}} = G^R(E+\omega\mathbf{p}_+) - G^A(E\mathbf{p}_-) \tag{2.51}$$

and $\Sigma^{R,A}(E\mathbf{p})$ is the self-energy of the corresponding Green's functions, we rewrite the Bethe–Salpeter equation in the form

$$\left\{\omega - \frac{1}{m}\mathbf{p}\cdot\mathbf{q} - \Sigma^R(E+\omega\mathbf{p}_+) + \Sigma^A(E\mathbf{p}_-)\right\}\phi^{RA}_{\mathbf{pp}'}(E\mathbf{q}\omega)$$
$$= \Delta G_{\mathbf{p}}\left\{\frac{1}{2\pi i}\delta(\mathbf{p}-\mathbf{p}') - \sum_{\mathbf{p}''} U^E_{\mathbf{pp}''}(\mathbf{q}\omega)\phi^{RA}_{\mathbf{p}''\mathbf{p}'}(E\mathbf{q}\omega)\right\}. \tag{2.52}$$

Bearing in mind that we are dealing with an electronic system at $T=0$, we will set $E=E_F$ at the outset. Summing both parts of (2.52) over $\mathbf{p}$ and $\mathbf{p}'$, and using the exact Ward identity [41]

$$\Sigma^R(E+\omega\mathbf{p}_+) - \Sigma^A(E\mathbf{p}_-) = \sum_{\mathbf{p}'} U^E_{\mathbf{pp}'}(\mathbf{q}\omega)\Delta G_{\mathbf{p}'}, \tag{2.53}$$

we find

$$\omega\phi^{RA}_{E_F}(\mathbf{q}\omega) - \tilde{\phi}^{RA}_{E_F}(\mathbf{q}\omega) = -N(E_F), \tag{2.54}$$

where

$$\tilde{\phi}^{RA}_{E_F}(\mathbf{q}\omega) = \sum_{\mathbf{pp}'} \frac{1}{m}(\mathbf{p}\cdot\mathbf{q})\phi^{RA}_{\mathbf{pp}'}(E_F\mathbf{q}\omega). \tag{2.55}$$

Analogously, multiplying (2.52) by $1/m(\mathbf{p}\cdot\mathbf{q})$ and summing again over $\mathbf{p}$, making use of (2.53) and the approximate representation [41]

$$\sum_{\mathbf{p}'} \phi^{RA}_{\mathbf{pp}'}(E_F\mathbf{q}\omega) \approx -\frac{\Delta G_{\mathbf{p}}}{2\pi i N(E_F)} \times \sum_{\mathbf{p}'\mathbf{p}''} \left\{1 + \frac{d}{\mathbf{p}_F^2\mathbf{q}^2}(\mathbf{p}\cdot\mathbf{q})(\mathbf{p}''\cdot\mathbf{q})\right\} \phi^{RA}_{\mathbf{p}''\mathbf{p}'}(E_F\mathbf{q}\omega), \tag{2.56}$$

in which we have left only the first two terms of the expansion in the corresponding angular variables,* we obtain the following expression, after several transformations:

$$\{\omega + M_{E_F}(\mathbf{q}\omega)\}\tilde{\phi}^{RA}_{E_F}(\mathbf{q}\omega) - \frac{2E_F}{dm}\mathbf{q}^2\phi^{RA}_{E_F}(\mathbf{q}\omega) = 0. \tag{2.57}$$

In this expression we have introduced a current relaxation kernel [41],

$$M_{E_F}(\mathbf{q}\omega) = 2i\gamma(E_F) + \frac{id}{2\pi N(E_F)\mathbf{p}_F^2\mathbf{q}^2} \times \sum_{\mathbf{pp}'} (\mathbf{p}\cdot\mathbf{q})\Delta G_{\mathbf{p}} U^{E_F}_{\mathbf{pp}'}(\mathbf{q}\omega)\Delta G_{\mathbf{p}'}(\mathbf{p}'\cdot\mathbf{q}). \tag{2.58}$$

In (2.58) we have used the simplest approximation (2.34) in writing out the explicit expressions for $\Sigma^{R,A}(E\mathbf{p})$. We can thus switch from the exact Bethe–Salpeter equation for $\phi^{RA}_{\mathbf{pp}'}(E_F\mathbf{q}\omega)$ to an approximate but *closed* system of equations (2.54) and (2.57), whose solution immediately gives the function $\phi^{RA}_{E_F}(\mathbf{q}\omega)$, which, according to the analysis in the preceding section, determine all the relevant characteristics of the system.

*The approximate validity (in the limits $\omega \to 0$ and $\mathbf{q} \to 0$) of (2.56) can easily be verified by summing both parts over $\mathbf{p}$, after multiplying them by $(\mathbf{p}\cdot\mathbf{q})$.

$$\phi^{RA}_{E_F}(\mathbf{q}\omega) = -N(E_F)\frac{\omega + M_{E_F}(\mathbf{q}\omega)}{\omega^2 + \omega M_{E_F}(\mathbf{q}\omega) - \dfrac{2E_F}{dm}\mathbf{q}^2}. \tag{2.59}$$

Let us examine how we can derive the localization phenomenon from these relations. We see that under the conditions of existence of the limit

$$R^2_{\mathbf{q}}(E_F) = -\frac{2E_F}{md}\lim_{\omega\to 0}\frac{1}{\omega M_{E_F}(\mathbf{q}\omega)} > 0 \tag{2.60}$$

we can extract from (2.59) a singular contribution of the type shown in (2.27)

$$\phi^{RA}_{E_F}(\mathbf{q}\omega) \underset{\omega\to 0}{\approx} -\frac{N(E_F)}{\omega}\frac{1}{1 - \dfrac{2E_F}{md}\dfrac{\mathbf{q}^2}{\omega M_{E_F}(\mathbf{q}\omega)}}$$

$$= -\frac{N(E_F)}{\omega}\frac{1}{1 + R^2_{\mathbf{q}}(E_F)\mathbf{q}^2} = -\frac{1}{\omega}\chi_{\mathbf{q}}(E_F)\chi_{-\mathbf{q}}(E_F). \tag{2.61}$$

where

$$\chi_{\pm\mathbf{q}}(E_F) = \frac{\sqrt{N(E_F)}}{1 + i\mathbf{q}R_{\mathbf{q}}(E_F)}. \tag{2.62}$$

These expressions satisfy all the general requirements of (2.29)–(2.31), in particular

$$A_{E_F}(\mathbf{q}) = \frac{1}{N(E_F)}\chi_{\mathbf{q}}(E_F)\chi_{-\mathbf{q}}(E_F) = \frac{1}{1 + R^2_{\mathbf{q}}(E_F)\mathbf{q}^2}$$

$$\underset{\mathbf{q}\to 0}{\approx} 1 - R^2_{\mathrm{loc}}(E_F)\mathbf{q}^2, \tag{2.63}$$

where the localization length is expressed as follows:

$$R^2_{\mathrm{loc}}(E_F) = -\frac{2E_F}{md}\lim_{\omega\to 0}\frac{1}{\omega M_{E_F}(0\omega)}. \tag{2.64}$$

It is also useful to introduce the characteristic frequency [41]

$$\omega_0^2(E_F) = -\lim_{\omega\to 0} \omega M_{E_F}(0\omega) > 0, \tag{2.65}$$

so that

$$R_{\text{loc}}(E_F) = \sqrt{\frac{2E_F}{md}}\, \omega_0^{-1}(E_F). \tag{2.66}$$

We see that the localization is equivalent to the requirement that the $\lim_{\omega\to 0} \omega M_{E_F}(0\omega)$ be finite, i.e., that $R_{\text{loc}}(E_F)$ or $\omega_0(E_F)$ be finite. In other words, it is equivalent to the requirement that the current relaxation kernel (2.58) be *divergent*: $\operatorname{Re} M_{E_F}(0\omega) = -\omega_0^2(E_F)/\omega$ in the limit $\omega \to 0$. Such a behavior of the relaxation kernel was regarded as the localization criterion in Refs. 41 and 49–53. We have derived it here from slightly more general considerations based on the localization criterion suggested above.

It is easy to see [41] that for small values of ω and $\mathbf{q}$ the retarded density–density response function $\chi^R(\mathbf{q}\omega)$ is given by

$$\chi^R(\mathbf{q}\omega) = \omega\phi_{E_F}^{RA}(\mathbf{q}\omega) + N(E_F) + 0(\omega, \mathbf{q}^2). \tag{2.67}$$

We then find from (2.59)

$$\chi^R(\mathbf{q}\omega) = -N(E_F)\frac{\dfrac{2E_F}{md}\mathbf{q}^2}{\omega^2 + \omega M_{E_F}(\mathbf{q}\omega) - \dfrac{2E_F}{md}\mathbf{q}^2}. \tag{2.68}$$

Ignoring ω^2 in the denominators of (2.59) and (2.68), we can write for small values of ω

$$\phi_{E_F}^{RA}(\mathbf{q}\omega) = -N(E_F)\frac{1}{\omega + iD_{E_F}(\mathbf{q}\omega)\mathbf{q}^2}, \tag{2.69}$$

$$\chi^R(\mathbf{q}\omega) = N(E_F)\frac{iD_{E_F}(\mathbf{q}\omega)\mathbf{q}^2}{\omega + iD_{E_F}(\mathbf{q}\omega)\mathbf{q}^2}. \tag{2.70}$$

In these expressions we have introduced, by definition, the *generalized* diffusion coefficient [41]

$$D_{E_F}(\mathbf{q}\omega) = i\frac{2E_F}{dm}\frac{1}{M_{E_F}(\mathbf{q}\omega)}. \tag{2.71}$$

We can then find from (2.13) and (2.68) a general expression for the conductivity

$$\sigma(\omega) = \frac{ne^2}{m}\frac{i}{\omega + M_{E_F}(0\omega)} \xrightarrow[\omega\to 0]{} e^2 D_{E_F}(00)N(E_F). \tag{2.72}$$

In the first part of this expression we assumed that $n/N(E_F) = 2E_F/d$, where n is the electron density. We see, in particular, that the usual metallic behavior (nonvanishing static conductivity at $T = 0$) corresponds to $M_{E_F}(0\omega \to 0) = i/\tau_{E_F}$, where τ_{E_F} serves as the generalized mean free time.

This discussion leads to the following behavior of the current relaxation kernel for $\mathbf{q} = 0$ and $\omega \to 0$:

$$M_{E_F}(0\omega) \simeq \begin{cases} \dfrac{i}{\tau_{E_F}}; & E_F > E_c \quad \text{(Metal)} \\ \dfrac{i}{\tau_{E_F}} - \dfrac{\omega_0^2(E_F)}{\omega}; & E_F < E_c \quad \text{(Insulator).} \end{cases} \tag{2.73}$$

Incorporating (2.73) into (2.59), using relation (2.18) and ignoring for the present the component $\operatorname{Im}\phi_E^{R\bar{R}}(\mathbf{q}\omega)$, which is nonsingular in the limits $\omega \to 0$ and $\mathbf{q} = 0$, we find the explicit expression for the Berezinskii–Gor'kov spectral density

$$\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_{\mathbf{p}} = \begin{cases} \dfrac{1}{\pi}\dfrac{D_{E_F}\mathbf{q}^2}{\omega^2 + (D_{E_F}\mathbf{q}^2)^2}; & E_F > E_c \\ A_{E_F}(\mathbf{q})\delta(\omega) + \dfrac{1}{\pi}\dfrac{D_{E_F}\mathbf{q}^2}{\omega^2 + [\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^2}; & E_F < E_c, \end{cases} \tag{2.74}$$

where [compare with Eq. (2.63)]

$$A_{E_F}(\mathbf{q}) = \frac{\omega_0^2(E_F)\tau_{E_F}}{\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2} = \frac{1}{1 + R_{\text{loc}}^2(E_F)\mathbf{q}^2}. \qquad (2.74')$$

In this expression we have introduced, by definition, the *renormalized* diffusion coefficient

$$D_{E_F} = \frac{2E_F}{dm}\tau_{E_F} = \frac{1}{d}V_F^2\tau_{E_F}, \qquad (2.75)$$

where v_F is the velocity of the electron at the Fermi surface.

Our task now is to find the behavior of the type of (2.73), from the *solution* of the general equations of the microscopic theory. After making use of approximation (2.34) for single-electron Green's functions, the current relaxation kernel $M_{E_F}(\mathbf{q}\omega)$ will be determined entirely by the approximation for the vertex $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$ in (2.58). The Vollhardt–Wolfle approximation [41] is based on the use of expression (2.36) for $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$, which is determined by the summation of the Langer–Neal diagrams (Fig. 4b). A transformation of the second term in (2.58) gives rise to the following contribution to $M_{E_F}(0\omega)$:

$$\delta M_{E_F}^c(0\omega) = -2\rho V^2 \sum_k \frac{1}{\omega + iD_0^{E_F}k^2}. \qquad (2.76)$$

Expression (2.76), however, falls short of achieving a localization. The basic idea behind the self-consistent theory of localization involves the replacement of the classical diffusion coefficient in the denominator of (2.76) by the generalized diffusion coefficient (2.71), which in turn is defined in terms of the current relaxation kernel [49–53]. As a result, we obtain the *self-consistent* equation for the kernal $M_{E_F}(0\omega)$ (Refs. 41 and 53)

$$M_{E_F}(\omega) = 2i\gamma(E_F)\left\{1 + \frac{1}{\pi N(E_F)}\sum_{|\mathbf{k}|<k_0}\frac{1}{-i\omega + D_{E_F}(\omega)k^2}\right\},$$
$$D_{E_F}(\omega) = i\frac{2E_F}{dm}\frac{1}{M_{E_F}(\omega)}. \qquad (2.77)$$

We will discuss in detail below how the choice of the cutoff momentum k_0 in (2.77) is made. The most rigorous derivation of Eq. (2.77), based on a general diagram analysis, in which the less singular components for $d \geq 2$ were discarded in the limit $\omega \to 0$, was made by Wölfle and Vollhardt [53]. We will analyze below several simplest corrections to (2.77).

Equation (2.77) was initially analyzed [41, 52] only for the two-dimensional case. The approximation based on the summation of the Langer–Neal diagrams was specially adapted for this case, since these diagrams lead to the contributions which are predominant in the limit $d \to 2$ (see the discussion below). We will analyze its solution, however, for $d > 2$, which was carried out independently in Refs. 53 and 54. In this case the results describing the location of the mobility edge in the band, as well as the behavior of all the principal physical quantities near it turn out, as we will see below, to be entirely justifiable and presumably qualitatively correct.

We see from the definition of $M_{E_F}(\mathbf{q}\omega)$ in (2.58) that since $\Delta G_{\mathbf{p}} \sim \operatorname{Im} G^R(E\mathbf{p}) \sim \delta(E_F - p^2/2m)$ [for $\gamma(E_F) \ll E_F$], the total momentum $\mathbf{k} = \mathbf{p} + \mathbf{p}'$ in $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$, which is defined by expression (2.36), being an integration momentum in (2.77) varies (in terms of the modulus) from zero to the momentum of order $2p_F$. On the other hand, expression (2.36) is, as was noted above, valid only for $|\mathbf{p} + \mathbf{p}'| < l^{-1}$, where $l \sim v_F \gamma^{-1}$ is the mean free path of an electron. We can therefore clearly see that the cutoff momentum in (2.77) is

$$k_0 \sim \operatorname{Min}\{p_F, l^{-1}\} \sim \operatorname{Min}\{\sqrt{2mE_F}; m^{(d+1)/2} \rho V^2 E_F^{(d-3)/2}\}. \quad (2.78)$$

It is evident that for $2 < d < 4$ the cutoff momentum is defined, in order of magnitude, by the Fermi momentum in the limit $E_F \to 0$ (the Fermi energy decreases toward the band edge) (Ref. 54; see also Refs. 49–51):

$$k_0 = x_0 p_F = x_0 \sqrt{2mE_F}, \quad (2.79)$$

where $x_0 = \text{const} \sim 1$–$2$. Such a choice of the cutoff is, in our opinion, unambiguous; however, the viewpoint of the authors of Refs. 41, 52, and 53 is not entirely clear in this regard. However, since $E_F \sim \gamma(E_F)$, we have $p_F \sim l^{-1}$ near the mobility edge, as was noted above, so that both choices are equivalent.

Switching in (2.77) to a dimensionless integration variable, we can write this equation in a form that is convenient for calculation

$$M_{E_F}(\omega) = 2i\gamma(E_F) + d\lambda x_0^{d-2} M_{E_F}(\omega) \int_0^1 dy y^{d-1} \frac{1}{y^2 - \dfrac{d\omega}{4x_0^2 E_F^2} M_{E_F}(\omega)}, \tag{2.80}$$

where

$$\lambda = \frac{\gamma(E_F)}{\pi E_F} = \left(\frac{m}{2\pi}\right)^{d/2} \frac{E_F^{(d/2)-1}}{\Gamma\left(\dfrac{d}{2}\right)} \rho V^2$$

$$= \frac{1}{(2\pi)^d \Gamma\left(\dfrac{d}{2}\right)} \left(\frac{E_F}{E_{sc}}\right)^{(d-4)/2} \tag{2.81}$$

is the dimensionless "coupling constant" of the theory in which we used the explicit form of the density of states of the free electrons,

$$N(E) = \left(\frac{m}{2\pi}\right)^{d/2} \frac{E^{(d/2)-1}}{\Gamma\left(\dfrac{d}{2}\right)},$$

as well as definition (2.49). The normal condition under which the perturbation theory is applicable implies that the inequality $\lambda \ll 1$ holds, i.e., $E_F \gg E_{sc}$ (for $d < 4$).

Assuming that $\omega = 0$ in (2.80) and analyzing the metallic regime in which, according to (2.73), Re $M_{E_F}(\omega = 0) = 0$ and Im $M_{E_F}(\omega = 0) = \tau_{E_F}^{-1}$, we find

$$\frac{i}{\tau_{E_F}} = 2i\gamma(E_F) + \frac{d}{d-2} \lambda x_0^{d-2} \frac{i}{\tau_{E_F}},$$

so that

$$\tau_{E_F} = \frac{1}{2\gamma(E_F)} \left\{1 - \frac{d}{d-2} \lambda x_0^{d-2}\right\}. \tag{2.82}$$

We then find from (2.72) the static conductivity

$$\sigma = \frac{ne^2}{m}\frac{1}{2\gamma(E_F)}\left\{1-\left(\frac{E_c}{E_F}\right)^{(4-d)/2}\right\}; \quad 2<d<4, \tag{2.83}$$

where

$$E_c = \left\{\frac{d}{d-2}\frac{x_0^{d-2}}{\Gamma\left(\frac{d}{2}\right)}(2\pi)^{-d/2}\right\}^{2/(4-d)} E_{sc}. \tag{2.84}$$

The energy E_{sc}, which is defined in (2.49), again appears in this case. We see from (2.83) that E_c serves as the mobility edge. We find that at $E_F \geqslant E_c$

$$\sigma \approx \frac{ne^2}{m}\frac{1}{2\gamma(E_F)}\left(\frac{4-d}{2}\right)\left(\frac{E_F-E_c}{E_c}\right); \quad 2<d<4, \tag{2.85}$$

so that the conductivity vanishes at the mobility edge linearly in $E_F - E_c$. The result in (2.84) coincides, to within a constant, with the estimate of E_c in (2.48), which is based on a different line of reasoning. For $d=3$ the threshold energy lies in the "strong-coupling" region $E_{sc} = m^3(\rho V^2)^2$, where our choice of diagrams is generally not correct [7, 35] and all the perturbation-theory diagrams must be taken into account. We clearly see this from the fact that the requirement $\lambda \ll 1$, according to (2.81), is equivalent to the condition $E \gg E_{sc}$. In the limit $d \to 2 E_c \to \infty$, consistent with the total localization in the case of arbitrarily small ($\lambda \ll 1$) disorder. The applicability of the perturbation theory in the limit $d \to 2$ will be discussed in greater detail below.

It is pertinent to analyze now the role of some diagrams which were not considered above. The irreducible vertex $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$ in (2.58) can be represented in a more complex form by supplementing the "cooperon" contribution (2.36) (Fig. 4b) by the "diffusion" contribution, i.e., by the diagrams (Fig. 5a) that contain the "diffusion" propagator [41, 42] (Fig. 5b, $\mathbf{q} \to 0$ and $\omega \to 0$):

$$\Gamma_{\mathbf{pp}'}^{E_F D}(\mathbf{q}\omega) = \frac{2\gamma(E_F)\rho V^2}{-i\omega + D_0^{E_F}\mathbf{q}^2}. \tag{2.86}$$

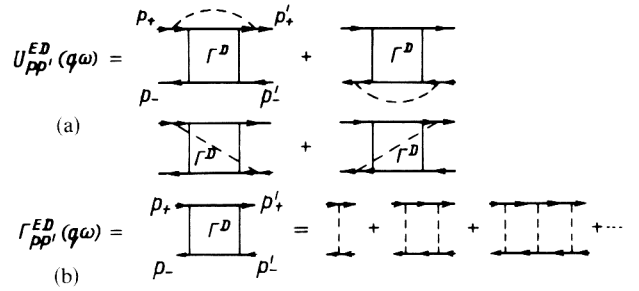

Figure 5. (a) A simplest "diffusion" contribution to the irreducible vertex $U^E_{\mathbf{pp}'}(\mathbf{q}\omega)$. (b) Diagrams for a diffusion "propagator" ("ladder" approximation).

The corresponding contribution to $M_{E_F}(\mathbf{q}\omega)$ in (2.58) is [41]

$$\delta M^D_{E_F}(\mathbf{q}\omega) = -\frac{d}{\mathbf{p}_F^2}\rho V^2 \sum_{\mathbf{k}} \frac{\mathbf{q}^2 + \mathbf{q}^{-2}(k\cdot\mathbf{q})}{\omega + iD_0^{E_F}k^2}. \tag{2.87}$$

Using this contribution and performing the self-consistency operation, we obtain the following equation for $M_{E_F}(\omega)$, instead of (2.77):

$$\begin{aligned} M_{E_F}(\omega) &= 2i\gamma(E_F)\left\{1 + \frac{1}{\pi N(E_F)}\sum_{|\mathbf{k}|<k_0}\frac{1 + k^2/2\mathbf{p}_F^2}{-i\omega + D_{E_F}(\omega)k^2}\right\} \\ D_{E_F}(\omega) &= i\frac{2E_F}{dm}\frac{1}{M_{E_F}(\omega)}. \end{aligned} \tag{2.88}$$

By analogy with (2.82), we then find for $\omega = 0$

$$\tau_{E_F} = \frac{1}{2\gamma(E_F)}\left\{1 - \left[\frac{d}{d-2} + \frac{x_0^2}{2}\right]x_0^{d-2}\lambda\right\} \tag{2.89}$$

in the metallic region. We see that the use of the diagrams in Fig. 5a changes the *constant* in (2.82). The cooperon contribution is dominant in the limit $d \to 2$, whereas at $d = 3$ the diffusion contribution is *of the same order of magnitude* as the cooperon contribution. The mobility edge shifts slightly: $d/d-2$ is replaced by

$d/d-2+x_0^2/2$ in (2.84). The "critical behavior," however, remains the same: conductivity goes down to zero in accordance with (2.85). We will therefore confine ourselves to approximation (2.77), which we feel "works" for $d \to 2$. At $d=3$, *all* the perturbation-theory diagrams must evidently be summed, both for $U_{\mathbf{pp}}^{E_F}(\mathbf{q}\omega)$ and for a single-electron Green's function, which is extremely important. The mobility edge $E_c \to \infty$ for $d \to 2$, and it may seem that we could restrict ourselves to the simplest approximation (2.34) in the case of a single-electron Green's function, since the condition $E_c \gg \gamma(E_c)$ in this case is satisfied. Actually the situation is, as we will see below, more complex even for $d \to 2$.

The use of approximation (2.36) for $U_{\mathbf{pp}}^{E_F}(\mathbf{q}\omega)$, along with approximation (2.34) for $G^{R,A}(E\mathbf{p})$, we might note, is generally in contradiction with Ward's identity (2.53). The use of "crossed" diagrams for $U_{\mathbf{pp}}^{E_F}(\mathbf{q}\omega)$ implies that the same diagrams must also be used for $\Sigma^{R,A}(E\mathbf{p})$ [see the derivation of (2.53) in Ref. 41]. At the same time, several examples of exactly solvable models [55–57], in which it was possible to sum the crossed diagrams (such as those in Fig. 4b) for $\Sigma^{R,A}(E\mathbf{p})$ show that the single-electron Green's function obtained in this manner has a structure radically different from that given by the simplest approximation (2.34). We also clearly see this from the fact that (2.34) does not give a correct single-electron state density such as the one shown in Fig. 1b. Specifically, Eq. (2.34) does not account for the "tail" in the density of states, which appears in the perturbation theory [55, 56] only when the crossed diagrams are taken into account (see the discussion below).

All these factors can, in our opinion, change the critical behavior such as that in (2.83), whereas the estimated location of the mobility edge (2.84) apparently remains the same (to within a constant) even in a more precise theory.

Turning our attention now to the region of localized states ($E_F < E_c$), we will seek a solution of Eq. (2.80) in the form of the second expression in (2.73) and find an equation, which determines $\omega_0^2(E_F)$, from the real part of (2.80) in the limit $\omega \to 0$,

$$1 = d\lambda x_0^{d-2} \int_0^1 dy y^{d-1} \frac{1}{y^2 + \dfrac{d\omega_0^2(E_F)}{4(x_0 E_F)^2}}. \tag{2.90}$$

Analogously, we find an equation for τ_{E_F} in the localization region from the imaginary part of (2.80) in the limit $\omega \to 0$

$$1 - 2\gamma(E_F)\tau_{E_F} = d\lambda x_0^{d-2} \int_0^1 dy y^{d+1} \frac{1}{\left[y^2 + \frac{d\omega_0^2(E_F)}{4(x_0 E_F)^2} \right]^2}. \quad (2.91)$$

Using (for $\omega_0^2 \to 0$) a simplest estimate of the integral in (2.90), we find

$$\begin{aligned} 1 &\simeq d\lambda x_0^{d-2} \int_{[(d\omega_0^2(E_F))/(4x_0^2 E_F^2)]^{1/2}}^1 dy y^{d-3} \\ &= \begin{cases} d\lambda x_0^{d-2} \dfrac{1}{d-2} \left\{ 1 - \left(\dfrac{d\omega_0^2(E_F)}{4x_0^2 E_F^2} \right)^{(d-2)/2} \right\}; & 2 < d < 4 \\ \lambda \ln \dfrac{2x_0^2 E_F}{\omega_0^2(E_F)}; & d = 2. \end{cases} \end{aligned} \quad (2.92)$$

Using (2.84), we then find

$$\omega_0^2(E_F) = \begin{cases} \dfrac{4}{d} x_0^2 E_F^2 \left\{ 1 - \left(\dfrac{E_F}{E_c} \right)^{(4-d)/2} \right\}^{2/(d-2)}; & 2 < d < 4 \\ 2x_0^2 E_F^2 \exp\left[-\dfrac{1}{\lambda} \right]; & d = 2. \end{cases} \quad (2.93)$$

The location of the mobility edge is given by the condition $\omega_0^2(E_F = E_c) = 0$. The integral in (2.90) can also be calculated exactly [54], then the first expression in (2.93) is multiplied by

$$\left[\frac{d}{d-2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(2 - \frac{d}{2}\right) \right]^{-2/(d-2)} \quad (\sim 1 \text{ for } d = 3),$$

while the second expression remains the same [53, 54]. Using the "unity representation" (2.90) in (2.91), we find a relationship between τ_{E_F} and $\omega_0^2(E_F)$:

$$2\gamma(E_F)\tau_{E_F} = d\lambda x_0^{d-2} \frac{d\omega_0^2(E_F)}{4x_0^2 E_F^2} \int_0^1 dy \frac{y^{d-1}}{\left[y^2 + \frac{d\omega_0^2(E_F)}{4x_0^2 E_F^2} \right]^2}$$

$$\approx d\lambda x_0^{d-2} \frac{1}{d-4} \left\{ \frac{d\omega_0^2(E_F)}{4x_0^2 E_F^2} - \left(\frac{d\omega_0^2(E_F)}{4x_0^2 E_F^2} \right)^{(d-2)/2} \right\}; \quad d < 4. \tag{2.94}$$

so that as $\omega_0^2(E_F) \to 0$, i.e., near the mobility edge we have

$$2\gamma(E_F)\tau_{E_F} \approx \begin{cases} \dfrac{d}{4-d} \lambda x_0^{d-2} \left(\dfrac{d\omega_0^2(E_F)}{4x_0^2 E_F^2} \right)^{(d-2)/2}; & 2 < d < 4 \\ \lambda \left\{ 1 - \dfrac{\omega_0^2(E_F)}{2x_0^2 E_F^2} \right\}; & d = 2. \end{cases} \tag{2.95}$$

We find from (2.66) and (2.93) the localization radius

$$R_{\text{loc}}(E_F) = \frac{1}{x_0 \sqrt{2mE_F}} \left\{ 1 - \left(\frac{E_F}{E_c} \right)^{(4-d)/2} \right\}^{-1/(d-2)}$$

$$\sim \frac{1}{\sqrt{2mE_F}} \left| \frac{E_F - E_c}{E_c} \right|^{-\nu}; \quad E_F \leqslant E_c \quad (2 < d < 4), \tag{2.96}$$

where the critical exponent of the localization radius is

$$\nu = \frac{1}{d-2}. \tag{2.97}$$

We see from (2.97) and (2.85) that this theory satisfies the Wegner scaling law for the critical exponent of the conductivity [25]:

$$t = (d-2)\nu. \tag{2.98}$$

The values of the critical exponents obtained here coincide with those obtained in the main approximation of the $\epsilon = d - 2$-expansion in the elementary scaling theory of localization [26, 58]. As was shown by Vollhardt and Wolfle [53], the basic equations of this scaling theory can be derived directly from the equations of the

self-consistent theory [53]. We feel, however, that these values of the critical exponents should not be taken too seriously, since they were determined by going beyond the limits of the perturbation theory, which was used as the basis for derivation of the basic equations of the theory.

Let us examine the dielectric constant as an example of a physical quantity calculated in the localized region. Using the relationship between the dielectric constant and the density–density response function [59]

$$\epsilon(\mathbf{q}\omega) = 1 - \frac{4\pi e^2}{\mathbf{q}^2}\chi^R(\mathbf{q}\omega), \tag{2.99}$$

we find from (2.68), (2.93), and (2.96)

$$\epsilon(0\omega \to 0) = 1 + \frac{\omega_p^2}{\omega_0^2(E_F)} = 1 + \kappa_D^2 R_{\text{loc}}^2(E_F) \sim \left|\frac{E_F - E_c}{E_c}\right|^{-2\nu}, \tag{2.100}$$

where $\omega_p^2 = 4\pi ne^2/m$ is the square of the plasma frequency, and $\kappa_D^2 = 4\pi e^2 N(E_F)$ is the inverse square of the screening length of the metal. Expression (2.100) gives the divergence of the dielectric constants as the insulator-metal transition is approached.

For $d = 2$ we find from (2.66) and (2.93) (Ref. 41)

$$R_{\text{loc}}(E_F) = \frac{1}{x_0\sqrt{2mE_F}} \exp\left\{\frac{\pi E_F}{m\rho V^2}\right\}. \tag{2.101}$$

Thus the localization length in this case is exponentially large for $E \geqslant m\rho V^2$, and we have a "weak localization." We feel it worthwhile to give detailed results on the frequency dependence of the conductivity for $d = 2$. (The equations for the frequency dependence of the conductivity for $d > 2$, which appear in this theory, may be found in Refs. 53 and 60.) A direct though slightly laborious analysis of Eq. (2.80) for $d = 2$ reveals that several frequency intervals, in which the conductivity behaves in a markedly different manner, can be singled out [54]. For extremely low frequencies $\omega \ll \lambda^{-1}\exp(-\lambda^{-1})\gamma(\lambda \ll 1)$ we find [41]

$$\sigma(\omega) \approx \frac{ne^2}{m}\frac{\gamma}{\lambda}\exp\left(\frac{2}{\lambda}\right)\frac{\omega^2}{2(x_0E_F)^4}. \tag{2.102}$$

At slightly higher frequencies $\lambda^{-1}\exp(-\lambda^{-1})\gamma \ll \omega \ll \lambda^{-2}\exp(-\lambda^{-1})\gamma$ we have

$$\sigma(\omega) \approx \frac{ne^2}{m}\exp\left(\frac{1}{\lambda}\right)\frac{\omega}{2(x_0 E_F)^2}. \tag{2.103}$$

With further increase of the frequency, we find that for $\lambda^{-2}\exp(-\lambda^{-1})\gamma \ll \omega \ll \lambda^2\gamma$ conductivity behaves in a logarithmic manner as obtained in Ref. 43:

$$\sigma(\omega) = \frac{ne^2}{m}\frac{1}{2\gamma}\left\{1 - \lambda \ln\frac{2\gamma}{\omega}\right\}. \tag{2.104}$$

Finally, for $\lambda^2\gamma \ll \omega \ll \gamma$ the self-consistent theory gives

$$\sigma(\omega) \approx \frac{ne^2}{m}\frac{1}{2\gamma}\left\{1 - \frac{\tilde{E}_c}{E_F}\right\}, \tag{2.105}$$

where $\tilde{E}_c \approx (m/\pi)\rho V^2 x^*$, with x^* defined as the biggest of the roots of the equation $x^* = \ln(x_0^2/\pi) + 2\ln x^*$ (solution exists for $x_0 > \sqrt{\pi}$). We see that the conductivity in this frequency range is essentially the same as the constant (frequency independent) metallic conductivity with the "mobility edge" $\tilde{E}_c$. This circumstance may conceivably reconcile certain discrepancies between the numerical data on two-dimensional conductivity obtained by different authors [7] and the concept of total two-dimensional localization: logarithmic corrections and the dielectric-like behavior are seen only at extremely low frequencies, although there is a broad (since λ is small) frequency range, within which a finite mobility edge $\tilde{E}_c$ can be "simulated" in the system. At $E_F \geq \tilde{E}_c$, the localization length, as we have seen, also begins to increase exponentially. However, $d = 2$ plays the role of the "lower critical dimensionality."

For $d > 4$ an analysis of Eq. (2.80) gives physically meaningless results [54]. The origin of these results is clearly traceable to the fact that, according to (2.81), the perturbation-theory expansion in this model is in the parameter $(E_F/E_{sc})^{(d-4)/2}$. Accordingly, at $d < 4$ the expansion diverges in the limit $E_F \to 0$, while at $d > 4$ it diverges in the limit $E_F \to \infty$; i.e., there is an "inversion" of the strong- and

weak-coupling regions on the energy axis. As a result, the conductivity vanishes, while the localization length becomes finite with *increasing* Fermi energy from the band edge [54]. At $d > 4$ such a behavior reveals a certain flaw of the model with a point interaction ("white noise" random-potential correlation) [61]. The limiting case $d = 4$ requires a careful analysis. Equation (2.80) implies that we always have a metallic behavior if the dimensionless coupling constant of the four-dimensional theory is small, $m^2 \rho V^2 \ll 1$. Equation (2.90) does not have any real solutions $\omega_0^2(E_F) > 0$. This seems to be true for any energy outside an exponentially small (over $m^2 \rho V^2 \ll 1$) vicinity of the band edge (see the analysis of $d = 4$ in Sec. 3). This situation changes if we assume that the cutoff parameter k_0 in (2.77) is defined by the effective range of the potential (or of the two-point correlation of random potentials), for example, rather than by the Fermi momentum. In this case it is possible to have $k_0 \ll p_F$. For $d < 4$ the results in this case are the same as those obtained above, but the mobility edge is given by the relation (for any $d > 2$!) (Ref. 54):

$$E_c = \frac{d}{d-2} \left(\frac{m}{2\pi} \right)^{d/2} \frac{\tilde{E}_0^{(d/2)-1}}{\Gamma\left(\frac{d}{2}\right)} \rho V^2 = \frac{d}{d-2} \frac{1}{\pi} \gamma(\tilde{E}_0), \quad (2.106)$$

where $\tilde{E}_0 = k_0^2/2m$. For $d \geqslant 4$ we find [54]

$$\sigma \simeq \frac{ne^2}{m} \frac{1}{2\gamma(E_F)} \frac{E_F - E_c}{E_c}; \quad E_F \geqslant E_c \quad (2.107)$$

$$\omega_0^2(E_F) \simeq \frac{4}{d} \frac{d-4}{d-2} \left\{ 1 - \frac{E_F}{E_c} \right\}; \quad E_F \leqslant E_c. \quad (2.108)$$

Accordingly, using (2.66) we find that the critical exponent of the localization length is $\nu = \frac{1}{2}$ for $d > 4$. Thus, we can refer to $d = 4$ as the "upper critical dimensionality" of the localization theory.

We have analyzed a free electron in a field of random scattering centers. The self-consistent theory can be generalized to describe an electron with an arbitrary energy spectrum in a lattice with an Anderson-type disorder [3]. Such a generalization is relevant, first of

all, in view of the availability of extensive literature on localization in the Anderson model [4–7]. The localization in the Anderson model was first analyzed within the context of the self-consistent theory by Götze [62] and Prelovšek [63]. Their analysis was based on Götze's formalism [49–51]. Kotov and Sadovskiĭ [64] performed such an analysis using the Vollhardt–Wölfle method. We will confine ourselves here to the discussion of the results of Ref. 64 for the case in which a lattice electron is described by the effective-mass approximation and the Anderson disorder is described by the Gaussian distribution with a width $\tilde{W}$. All the relevant results can then be determined, from the equations given above, by a simple substitution:

$$m \to m^* = \frac{1}{2Va^2}; \qquad \rho V^2 \to \tilde{W}^2 \Omega_0, \tag{2.109}$$

where m^* is the effective mass of the lattice electron for a strong-coupling approximation (V is the transfer integral between nearest neighbors, and a is the lattice constant), and Ω_0 is the volume per lattice site; we will restrict ourselves here to the lattices of cubic symmetry. A more general analysis was carried out by Kotov and Sadovskiĭ [64]. For the static conductivity in the metallic region, by analogy with (2.82) and (2.83), we then find

$$\sigma = \frac{e^2}{2\pi d} \frac{v_F^2}{\tilde{W}^2 \Omega_0} \left\{ 1 - \frac{d}{d-2} \lambda x_0^{d-2} \right\}. \tag{2.110}$$

In this expression the dimensionless "coupling constant" is analogous to that in (2.81):

$$\lambda = \left(\frac{m^*}{2\pi} \right)^{d/2} \frac{\tilde{W}^2 \Omega_0}{\Gamma\left(\frac{d}{2} \right)} E_F^{(d-4)/2}, \tag{2.111}$$

where E_F is the Fermi energy reckoned from the band edge. We easily see that the mobility edge is determined by

$$\left(\frac{\tilde{W}}{V} \right)^2 = \frac{d-2}{2} \Gamma\left(\frac{d}{2} \right) \left(\frac{m^*}{2\pi} \right)^{-d/2} \frac{x_0^{2-d}}{V^2 \Omega_0} E_F^{(4-d)/2}. \tag{2.112}$$

This is the condition under which the braces in (2.110) vanish. If the "disorder" is specified $(\tilde{W}/V)$, Eq. (2.112) gives $E_F = E_c$, the location of the mobility edge in the band. If the Fermi energy is specified, we find from (2.112) the critical ratio $(\tilde{W}/V)_c$, which corresponds to the electron localization at the Fermi surface. In particular, for a half-filled band $E_F = ZV$ (Z is the number of nearest neighbors in the lattice), we find

$$\left(\frac{\tilde{W}}{V}\right)_c^2 = \frac{d-2}{d}\,\Gamma\left(\frac{d}{2}\right)\left(\frac{m^* V}{2\pi}\right)^{-d/2}\frac{x_0^{2-d}}{\Omega_0}\,Z^{(4-d)/2} \tag{2.113}$$

and for $(\tilde{W}/V) \lessgtr (\tilde{W}/V)_c$, for the mobility edge, we have (2.84); where for the model which we are analyzing we have

$$E_{sc} = (m^*)^{d/(4-d)}(\Omega_0)^{2/(4-d)}V^{4/(4-d)}\left(\frac{\tilde{W}}{V}\right)^{4/(4-d)} \sim V\left(\frac{\tilde{W}}{V}\right)^{4/(4-d)}. \tag{2.114}$$

In comparing these results with the literature on the Anderson model, we should bear in mind that our parameter $\tilde{W}^2$ represents the dispersion of the Gaussian energy distribution at the lattice sites. The dispersion is $W^2/12$ for a uniform distribution with a width W, used in Ref. 3. For the "Anderson" critical ratio we therefore find $(W/V)_c^2 = 12(\tilde{W}/V)_c^2$. This procedure gives only an approximate description of Anderson's disorder, since all the perturbation-theory diagrams connected with the higher-order cumulants of the Anderson random field have been ignored here. Table I gives the critical disorder for the total localization of all states in the band. The critical disorder was determined from (2.113) for various three-dimensional lattices of cubic symmetry, for two values of the dimensionless cutoff parameter x_0. Despite the obvious crudeness of the theory, we find our results to be in extremely good agreement with the results of the numerical calculations for the simple cubic lattice: $(W/V)_c \approx 15$ (Ref. 65), $(W/V)_c = 19 \pm 0.5$ (Ref. 66), and $(W/V)_c = 16 \pm 0.5$ (Ref. 67) for the "Anderson" disorder. We also find our results to be in excellent agreement with the results of a most accurate analysis of the localization, within the context of Anderson's method, performed by Licciardello and Economou [68]: $(W/V)_c \approx 14.5$. Our results are

Table I. Critical Disorder Corresponding to the Localization of All the States in the Band for a Gaussian Energy Distribution at the Lattice Sites $\left(\frac{\tilde{W}}{V}\right)_c$ and for a Homogeneous Anderson Distribution $\left(\frac{W}{V}\right)_c$ in Cubic Symmetry Lattices.

Lattice	Z	Ω_0	$\left(\frac{\tilde{W}}{V}\right)_c$ $x_0 = 1$	$\left(\frac{\tilde{W}}{V}\right)_c$ $x_0 = 2$	$\left(\frac{W}{V}\right)_c$ $x_0 = 1$	$\left(\frac{W}{V}\right)_c$ $x_0 = 2$
Simple cubic lattice	6	a^3	5.67	4.01	19.67	13.91
Body-centered-lattice	8	$\frac{a^3}{2}$	8.63	6.10	29.88	21.13
Face-centered-lattice	12	$\frac{a^3}{4}$	13.50	9.55	46.78	33.08

also in reasonably good agreement with the only result (known to us) of a numerical analysis of the Gaussian disorder, $(W/V)_c \simeq 7$ [69]. We know of no numerical calculations for bcc and fcc lattices.

Let us now consider the results for the hypercubic lattice in a d-dimensional space. In particular, for a static conductivity of a half-filled band we find from (2.110) $(2 < d < 4)$

$$\sigma = \sigma_{mm} \frac{\left|\frac{\tilde{W}}{V} - \left(\frac{\tilde{W}}{V}\right)_c\right|}{\left(\frac{\tilde{W}}{V}\right)_c}; \quad \left(\frac{\tilde{W}}{V}\right)^2 \leqslant \left(\frac{\tilde{W}}{V}\right)_c^2$$

$$= \frac{d-2}{d} \Gamma\left(\frac{d}{2}\right) (4\pi)^{d/2} x_0^{2-d} Z^{(4-d)/2}, \tag{2.115}$$

where

$$\sigma_{mm} = \frac{4Z}{\pi d} \frac{e^2}{a^{d-2}} \left(\frac{V}{\tilde{W}}\right)_c^2, \tag{2.116}$$

essentially in agreement with the "minimal metallic conductivity" estimates by Mott [1, 2]. Specifically, in a model with "Anderson"

disorder for $d = 3$ we would have $\sigma_{mm} \simeq 0.013 e^2/\hbar a \simeq 10^2\ \Omega^{-1}\ \mathrm{cm}^{-1}$ with $a = 3$ Å. Curiously enough, in the limit $d \to 2$ we would have $\sigma_{mm} \sim e^2/a^{d-2} 1/d - 2 \to \infty$, since $(\tilde{W}/V)_c \to 0$ (2.113), which indicates a trend toward a total localization at $d = 2$.

For a characteristic frequency (2.65) in this model we find

$$\omega_0^2(E_F = ZV) \approx \frac{4}{d} Z^2 V^2 x_0^2 \left\{ 1 - \frac{\left(\frac{\tilde{W}}{V}\right)_c^2}{\left(\frac{\tilde{W}}{V}\right)^2} \right\}^{2/(d-2)}$$

$$\approx \frac{4}{d} Z^2 V^2 x_0^2 \left\{ 2 \frac{\frac{\tilde{W}}{V} - \left(\frac{\tilde{W}}{V}\right)_c}{\left(\frac{\tilde{W}}{V}\right)_c} \right\}^{2/(d-2)} ; \quad \frac{\tilde{W}}{V} \geqslant \left(\frac{\tilde{W}}{V}\right)_c ; \tag{2.117}$$

hence,

$$R_{\mathrm{loc}}(E_F = ZV) \simeq a \left\{ 2 \frac{\frac{\tilde{W}}{V} - \left(\frac{\tilde{W}}{V}\right)_c}{\left(\frac{\tilde{W}}{V}\right)_c} \right\}^{-(1/(d-2))} \tag{2.118}$$

for the localization length at the band center. We see that the critical index of the localization length is given by (2.97), even for localization at the band center. The necessary refinements resulting from calculation of the displacement of the original band edge by a random field are given in Ref. 64. The role of this displacement in the determination of the quantity $(\tilde{W}/V)_c$ is insignificant. The equations for all physical quantities in a two-dimensional Anderson model can also be easily written.

3. Field Theory Treatment of Localization

3.1. Effective Field Theory of an Electron in a Disordered System

A systematic analysis of the region near the mobility edge would be reasonable to carry out by constructing an effective field theory

corresponding to the Anderson transition. This can be done by a method used in the analysis of the critical phenomena in the phase-transition theory [21–23]. The original ideas used for a scaling description of the mobility edge, which were advanced immediately after the development of the modern theory of critical phenomena, were formulated virtually at the same time by several authors [24, 25, 70–74]. A rather extensive literature on this subject is now available and several effective Lagrangians have been suggested for an electron near the mobility edge [7, 10]. We will examine the earlier field theory approach [24, 72], since it emphasizes rather than obscures, in our view, the difficulties encountered in a theoretical description of the problem of interest.

We will again consider the behavior of a free electron in a field of random scattering centers. If we confine ourselves to the Gaussian statistics of a random field produced by these centers, which is a valid approach if we are dealing with [75] the limit of $V \to 0$, $\rho \to \infty$, and $\rho V^2 \to \text{const}$, we easily see that the Edwards diagram expansion [11, 33, 42] for the average Green's function $G^{R,A}(E\mathbf{p})$ is generated by the standard series expansion in the perturbation theory [22] of the propagator (Green's function) of the scalar $O(n)$ symmetric Euclidean field theory with a Lagrangian [7],

$$L(\mathbf{r}) = \frac{1}{2}\sum_{j=1}^{n}\left\{\frac{1}{2m}(\nabla\phi_j)^2 - \underset{\delta\to 0^+}{(E \pm i\delta)}\phi_j^2\right\} - \frac{1}{8}\rho V^2\left(\sum_{j=1}^{n}\phi_j^2\right)^2, \quad (3.1)$$

where m is the electron mass, and E is the electron energy. In this case the number of field components $n \to 0$ (at the end of the calculations), allowing the elimination of the "loop" diagrams which are missing in the Edwards diagram technique. This method, proposed by de Gennes [76] and developed by des Cloizeaux [77], was successfully used in the scaling theory of polymers with excluded volume. The use of an appropriate formalism has made it possible [24] to develop a scaling model of the mobility edge within the context of the Anderson method.

The Green's function corresponding to Lagrangian [3.1] can be formally determined by a functional integral (a minus sign was chosen in order to match this expression with the standard form of

the Green's function for a free electron),

$$G(\mathbf{r}-\mathbf{r}'E) = -\frac{1}{Z}\lim_{n\to 0}\frac{1}{n}\sum_{j=1}^{n}\int\{\delta\phi(\mathbf{r})\}\phi_j(\mathbf{r})\phi_j(\mathbf{r}')\exp\{-S[\phi]\}, \quad (3.2)$$

where

$$S[\phi] = \int d^d\mathbf{r}L(\mathbf{r}) \quad (3.3)$$

is the action functional of the field theory (3.1), and the normalization integral is

$$Z = \int\{\delta\phi(\mathbf{r})\}\exp\{-S[\phi]\}. \quad (3.4)$$

In a formal calculation the functional integral in (3.2) diverges because of the "incorrect" sign of the coupling constant (attraction) [24]. This indicates that the ground state in the field theory (3.1) is unstable. For this reason, this functional integral should be understood as *an analytic continuation over the coupling constant* from an arbitrary constant $g > 0$ to a "physical" constant $g = -\rho V^2 < 0$. We will discuss this problem more thoroughly below. For now we will treat expression (3.2) simply as an abridged notation of the diagram rules in the perturbation theory for field theory (3.1).

We should point out an important feature of the method we are considering. In (3.1) there are no random parameters, and an averaging procedure in it has already been performed. For this reason, expression (3.1) is said to be an "effective" Lagrangian. On the other hand, such a convenience exacts its "toll": A calculation of a two-particle average Green's function requires a *different* effective Lagrangian. We saw that a solution of the localization problem generally requires knowledge of correlators like those in (2.19) and (2.24), which include single-electron Green's functions with different energy variables. We can easily write down an effective Lagrangian which immediately "generates" a diagram technique for a two-particle Green's function (see Fig. 3). In this case we need a Lagrangian of *two* interacting scalar fields ϕ (n components, $n\to 0$) and φ (m components, $m\to 0$) which has $O(n)\times O(m)$ symmetry [7, 74].

$$L(\mathbf{r}) = \frac{1}{2}\sum_{j=1}^{n}\left\{\frac{1}{2m}(\nabla\phi_j)^2 - (E+\omega+i\delta)\phi_j^2\right\}$$
$$+\frac{1}{2}\sum_{i=1}^{m}\left\{\frac{1}{2m}(\nabla\varphi_i)^2 - (E-i\delta)\varphi_i^2\right\}$$
$$-\frac{1}{8}\rho V^2\left\{\left(\sum_{j=1}^{n}\phi_j^2\right)^2 + \left(\sum_{i=1}^{m}\varphi_i^2\right)^2\right.$$
$$\left.+2\sum_{j=1}^{n}\sum_{i=1}^{m}\phi_j^2\varphi_i^2\right\}; \quad \delta\to 0^+. \tag{3.5}$$

The appearance of a Langrangian of two interacting fields implies that there are certain complications arising from the transition to a two-particle Green's function. However, the principal difficulty of the theory, associated with the "incorrect" sign of the coupling constant, is nonetheless seen, already in, in (3.1).

First, we will determine at the elementary level the range of applicability of the perturbation theory, similar to that how it was done in the theory of critical phenomena during the derivation of the so-called Ginzburg criterion which determines the size of the critical region, where the order-parameter fluctuations are significant [78]. We will follow generally the procedure used in Ref. 35. The self-energy part of the Green's function in a "one-loop" approximation, which corresponds to the summation of the diagrams like those in Fig. 4a, is given by

$$\Sigma(E_0) = \rho V^2\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{E_0 - \Sigma(E_0) - \dfrac{\mathbf{p}^2}{2m}}, \tag{3.6}$$

where E_0 is the "bare" energy of a free electron. Let us determine the renormalized energy E by the relation

$$E(E_0) = E_0 - \operatorname{Re}\Sigma(E_0). \tag{3.7}$$

We can then rewrite (3.6) in the form

$$E_0 - E(E_0) + i\operatorname{Im}\Sigma(E_0) = \rho V^2\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{E(E_0) - \dfrac{\mathbf{p}^2}{2m} - i\operatorname{Im}\Sigma(E_0)}. \tag{3.8}$$

Let us now determine in this approximation the renormalized (displaced) band edge E_{0c} (Refs. 35 and 64) from the condition under which the density of states vanishes

$$N(E_0) = \mp \frac{1}{\pi} \int \frac{d^d\mathbf{p}}{(2\pi)^d} \operatorname{Im} G^{R,A}(E_0\mathbf{p}) \xrightarrow[E_0 \to E_{0c}]{} 0. \tag{3.9}$$

we have

$$\operatorname{Im} G^{R,A}(E_0\mathbf{p}) = \begin{cases} \mp \pi\delta\left[E(E_0) - \dfrac{\mathbf{p}^2}{2m}\right]; & \operatorname{Im}\Sigma(E_0) = 0 \\ \dfrac{\operatorname{Im}\Sigma(E_0)}{\left[E(E_0) - \dfrac{\mathbf{p}^2}{2m}\right]^2 + [\operatorname{Im}\Sigma(E_0)]^2}; & \operatorname{Im}\Sigma(E_0) \neq 0. \end{cases} \tag{3.10}$$

To satisfy (3.9) the necessary condition $\operatorname{Im}\Sigma(E_0 \to E_{0c}) \to 0$ must hold; furthermore, we must require that

$$N(E_0 \to E_{0c}) = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\, \delta\left(E(E_0) - \frac{\mathbf{p}^2}{2m}\right) \xrightarrow[E_0 \to E_{0c}]{} 0, \tag{3.11}$$

which is equivalent to the condition

$$E(E_0) \xrightarrow[E_0 \to E_{0c}]{} 0. \tag{3.12}$$

We can then find from (3.8) the equation which determines E_{0c},

$$E_{0c} = -\rho V^2 \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{1}{\mathbf{p}^2/2m} \equiv -\rho V^2 D(0) = -\rho V^2 2mS_d \frac{p_0^{d-2}}{d-2}, \tag{3.13}$$

where $S_d = 2^{-(d-1)}\pi^{-d/2}1/\Gamma(d/2)$, and $p_0 = a^{-1}$ is the cutoff momentum on the order of the reciprocal minimum length in the problem determined by the correlation radius of the random potentials, i.e., the length at which the point like interaction in (3.1) becomes

spread-out. Our definition of a displaced band edge differs from that of Brezin and Parisi [79] but is consistent with the definition of the displacement of the phase-transition point due to fluctuations [78]. Let us now consider $G^{-1}(E_0 p = 0) = E_0 - \mathrm{Re}\,\Sigma(E_0) - i\,\mathrm{Im}\,\Sigma(E_0)$ and subtracting $E_{0c} + \rho V^2 D(0) = 0$, we find

$$E_0 - E_{0c} - \mathrm{Re}\,\Sigma(E_0) - \rho V^2 D(0) = E_0 - \mathrm{Re}\,\Sigma(E_0) = E(E_0). \quad (3.14)$$

Ignoring $\mathrm{Im}\,\Sigma(E_0)$ in (3.8), we find

$$\mathrm{Re}\,\Sigma(E_0) \simeq \rho V^2 \int \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{E(E_0) - \mathbf{p}^2/2m} \equiv \rho V^2 D[E(E_0)]. \quad (3.15)$$

We then derive from (3.14) an equation which relates the renormalized energy $E(E_0)$ to the distance from the band edge in terms of the bare energy $E_0 - E_{0c}$,

$$E_0 - E_{0c} - \rho V^2 \{D[E(E_0)] - D(0)\} = E(E_0). \quad (3.16)$$

The "Ginzburg criterion" (Ref. 35) follows (by analogy with the requirement in Ref. 78) from the requirement that the following very simple equation hold:

$$E \simeq E_0 - E_{0c}. \quad (3.17)$$

This equation implies that the renormalized energy is equal to the electron energy reckoned from the displaced (in a one-loop approximation) band edge. Clearly, Eq. (3.17) holds for $(2 < d < 4)$ if

$$|E| \gg \rho V^2 |D(E) - D(0)| = \rho V^2 \pi m S_d (2mE)^{(d/2)-1} \frac{1}{\left|\sin \pi\left(\frac{d}{2} - 1\right)\right|},$$

or for

$$|E| \gg \left\{\frac{B_d}{\left|\sin \frac{\pi}{2} d\right|}\right\}^{2/(4-d)} E_{sc}, \quad (3.18)$$

where $B_d = 2^{-d/2}\pi^{1-d/2}/\Gamma(d/2)$, and $E_{sc} = m^{d/(4-d)}(\rho V^2)^{2/(4-d)}$ [see Eq. (2.49)]. In the limit $d \to 2$ we have $|\sin(\pi/2)d| \to (\pi/2)(d-2)$, while $(2/\pi)B_d = A_d$ which is given in Eq. (1.48). We see that (3.18) is equivalent to (1.48). In the limit of $d \to 2$ condition (3.18) is more stringent than the simple requirement $E \gg \gamma(E)$ (or $E \gg E_{sc}$) which we regarded above as the condition under which the perturbation theory can be used. Omission of $\operatorname{Im}\Sigma(E_0)$ in (3.15) is fully justifiable under the conditions of (3.18). The simplest approximation (2.34), in which the variable E is taken to mean the distance from the band edge displaced by the random field, is valid only under these conditions. According to (2.48) and (2.84), the mobility edge is situated in the "Ginzburg critical region" (3.18), where the higher-orders of perturbation theory are important, even in the limit of $d \to 2$ (Ref. 35) and the "naive" hopes expressed above, for example, in determining the region where the self-consistent theory can be used are, in fact, not justifiable. The divergence of region (3.18) in the limit of $d \to 2$ is analogous to the divergence of the "Ginzburg critical region" in the problem of phase transition in two dimensions [78].

In our analysis of the field theory (3.1) we understand E [in (3.1), (3.5), etc.] everywhere to mean the electron energy (3.14)–(3.17) renormalized in a one-loop approximation. This assumption allows us to eliminate from our analysis the real part of the diagram like the one in Fig. 4a, which determines the displacement of the band edge. Formally, this can be accomplished by incorporating into (3.1) an appropriate counterterm for "mass" renormalization: $\delta E = E - E_0 = -\operatorname{Re}\Sigma(E_0)$.

We will examine the cases of $d = 2$ and 4 separately. For $d = 4$ the analysis is essentially the same as the preceding one. The displacement of the band edge, according to (3.13), is

$$E_{0c} = \rho V^2 \frac{m}{8\pi^2}\frac{p_0^2}{2} = \rho V^2 \frac{m^2}{8\pi^2}\tilde{E}_0, \tag{3.19}$$

where $\tilde{E}_0 = p_0^2/2m$ is the energy associated with the momentum cutoff. Equation (3.16) for $d = 4$ gives

$$E = E_0 - E_{0c} + 2\rho V^2 \frac{m^2}{2\pi^2} E \ln\frac{\sqrt{2mE}}{p_0}. \tag{3.20}$$

Condition $E \gg \gamma(E)$ for $d = 4$ reduces to the requirement $(m^2/4\pi)\rho V^2 \ll 1$, i.e., to the requirement that the dimensionless coupling constant be small. The requirement that $E \simeq E_0 - E_{0c}$ leads, according to (3.20), to the following inequality

$$E \gg \tilde{E}_0 \exp\left\{-\frac{4\pi^2}{m^2}\frac{1}{\rho V^2}\right\}. \tag{3.21}$$

This expression defines the "Ginzburg critical region" of the four-dimensional theory, which is exponentially small over the coupling constant.

In the case of $d = 2$ the situation is more complicated. The displacement of the band edge, determined from (3.13), becomes infinite and from (3.15) we find

$$\operatorname{Re}\Sigma(E\mathbf{p}) \simeq \rho V^2 \frac{m}{2\pi} \ln\left|\frac{\tilde{E}_0}{E}\right|; \qquad E \ll \tilde{E}_0 = \frac{\mathbf{p}_0^2}{2m}. \tag{3.22}$$

The correction to E_0 in (3.7) is appreciable everywhere at $E \ll \tilde{E}_0$, implying that formally the perturbation theory cannot be used at $d = 2$. We therefore understand $d = 2$ everywhere to mean $d = 2 + \epsilon$ when (3.13) and (3.18) are clearly defined.

The higher-order corrections to the perturbation theory in the field theory (3.1) can easily be examined in the so-called "parquet" approximation [72]. In this approximation, as we know, the theory of critical phenomena for $d = 4$ or $d = 4 - \epsilon$ can be analyzed completely, because the "parquet" comprises the dominating sequence of diagrams for $d \sim 4$. This circumstance, by itself, does not depend on the sign of the interaction constant. We will write out the appropriate equations for the field theory (3.1), incorporating the new variable $-E = \tau > 0$, and carry out an analytical continuation $\tau \to -E \mp i\delta$, where $E > 0$, only at the end of the calculations. This is a convenient way of analysis, since it allows us to analyze the logarithmic integrals in the usual manner. Let us consider the simplest corrections to the initial interaction $-\rho V^2$, which are represented by the sum of the diagrams in Fig. 6 (the rules governing this diagram technique are found, for example, in the book by Shang-keng Ma [22]). The result, of course, depends on the choice of

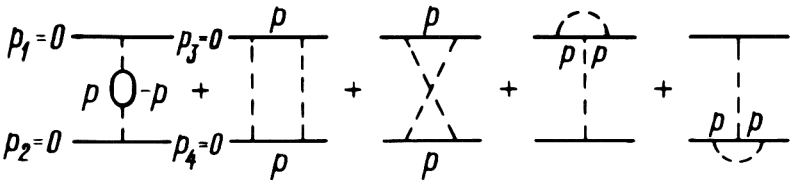

Figure 6. The first-order corrections (second-order diagrams) to the bare interaction in field theory (3.1).

momenta at the external legs. Let us assume, for definiteness, that all the external momenta vanish. We thus find

$$\Gamma(\mathbf{p}=0\tau) \simeq -\rho V^2\left\{1+\rho V^2\left(\frac{n}{2}+4\right)\int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{(\mathbf{p}^2/2m+\tau)^2}+\cdots\right\}$$

$$\approx -\rho V^2\{1+\tfrac{1}{2}(n+8)us+\cdots\} \underset{n\to 0}{=} -\rho V^2\{1+4us+\cdots\}.$$

We have introduced a dimensionless interaction constant,

$$u=\frac{m^2}{2\pi^2}\,a^{-\epsilon}\rho V^2 \tag{3.23}$$

and a "logarithmic" variable

$$s=\begin{cases}\ln\dfrac{p_0}{\sqrt{2m\tau}}; & d=4\\[2ex] \dfrac{1}{\epsilon}\left\{\left(\dfrac{p_0}{\sqrt{2m\tau}}\right)^{\epsilon}-1\right\}; & d=4-\epsilon.\end{cases} \tag{3.24}$$

We see that the parameter us should be used to select the diagrams of perturbation theory. In the limit $\tau\to 0$, we can easily have a situation when $us\gg 1$, i.e., when the corrections are larger than the initial interaction. The procedure by which the "parquet" diagrams are summed singles out the "main logarithms," i.e., the contributions proportional to $\sim(us)^n$, and omits the contributions of the type $u^k(us)^n$ $(k\geqslant 1,\ u\ll 1)$. Since a detailed account of the "parquet" formalism is outside the scope of this review [80–82], we will simply

summarize its salient point [72]. The total vertex $\Gamma(s)$, which is rendered dimensionless through multiplication by $m^2 a^{-\epsilon}/2\pi^2$, is determined in the "parquet" approximation by an integral equation of the type

$$\Gamma(s) = -u - \frac{1}{2}(n+8)\int_0^s dt\Gamma^2(t), \tag{3.25}$$

which is equivalent to the differential equation

$$\frac{d\Gamma(s)}{ds} = -\frac{1}{2}(n+8)\Gamma^2(s) \tag{3.26}$$

with the initial condition $\Gamma(s=0) = -u$. The solution of this equation yields

$$\Gamma(s) = \frac{-u}{1-\frac{1}{2}(n+8)us} \underset{n=0}{=} \frac{-u}{1-4us}. \tag{3.27}$$

We see that in the limit $\tau \to 0$ the vertex $\Gamma(s)$ has a ghost pole at $s_c = \frac{1}{4}u$, indicating that the perturbation theory is inapplicable at $s \gtrsim u^{-1}$. In field-theory-renormalization-group terms, such a behavior corresponds to an increase of the invariant charge of the theory in the limit $s \to s_c \sim u^{-1}$, a situation which indicates that the perturbation theory does not hold [83, 84]. In the approximation under consideration, Eq. (3.26) is, in fact, the Gell-Mann–Low equation for an invariant charge. Equation (3.27) indicates that the theory in question possesses *asymptotic freedom* (the "zero-charge" situation is realized in the theory of critical phenomena). The perturbation theory is valid only at high energies and high momenta. An analytic continuation $\tau \to -E \mp i\delta$ yields

$$\Gamma(E) = -\frac{u}{1-\frac{4u}{\epsilon}\left\{\left(-\frac{2mE}{p_0^2}\right)^{-\epsilon/2}-1\right\}}$$

$$\approx -\frac{u}{1-\frac{4u}{\epsilon}\left\{\left(\frac{2m|E|}{p_0^2}\right)^{-\epsilon/2}-1\right\} \pm iu\left(\frac{2m|E|}{p_0^2}\right)^{-\epsilon/2}\theta(E)\frac{\pi}{2}};$$

$$d = 4 - \epsilon. \tag{3.28}$$

Analogously, we have

$$\Gamma(E) = -\frac{u}{1 - 4u \ln \dfrac{p_0}{\sqrt{2m|E|}} \pm iu \dfrac{\pi}{2} \theta(E)}; \quad d = 4. \tag{3.29}$$

We see that the ghost pole remains in this case, although it lies on the negative part of the energy axis (where there are no electronic states in this approximation [72])

$$E \simeq -E^* = -\frac{p_0^2}{2m} \left(\frac{4u}{\epsilon}\right)^{2/\epsilon} = -\left(\frac{2}{\pi^2(4-d)}\right)^{2/(4-d)} E_{sc}; \quad d = 4 - \epsilon$$

$$E = -E^* = -\frac{p_0^2}{2m} \exp\left\{-\frac{\pi^2}{m^2} \frac{1}{\rho V^2}\right\}; \quad d = 4. \tag{3.30}$$

Thus, the meaning of Ginzburg conditions (3.18) and (3.21) is that the perturbation theory can be used to analyze the energies far from the ghost pole. The perturbation theory is not valid at $E \geqslant E^* \sim E_{sc} (d < 4)$, since, as we can easily see from Eqs. (3.28) and (3.29),

$$|\Gamma(E = +E^*)| \simeq \frac{8}{\pi\epsilon} \gg 1$$

$$|\Gamma(E = +E^*)| = \frac{2}{\pi} \sim 1, \tag{3.31}$$

so that the effective interaction becomes strong. We might note, however, that incorporation of the "imaginary" component into the denominators of (3.28) and (3.29) exceeds the acceptable accuracy, since the parquet approximation accounts only for the large logarithmic ($\sim u/\epsilon$, $\epsilon \ll 1$) contributions, whereas the imaginary contribution is of the order of $\sim u$.

Thus a physically correct description of the effective interaction in the field theory (3.1) cannot be obtained by using the parquet approximation. Nonphysical singularities appear at $E \sim E_{sc}$ during the calculation of physical quantities such as the density of states, and the energy $\sim E_{sc}$ determines the width of the "Ginzburg critical

region" of our problem. In this region the perturbation theory breaks down. At $E \gg E_{sc}$ the density of states can be expanded in a series over the parameter $(E/E_{sc})^{(d-4)/2}$ [72, 73].

Let us now briefly examine the Lagrangian of two interacting fields (3.5) which is directly connected with the localization problem. In the theory of phase transitions the parquet equations were analyzed in an analogous problem in Ref. 85. In the differential form in terms of the logarithmic variable

$$S = \frac{1}{\epsilon}\left\{\left(\frac{p_0}{\max(\sqrt{2m\tau}, \sqrt{2m\tau'})}\right)^{\epsilon} - 1\right\}$$

we can write these equations for our problem in the form [$\tau = -(E+\omega) > 0$, $\tau' = -E > 0$]

$$\begin{aligned}
\frac{d\Gamma_\phi}{ds} &= -\frac{1}{2}(n+8)\Gamma_\phi^2 - \frac{1}{2}m\Gamma^2, \\
\frac{d\Gamma_\varphi}{ds} &= -\frac{1}{2}(m+8)\Gamma_\varphi^2 - \frac{1}{2}n\Gamma^2, \\
\frac{d\Gamma}{ds} &= -\frac{1}{2}\{(n+2)\Gamma_\phi + (m+2)\Gamma_\varphi + 4\Gamma\}\Gamma,
\end{aligned} \tag{3.32}$$

where Γ_ϕ describes the interaction ϕ^4, $\Gamma_\varphi - \varphi^4$ [4], and Γ describes the interaction of the type $\phi^2\varphi^2$, which is of main interest to us. For $m = 0$ and $n = 0$ the equations in (3.32) can be rewritten in simpler form

$$\begin{aligned}
\frac{d\Gamma_\phi}{ds} &= -4\Gamma_\phi^2; \qquad \frac{d\Gamma_\varphi}{ds} = -4\Gamma_\varphi^2, \\
\frac{d\Gamma}{ds} &= -(\Gamma_\phi + \Gamma_\varphi)\Gamma - 2\Gamma^2.
\end{aligned} \tag{3.33}$$

The equations for Γ_ϕ and Γ_φ become separate equations and coincide with (3.26) (at $n = 0$); the solution of these equations is given by (3.27):

$$\Gamma_\phi = \Gamma_\varphi = -\frac{u}{1 - 4us}. \tag{3.34}$$

The equation for Γ, which can then be written in the form

$$\frac{d\Gamma}{ds} = -\frac{2u\Gamma}{1-4us} + 2\Gamma^2; \quad \Gamma(0) = -u \tag{3.35}$$

reduces to a Riccati equation. The solution of this equation immediately yields

$$\Gamma = \Gamma_\phi = \Gamma_\varphi = -\frac{u}{1-4us}. \tag{3.36}$$

This result is a direct consequence of the "decoupling" of the equations in (3.32) due to the limit $n \to 0$ and $m \to 0$. This result is also evident from the fact that in the limit $\omega \to 0$ and $\delta \to 0$ Lagrangian (3.5) describes the $O(n+m)$ symmetric theory. Therefore, a naive expectation that the solution of a system of coupled equations (3.32) can lead to a behavior different from the nonphysical result like that in (3.27) is unjustifiable. The "strong-coupling" region is retained in the theory, which again indicates that the perturbation theory is basically inapplicable in the energy region of interest. The results like those in (3.33)–(3.35) were obtained by Nitzan *et al.* [74] in terms of Wilson's recurrence formulas as an indication of the absence of fixed points of Lagrangian (3.5) for the values of the parameters which correspond to the problem of an electron in a random field.

The difficulties discussed above are, in our view, the key difficulties in the description of the energy region near the mobility edge, rather than being a consequence of the use of an "illegitimate" effective Lagrangian. Any other theory must encounter these difficulties sooner or later in a systematic analysis and find a method for their solution. Unfortunately, this problem is either ignored or bypassed in most present-day studies by making use of simplest approximations (for a single-particle Green's function, for example) outside their range of applicability.

3.2. Analytic Properties over the Coupling Constant and Instantons

In the analysis of the preceding section we have, in fact, attempted to calculate in a rather naive way the functional integral of the type

(3.2). We understood this integral to mean an abbreviated notation of the diagram rules for the perturbation theory. The results we obtained show that this approach has serious flaws in the region of low energies and particularly negative energies, where the perturbation theory gives rise to the appearance of a ghost pole in the effective interaction [(3.28) and (3.29)] and other nonphysical results. It is an established fact, on the other hand, that an analysis of the behavior of an electron in a Gaussian random field of the scattering centers in the negative energy region leads to the existence of a "tail" in the density of states (Refs. 6 and 8) which is associated with the localization of electrons in the deep fluctuations of the random field. This situation typically is marked by the appearance (in the density of states) of nonanalytic contributions in terms of the coupling constant [86–89], which cannot be accounted for by the standard perturbation theory. The functional integrals such as those in (3.2) can be properly analyzed only by using the method of *analytic continuation over the coupling constant*. A corresponding method based on the calculation of the integrals by the method of steepest descent was suggested (in connection with another problem) by Langer [90]. In a field theory approach to localization this method was used for the first time by Cardy [91] and Sadovskiĭ [92]. Let us discuss the main results of these studies.

Before considering the functional integral, it would be useful to analyze the analytic continuation by using as an example an ordinary integral of the type [93–95]

$$Z(g) = \int_{-\infty}^{\infty} d\phi \exp\left\{-\frac{1}{2}\phi^2 - \frac{1}{8} g\phi^4\right\}, \tag{3.37}$$

which determines the "zero-dimensional" analog of the functional integral (field theory) we are examining. We assume that $g = \lambda e^{i\theta}$, where $\lambda \in \mathrm{Re}$ and $\lambda > 0$. For $\theta = 0$ converging integral (3.37) can be expressed by means of a modified Bessel function

$$Z(\lambda) = \frac{1}{\sqrt{\lambda}} \exp\left(\frac{1}{4\lambda}\right) K_{1/4}\left(\frac{1}{4\lambda}\right). \tag{3.38}$$

At $\lambda \ll 1$ this expression yields $Z(\lambda \ll 1) \approx \sqrt{2\pi}$. We wish to extend

(3.37) analytically from $\theta = 0$ to $\theta = \pm\pi$, i.e., from $g = \lambda > 0$ to $g = -\lambda < 0$. We must therefore substitute the variable $\tilde{\phi} = \lambda^{1/2}\phi$ into (3.37) and find

$$Z(\lambda e^{i\theta}) = \lambda^{-1/2} \int d\tilde{\phi} \exp\left\{-\frac{1}{\lambda} S[\phi]\right\}, \tag{3.39}$$

where

$$S[\tilde{\phi}] = \tfrac{1}{2}\tilde{\phi}^2 + \tfrac{1}{8} e^{i\theta}\tilde{\phi}^4. \tag{3.40}$$

At $\theta = 0$ the integration in (3.39) is over the real axis of $\tilde{\phi}$. If the value of θ changes, then the integration path should be rotated (as a result of the substitution $\tilde{\phi} \to e^{-i\theta/4}\tilde{\phi}$) through an angle $-\theta/4$ in such a way that the integral would remain converging. The integration paths corresponding to $\theta = \pm\pi$ are illustrated in Fig. 7a. Because of the presence of the factor $\lambda^{-1}(\lambda \ll 1)$ in the exponential function in (3.39), the integral can be computed by the method of steepest descent. The saddle points are determined by the condition

$$\frac{\partial S[\tilde{\phi}]}{\partial \tilde{\phi}} = \tilde{\phi} - \frac{1}{2}\tilde{\phi}^3 = 0, \tag{3.41}$$

which yields $\tilde{\phi}_0 = 0$ and $\tilde{\phi}_{1,2} = \pm\sqrt{2}$. The main contribution to $Z(e^{i\pi}\lambda)$ comes from the saddle point $\tilde{\phi}_0 = 0$, and the integration

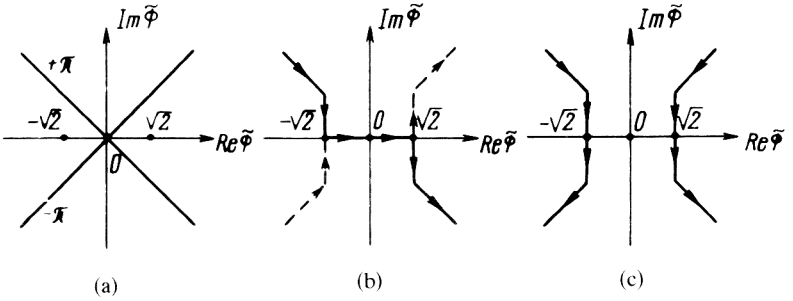

Figure 7. The integration paths in the complex plane $\tilde{\phi}$ in (3.39). (a) The initial paths corresponding to $\theta = \pm\pi$; (b) the paths for calculating the main contribution, Re $Z(-\lambda)$; (c) the path for calculating the discontinuity, $\Delta Z(-\lambda) = \text{Im}\, Z(-\lambda)$.

path can be deformed in such a way that it would pass through this saddle point as it moves along the real axis (Fig. 7b). We find Re $S[\tilde{\phi}] > \text{Re } S[0]$ everywhere along this path. We then find

$$Z(e^{i\pi}\lambda) \approx \lambda^{1/2} \int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2\lambda} x^2\right) + O(\lambda) = \sqrt{2\pi} + O(\lambda). \tag{3.42}$$

Analogously we can also analyze $Z(e^{-i\pi}\lambda)$. Finally, we find

$$\text{Re } Z(-\lambda) = \tfrac{1}{2}\{Z(e^{i\pi}\lambda) + Z(e^{-i\pi}\lambda)\} \approx \sqrt{2\pi}, \tag{3.43}$$

consistent with asymptotic value of (3.38) at $\lambda \ll 1$. The situation changes if we consider the *discontinuity* at the negative part of the real axis of the "coupling constant":

$$\Delta Z(-\lambda) \equiv \frac{1}{2i}\{Z(e^{i\pi}\lambda) - Z(e^{-i\pi}\lambda)\} = \text{Im } Z(-\lambda). \tag{3.44}$$

The integration path for $Z(e^{i\pi}\lambda)$ *minus* the integration path for $Z(e^{-i\pi}\lambda)$ transforms into the integration paths illustrated in Fig. 7c (compare with those shown in Fig. 7b). These contours go through the "nontrivial" saddle points, $\tilde{\phi}_{1,2} = \pm\sqrt{2}$, but the "trivial" point, $\tilde{\phi}_0 = 0$, *drops out*. At all the remaining points of these contours we would again have Re $S[\tilde{\phi}] > \text{Re } S[\pm\sqrt{2}]$. Thus the principal contribution to $\Delta Z(-\lambda)$ comes from the "non-trivial" saddle points,

$$\begin{aligned}
\Delta Z(-\lambda) &\approx -\frac{1}{i}\lambda^{-1/2}\exp\left(-\frac{1}{2\lambda}\right)\int_{-i\infty}^{+i\infty} dx \exp\left(\frac{1}{\lambda}x^2\right) + O(\lambda) \\
&= -\lambda^{-1/2}\exp\left(-\frac{1}{2\lambda}\right)\int_{-\infty}^{\infty} dz \exp\left(-\frac{1}{\lambda}Z^2\right) - O(\lambda) \\
&= -\sqrt{\pi}\exp\left(-\frac{1}{2\lambda}\right)\{1 + O(\lambda)\}.
\end{aligned} \tag{3.45}$$

In summary, the $Z(g)$ function, given by integral (3.37), is *an analytic function in the complex g plane with a cut along the negative part of the real axis*. The discontinuity at the cut, which is

nonanalytic in terms of the coupling constant, *is determined by the contribution from the nontrivial saddle points.*

Let us now consider a certain function of a complex variable $G(g)$, which is analytic in the complex plane of the g variable, with a cut along the negative part of the real axis. According to the Cauchy theorem, we then find

$$G(g) = \frac{1}{2\pi i} \oint dz \frac{G(z)}{z - g}$$

where the contour C is illustrated in Fig. 8. In the limit $R \to \infty$, under the assumption that the integrand falls off at infinity in the complex plane rapidly enough [a situation which, incidentally, does not occur in example (3.37)!], we find a dispersion relation in terms of the coupling constant,

$$G(g) = \frac{1}{\pi} \int_{-\infty}^{0} dz \frac{\Delta G(z)}{z - g} = \frac{1}{\pi} \int_{-\infty}^{0} dz \frac{\operatorname{Im} G(z)}{z - g}, \tag{3.46}$$

where

$$\Delta G(g) = \frac{1}{2i} \{G(g + i\epsilon) - G(g - i\epsilon)\} = \operatorname{Im} G(g) \tag{3.47}$$

if $g \in \mathrm{Re}$, $\epsilon \to 0^+$; $\Delta G(g) \neq 0$ if $g < 0$, and $\Delta G(g) = 0$ if $g > 0$. This dispersion relation makes it possible to reconstruct the $G(g)$ function in the entire complex g plane from the known discontinuity at the cut directed along the negative part of the real axis. The accuracy of the

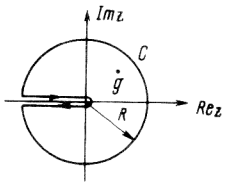

Figure 8. The path of integration which is used in the derivation of the dispersion relation over the coupling constant.

corresponding results is determined solely by the accuracy of the calculation of the discontinuity at the cut.

This analysis can be extended to the calculation of the functional integrals such as those in (3.2). These calculations involve the following general procedure.

1. We seek the "stationary points" of the classical action $S[\phi]$, defined in (3.3) as the solution of the classical equations of the field theory, $\delta S[\phi_{cl}]/\delta\phi_{cl} = 0$, with a *finite* action: $S[\phi_{cl}] < \infty$.

2. We expand the action in powers of $\phi(\mathbf{r}) - \phi_{cl}(\mathbf{r})$ and calculate the corresponding functional integrals if only in a Gaussian approximation.

Such a procedure, implemented near a trivial classical solution, $\phi_{cl} = 0$, clearly leads to the conventional perturbation theory. We will see that in complete analogy with example (3.37) examined above, field theory (3.1) with the coupling constant $g = -\rho V^2 < 0$ has nontrivial solutions, $\phi_{cl}(\mathbf{r}) \neq 0$, with a finite action (instantons) which determine [by analogy with (3.45)] the discontinuity of the Green's function at the cut along the negative part of the real axis in the complex plane of the coupling constant. The Green's function of the theory of interest has an essential singularity at $g = 0$ and is analytic in the complex g plane with a cut along the negative part of the real axis. This function satisfies dispersion relation (3.46).

The presence of an essential singularity in the Green's function at zero of the coupling constant was pointed out for the first time in the quantum field theory by Dyson [96], who used electrodynamics as an example to illustrate his conclusion. The described mathematical procedure, suggested for the first time by Langer [90], was used by Lipatov [97] to study the asymptotic properties of the perturbation-theory series in the quantum-field theory. The dispersion relation for the coupling constant was initially used by Bender and Wu [98] in a problem of anharmonic oscillator and by Bogomolny [99] and Dorfel *et al.* [100] in the field theory. Additional details may be found in the review articles by Kazakov and Shirkov [94] and Zinn-Justin [95].

Let us examine the nonlinear solutions with a finite action (instantons) in field theory (3.1). The corresponding classical field equations are

$$\frac{1}{2m}\nabla^2\phi_j = -E\phi_j - \frac{1}{2}\rho V^2\phi_j\Big(\sum_{j=1}^{n}\phi_j^2\Big). \tag{3.48}$$

We will seek an instanton solution of the form

$$\phi_j^{\mathrm{cl}}(\mathbf{r}) = \phi_{\mathrm{cl}}(\mathbf{r}) u_j,$$

where $\mathbf{u}$ is a unit vector in the "isotopic-spin" space of the $O(n)$-symmetric theory under consideration: $\mathbf{u}^2 = 1$. Confining ourselves to the spherically symmetric solutions, which correspond to the minimum of action [95, 101–103], we find from (3.48)

$$\frac{1}{2m}\left\{\frac{d^2\phi_{\mathrm{cl}}}{dr^2} + \frac{d-1}{r}\frac{d\phi_{\mathrm{cl}}}{dr}\right\} = -E\phi_{\mathrm{cl}} - \frac{1}{2}\rho V^2 \phi_{\mathrm{cl}}^3. \tag{3.49}$$

The trivial solution $\phi_{\mathrm{cl}} = 0$ is self-evident. We are concerned with the nontrivial solutions of (3.49) with a finite action (instantons), i.e., such solutions that yield converging action integral (3.3). Using the results obtained by Coleman [103] and Makhankov [104], we can show that the solutions which we are seeking for this equation exist only for $d < 4$. (At $E = 0$ a conformally invariant solution is possible even for $d = 4$ [97].) Least difficult, however, is a simple qualitative analysis [92] based on a method proposed by Finkel'stein *et al.* [105] more than three decades ago. Introducing dimensionless variables

$$\phi_{\mathrm{cl}}(\mathbf{r}) = \sqrt{\frac{2|E|}{\rho V^2}}\,\chi(t); \qquad r = \frac{t}{\sqrt{2m|E|}}, \tag{3.50}$$

we rewrite (3.49) in the form

$$\frac{d^2\chi}{dt^2} + \frac{d-1}{t}\frac{d\chi}{dt} = \pm\chi - \chi^3, \tag{3.51}$$

where the upper sign corresponds to $E < 0$, and the lower sign corresponds to $E > 0$. Equation (3.51) is an equation of motion of a particle of unit mass in a potential (Fig. 9a)

$$U(\chi) = \pm\tfrac{1}{2}\chi^2 + \tfrac{1}{4}\chi^4, \tag{3.52}$$

which is subjected to a time-dependent frictional force. Clearly, we are seeking solutions that satisfy the initial conditions,

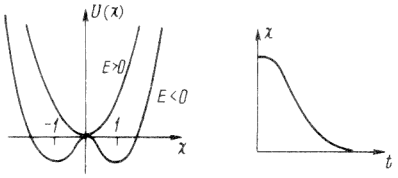

Figure 9. (a) The potential energy corresponding to the equation of motion (3.51). The "particle" is moving down an incline, with friction diminishing with time as $\sim 1/t$. (b) A qualitative form minimum-action instanton solution.

$$\chi(t=0) = \text{const}; \qquad \left.\frac{d\chi}{dt}\right|_{t=0} = 0. \tag{3.53}$$

The asymptotic behavior of the solutions of (3.51) at $t \gg 1$ can easily be determined by linearizing this equation near the extremal values of $U(\chi)$. An instanton with minimum action can occur only at $E < 0$, and is a solution (Fig. 9b) corresponding to the motion at which a "particle" rolls down an incline of $U(\chi)$ as $t \to \infty$ and stops at the point $\chi = 0$. The asymptotic form of this solution at $t \gg 1$ is

$$\chi(t) \sim t^{(1-d)/2} \exp(-t). \tag{3.54}$$

At $E > 0$ there are no solutions with finite action (3.3), and for the solution given by (3.53) and (3.54), with the help of (3.50), we easily find

$$S[\phi_{\text{cl}}] = \int d^d\mathbf{r}\, \mathscr{L}(\mathbf{r} \mid \phi_{\text{cl}}(\mathbf{r})) = \mathscr{A}_d \frac{m^{-d/2}}{\rho V^2} |E|^{2-(d/2)}$$

$$= \mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}; \qquad E_{sc} = m^{d/(4-d)}(\rho V^2)^{2/(4-d)}, \tag{3.55}$$

where the constant $\mathscr{A}_d$, which depends solely on the dimensionality of the space, is determined by the dimensionless integrals of $\chi(t)$. This constant can be determined by numerically integrating (3.51) with initial conditions (3.53). Such a problem, however, is of limited value, since model (3.1), which corresponds to a random Gaussian

field with a point like correlation, becomes meaningless at short ranges (small values of t), where the exact solution is now known. In a real system (in a lattice with impurities, for example) another effective Lagrangian "is operating" at small distances. This situation gives rise to a *physical* problem of the instanton core (see Ref. 106, for example, and the discussion below). Asymptotic expression (3.54) was also derived independently by Vrezin and Parisi [107].

The existence of instanton solutions in field theory (3.1) is a general property of field theories with an unstable vacuum [101, 102]. The action (3.3) in field theory (3.1) is not bounded from below for the arbitrary variations of the field $\phi(\mathbf{r})$. It has been rigorously shown [95], however, that an instanton accounts for the absolute minimum of $S[\phi_{\text{cl}}]$ in the class of functions that satisfy the classical equations $\delta S[\phi_{\text{cl}}]/\delta\phi_{\text{cl}} = 0$.

The instanton solutions give rise, at $E < 0$, to new contributions to the single-electron Green's function; specifically, they account, as we can see, for the fact that its imaginary part is nonvanishing even at $E < 0$, which corresponds to the formation of the density of states tail [6, 8, 86–89]. We will follow below primarily the procedure used in Ref. 92 (see also Ref. 91). Given that we are interested in the effect produced by the instanton solution, we will write the field $\phi_j(\mathbf{r})$ in the form

$$\phi_j(\mathbf{r}) \approx \phi_{\text{cl}}(\mathbf{r})u_j + \varphi_j(\mathbf{r}). \tag{3.56}$$

Having in mind the main contribution over the coupling constant (2.81) and confining ourselves to terms quadratic in $\varphi_j(\mathbf{r})$ in the action, we find a single-instanton contribution to the single-electron Green's function in the following schematic form [91]:

$$G(\mathbf{r}-\mathbf{r}'\,|\,-\rho V^2) \sim Z_0^{-1} \exp[-S[\phi_{\text{cl}}]] \int \{\delta\phi(\mathbf{r})\}\phi_{\text{cl}}(\mathbf{r})\phi_{\text{cl}}(\mathbf{r}') \times \exp\left\{-\frac{1}{2}\int d^d\mathbf{r}\varphi_i M_{ik}\varphi_k\right\}\left\{1 + O\left[\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right]\right\}, \tag{3.57}$$

where the factor

$$\exp\{-S[\phi_{\text{cl}}]\} = \exp\left\{-\mathscr{A}_d \frac{m^{-d/2}}{\rho V^2}|E|^{2-(d/2)}\right\} = \exp\left\{-\mathscr{A}_d\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\} \tag{3.58}$$

is nonanalytic in terms of the coupling constant and therefore cannot be determined from the conventional perturbation theory. Normalization integral (3.4) can be adequately described by Z_0, i.e., by a zeroth order approximation and

$$M_{ik} = M_l u_i u_k + M_T(\delta_{ik} + u_i u_k), \tag{3.59}$$

$$M_L = -\frac{1}{2m}\nabla^2 - E - \frac{3}{2}\rho V^2 \phi_{\text{cl}}^2(\mathbf{r}), \tag{3.60}$$

$$M_T = -\frac{1}{2m}\nabla^2 - E - \frac{1}{2}\rho V^2 \phi_{\text{cl}}^2(\mathbf{r}). \tag{3.61}$$

Introducing orthonormal sets of eigenfunctions of the operators M_L and M_T

$$M_L \psi_n^L = \lambda_n^L \psi_n^L, \tag{3.62}$$

$$M_T \psi_m^T = \lambda_m^T \psi_m^T, \tag{3.63}$$

we can expand the field $\phi(\mathbf{r})$ in these eigenfunctions

$$\boldsymbol{\phi}(\mathbf{r}) = \phi_L(\mathbf{r})\mathbf{u} + \boldsymbol{\varphi}_T(\mathbf{r}), \tag{3.64}$$

where

$$\phi_L(\mathbf{r}) = \phi_{\text{cl}}(\mathbf{r}) + \sum_n \mathbf{c}_n^L \psi_n^L(\mathbf{r}), \tag{3.65}$$

$$\boldsymbol{\varphi}_T(\mathbf{r}) = \sum_m \mathbf{c}_m^T \psi_m^T(\mathbf{r}). \tag{3.66}$$

We assume that φ^T is orthogonal to $\mathbf{u}$ in the isotopic-spin space. In (3.57) we then set

$$\int \{\delta\phi(\mathbf{r})\} \ldots = \prod_n \int d\mathbf{c}_n^L \prod_m \int d\mathbf{c}_m^T \ldots. \tag{3.67}$$

However, the resulting Gaussian integral should be calculated with

some caution. Let us qualitatively analyze Eqs. (3.62) and (3.63). We clearly see that these equations can be written as Schrödinger equations with the potentials

$$V_L(\mathbf{r}) = -E - \tfrac{3}{2}\rho V^2 \phi_{\mathrm{cl}}^2(\mathbf{r}) = -E - 3|E|\chi^2(\mathbf{r}), \tag{3.68}$$

$$V_T(\mathbf{r}) = -E - \tfrac{1}{2}\rho V^2 \phi_{\mathrm{cl}}^2(\mathbf{r}) = -E - |E|\chi^2(\mathbf{r}). \tag{3.69}$$

The qualitative behavior of these potentials is illustrated in Fig. 10 and the instanton solution of $\chi(\mathbf{r})$ is shown in Fig. 9b. We see that these potentials are attractive in nature ($-E > 0$ is an irrelevant constant). These potentials, and hence the eigenvalues λ_n^L and λ_m^T, do not depend on the coupling constant but are proportional to $|E|$. The structure of the spectrum can easily be understood on the basis of symmetry considerations.

A. Translational Zero Modes. If $\phi_{\mathrm{cl}}(\mathbf{r})$ is the solution of classical field equations (3.48) and (3.49), then $\phi_{\mathrm{cl}}(\mathbf{r} + \mathbf{R}_0)$ is also a solution by virtue of the translational symmetry of the problem (no condition is imposed on the location of the instanton "center" in the system). Let us consider an infinitesimal translation $\delta\mathbf{R}_0$. We then find

$$\phi_{\mathrm{cl}}(\mathbf{r} + \delta\mathbf{R}_0) \simeq \phi_{\mathrm{cl}}(\mathbf{r}) + \delta\mathbf{R}_{0\mu} \frac{\partial \phi_{\mathrm{cl}}}{\partial x_\mu}. \tag{3.70}$$

At the same time, we can treat Eq. (3.70) as a "perturbation" near the instanton solution $\phi_{\mathrm{cl}}(\mathbf{r})$ and write it as expansion (3.65)

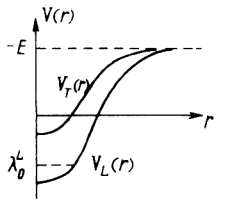

Figure 10. The potential energy in Schrödinger equations (3.62) and (3.63) expressed in qualitative form. λ_0^L is the lowest eigenvalue of the operator M_L.

$$\delta\phi_L(\mathbf{r}) = \delta\mathbf{R}_{0\mu}\frac{\partial\phi_{\rm cl}}{\partial x_\mu} = \delta\mathbf{c}_1^{L\mu}\psi_1^{L\mu}(\mathbf{r}), \tag{3.71}$$

since it becomes quite obvious after a direct substitution of $\phi_{\rm cl}(\mathbf{r}+\mathbf{R}_0)$ into Eq. (3.48) and subsequent differentiation of it with respect to $\mathbf{R}_0$ that

$$\frac{1}{2m}\nabla^2\frac{\partial\phi_{\rm cl}}{\partial\mathbf{R}_{0\mu}} + E\frac{\partial\phi_{\rm cl}}{\partial\mathbf{R}_{0\mu}} + \frac{3}{2}\rho V^2\phi_{\rm cl}^2\frac{\partial\phi_{\rm cl}}{\partial\mathbf{R}_{0\mu}} = 0, \tag{3.72}$$

so that $\partial\phi_{\rm cl}/\partial x_\mu$ is an eigenfunction of the operator M_L with a *zero* eigenvalue ($\lambda_1^{L\mu} = 0$). Clearly, $\psi_1^{L\mu} \sim \partial\phi_{\rm cl}/\partial x_\mu$. Requiring that the standard normalization condition be satisfied, $\int d^d\mathbf{r}|\psi_1^{L\mu}(\mathbf{r})|^2 = 1$, we immediately find

$$\psi_1^{L\mu}(\mathbf{r}) = \left\{\int d^d\mathbf{r}\left(\frac{\partial\phi_{\rm cl}}{\partial x_\mu}\right)^2\right\}^{-1/2}\frac{\partial\phi_{\rm cl}}{\partial x_\mu}. \tag{3.73}$$

The translational zero mode $\psi_1^{L\mu}(\mathbf{r})$ is obviously d-fold degenerate ($\mu = 1, \ldots, d$ is the dimensionality of the space). We clearly see from the fact that there are d zero eigenvalues, $\lambda_1^{L\mu} = 0$, that Schrödinger equation (3.62) has at least one *negative* eigenvalue λ_0^L: The ground state of a quantum-mechanical system cannot be degenerate. Since the translational modes are d-fold degenerate — they correspond to a "p-type" level [i.e., they are transformed according to the rotation-group representation $O(d)$ with $l = 1$], we clearly see that there is only a *single* state $\psi_0^L(\mathbf{r})$ with $\lambda_0^L < 0$ corresponding to $l = 0$ (the s-type state) in the attractive potential $V_L(\mathbf{r})$. A more rigorous proof of the uniqueness of this state may be found in Ref. 95.

It becomes clear from (3.71) and (3.73) that integration over $dc_1^{L\mu}$ is equivalent to integration over the "collective variable" $d\mathbf{R}_{0\mu}$, i.e., instead of integrating over $dc_1^{L\mu}$ in functional measure (3.67), we can integrate over the "instanton center" $\mathbf{R}_{0\mu}$, given that the integrand in (3.57) corresponds to an *arbitrary* instanton center. A conversion to integration over $\mathbf{R}_{0\mu}$ can be realized according to a rule, which is clearly evident from (3.71) and (3.73) (Ref. 102):

$$d\mathbf{c}_1^{L\mu} \to d\mathbf{R}_{0\mu}\left\{\int d^d\mathbf{r}\left(\frac{\partial\phi_{\rm cl}}{\partial x_\mu}\right)^2\right\}^{1/2} \tag{3.74}$$

for each one of the d translational modes.

B. Rotational Zero Modes. By complete analogy with the preceding analysis, we easily see that the existence of a classical solution of Eq. (3.48) of the type $\phi_{\mathrm{cl}}(\mathbf{r})\mathbf{u}$ implies that $\phi_{\mathrm{cl}}(\mathbf{r})\mathbf{u}'$ is also its solution, where $\mathbf{u}'$ can be obtained from $\mathbf{u}$ by an arbitrary rotation in the n-dimensional "isotopic" space. Clearly, $\phi_{\mathrm{cl}}(\mathbf{r})$ is an eigenfunction of the operator M_T in (3.63) with *zero* eigenvalue, $\lambda_0^T = 0$; $M_T\phi_{\mathrm{cl}} = 0$ coincides with (3.49). This level, which is $(n-1)$-fold degenerate [in the n-dimensional space an arbitrary rotation of the vector is determined by the $(n-1)$ angle], becomes in the limit $n \to 0$ a nondegenerate (and a ground state) level for the operator M_T. A normalized eigenfunction of the rotational zero mode is

$$\psi_0^T(\mathbf{r}) = \left\{ \int d^d\mathbf{r}\, \phi_{\mathrm{cl}}^2(\mathbf{r}) \right\}^{-1/2} \phi_{\mathrm{cl}}(\mathbf{r}). \tag{3.75}$$

In the case of an infinitesimal rotation $\mathbf{u} \to \mathbf{u} + \delta\mathbf{u}$ and $\mathbf{u} \cdot \delta\mathbf{u} = 0$, we would have

$$\begin{aligned} \delta\phi_T(\mathbf{r}) &= \phi_{\mathrm{cl}}(\mathbf{r})\delta\mathbf{u} = \delta\mathbf{c}_0^T \psi_0^T(\mathbf{r}) \\ &= \delta\mathbf{u} \left\{ \int d^d\mathbf{r}\, \phi_{\mathrm{cl}}^2(\mathbf{r}) \right\}^{1/2} \psi_0^T(\mathbf{r}). \end{aligned} \tag{3.76}$$

We have incorporated (3.66) into the second relation in (3.76) and (3.75) into the third relation. We see that in functional measure (3.67) it is possible to transform from integration over $d\mathbf{c}_0^T$ to integration over the "collective coordinate" $d\mathbf{u}$ (the direction in the "istopic" space) by relating the integrand in (3.57) to this arbitrary direction specified by the vector $\mathbf{u}$. This transformation to integration over the direction of $\mathbf{u}$ is accomplished by applying the rule

$$d\mathbf{c}_{0j}^T \to d\mathbf{u}_j \left\{ \int d^d\mathbf{r}\, \phi_{\mathrm{cl}}^2(\mathbf{r}) \right\}^{1/2} \tag{3.77}$$

for each one of the $(n-1)$-rotational zero modes which corresponds to $j = 1, \ldots, n-1$ directions of the vector $\mathbf{u}$.

As a result, we can write the measure in functional integral (3.57) in the following way:

$$\int \{\delta\phi(\mathbf{r})\} \ldots = \int d^d\mathbf{R}_0 \left\{ \int d^d\mathbf{r} \left(\frac{\partial \phi_{\mathrm{cl}}}{\partial x_\mu} \right)^2 \right\}^{d/2} \prod_{n \neq 1} \int d\mathbf{c}_n^L$$

$$\times \int_{\mathbf{u}^2=1} d\mathbf{u} \left\{ \int d^d\mathbf{r} \phi_{\mathrm{cl}}^2(\mathbf{r}) \right\}^{(n-1)/2} \prod_{m \neq 0} \int d\mathbf{c}_m^T \ldots . \quad (3.78)$$

The presence of Jacobians for transformation to the collective coordinates shown in the braces is extremely important. This heuristic derivation of the transformation from (3.67) to (3.78) is based on the work of Callan and Coleman [102].

Let us now return to the question of the *negative* eigenvalue, $\lambda_0^L < 0$, of the operator M_L. The eigenvalue-related integral over dc_0^L in (3.57) formally can be written in the form

$$\int d\mathbf{c}_0^L \exp\left\{ -\frac{1}{2} \lambda_0^L (c_0^L)^2 \right\} = \int d\mathbf{c}_0^L \exp\{|\lambda_0^L|(\mathbf{c}_0^L)^2\}. \quad (3.79)$$

This integral should again be thought of as an analytic continuation from positive λ_0^L to negative λ_0^L. By analogy with the analysis in the preceding section, let us turn to integration along the imaginary axis [90] and find

$$\int_{-i\infty}^{+i\infty} d\mathbf{c}_0^L e^{|\lambda_0^L|(c_0^L)^2} = \pm i \int_{-\infty}^{\infty} dx e^{-|\lambda_0^L|x^2} = \pm i \left(\frac{\pi}{|\lambda_0^L|} \right)^{1/2}. \quad (3.80)$$

We see that by analogy with the example in the preceding section, functional integral (3.57) corresponding to the "nontrivial" steepest descent solutions determines the *discontinuity* of the Green's function at the cut in the complex plane of the coupling constant along the negative part of the real axis, i.e., Im $G(\mathbf{r}-\mathbf{r}'|-\rho V^2)$, Thus (3.57) reduces to

$$\mathrm{Im}\ G(\mathbf{r}-\mathbf{r}'|-\rho V^2) \sim Z_0^{-1} \exp\{-S[\phi_{\mathrm{cl}}]\} \left\{ 1 + O\left(\frac{|E|}{E_{sc}} \right)^{2-(d/2)} \right\}$$

$$\times \left\{ \int d^d\mathbf{r} \left(\frac{\partial \phi_{\mathrm{cl}}}{\partial x_\mu} \right)^2 \right\}^{d/2} \left\{ \int d^d\mathbf{r} \phi_{\mathrm{cl}}^2(\mathbf{r}) \right\}^{(n-1)/2}$$

$$\times \{|\mathrm{Det}'\ M_L|\}^{-1/2} \{\mathrm{Det}'\ M_T\}^{-(1/2)(n-1)}$$

$$\times \int d^d\mathbf{R}_0 \int_{\mathbf{u}^2=1} d\mathbf{u} \phi_{\mathrm{cl}}(\mathbf{r}-\mathbf{R}_0) \phi_{\mathrm{cl}}(\mathbf{r}-\mathbf{R}_0), \quad (3.81)$$

where the prime on the determinants denotes that the zero eigenvalues corresponding to the translational and rotational modes have been eliminated from the products $\Pi_n \lambda_n^L$ and $\Pi_m \lambda_m^T$ that determine these determinants.

Using (3.50), we can easily obtain a dimensional estimate of the Jacobians:

$$J_L[\phi_{cl}] \equiv \int d^d\mathbf{r} \left(\frac{\partial \phi_{cl}}{\partial x_\mu}\right)^2 \sim mS[\phi_{cl}] \sim m^{1-(d/2)} \frac{|E|^{(4-d)/2}}{\rho V^2}, \quad (3.82)$$

$$J_T[\phi_{cl}] \equiv \int d^d\mathbf{r} \phi_{cl}^2(\mathbf{r}) \sim |E| S[\phi_{cl}] \sim m^{-d/2} \frac{|E|^{(2-d)/2}}{\rho V^2}. \quad (3.83)$$

We then immediately find from (3.81), for $n \neq 0$,

$$\begin{aligned} \text{Im } G(\mathbf{r}-\mathbf{r}'|-\rho V^2) \sim Z_0^{-1}\{|\text{Det}' M_L|\}^{-1/2}\{\text{Det } M_T\}^{-(1/2)(n-1)} \\ \times m^{(d/2)[1-(d/2)]-(d/4)(n+1)} \frac{|E|^{(d/4)(4-d)+(1/4)(2-d)(n+1)}}{(\rho V^2)^{(d+n+1)/2}} \\ \times \exp\left\{-\mathscr{A}_d \frac{m^{-d/2}}{\rho V^2} |E|^{2-(d/2)}\right\} \\ \times \left\{1 + O\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\} \int d^d\mathbf{t}_0 \chi(\mathbf{t}-\mathbf{t}_0)\chi(\mathbf{t}'-\mathbf{t}_0). \end{aligned} \quad (3.84)$$

This expression can be simplified by means of a dimensional analysis based on the fact that the eigenvalues λ_n^L and λ_m^T are proportional to $|E|$. We then find in the limit of $n \to 0$

$$\begin{aligned} \text{Im } G(\mathbf{r}-\mathbf{r}'|-\rho V^2) \sim \frac{m^{(d/4)(1-d)}|E|^{(d/4)(5-d)}}{(\rho V^2)^{(d+1)/2}} \exp\left\{-\mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\} \\ \times \int d^d\mathbf{t}_0 \chi(\mathbf{t}-\mathbf{t}_0)\chi(\mathbf{t}'-\mathbf{t}_0) \\ \times \left\{1 + O\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\}. \end{aligned} \quad (3.85)$$

Such a result was obtained by Cardy [91]. Here, however, the

dimensionless determinants must be calculated. This requires finding the spectrum of the eigenvalues of Eqs. (3.62) and (3.63) and carrying out a renormalization (at $d \geq 2$ the determinants reveal some divergences [107]). We will, however, use a method proposed by Sadovskiĭ [92]. This method is based on the use of the dispersion relation over the coupling constant and upon matching with the problem of phase transitions. Recalling that the eigenvalues λ_n^L and λ_m^T are independent of the coupling constants ρV^2, so that both $\mathrm{Det}' M_L$ and $\mathrm{Det}' M_T$ are also independent of it, and ignoring corrections of order $(|E|/E_{sc})^{2-(d/2)}$, we can rewrite (3.84) in the momentum representation [92, 100]

$$\mathrm{Im}\, G(E\mathbf{p}|-\rho V^2) \simeq C(E\mathbf{p}) \frac{1}{(\rho V^2)^{(d+n+1)/2}} \times \exp\left\{-\mathscr{A}_d \frac{m^{-d/2}}{\rho V^2} |E|^{2-(d/2)}\right\}, \quad (3.86)$$

where $C(E\mathbf{p})$ *does not depend* on the coupling constant. At $n = 0$, we would have

$$\mathrm{Im}\, G(E\mathbf{p}|-\rho V^2) = C(E\mathbf{p}) \frac{1}{(\rho V^2)^{(d+1)/2}} \exp\left\{-\frac{A(E)}{\rho V^2}\right\}. \quad (3.87)$$

We have introduced in this expression

$$A(E) = \mathscr{A}_d m^{-d/2} |E|^{2-(d/2)}. \quad (3.88)$$

The condition for the applicability of these equations is $S[\phi_{\mathrm{cl}}] \gg 1$ when the method of steepest descent "works" well in the calculation of the functional integral. In other words, the condition

$$\frac{A(E)}{\rho V^2} = \mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)} \gg 1; \quad |E| \gg E_{sc} \quad (3.89)$$

must be satisfied. Thus the condition for the applicability of the "perturbation theory" near the instanton solution is qualitatively the same as that for the conventional perturbation theory in the region

$E > 0$. The energy region of width $\sim 2E_{sc}$ around $E = 0$ is the analog of the Ginzburg critical region in the theory of critical phenomena.

If $\operatorname{Im} G(E\mathbf{p} \mid -\rho V^2)$, i.e., the discontinuity at the cut in the complex plane of the coupling constant is known, we can determine the Green's function from dispersion integral (3.46)

$$G(E\mathbf{p} \mid g) = \frac{1}{\pi} C(E\mathbf{p}) \int_{-\infty}^{0} dz \frac{\exp\left\{-\dfrac{A(E)}{z}\right\}}{(z-g)(-z)^{(d+1)/2}}, \tag{3.90}$$

where g is an arbitrary coupling constant in the complex plane. For an electron in a random field we have $g \to -\rho V^2 < 0$. Integral (3.90) can be calculated

$$G(E\mathbf{p} \mid g) = -\frac{1}{\pi} C(E\mathbf{p}) g^{-(d+1)/2} \times \exp\left\{\frac{A(E)}{g}\right\} \Gamma\left(\frac{d+1}{2}\right) \Gamma\left(\frac{1-d}{2}; \frac{A(E)}{g}\right), \tag{3.91}$$

where $\Gamma(\alpha, x) = \int_x^\infty dt t^{\alpha-1} e^{-t}$ is an incomplete gamma function.

It can immediately be seen from expression (3.87) that the exponential factor in the density of states is correct [8, 86–89]. We see that this factor is determined completely by the instanton solutions in the field theory (3.1). The necessity of analyzing the regions $E > 0$ and $E < 0$ from different viewpoints arises automatically, since the instantons exist only at $E < 0$. Another advantage of this method is that it allows a direct calculation of the coefficient of the exponential function $C(E\mathbf{p})$. Here we must point out [92] that at $g > 0$ Green's functions (3.90) and (3.91) describe the correlator of the *stable* field theory (the theory of the second-order phase transitions). Far from the critical region this correlator is well known [21–23]: This is the usual Ornstein–Zernike correlator. In our case, at $|E| \gg E_{sc}$ we would have

$$G(E\mathbf{p} \mid g > 0) \approx -\frac{1}{|E| + \mathbf{p}^2/2m}. \tag{3.92}$$

More precisely, this expression is valid if a stronger inequality of the

type in (3.18) is satisfied; however, this refinement is important only in the limit $d \to 2$. We understand E everywhere to mean the renormalized energy, i.e., the distance from the displaced band edge (or the "temperature" reckoned from the fluctuation-displaced transition temperature in the theory of critical phenomena), which is calculated in a one-loop approximation. Using the asymptotic form of the incomplete gamma function: $\Gamma(\alpha, x)_{x \gg 1} \approx x^{\alpha-1} e^{-x} \times \{1 + O(x^{-1})\}$, we find from (3.91) ($|E| \gg E_{sc}$)

$$G(E\mathbf{p} \mid g > 0) \approx -\frac{1}{\pi} \Gamma\left(\frac{d+1}{2}\right) \{A(E)\}^{-(d+1)/2} C(E\mathbf{p}). \quad (3.93)$$

Comparing (3.92) and (3.93), we find

$$C(E\mathbf{p}) \underset{|E| \gg E_{sc}}{\approx} \frac{\pi \mathscr{A}_d^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} m^{-(d/4)(d+1)} \frac{|E|^{(d+1)[1-(d/4)]}}{|E| + \frac{\mathbf{p}^2}{2m}}. \quad (3.94)$$

As a result, we find the following expression for the imaginary part of the Green's function of an electron, at $|E| \gg E_{sc}$,

$$\operatorname{Im} G^{R,A}(E\mathbf{p} \mid -\rho V^2) \approx \pm \frac{\pi \mathscr{A}_d}{\Gamma\left(\frac{d+1}{2}\right)} \frac{|E|^{(d+1)[1-(d/4)]}}{|E| + \frac{\mathbf{p}^2}{2m}} \frac{1}{(m^{d/2} \rho V^2)^{(d+1)/2}}$$

$$\times \exp\left\{-\frac{A(E)}{\rho V^2}\right\} \left\{1 + O\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\}. \quad (3.95)$$

We can then calculate the density of the electronic states in the "tail" region, including the coefficient of the exponential function. We thus find ($E < 0$, $|E| \gg E_{sc}$)

$$N(E) = -\frac{1}{\pi} \int \frac{d^d \mathbf{p}}{(2\pi)^d} \operatorname{Im} G^R(E\mathbf{p} \mid -\rho V^2) \approx S_d \frac{\mathscr{A}_d^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}$$

$$\times \frac{|E|^{(d+1)[1-(d/4)]}}{(m^{d/2} \rho V^2)^{(d+1)/2}} \exp\left\{-\frac{A(E)}{\rho V^2}\right\} \int_0^{p_0} dp p^{d-1} \frac{1}{|E| + \frac{\mathbf{p}^2}{2m}}, \quad (3.96)$$

where $S_d = 2^{-(d-1)}\pi^{-d/2}/\Gamma(d/2)$. At $d = 1$ we can perform the limit, $p_0 = a^{-1} \to \infty$. We find from (3.96)

$$N(E) = S_1 \frac{\pi \mathscr{A}_1}{\sqrt{2}} \frac{|E|}{\rho V^2} \exp\left\{ -\mathscr{A}_1 \frac{|E|^{3/2}}{m^{1/2}\rho V^2} \right\}. \tag{3.97}$$

The constant $\mathscr{A}_1 = \frac{4}{3}\sqrt{2}$ (Ref. 86) [at $d = 1$ Eq. (3.49) can be solved exactly [90]] is in agreement with Halperin's [108] exact result within a factor of $3/\pi$. This demonstrates the precision of the method based upon neglecting the corrections to the result (3.87) obtained by using the method of steepest descent, while performing calculations based on the dispersion relation over the coupling constant. At $d \geqslant 2$ the divergence of the integral in (3.86) is cut off at the momenta $\sim a^{-1}$ which are associated with the reciprocal radius of the random-field correlator. Our analysis is suitable for the energies $|E| \ll \tilde{E}_0 = p_0^2/2m$. For $2 < d < 4$ the density of states tail is given by

$$N(E) \underset{E_{sc} \ll |E| \ll \tilde{E}_0}{\approx} S_d (\mathscr{A}_d)^{(d+1)/2} \frac{2m}{\Gamma\left(\frac{d+1}{2}\right)} \frac{(2m\tilde{E}_0)^{(d-2)/2}}{d-2} \times \left(\frac{|E|}{E_{sc}}\right)^{(d+1)[1-(d/4)]} \exp\left\{ -\mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)} \right\}. \tag{3.98}$$

Similarly, at $d = 2$ we would have

$$N(E) \approx \text{Const} \frac{|E|^{3/2}}{m^{1/2}(\rho V^2)^{3/2}} \ln \frac{\tilde{E}_0}{|E|} \exp\left\{ -\mathscr{A}_2 \frac{|E|}{m\rho V^2} \right\}. \tag{3.99}$$

We should point out, however, that the range of applicability of Eq. (3.99) "goes off" to infinity in accordance with the inequality (3.18), which determines the range of applicability of the Ornstein–Zernike approximation.

At $|E| \gg \tilde{E}_0$ (a situation which corresponds to the assumption that the random potential correlation radius is large) the density of states tail can be determined by a semiclassical approximation [6, 55] or from a "lattice" instanton [106], which gives for any dimensionality of space a Gaussian asymptotic expression

$$N(E) \sim \exp\left\{-\frac{a^d}{\rho V^2} E^2\right\}. \tag{3.100}$$

Thus, at $d=3$ from (3.98) and Ref. 55 we find the following expression for the density of states in the tail region:

$$N(E) \underset{(E<0)}{\approx} \begin{cases} \dfrac{\mathscr{A}_3^2}{2\pi^2} \dfrac{|E|\tilde{E}_0^{1/2}}{m^{3/2}(\rho V^2)^2} \exp\left\{-\mathscr{A}_3 \dfrac{|E|^{1/2}}{m^{3/2}\rho V^2}\right\}; & E_{sc} \ll |E| \ll \tilde{E}_0 \\ \dfrac{m^{3/2}}{4\pi^2 a^3} \dfrac{\rho V^2}{E^{3/2}} \exp\left\{-\dfrac{a^3}{\rho V^2} E^2\right\}; & |E| \gg \tilde{E}_0. \end{cases} \tag{3.101}$$

The expression for the density of states can also be derived from Cardy's result (3.85). We find [91]

$$N(E) = \operatorname{Im} G^R(\mathbf{r}-\mathbf{r}|-\rho V^2) \sim m^{(d/4)(1-d)} \frac{|E|^{(d/4)(5-d)}}{(\rho V^2)^{(d+1)/2}} \int_0^\infty dt t^{d-1} \chi^2(t) \times \exp\left\{-\mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\}. \tag{3.102}$$

This expression is in agreement with (3.98) and (3.99) with respect to the dimensionality, but the energy dependence $|E|$ turns out to be different. To understand the reasons for this difference, we will Fourier transform (3.85) to the momentum space. We then find

$$\operatorname{Im} G(E\mathbf{p}|-\rho V^2) \sim m^{-(d/4)(d+1)} \frac{|E|^{(d+1)[1-(d/4)]-1}}{(\rho V^2)^{(d+1)/2}} \mathscr{F}\left(\frac{p}{\sqrt{2m|E|}}\right) \times \exp\left\{-\mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\}, \tag{3.103}$$

where

$$\mathscr{F}(\mathbf{z}) = \int d^d(\mathbf{t}-\mathbf{t}') e^{-i\mathbf{z}(\mathbf{t}-\mathbf{t}')} \int d^d\mathbf{t}_0 \chi(\mathbf{t}-\mathbf{t}_0)\chi(\mathbf{t}'-\mathbf{t}_0). \tag{3.104}$$

A comparison of (3.103) and (3.95) shows that our approximation

specifies the function $\mathcal{F}(\mathbf{z}) = (1 + z^2)^{-1}$ by matching with the Ornstein–Zernike result. Direct calculations clearly show that the use of asymptotic form (3.54) of the instanton solution of (3.104) accounts for this behavior of the $\mathcal{F}(\mathbf{z})$ function. The results of Refs. 91 and 92 for the *imaginary part of the Green's function* are therefore equivalent in the range of their applicability. Our method, however, at once yields the renormalized (finite) expression for Im $G(E\mathbf{p})$, whereas Eqs. (3.85) and (3.103) generally contain an additional dimensionless product of eigenvalues (3.62) and (3.63), which accounts for the infinite constant (which is cancelled out by the action-renormalization counterterm [95, 107]). The use of Eq. (3.103) in the calculation of the density of states gives rise to the integral

$$\int d^d\mathbf{p}\,\mathcal{F}\left(\frac{\mathbf{p}}{\sqrt{2m|E|}}\right) \sim (2m|E|)^{d/2} \int_0^{p_0/\sqrt{2m|E|}} dz z^{d-1} \frac{1}{1+z^2}$$

$$\sim m^{d/2}|E|^{(d/2)} f\left(\frac{\tilde{E}_0}{|E|}\right). \tag{3.105}$$

If we could assume that $f(x)$ in (3.105) is a dimensionless constant, we would obtain Cardy's result (3.102) for the density of states. However *a dimensionless function* of the ratio $\tilde{E}_0/|E|$ and our results for the density of states appear in (3.105) in the "Ornstein–Zernike approximation." This situation points out to a well-known fact that the Ornstein–Zernike approximation cannot be used at short distances or for the calculation of the mean-square fluctuation [109]. The "physical cutoff" of the divergence at short distances in our case occurs at a length $\sim a$ of the random-field correlation radius, which imposes a limitation on the energy region $|E| \ll \tilde{E}_0$ mentioned above. Cardy [91] has not addressed this problem formally, since the integral $\int_0^\infty dt t^{\alpha-1}\chi^2(t)$ is finite, because the instanton contribution to the action is finite. As we have indicated above, however, the model with a point correlation of the Gaussian random field, which reduces to field theory (3.1), is not a realistic model at short distances, where the physical cutoff mechanism is always "operating." This mechanism is attributable to the fact that the random-field correlation radius in a real system is finite (in the case of a lattice, this distance is on the order of the interatomic range). There is, correspondingly, the problem involving the instanton "core" (Ref. 106), and the effective

Lagrangian and classical equations at short ranges change in such a way as to reproduce the semiclassical (Gaussian) asymptotic form of the density of states tail (3.100) at $|E| \gg \tilde{E}_0$ [6, 55, 106]. In this respect, our results for the density of states seem to correspond more closely to the physical formulation of the problem, whereas Cardy [91] analyzed a different, more formal model.

3.3. Instantons and Localization

It is evident from the analysis which we have carried out that localization is closely related to the appearance in the relevant energy region of nonlinear finite-action solutions (instantons) of classical equations of the effective field theory which can be associated with the problem of an electron in a random field. Let us now analyze how a two-particle Green's function is calculated and show, following Refs. 35 and 91, that the instanton solutions of the effective field theory lead to the appearance of a singular contribution of the type (2.26), which was discussed above in connection with the general criterion of localization. We will establish a direct connection between instantons and localization and verify indirectly the general relations obtained during the analysis of this criterion.

As we have indicated above, a calculation of the average two-particle Green's function in a Gaussian random field with a "white-noise" correlation (point interaction) requires an analysis of the effective field theory determined by the $O(n) \times O(m)$ symmetric $(n \to 0,\ m \to 0)$ Lagrangian (3.5) of two interacting fields. We can derive from (3.5) the classical equations

$$\frac{1}{2m}\nabla^2\phi_j = -(E+\omega)\phi_j - \frac{1}{2}\rho V^2\phi_j\left(\sum_{j=1}^{n}\phi_j^2\right) - \frac{1}{2}\rho V^2\phi_j\left(\sum_{i=1}^{m}\varphi_i^2\right)$$

$$\frac{1}{2m}\nabla^2\varphi_i = -E\varphi_i - \frac{1}{2}\rho V^2\varphi_i\left(\sum_{i=1}^{m}\varphi_i^2\right) - \frac{1}{2}\rho V^2\varphi_i\left(\sum_{j=1}^{n}\phi_j^2\right). \tag{3.106}$$

In general, an analysis of this system of nonlinear differential equations is complex and to the best of my knowledge has not been performed by anyone. At $E < 0$ and $E + \omega > 0$, however, a simple

qualitative analysis shows that Eq. (3.107) has a spherically symmetric instanton solution of the type

$$\varphi_i^{\mathrm{cl}}(\mathbf{r}) = \varphi_{\mathrm{cl}}(\mathbf{r}) e_i; \qquad \phi_j^{\mathrm{cl}}(\mathbf{r}) = 0, \tag{3.107}$$

where $\varphi_{\mathrm{cl}}(\mathbf{r})$ is again given by an equation of the type shown in (3.50)

$$\varphi_{\mathrm{cl}}(\mathbf{r}) = \sqrt{\frac{2|E|}{\rho V^2}}\, \chi(t); \qquad r = \frac{t}{\sqrt{2m|E|}}. \tag{3.108}$$

Here $\chi(t)$ is defined by (3.53) and (3.54). In Eq. (3.107) $\mathbf{e}$ is a unit (m-component) "isotopic" vector for the field φ. Although the system of equations in (3.106) probably has other finite-action solutions, we need only (3.107), which appears first on the energy scale. The two-particle Green's function is defined by the functional integral

$$\begin{aligned} &\langle G^{\mathrm{R}}(\mathbf{r}\mathbf{r}'; E + \omega + i\delta) G^{A}(\mathbf{r}'\mathbf{r}; E - \delta)\rangle \\ &\quad = \lim_{n\to 0,\, m\to 0} Z^{-1} \int \{\delta\phi(\mathbf{r})\} \int \{\delta\varphi(\mathbf{r})\} \\ &\qquad \times \phi_j(\mathbf{r})\phi_j(\mathbf{r}')\varphi_i(\mathbf{r}')\varphi_i(\mathbf{r}) \exp\{-S[\phi, \varphi]\}, \end{aligned} \tag{3.109}$$

where

$$Z = \int \{\delta\phi(\mathbf{r})\} \int \{\delta\varphi(\mathbf{r})\} \exp\{-S[\phi, \varphi]\} \tag{3.110}$$

is a normalization integral, and $S[\phi, \varphi]$ is an action which corresponds to Lagrangian (3.5). By analogy with the analysis of a single-electron Green's function which we carried out above, we will analyze in (3.109) the contribution associated with the Gaussian fluctuations around a classical solution of (3.107). We then find, just as in (3.81),

$$\langle G^R(\mathbf{r}\mathbf{r}'; E+\omega+i\delta)G^R(\mathbf{r}'\mathbf{r}; E-i\delta)\rangle \sim Z_0^{-1}\exp\{-S[\varphi_{\rm cl}]\}J_L^{d/2}[\varphi_{\rm cl}]$$
$$\times J_T^{(m-1)/2}[\varphi_{\rm cl}]\int d^d\mathbf{R}_0\int_{\mathbf{e}^2=1} d\mathbf{e}\,\varphi_{\rm cl}(\mathbf{r}'-\mathbf{R}_0)\varphi_{\rm cl}(\mathbf{r}-\mathbf{R}_0)$$
$$\times\int\{\delta\phi(\mathbf{r})\}\int\{\delta\varphi(\mathbf{r})\}'\phi_j(\mathbf{r})\phi_j(\mathbf{r}')\exp\{-S_0[\phi,\varphi]\}, \tag{3.111}$$

where $S[\varphi_{\rm cl}]$ is the same as in (3.55), $J_L[\varphi_{\rm cl}]$ and $J_T[\varphi_{\rm cl}]$ were defined in (3.82) and (3.83), the normalization integral was again taken at $\rho V^2=0$, and $S[\phi,\varphi]$ denotes the action corresponding to the Gaussian fluctuations near instanton solution (3.107) [$\varphi(\mathbf{r})$ now denotes a deviation from $\varphi_{\rm cl}(\mathbf{r})$]

$$S_0[\phi,\varphi]=\int d^d\mathbf{r}\{\mathscr{L}_0(\phi)+\mathscr{L}_0(\varphi)\}, \tag{3.112}$$

where

$$\mathscr{L}_0(\phi)=\sum_i \phi_i(M_T-\omega-i\delta)\phi_i, \tag{3.113}$$

$$\mathscr{L}_0(\varphi)=\sum_{ij}\varphi_i(M_T+i\delta)(\delta_{ij}-e_ie_j)\varphi_j+\sum_{ij}\varphi_L(M_L+i\delta)e_ie_j\varphi_j. \tag{3.114}$$

Here M_L and M_T are the operators which were introduced in (3.60) and (3.61). The prime on the functional integral in (3.111) means that the zero eigenvalues of the operators M_L and M_T (zero modes), which are taken into account in the integration over the "collective" variables $\mathbf{R}_0$ and $\mathbf{e}$, have been eliminated.

Incorporating eigenfunctions (3.62) and (3.63), and calculating in (3.111) the Gaussian functional integral over the field $\phi(\mathbf{r})$, we find

$$\int\{\delta\phi(\mathbf{r})\}\phi_j(\mathbf{r})\phi_j(\mathbf{r}')\exp\{-S_0[\phi,\varphi]\}$$
$$\sim\sum_k\frac{\psi_k^T(\mathbf{r}-\mathbf{R}_0)\psi_k^T(\mathbf{r}'-\mathbf{R}_0)}{(\lambda_k^T-\omega-i\delta)^{1+(n/2)}}$$
$$\xrightarrow[n\to 0]{}\frac{\psi_0^T(\mathbf{r}-\mathbf{R}_0)\psi_0^T(\mathbf{r}'-\mathbf{R}_0)}{\omega+i\delta}+\sum_{k\neq 0}\ldots, \tag{3.115}$$

where the normalized eigenfunction of the lowest level, associated with operator $M_T(\lambda_0^T = 0$—the "rotational" zero mode) is given by (3.75). Accordingly, a singular (in the limit of $\omega \to 0$) contribution to a two-particle Green's function ($m \to 0$, $n \to 0$) arises from (3.111):

$$\langle G^R(\mathbf{r}\mathbf{r}'; E + \omega + i\delta) G^A(\mathbf{r}'\mathbf{r}; E - i\delta) \rangle \sim \frac{i}{\omega + i\delta} \exp\{-S[\varphi_{\text{cl}}]\} Z_0^{-1}$$
$$\times J_L^{d/2}[\varphi_{\text{cl}}] J_T^{1/2}[\varphi_{\text{cl}}] \{|\text{Det}' M_L|\}^{-1/2} \{\text{Det}' M_T\}^{1/2}$$
$$\times \int d^d \mathbf{R}_0 \varphi_{\text{cl}}^2(\mathbf{r} - \mathbf{R}_0) \varphi_{\text{cl}}^2(\mathbf{r}' - \mathbf{R}_0). \tag{3.116}$$

Here there are no zero eigenvalues in Det' M_L and Det' M_T. The factor i arises from a single negative eigenvalue of the operator M_L. An expression equivalent to (3.116) was initially obtained (for $\omega = 0$) by Cardy [91] (see also Ref. 110). We have based our analysis of the derivation given by Sadovskiĭ [35].

The singular contribution, we might note, is related to the existence of the "rotational zero mode," i.e., to the symmetry of the system. It is therefore justifiable to assume that this contribution does not vanish when higher-order corrections are taken into account.

The result (3.116) is in complete agreement with the form of the two-particle Green's function in the energy region corresponding to the localized states suggested in (2.26) and (2.27). In particular, if we transform in (3.116) to a Fourier representation using

$$\tilde{\chi}_{\mathbf{q}} = \int d^d \mathbf{r} e^{-i\mathbf{q}\mathbf{r}} \varphi_{\text{cl}}^2(\mathbf{r}), \tag{3.117}$$

we find for the "localization amplitude" (2.20)

$$A_E(\mathbf{q}) \sim \chi_{\mathbf{q}} \chi_{-\mathbf{q}}, \tag{3.118}$$

consistent with (2.29). Introducing the Fourier representation of the instanton

$$\varphi_{\mathbf{q}}^{\text{cl}} = \int d^d \mathbf{r} e^{-i\mathbf{q}\mathbf{r}} \varphi_{\text{cl}}(\mathbf{r}), \tag{3.119}$$

we see that

$$\tilde{\chi}_{\mathbf{q}} = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\, \varphi_{\mathbf{p}}^{\mathrm{cl}} \varphi_{\mathbf{q}-\mathbf{p}}^{\mathrm{cl}}. \tag{3.120}$$

Comparing this expression with (2.26)–(2.28), we find

$$\psi_{\mathbf{p}}^{\mathbf{q}}(E) \sim \varphi_{\mathbf{p}}^{\mathrm{cl}}(E)\varphi_{\mathbf{q}-\mathbf{p}}^{\mathrm{cl}}(E). \tag{3.121}$$

Thus this analysis is, in fact, a justification of the form of the two-particle Green's function proposed in (2.26). Taking into account (2.21) and (2.22) and the explicit instanton contribution to the density of states which follows from (3.81)

$$N(E) \sim Z_0^{-1} \exp\{-S[\varphi_{\mathrm{cl}}]\} J_L^{d/2}[\varphi_{\mathrm{cl}}] J_T^{-1/2}[\varphi_{\mathrm{cl}}] \{|\mathrm{Det}'\, M_L|\}^{-1/2}$$
$$\times \{\mathrm{Det}'\, M_T\}^{1/2} \int d^d\mathbf{R}_0 \varphi_{\mathrm{cl}}^2(\mathbf{r}-\mathbf{R}_0), \tag{3.122}$$

we immediately find from (3.116) the following expression for $A_E(\mathbf{r}-\mathbf{r}')$, which is valid to within a dimensionless constant:

$$A_E(\mathbf{r}-\mathbf{r}') \sim \int d^d\mathbf{R}_0 \varphi_{\mathrm{cl}}^2(\mathbf{r}-\mathbf{R}_0)\varphi_{\mathrm{cl}}^2(\mathbf{r}'-\mathbf{R}_0) \left\{\int d^d\mathbf{r}\varphi_{\mathrm{cl}}^2(\mathbf{r})\right\}^{-1}. \tag{3.123}$$

For the "probability of return" A_E (2.22) we find from (3.123)

$$A_E \sim \int d^4\mathbf{r}\varphi_{\mathrm{cl}}^4(\mathbf{r}) \left\{\int d^d\mathbf{r}\varphi_{\mathrm{cl}}^2(\mathbf{r})\right\}^{-1} \sim (m|E|)^{d/2}. \tag{3.124}$$

Incorporating now asymptotic expression (3.54), we easily see that

$$A_E(\mathbf{r}-\mathbf{r}') \sim \exp\{-|\mathbf{r}-\mathbf{r}'|R_{\mathrm{loc}}^{-1}(E)\}, \tag{3.125}$$

where

$$R_{\mathrm{loc}}(E) \sim \{2m|E|\}^{-1/2} \sim (m|E|)^{-\nu}; \quad \nu = \tfrac{1}{2}. \tag{3.126}$$

Thus, at $|E| \gg E_{sc}$ (in the depth of the tail) we have a "classical"

result for the localization length which is determined simply by the instanton "radius." We see from (3.124) and (3.126) that $A_E \sim R_{\text{loc}}^{-d}$, consistent with the a simple estimate mentioned in Sec. 1.

Accordingly, an instanton method, in contrast with the analysis in Sec. 1 which is based on the use of a homogeneous Bethe–Salpeter equation, describes an entire energy region corresponding to the localized states and, in contrast with the self-consistent localization theory (Sec. 2), it describes correctly the asymptotic behavior of the tail region in the density of states. Several other results can also be obtained within the framework of this method. For example, on the basis of its modification proposed in Ref. 111, Houghton *et al.* [112] carried out a rigorous analysis of the dynamic conductivity and derived the well-known Mott law: $\sigma(\omega) \sim \omega^2(\ln \omega)^{d+1}$ (Ref. 1) for the conductivity along the localized states.

Let us consider the relationship between the two methods of determining the singular part of a two-particle Green's function—the method based on the use of a homogeneous Bethe–Salpeter equation (Sec. 1) and the instanton method. We will show that in general each description arises in a natural way as a manifestation of a *different* instability of the system within the framework of the effective-action formalism for composite fields [35].

The effective action for this system of fields ϕ and φ is a functional Γ of the "classical" (average) fields ϕ_{cl} and φ_{cl} and the corresponding Green's functions which satisfies the following variational principle [113]:

$$\frac{\delta\Gamma}{\delta\phi_{\text{cl}}(\mathbf{r})} = 0; \qquad \frac{\delta\Gamma}{\delta\varphi_{\text{cl}}(\mathbf{r})} = 0; \qquad \frac{\delta\Gamma}{\delta G(\mathbf{r}, \mathbf{r}')} = 0. \tag{3.127}$$

The functional can be determined from a double Legendre transformation of the generating functional of the "classical" fields and Green's functions of the field theory under consideration [113]. The effective-action formalism is especially fit for the analysis of the symmetry breaking with respect of the fields ϕ and φ and the corresponding Green's functions G. An analogous formalism in the many-body problem was analyzed by several authors [114–117].

For brevity, we will use the matrix notations

$$\Phi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}; \qquad \Phi^+ = (\phi\varphi), \tag{3.128}$$

$$\hat{G} = \begin{bmatrix} G_{\phi\phi} & G_{\phi\varphi} \\ G_{\varphi\phi} & G_{\varphi\varphi} \end{bmatrix}; \qquad G_{\phi\varphi} = G_{\varphi\phi}. \tag{3.129}$$

Introducing the matrix (the δ symbols correspond to the isotopic field indices)

$$\hat{G}_0^{-1}(\mathbf{r}\mathbf{r}') = \begin{bmatrix} \left\{\frac{1}{2m}\nabla^2 - (E + \omega + i\delta)\right\}\delta_{ij} & 0 \\ 0 & \left\{\frac{1}{2m}\nabla^2 - (E - i\delta)\right\}\delta_{ij} \end{bmatrix} \delta(\mathbf{r} - \mathbf{r}'), \tag{3.130}$$

we can rewrite Lagrangian (3.5) in a compact form

$$\mathscr{L}(\mathbf{r}) = \frac{1}{2}\,\mathrm{Sp}\int d^d\mathbf{r}'\Phi^+(\mathbf{r})\hat{G}_0^{-1}(\mathbf{r}\mathbf{r}')\Phi(\mathbf{r}') - \frac{1}{8}\rho V^2(\mathrm{Sp}\,\Phi^+\Phi)^2 \tag{3.131}$$

Introducing the external sources

$$J = (J_\phi J_\varphi); \qquad \hat{K} = \begin{bmatrix} K_{\phi\phi} & K_{\phi\varphi} \\ K_{\varphi\phi} & K_{\varphi\varphi} \end{bmatrix}, \tag{3.132}$$

we can write the generating functional of the "classical" (average) fields and Green's functions as follows:

$$\begin{aligned} Z\{J, K\} &\equiv \exp\{-W(J, K)\} \\ &= \int \{\delta\Phi\} \exp\left\{-S[\Phi] - \mathrm{Sp}\int d^d\mathbf{r}J(\mathbf{r})\Phi(\mathbf{r})\right. \\ &\quad \left. -\frac{1}{2}\,\mathrm{Sp}\int d^d\mathbf{r}\int d^d\mathbf{r}'\Phi^+(\mathbf{r})\hat{K}(\mathbf{r}\mathbf{r}')\Phi(\mathbf{r}')\right\}, \end{aligned} \tag{3.133}$$

where the action is

$$S[\Phi] = \mathrm{Sp} \int d^d\mathbf{r} \int d^d\mathbf{r}' \Phi^+(\mathbf{r}) \hat{G}_0^{-1}(\mathbf{r}\mathbf{r}') \Phi(\mathbf{r}') + S_{\mathrm{int}}[\Phi]. \quad (3.134)$$

We then find

$$\frac{\delta W\{J, K\}}{\delta J_\phi(\mathbf{r})} = \langle \phi(\mathbf{r}) \rangle \equiv \phi_{\mathrm{cl}}(\mathbf{r}); \qquad \frac{\delta W\{J, K\}}{\delta J_\varphi(\mathbf{r})} = \langle \varphi(\mathbf{r}) \rangle = \varphi_{\mathrm{cl}}(\mathbf{r}). \quad (3.135)$$

In this expression the angular brackets denote the functional averaging. Equation (3.135) essentially defines the average ("classical") fields. Analogously, we find

$$\begin{aligned}
\frac{\delta W\{J, K\}}{\delta K_{\phi\phi}(\mathbf{r}\mathbf{r}')} &= \frac{1}{2}\{\phi_{\mathrm{cl}}(\mathbf{r})\phi_{\mathrm{cl}}(\mathbf{r}') + G_{\phi\phi}(\mathbf{r}\mathbf{r}')\}, \\
\frac{\delta W\{J, K\}}{\delta K_{\varphi\varphi}(\mathbf{r}\mathbf{r}')} &= \frac{1}{2}\{\varphi_{\mathrm{cl}}(\mathbf{r})\varphi_{\mathrm{cl}}(\mathbf{r}') + G_{\varphi\varphi}(\mathbf{r}\mathbf{r}')\}, \\
\frac{\delta W\{J, K\}}{\delta K_{\phi\varphi}(\mathbf{r}\mathbf{r}')} &= \frac{1}{2}\{\phi_{\mathrm{cl}}(\mathbf{r})\varphi_{\mathrm{cl}}(\mathbf{r}') + G_{\phi\varphi}(\mathbf{r}\mathbf{r}')\} = \frac{\delta W\{J, K\}}{\delta K_{\varphi\phi}(\mathbf{r}'\mathbf{r})}.
\end{aligned} \quad (3.136)$$

The effective-action functional which we are seeking can be determined from a double Legendre transformation [113]

$$\begin{aligned}
\Gamma(\Phi_{\mathrm{cl}}, \hat{G}) = {} & W\{J, K\} - \mathrm{Sp} \int d^d\mathbf{r} J(\mathbf{r}) \Phi_{\mathrm{cl}}(\mathbf{r}) \\
& - \frac{1}{2} \mathrm{Sp} \int d^d\mathbf{r} \int d^d\mathbf{r}' \Phi_{\mathrm{cl}}^+(\mathbf{r}) \hat{K}(\mathbf{r}\mathbf{r}') \Phi_{\mathrm{cl}}(\mathbf{r}') \\
& - \frac{1}{2} \int d^d\mathbf{r} \int d^d\mathbf{r}' \hat{K}(\mathbf{r}\mathbf{r}') \hat{G}(\mathbf{r}'\mathbf{r}),
\end{aligned} \quad (3.137)$$

where the sources J and $\hat{K}$ are expressed in terms of Φ_{cl} and $\hat{G}$ with the help of Eqs. (3.135) and (3.136). We then easily see that

$$\frac{\delta\Gamma(\Phi_{\mathrm{cl}}, \hat{G})}{\delta\phi_{\mathrm{cl}}(\mathbf{r})} = -J_{\phi}(\mathbf{r}) - \int d^{d}\mathbf{r}' K_{\phi\phi}(\mathbf{r}\mathbf{r}')\phi_{\mathrm{cl}}(\mathbf{r}'),$$
$$\frac{\delta\Gamma(\Phi_{\mathrm{cl}}, \hat{G})}{\delta\varphi_{\mathrm{cl}}(\mathbf{r})} = -J_{\varphi}(\mathbf{r}) - \int d^{d}\mathbf{r}' K_{\varphi\varphi}(\mathbf{r}\mathbf{r}')\varphi_{\mathrm{cl}}(\mathbf{r}'), \tag{3.138}$$

$$\frac{\delta\Gamma(\Phi_{\mathrm{cl}}, \hat{G})}{\delta G_{\phi\phi}(\mathbf{r}\mathbf{r}')} = -\frac{1}{2}K_{\phi\phi}(\mathbf{r}\mathbf{r}'); \qquad \frac{\delta\Gamma(\Phi_{\mathrm{cl}}, \hat{G})}{\delta G_{\varphi\varphi}(\mathbf{r}\mathbf{r}')} = -\frac{1}{2}K_{\varphi\varphi}(\mathbf{r}\mathbf{r}'),$$
$$\frac{\delta\Gamma(\Phi_{\mathrm{cl}}, \hat{G})}{\delta G_{\phi\varphi}(\mathbf{r}\mathbf{r}')} = -\frac{1}{2}K_{\phi\varphi}(\mathbf{r}\mathbf{r}'). \tag{3.139}$$

If we set the external sources J and $\hat{K}$ equal to zero, we see that the functional Γ determined in this manner satisfies variational principle (3.127). This functional is [113] the generating functional of the Green's functions which are irreducible in a two-particle channel. According to Cornwall *et al.* [113], after a straightforward extension to the two-field system, we obtain

$$\Gamma(\Phi_{\mathrm{cl}}, \hat{G}) = S[\Phi_{\mathrm{cl}}] - \tfrac{1}{2}\,\mathrm{Tr}\ln\hat{G}^{-1} - \tfrac{1}{2}T\{\tilde{G}^{-1}\hat{G} - 1\} + \Phi(\Phi_{\mathrm{cl}}, \hat{G}), \tag{3.140}$$

where Tr and ln are to be understood in the functional context [113]; specifically, Tr includes all the necessary integrations, and $\mathrm{Tr}\ln\hat{G} = \ln\mathrm{Det}\,G$. In Eq. (3.140) $\tilde{G}^{-1}$ denotes the reciprocal Green's function matrix in a classical field

$$\tilde{G}^{-1}(\mathbf{r}, \mathbf{r}') = \begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{bmatrix} \delta(\mathbf{r} - \mathbf{r}'), \tag{3.141}$$

where

$$A_{ij} = \left\{ -\frac{1}{2m}\nabla^2 - (E + \omega + i\delta) - \frac{1}{2}\rho V^2(\phi_{\mathrm{cl}}^2 + \varphi_{\mathrm{cl}}^2) \right\} \delta_{ij} - \rho V^2 \phi_{\mathrm{cl}_i}\phi_{\mathrm{cl}_j},$$
$$B_{ij} = -\rho V^2 \phi_{\mathrm{cl}_j}\varphi_{\mathrm{cl}_i}; \qquad C_{ij} = -\rho V^2 \varphi_{\mathrm{cl}_i}\phi_{\mathrm{cl}_j}, \tag{3.142}$$
$$D_{ij} = \left\{ -\frac{1}{2m}\nabla^2 - (E - i\delta) - \frac{1}{2}\rho V^2(\phi_{\mathrm{cl}}^2 + \varphi_{\mathrm{cl}}^2) \right\} \delta_{ij} - \rho V^2 \varphi_{\mathrm{cl}_i}\varphi_{\mathrm{cl}_j},$$

$$\phi_{\mathrm{cl}}^2 = \sum_{j=1}^{n} \phi_{\mathrm{cl}_j}^2; \qquad \varphi_{\mathrm{cl}}^2 = \sum_{i=1}^{m} \varphi_{\mathrm{cl}}^2.$$

The functional $\Phi(\Phi_{cl}, \hat{G})$ can be calculated in the following way [113]. In the classical action $S[\Phi]$ the field Φ must be "displaced" by the amount Φ_{cl}. The new action $S[\Phi_{cl}+\Phi]$ would then determine the new interaction vertices which depend on Φ_{cl}. The functional $\Phi(\Phi_{cl}, \hat{G})$ would then be determined by all the vacuum diagrams, which are irreducible in a two-particle channel, of the theory with the action $S[\Phi_{cl}+\Phi]$ and by the propagators which are equal to the matrix Green's functions $\hat{G}$. In other words, only those vacuum diagrams need be retained which give the irreducible self-energy diagrams after cutting any one of the lines denoting a Green's function. Because of the dependence of the vertices on $\Phi_{cl}(\mathbf{r})$, the translational invariance is generally missing. Thus the functional $\Phi(\Phi_{cl}, \hat{G})$ satisfies the condition

$$\frac{\delta\Phi}{\delta\hat{G}} = \frac{1}{2}\hat{\Sigma}, \tag{3.143}$$

and the equation

$$\frac{\delta\Gamma}{\delta\hat{G}} = \frac{1}{2}\hat{G}^{-1} - \frac{1}{2}\tilde{G}^{-1} + \frac{1}{2}\hat{\Sigma} = 0 \tag{3.144}$$

is just the matrix Dyson equation for the corresponding Green's functions. The matrix $\hat{\Sigma}$ is comprised of irreducible self-energies with "dressed" internal lines.

Let us first examine the "normal" phase, in which $\phi_{cl} = \varphi_{cl} = 0$ and only the Green's functions $G_{\phi\phi}$ and $G_{\varphi\varphi}$ are nonvanishing. In this case, Eq. (3.140) essentially reduces to

$$\Gamma(\hat{G}) = \Phi(\hat{G}) - \tfrac{1}{2}\operatorname{Tr}\ln\hat{G}^{-1} - \tfrac{1}{2}\operatorname{Tr}\{\hat{G}_0^{-1}\hat{G} - 1\}. \tag{3.145}$$

Matrix (3.141) reduces to a simple form (3.130). All the expressions are translationally invariant by virtue of $\phi_{cl} = \varphi_{cl} = 0$.

A stable system must satisfy the condition $\delta^2\Gamma > 0$ for any variations of Φ_{cl} and $\hat{G}$. Let us consider its stability relative to arbitrary variations of the *Green's functions* in the "normal" phase. We easily find

$$\frac{\delta^2\Gamma}{\delta G_{\phi\phi}\delta G_{\phi\phi}} = -\frac{1}{2}G_{\phi\phi}^{-2} + \frac{1}{2}\frac{\delta\Sigma_{\phi\phi}}{\delta G_{\phi\phi}} = -\frac{1}{2}G_{\phi\phi}^{-2} + \frac{1}{2}U_{\phi\phi\phi\phi} \tag{3.146}$$

$$\frac{\delta^2\Gamma}{\delta G_{\phi\varphi}\delta G_{\phi\varphi}} = \frac{1}{2}\frac{\delta G^{-1}_{\phi\phi}}{\delta G_{\phi\varphi}} + \frac{1}{2}\frac{\delta \Sigma_{\phi\varphi}}{\delta G_{\phi\varphi}} = -\frac{1}{2} G^{-1}_{\phi\phi} G^{-1}_{\varphi\varphi} + \frac{1}{2} U_{\phi\varphi\phi\varphi}, \quad (3.147)$$

etc. Here U are the vertices which are irreducible in the corresponding two-particle channels. The first term in (3.147) can be determined by directly calculating the appropriate derivative and then transforming to the "normal" phase ($G_{\phi\varphi} = G_{\varphi\phi} = 0$). Figure 11a is a graphic representation of the variation of the self-energies resulting from the variation of the Green's functions, which was used in the derivation of (3.146) and (3.147). Clearly, we are concerned with the stability of the system with respect to the variations $\delta G_{\phi\varphi}$. In a stable system we would have

$$\mathrm{Tr}\ \delta G_{\phi\varphi} \frac{\delta^2\Gamma}{\delta G_{\phi\varphi}\delta G_{\varphi\phi}} \delta G_{\phi\varphi} \geqslant 0. \quad (3.148)$$

Using $\delta G_{\phi\varphi} = G_{\phi\phi}\psi_{\phi\varphi}G_{\varphi\varphi}$ (see Fig. 11b) in (3.148), we find, with the help of (3.147), that the stability threshold of the "normal" phase

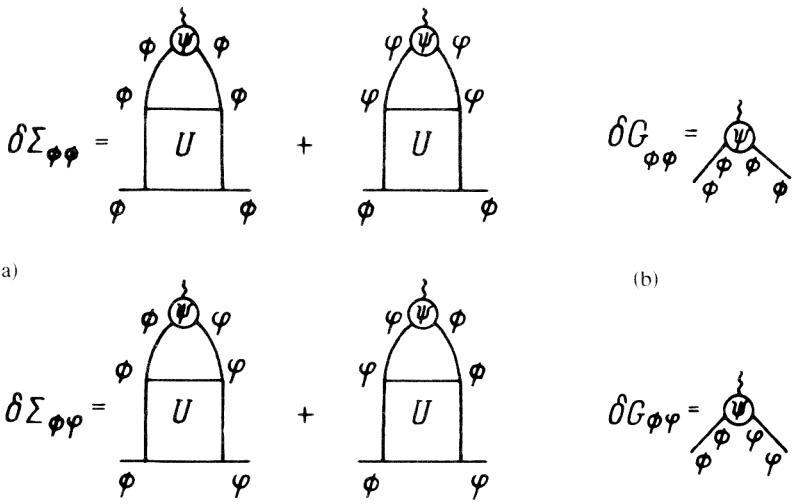

Figure 11. (a) Variation of self-energy as a result of arbitrary variation of the Green's function. (b) Graphic representation of the arbitrary variation of the Green's functions (variation of the external source, $\delta K_{\phi\varphi} \equiv \psi_{\phi\varphi}$).

relative to this variation is determined by the condition

$$\mathrm{Tr}\, G_{\varphi\varphi}\psi_{\varphi\phi}G_{\phi\phi}U_{\phi\varphi\varphi\phi}G_{\phi\phi}\psi_{\phi\varphi}G_{\varphi\varphi} - \mathrm{Tr}\, G_{\varphi\varphi}\psi_{\varphi\phi}\psi_{\phi\varphi}G_{\phi\phi} = 0 \quad (3.149)$$

which is graphically illustrated in Fig. 12a. It is fairly clear that the stability condition is broken in the case where a nontrivial solution of a homogeneous Bethe–Salpeter equation with $\omega = 0$ appears (Fig. 12b). We have seen, in fact, using a specific example which was analyzed in Sec. 1, that (at $\omega = 0$) the operator $U_{\phi\varphi\varphi\phi}$ appearing in (3.149) is a Hermitian, and that its eigenvalues are real. The stability condition is equivalent to the requirement that the quadratic form be positive

$$\mathrm{Tr}\, \psi_{\varphi\phi}X_{\phi\varphi}\psi_{\varphi\phi} > 0, \quad (3.150)$$

where

$$X_{\phi\varphi} = G_{\varphi\varphi}G_{\phi\phi}U_{\phi\varphi\varphi\phi}G_{\phi\phi}G_{\varphi\varphi} - G_{\varphi\varphi}G_{\phi\phi}. \quad (3.151)$$

At $\omega = 0$, it is clear that $G_{\phi\phi} = G^{*}_{\varphi\varphi}$. Introducing a system of eigenfunctions of the operator $X_{\phi\varphi}$

$$X_{\phi\varphi}\psi^{\nu}_{\varphi\phi} = \lambda_{\nu}\psi^{\nu}_{\varphi\phi}; \qquad \lambda_{\nu} \in \mathrm{Re}, \quad (3.152)$$

we find that the nontrivial solution of a homogeneous Bethe–Salpeter equation is equivalent to the existence of the eigenvalue $\lambda_0 = 0$. We then find that the choice of $\psi_{\varphi\phi} \sim \psi^{0}_{\varphi\phi}$ violates condition (3.150). A similar analysis in case of ordinary phase transitions was carried out by Morandi [117]. Accordingly, our analysis shows that the appearance of a nontrivial solution of the Bethe–Salpeter equation

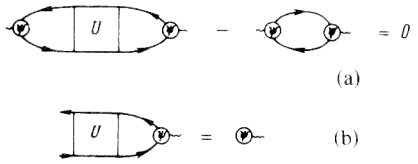

Figure 12. Graphic representation of Eq. (3.149) (a) and of the homogeneous Bethe–Salpeter equation derived from it (b).

determines the stability threshold of the "normal" phase. Here we have in mind the stability relative to the arbitrary variations $\delta G_{\phi\varphi}$. An expansion of the functional $\Gamma(\hat{G})$ in (3.145) in powers of $\delta G_{\phi\varphi} \sim \psi_{\phi\varphi}$ gives, in principle, a method of analyzing the corresponding "condensed" phase. The quantity $\psi_{\phi\varphi}$ in this case is the *order parameter*.

The first two equations in (3.127) are generalizations of the classical field equations given in (3.106). We are concerned with the case in which they acquire nontrivial solutions of the type shown in (3.107). Matrix (3.141) in this case reduces to

$$\tilde{G}^{-1}(\mathbf{r}\mathbf{r}') = \begin{bmatrix} (M_T - \omega - i\delta)\delta_{ij} & 0 \\ 0 & (M_L + i\delta)e_i e_j + (M_T + i\delta)(\delta_{ij} - e_i e_j) \end{bmatrix} \delta(\mathbf{r} - \mathbf{r}'), \tag{3.153}$$

where the operators M_L and M_T were introduced above in (3.60) and (3.61). A simplest approximation for effective action $\Gamma(\Phi_{\text{cl}}, \hat{G})$ reduces to the omission of the term $\Phi(\Phi_{\text{cl}}, \hat{G})$ in (3.140). In this case we see from (3.143) that $\hat{\Sigma} = 0$, i.e., $\hat{G} = \tilde{G}$. To avoid confusion we emphasize that these Green's functions are not related to the average Green's functions for an electron in a random field. Equation (3.140) then yields

$$\Gamma(\varphi_{\text{cl}}) = S[\varphi_{\text{cl}}] - \tfrac{1}{2}\,\text{Tr ln}\,\tilde{G}^{-1}_{\phi\phi} - \tfrac{1}{2}\,\text{Tr ln}\,\tilde{G}^{-1}_{\varphi\varphi}, \tag{3.154}$$

and the the equation $\delta\Gamma/\delta\varphi_{\text{cl}} = 0$ reduces to the generalized equation for an instanton

$$-\frac{1}{2m}\nabla^2\varphi_{\text{cl}} - E\varphi_{\text{cl}} - \frac{1}{2}\rho V^2\varphi_{\text{cl}}^3 + \frac{\delta\Gamma_1(\varphi_{\text{cl}})}{\delta\varphi_{\text{cl}}} = 0, \tag{3.155}$$

where we set

$$\begin{aligned}\Gamma_1(\varphi_{\text{cl}}) &= -\tfrac{1}{2}\,\text{Tr ln}\,\tilde{G}^{-1}_{\phi\phi} - \tfrac{1}{2}\,\text{Tr ln}\,\tilde{G}^{-1}_{\varphi\varphi} \\ &= -\tfrac{1}{2}\,\text{Tr ln Det}\,\tilde{G}^{-1}_{\phi\phi} - \tfrac{1}{2}\,\text{Tr ln Det}\,\tilde{G}^{-1}_{\varphi\varphi}.\end{aligned} \tag{3.156}$$

In the last equation Det is taken over the isotopic indices, while Tr is

taken over the arguments of the Green's functions. We find from (3.153)

$$\text{Det } \tilde{G}^{-1} = [\text{Det}(M_T - \omega - i\delta)]^n \text{ Det } M_L [\text{Det } M_T]^{m-1}. \quad (3.157)$$

Hence, we find

$$\Gamma_1(\varphi_{\text{cl}}) = -n \sum_k (\lambda_k^T - \omega - i\delta) - \sum_k (\lambda_k^L + i\delta) - (m-1) \sum_k (\lambda_k^T + i\delta)$$

$$\underset{n\to 0,\, m\to 0}{=} -\sum_k \lambda_k^L + \sum_k \lambda_k^T. \quad (3.158)$$

In other words ($\ln \text{Det } M_{L,T} = \text{Tr} \ln M_{L,T}$),

$$\Gamma_1(\varphi_{\text{cl}}) = -\frac{1}{2} \text{Tr} \ln \left\{ 1 - \left[-\frac{\nabla^2}{2m} - E \right]^{-1} \frac{3}{2} \rho V^2 \varphi_{\text{cl}}^2 \right\}$$

$$+ \frac{1}{2} \text{Tr} \ln \left\{ 1 - \left[-\frac{\nabla^2}{2m} - E \right]^{-1} \frac{1}{2} \rho V^2 \varphi_{\text{cl}}^2 \right\}. \quad (3.159)$$

Expanding (3.159) in a series in powers of $\rho V^2 \varphi_{\text{cl}}^2$, we see [95] that $\Gamma_1(\varphi_{\text{cl}})$ is the result of the summation of *one-loop* corrections to the classical action. Since we are dealing here with the first-order term in $\rho V^2 \varphi_{\text{cl}}^2$, we find

$$\Gamma_1^{(1)}(\varphi_{\text{cl}}) = -\frac{1}{2} \text{Tr} \left\{ \left[-\frac{\nabla^2}{2m} - E \right]^{-1} \rho V^2 \varphi_{\text{cl}}^2 \right\}$$

$$= \frac{1}{2} \rho V^2 \int d^d \mathbf{r} \varphi_{\text{cl}}^2(\mathbf{r}) \int \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{E - \mathbf{p}^2/2m}$$

$$\equiv -\frac{1}{2} \delta E \int d^d \mathbf{r} \varphi_{\text{cl}}^2(\mathbf{r}), \quad (3.160)$$

where $\delta E = E - E_0 = -\text{Re } \Sigma(E)$ is the "mass" renormalization in one-loop approximation, which was discussed in Sec. 3. Thus the last term in (3.155) can be dropped if we assume at the outset (as we have done everywhere) that E is the renormalized energy in the one-loop approximation. The second-order term in the expansion of $\Gamma_1(\varphi_{\text{cl}})$

gives rise to a one-loop correction to the coupling constant, while the higher-order terms seems to be unimportant insofar as their effect on the instanton solutions is concerned. Thus the results of the instanton approach analyzed above are reproduced in this approximation. The appearance of a nontrivial solution of $\varphi_{\mathrm{cl}}(\mathbf{r})$ implies that there is a "*local*" symmetry breaking over the field φ, which is different from the symmetry breaking that occurs in ordinary phase transitions. The corresponding translational and "rotational" symmetry breaking is "masked" by integration over the collective coordinates. This integration is an essential part of the calculation of the physical correlators. The condition under which the approximation we are considering can be used reduces to inequality (3.18) which was examined above. This inequality determines the energy region in which the variable E retains the simple meaning of energy reckoned from the displaced (in the one-loop approximation) band edge, and the renormalization of the coupling constant is irrelevant. We see that the critical energy at which the instanton solutions appear ($E = 0$) falls within the energy region where the nontrivial solutions of the homogeneous Bethe–Salpeter equation (2.33) appear and in which the approximations used above break down. Thus, the effective-action formalism leads in a natural way to instabilities of the "normal" (metallic) phase linked both to the appearance of a nontrivial solution of the homogeneous Bethe–Salpeter equation and to the instanton solutions. These two types of instabilities remain independent within the framework of approximations used above. This might possibly indicate that there are *two types* of electron localization. It is clear, on the other hand, that in the higher-order approximations these instabilities can be closely related in the effective-action formalism and a complete solution of the problem of their interplay requires an actual "penetration" into the "strong-coupling" region.

The energy region $|E| \leqslant [1/(d-2)]^{2/(4-d)} E_{sc}$, $E_{sc} = m^{d/(4-d)} (\rho V^2)^{2/(4-d)}$ remains outside the limits of all the methods to solve the localization problem discussed above. At the same time, we must stress that this energy region is of principal interest from the viewpoint of constructing a scaling model of the Anderson transition, since the mobility edge belongs to this region. This difficulty, which stems from the fact that neither the conventional perturbation theory nor a one-instanton approximation can be used in this case, is

a fundamental difficulty which accounts for the main difference between the localization theory and the theory of critical phenomena. A systematic solution of this problem, to the best of my knowledge, is yet to be obtained, although several attempts toward this end were discussed in my previous review article [7]. Immediately after the publication of a well-known paper by Wegner [118], considerable attention has been devoted to the development of various ways of reducing the problem of the electron in a random field, near the mobility edge, to the analysis of different variants of the nonlinear σ model [119–126]. Since different formalisms have been used, the results are not always in complete agreement with each other. These studies, nonetheless, give a general picture of the localization, equivalent to that of the elementary scaling theory [26, 58]. These attempts are therefore frequently viewed as a quantitative justification of the scaling concept in localization theory. This approach is basically an attempt to construct an effective Lagrangian for the description of the region around the mobility edge directly in terms of a two-particle Green's function, along the lines initially suggested by Aharony and Imry [127]. This theory is based to some degree on the assumption that a single-particle Green's function for an electron near a mobility edge has no singularities and can, in fact, be described correctly in the simplest approximations of the perturbation theory. The resulting critical behavior is essentially in agreement with the corresponding results of the self-consistent theory of localization (see Sec. 2), although the role of corrections in the expansion in $\epsilon = d - 2$ can generally be estimated for the critical indices like those in (2.97) and (2.98). Although the studies of the region near the mobility edge based on the formalism of nonlinear σ models have contributed to our understanding of localization, this contribution, in this author's view, is relatively small. Because of the complexity of the formalism of the nonlinear σ models, in contrast with the self-consistent theory, the physical quantities near the mobility edge cannot be calculated explicitly, and the specific results reduce solely to the determination of some critical indices. These studies, as well as the self-consistent theory of localization, in our view, basically ignore the complex problem associated with the appearance of the strong-coupling region near the mobility edge, where the *functional form* of a single-particle Green's function can change substantially (although

there can conceivably be no singularities, e.g., in the density of states associated with the mobility edge). This situation, as we have seen above, also applies to the case $d = 2 + \epsilon$, where some claims to describe the region near the mobility edge in terms of the perturbation theory were made. Furthermore, the results obtained in the known studies based on the σ models correspond only to the metallic region, in which respect they lose ground even to the self-consistent theory. It has not yet been possible to incorporate into these models the study of the instanton effects, whose importance was demonstrated above. For this reason, we prefer the method based on effective Lagrangians given in (3.1) and (3.5), which makes it possible to establish in each case a connection with the known results of the standard approximations and which does not obscure the difficulties associated with the inapplicability of both the standard perturbation theory and the perturbation theory near the instanton solution (one-instanton approximation) near the mobility edge. The mobility edge can hopefully be described in terms of this formalism, although such an attempt may conceivably require the use of completely new ideas and methods which are not related to the known versions of the perturbation theory. The problem we are discussing is thus linked with most urgent problems of the modern field theory, such, for example, as the problem of quark confinement.

4. Electron–Electron Interaction in the Localization Theory

4.1. Hartree–Fock Approximation. Localization Contributions

Localization is usually studied neglecting the electron–electron interaction, although its importance in the description of the metal-insulator transitions in disordered systems has been established long ago [2]. This fact has recently been confirmed in the studies of "dirty" metals [128, 129] (see also Ref. 9) and also in the analysis of the problem of "Coulomb gap" of the Fermi level for the electron system with strongly localized states [6, 130–133]. Several studies, in which an attempt was made to analyze the effects of electron–electron interaction as the metal-insulator transition in disordered systems is approached, have recently been published [134–139]. In all of these works the authors have analyzed only the metallic

region near the Anderson (or Mott) transition, without focussing any attention on the insulator part of the transition. In addition to the studies already mentioned [6, 130–139], the electron–electron interaction in the case of localized electrons was considered in connection with the early attempts to build a theory of "Fermi-glass" (Refs. 140–142). All these works have shown that correlations play an important if not the decisive role in the description of a metal-insulator transition in disordered systems. The results of these studies, on the other hand, contradict each other to a large extent, so that this problem is far from being solved. The answer to such a basic question as to whether the localization persists in systems with interaction remains unclear. The difficulties discussed above, which arise in the theoretical description of the Anderson transition even in a single-electron approximation, make the problem even less tractable. In this case it would be justifiable to analyze first the case of a weak interaction in a highly disordered system in order to identify the physical properties most strongly affected by correlations. Following principally the studies of Katsnel'son and Sadovskiĭ [143, 144], we will therefore examine the first-order corrections of the perturbation theory over interaction to the density of states and some other characteristics of the system near the Anderson transition, focusing particular attention on the region of localized states.* In this sense, we will attempt to extend and generalize the known results of Aronov and Al'tshuler [9, 128, 129] from metallic to insulating region. We assume that Anderson's single-electron problem is solved and that the concept of localization applies to a system with interaction. By making this assumption, which is central to our analysis, we can hope to justify the use of only the first-order corrections in terms of the interaction, although higher-order corrections in terms of the interaction, as we will see below, must be studied thoroughly in the immediate vicinity of the mobility edge.

If we assume that the single-electron problem is solved, we can again introduce the complete orthonormal set of exact wave functions $\varphi_\nu(\mathbf{r})$ and the corresponding exact eigenvalues ϵ_ν of the energy of an electron in a random field of a disordered system

*Some further developments, concerning in particular the possible interplay between localization and superconductivity can be found in Ref. 157.

which we have examined in Sec. 1. These functions and energies can correspond to both the localized and delocalized states. Let us consider a causal single-electron Green's function as it is expressed in these exact eigenfunctions and, in particular, its diagonal matrix element,

$$G_{\nu\nu}(\epsilon) = \left\langle \nu \left| \frac{1}{\epsilon - H + i\delta \operatorname{sign} \epsilon} \right| \nu \right\rangle, \tag{4.1}$$

where H is a complete Hamiltonian which takes the electron–electron interaction into account, and ϵ is the energy reckoned from the Fermi level. The interaction is taken into account by introducing the corresponding self-energy [145, 146] (see also Refs. 140–142), $\Sigma_\nu(\epsilon) = \Delta_\nu(\epsilon) - i\Gamma_\nu(\epsilon) \operatorname{sign} \epsilon$,

$$G_{\nu\nu}(\epsilon) = \frac{1}{\epsilon - \epsilon_\nu - \Sigma_\nu(\epsilon)}. \tag{4.2}$$

The feasibility of introducing such a self-energy in the representation of arbitrary quantum numbers ν was discussed extensively by Migdal [145]. In accordance with the standard procedure [145, 146], let us introduce the renormalized energy $\tilde{\epsilon}_\nu$ as a solution of the equation

$$\tilde{\epsilon}_\nu - \epsilon_\nu - \Delta_\nu(\tilde{\epsilon}_\nu) = 0, \tag{4.3}$$

and for $\epsilon \sim \tilde{\epsilon}_\nu$ we can write (4.2) in the form

$$G_{\nu\nu}(\epsilon) \simeq \frac{Z_\nu}{\epsilon - \tilde{\epsilon}_\nu + i\gamma_\nu \operatorname{sign} \epsilon}, \tag{4.4}$$

where

$$Z_\nu = \left\{ 1 - \frac{\partial \Delta_\nu(\epsilon)}{\partial \epsilon} \right\}^{-1}_{\epsilon = \tilde{\epsilon}_\nu}; \qquad \gamma_\nu = Z_\nu \Gamma_\nu(\epsilon = \tilde{\epsilon}_\nu). \tag{4.5}$$

Let us introduce, in accordance with Ref. 146, the "self-energy" $\tilde{\Sigma}_E(\epsilon)$, which was averaged over an isoenergetic surface $E = \epsilon_\nu$ and

over the configurations of the random field of a disordered system,

$$\tilde{\Sigma}_E(\epsilon) = \tilde{\Delta}_E(\epsilon) - i\tilde{\Gamma}_E(\epsilon) = \frac{1}{N_0(E)}\left\langle \sum_\nu \delta(E-\epsilon_\nu)\sum\nolimits_\nu(\epsilon)\right\rangle, \quad (4.6)$$

where $N_0(E)$ is the single-electron (average) density of states.

We will consider next a single-electron density of states, in which the interaction is taken into account. This density of states is usually given by

$$N(E) = -\frac{1}{\pi}\left\langle \sum_\nu \operatorname{Im} G^R_{\nu\nu}(E)\right\rangle. \quad (4.7)$$

Under the assumption that the interaction corrections are weak in the sense that the inequalities $\gamma_\nu \ll \epsilon_\nu \sim \tilde{\epsilon}_\nu$ are satisfied, we find in a first approximation, after some simple calculations using (4.4)–(4.6) in (4.7).

$$\frac{\delta N(E)}{N_0(E)} = \frac{N(E)-N_0(E)}{N_0(E)} \approx -\frac{\partial\tilde{\Delta}_E(\tilde{\epsilon}_\nu)}{\partial E} + \frac{\partial\tilde{\Delta}_E(\tilde{\epsilon}_\nu)}{\partial\tilde{\epsilon}_\nu}. \quad (4.8)$$

For reasons that will be clarified below, we call the quantity

$$\frac{\delta\tilde{N}(E)}{N_0(E)} = -\frac{\partial\tilde{\Delta}_E(\epsilon_\nu)}{\partial E} \quad (4.9)$$

a correction to the "thermodynamic density of states." This correction was analyzed for the first time by Abraham *et al.* [146] (see also Ref. 136). Here the different ways to define the density of states in a system with interactions are reflected (compare with Refs. 130 and 131).

Let us examine a model problem, in which the electron–electron interaction is described by a repulsive static potential with a finite range,

$$H_{\text{int}} = \frac{1}{2}\int d^d\mathbf{r}\int d^d\mathbf{r}' \sum_{\mu\nu\mu'\nu'} \varphi^*_\nu(\mathbf{r}')\varphi^*_\nu(\mathbf{r})v(\mathbf{r}-\mathbf{r}')$$
$$\times \varphi_{\mu'}(\mathbf{r})\varphi_{\mu'}(\mathbf{r}')a^+_\mu a^+_\nu a_{\mu'}a_{\nu'}. \quad (4.10)$$

Analyzing the Hartree–Fock diagrams (Fig. 13), we then find

$$\Sigma_{\mu}^{H} = \int d^{d}\mathbf{r} \int d^{d}\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \sum_{\nu} f_{\nu} \varphi_{\mu}^{*}(\mathbf{r}') \varphi_{\nu}^{*}(\mathbf{r}) \varphi_{\nu}(\mathbf{r}) \varphi_{\mu}(\mathbf{r}')$$
$$\Sigma_{\mu}^{F} = -\int d^{d}\mathbf{r} \int d^{d}\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \sum_{\nu} f_{\nu} \varphi_{\mu}^{*}(\mathbf{r}') \varphi_{\nu}^{*}(\mathbf{r}) \varphi_{\mu}(\mathbf{r}) \varphi_{\nu}(\mathbf{r}'), \tag{4.11}$$

where $f_{\nu} = f(\epsilon_{\nu})$ is the Fermi distribution function. Accordingly, we find from (4.6)

$$\tilde{\Sigma}_{E}^{H} = \int_{-\infty}^{\infty} d\omega f(E+\omega) \int d^{d}\mathbf{r} \int d^{d}\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \langle\!\langle \rho_{E}(\mathbf{r}) \rho_{E+\omega}(\mathbf{r}') \rangle\!\rangle^{H}$$
$$\tilde{\Sigma}_{E}^{F} = -\int_{-\infty}^{\infty} d\omega f(E+\omega) \int d^{d}\mathbf{r} \int d^{d}\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \langle\!\langle \rho_{E}(\mathbf{r}) \rho_{E+\omega}(\mathbf{r}') \rangle\!\rangle^{F}. \tag{4.12}$$

In these expressions we have introduced the spectral densities,

$$\langle\!\langle \rho_{E}(\mathbf{r}) \rho_{E+\omega}(\mathbf{r}') \rangle\!\rangle^{H} = \frac{1}{N_{0}(E)} \left\langle \sum_{\mu\nu} \delta(E-\epsilon_{\mu}) \delta(E+\omega-\epsilon_{\nu}) \varphi_{\mu}^{*}(\mathbf{r}') \varphi_{\mu}(\mathbf{r}') \varphi_{\nu}^{*}(\mathbf{r}) \varphi_{\nu}(\mathbf{r}) \right\rangle \tag{4.13}$$

$$\langle\!\langle \rho_{E}(\mathbf{r}) \rho_{E+\omega}(\mathbf{r}') \rangle\!\rangle^{F} = \frac{1}{N_{0}(E)} \left\langle \sum_{\mu\nu} \delta(E-\epsilon_{\mu}) \delta(E+\omega-\epsilon_{\nu}) \varphi_{\mu}^{*}(\mathbf{r}') \varphi_{\nu}^{*}(\mathbf{r}) \varphi_{\mu}(\mathbf{r}) \varphi_{\nu}(\mathbf{r}') \right\rangle. \tag{4.14}$$

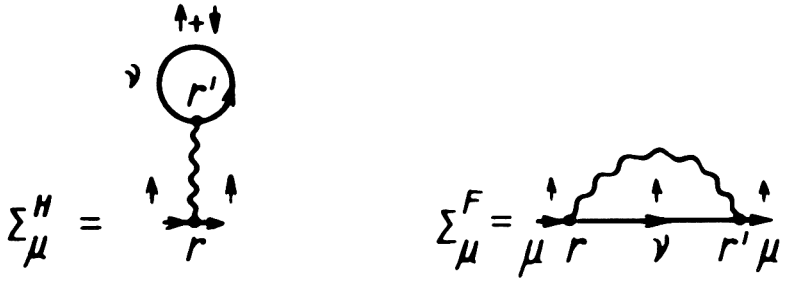

Figure 13. The Hartree–Fock diagrams for the self-energy.

Spectral density (4.14), introduced for the first time by Berezinskiĭ and Gor'kov [34], and was analyzed in Sec. 1 in connection with the general criterion of localization. Applying the same line of reasoning to spectral density (4.13) as was applied there to the contribution of $\epsilon_\mu = \epsilon_\nu$, we easily see that this spectral density acquires a δ-function in ω term in the region of localized states ($E < E_c$):

$$\begin{aligned} \langle\!\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\!\rangle^H &= A_E(\mathbf{r}-\mathbf{r}')\delta(\omega) + \rho_E^H(\omega, \mathbf{r}-\mathbf{r}') \\ \langle\!\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\!\rangle^F &= A_E(\mathbf{r}-\mathbf{r}')\delta(\omega) + \rho_E^F(\omega, \mathbf{r}-\mathbf{r}'), \end{aligned} \tag{4.15}$$

where, as in (1.9),

$$A_E(\mathbf{r}-\mathbf{r}') = \frac{1}{N_0(E)}\left\langle \sum_\mu \delta(E-\epsilon_\mu)|\varphi_\mu(\mathbf{r}')|^2|\varphi_\mu(\mathbf{r})|^2 \right\rangle > 0; \quad E < E_c \tag{4.16}$$

is the generalized inverse participation ratio.

Incorporating (4.15) into (4.12), we find the following contributions to $\tilde{\Sigma}_E$ which are associated with the appearance of localized states in the system

$$\begin{aligned} \tilde{\Sigma}_{E_{\text{loc}}}^{H,F} = \tilde{\Delta}_{E_{\text{loc}}}^{H,F} &= \pm f(E) \int d^d\mathbf{r} \int d^d\mathbf{r}'\, v(\mathbf{r}-\mathbf{r}') A_E(\mathbf{r}-\mathbf{r}') \\ &= \pm f(E) \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(-\mathbf{q}) A_E(\mathbf{q}). \end{aligned} \tag{4.17}$$

In the case of point-like interaction $v(\mathbf{r}-\mathbf{r}') = v_0\delta(\mathbf{r}-\mathbf{r}')$ we find

$$\tilde{\Sigma}_{E_{\text{loc}}}^{H,F} = \pm f(E) v_0 \int \frac{d^d\mathbf{q}}{(2\pi)^d} A_E(\mathbf{q}) = \pm f(E) v_0 A_E, \tag{4.18}$$

where the quantity A_E (2.22) is proportional to the total probability of return of an electron to the starting point during an infinite time. We note that because of the property

$$\langle\!\langle \rho_E(\mathbf{r})\rho_{E'}(\mathbf{r})\rangle\!\rangle^H = \langle\!\langle \rho_E(\mathbf{r})\rho_{E'}(\mathbf{r})\rangle\!\rangle^F, \tag{4.19}$$

which is evident from (4.13) and (4.14), the "regular" contributions

to $\tilde{\Sigma}_E^H$ and $\tilde{\Sigma}_E^F$, which are associated with $\rho_E^{H,F}(\omega, \mathbf{r}-\mathbf{r}')$ in (4.15), are also equal (and opposite in sign) in the case of point-like interaction.

In the case of spin-zero electrons the Hartree–Fock contributions in (4.18) cancel each other. After taking the spin into account, the Hartree contribution acquires an additional factor of 2 which is connected with the summing over the spin directions in the electron loop. As a result, we find a nonvanishing "localization" contribution

$$\tilde{\Sigma}_{E_{\text{loc}}}^{H+F} = \tilde{\Delta}_{E_{\text{loc}}}^{H+F} = f(E) v_0 A_E. \tag{4.20}$$

To avoid ambiguity, we will write out the equations for the point-like interaction. We will now take into account that the main energy dependence in (4.20) is determined by the Fermi function, which varies sharply near the Fermi energy. At $E \approx E_F$ the quantity A_E may be assumed to be a constant (a smooth function of E). This assumption may generally turn out to be incorrect near the mobility edge where A_E vanishes. The corresponding critical index, strictly speaking, is not known, although we can deduce from estimates like those in (3.124) that $\partial A_E/\partial E \to 0$ also in the limit of $E \to E_c$. We then find from (4.9) and (4.20)

$$\frac{\delta \tilde{N}_{\text{loc}}(E)}{N_0(E)} = -\frac{\partial \tilde{\Delta}_{E_{\text{loc}}}^{H+F}}{\partial E} \approx v_0 A_{E_F} \left(-\frac{\partial f(E)}{\partial E}\right). \tag{4.21}$$

In the "total" density of states determined by (4.7) and (4.8) the singular (localization) contribution (4.21) is cancelled by the second term in (4.8),

$$\begin{aligned}
\frac{\partial \tilde{\Delta}_{E_{\text{loc}}}^{H+F}}{\partial \tilde{\epsilon}_\nu} \simeq \frac{\partial \tilde{\Delta}_{E_{\text{loc}}}^{H+F}}{\partial \epsilon_\nu} &= \frac{1}{N_0(E)} \left\langle \sum_\nu \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \right. \\
&\quad \left. \times \frac{\partial f_\nu}{\partial \epsilon_\nu} \delta(E-\epsilon_\nu) |\varphi_\nu(\mathbf{r})|^2 |\varphi_\nu(\mathbf{r}')|^2 \right\rangle \\
&= \frac{\partial f(E)}{\partial E} \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') A_E(\mathbf{r}-\mathbf{r}') \\
&= v_0 A_E \frac{\partial f(E)}{\partial E}.
\end{aligned} \tag{4.22}$$

The localization contribution is retained in the "thermodynamic" density of states (4.9). This circumstance, as we will see below, accounts for the peculiar behavior of several thermodynamic quantities.

To have a clearer understanding of the physical meaning of the localization contribution to $\tilde{\Sigma}_E^{H+F}$ we should note that it is actually connected with the interaction of electrons which are in the same quantum state ν. We see from Fig. 13 that the contributions from the interaction of electrons with the same spin directions (indicated by the arrows in Fig. 13) are canceled out completely in the sum of these diagrams, and $\tilde{\Sigma}_E^{H+F}$ is determined by the interaction of two electrons with opposite spins which are in the same state ν, i.e., by the effective "Hubbard"-type interaction

$$H_{\text{eff}} = \frac{1}{2}\sum_{\nu\sigma}\int d^d\mathbf{r}\int d^d\mathbf{r}'\, v(\mathbf{r}-\mathbf{r}')|\varphi_\nu(\mathbf{r})|^2|\varphi_\nu(\mathbf{r}')|^2 n_{\nu\sigma}n_{\nu-\sigma}, \quad (4.23)$$

where $n_{\nu\sigma}$ is the operator of the number of electrons in the state ν with a spin σ. Using a simplest estimate of the quantity A_E, we find ($E_F < E_c$)

$$\tilde{\Delta}_{E_{\text{loc}}}^{H+F} \sim \begin{cases} v_0\left(\dfrac{1}{R_{\text{loc}}(E)}\right)^d; & E < E_F \\ 0; & E > E_F. \end{cases} \quad (4.24)$$

Comparing our results with the familiar qualitative estimates of Mott [2, 147], we find that they correspond to the formation of the "band" of singly occupied electron states of width $\tilde{\Delta}_{E_{\text{loc}}}^{H+F}$, which appear below the Fermi level. Similar arguments were also advanced by Kamimura [148] and Berezinskiĭ and Gor'kov [149]. Clearly, the single spins at the levels of this band account for the additional contribution to the paramagnetic susceptibility. Since the number of unpaired spins in order of magnitude is

$$\mathcal{N} \approx \int_{E_F-\tilde{\Delta}_{E_{\text{loc}}}^{H+F}}^{E_F} dE N_0(E) \approx N_0(E_F)\tilde{\Delta}_{E_{\text{loc}}}^{H+F} = v_0 N_0(E_F) A_{E_F}, \quad (4.25)$$

we find the Curie law (μ_B is the Bohr magneton)

$$\delta\chi^{\text{loc}} \sim \frac{1}{T} N\mu_B^2 = \frac{v_0}{T} N_0(E_F) A_{E_F} \mu_B^2. \tag{4.26}$$

This result can also be easily derived from a direct analysis of the Hartree–Fock corrections to the thermodynamic potential which are determined by the diagrams shown in Fig. 14. Carrying out some straightforward calculations, we find ($h = \mu_B H$, where H is the external magnetic field)

$$\langle \delta\Omega_H \rangle = \frac{1}{2} \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \times \sum_{\sigma\sigma'} f(E-\sigma h) f(E'-\sigma' h) \langle\!\langle \rho_E(\mathbf{r}) \rho_{E'}(\mathbf{r}') \rangle\!\rangle^H, \tag{4.27}$$

$$\langle \delta\Omega_F \rangle = -\frac{1}{2} \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \sum_{\sigma} f(E-\sigma h) \times [1 - f(E'-\sigma h)] \langle\!\langle \rho_E(\mathbf{r}) \rho_{E'}(\mathbf{r}') \rangle\!\rangle^F. \tag{4.28}$$

The term with unity in the square brackets in (4.28) reduces to an unessential constant which renormalizes the chemical potential,

$$\int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \left\langle \sum_{\mu\nu} f_\mu \varphi_\mu^*(\mathbf{r}') \varphi_\mu(\mathbf{r}) \varphi_\nu^*(\mathbf{r}') \varphi_\nu(\mathbf{r}) \right\rangle = v(0) \int d^d\mathbf{r} \left\langle \sum_{\mu} f_\mu |\varphi_\mu(\mathbf{r})|^2 \right\rangle = v(0) \sum_{\mu} f_\mu = Nv(0), \tag{4.29}$$

where N is the total number of electrons. Incorporating (4.12), we then find

$$\langle \delta\Omega_{H,F} \rangle = \int_{-\infty}^{\infty} dE f(E) N_0(E) \tilde{\Sigma}_E^{H,F}. \tag{4.30}$$

Figure 14. The Hartree–Fock corrections to the thermodynamic potential.

Integrating by parts in (4.30), we get

$$\delta\Omega = \langle \delta\Omega_H \rangle + \langle \delta\Omega_F \rangle = TN_0(E_F) \int_{-\infty}^{\infty} dE \frac{\partial \tilde{\Delta}_E^{H+F}}{\partial E} \ln[1 + e^{-E/T}]. \quad (4.31)$$

The term "thermodynamic density of states," which was introduced above in connection with (4.9), can be understood by comparison of (4.31) with the familiar expression for the thermodynamic potential of free electrons

$$\Omega = -T \int_{-\infty}^{\infty} dE N(E) \ln[1 + e^{-E/T}]. \quad (4.32)$$

Calculating the magnetic susceptibility $\chi = -\langle \partial^2 \Omega / \partial H^2 \rangle_{H=0}$, we find, after a straightforward differentiation of (4.27) and (4.28),

$$\delta\chi_H = -4\mu_B^2 \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r} - \mathbf{r}') \int_{-\infty}^{\infty} dE \frac{\partial^2 f(E)}{\partial E^2} \int_{-\infty}^{\infty} dE' f(E') \times \langle\langle \rho_E(\mathbf{r}) \rho_{E'}(\mathbf{r}') \rangle\rangle^H, \quad (4.33)$$

$$\delta\chi_F = 2\mu_B^2 \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r} - \mathbf{r}') \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \left\{ \frac{\partial^2 f(E)}{\partial E^2} f(E') + \frac{\partial f(E)}{\partial E} \frac{\partial f(E')}{\partial E'} \right\} \langle\langle \rho_E(\mathbf{r}) \rho_{E'}(\mathbf{r}') \rangle\rangle. \quad (4.34)$$

Integrating by parts the first term in (4.34), we see that $\delta\chi_F = 0$ in the limit of $T \to 0$. Finally, the localization contribution to the magnetic susceptibility is

$$\begin{aligned} \delta\chi_{\text{loc}} &= -4\mu_B^2 \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r} - \mathbf{r}') \\ &\quad \times \int_{-\infty}^{\infty} dE N_0(E) A_E(\mathbf{r} - \mathbf{r}') \frac{\partial^2 f(E)}{\partial E^2} f(E) \\ &\approx 4N_0(E_F) A_{E_F} v_0 \mu_B^2 \int_{-\infty}^{\infty} dE \left(\frac{\partial f(E)}{\partial E} \right)^2 \\ &= \frac{2\mu_B^2}{3} \frac{v_0}{T} N_0(E_F) A_{E_F}; \end{aligned} \quad (4.35)$$

consistent with the estimate in (4.26). This result fully confirms the qualitative arguments for the existence of singly occupied states below the Fermi level. A result like that in (4.35) was obtained by Berezinskiĭ and Gor'kov [149] in a one-dimensional model. Clearly, the interaction of these single spins generally becomes important in the limit of $T \to 0$ [2, 147].

The singular (localization) part of the thermodynamic potential is

$$\begin{aligned}\delta\Omega_{\text{loc}} &= \langle \delta\Omega_H \rangle_{\text{loc}} + \langle \delta\Omega_F \rangle_{\text{loc}} \\ &= \int d^d\mathbf{r} \int d^d\mathbf{r}' \int_{-\infty}^{\infty} dE v(\mathbf{r}-\mathbf{r}') A_E(\mathbf{r}-\mathbf{r}') N_0(E) f^2(E) \\ &= \int_{-\infty}^{\infty} dE \int \frac{d^d\mathbf{q}}{(2\pi)^d} v(-\mathbf{q}) A_E(\mathbf{q}) N_0(E) f^2(E). \end{aligned} \tag{4.36}$$

The corresponding contributions to the entropy and specific heat can easily be found:

$$S_{\text{loc}} = -\frac{\partial}{\partial T}\delta\Omega_{\text{loc}} = -\int_{-\infty}^{\infty} dE N_0(E) \frac{\partial}{\partial T} f^2(E) \int \frac{d^d\mathbf{q}}{(2\pi)^d} v(-\mathbf{q}) A_E(\mathbf{q})$$

$$\xrightarrow[T\to 0]{} v_0 N_0(E_F) A_{E_F} \tag{4.37}$$

$$C_{\text{loc}} = T\frac{\partial S_{\text{loc}}}{\partial T} \approx -\frac{\pi^2}{3} T v_0 \frac{\partial}{\partial E_F}\{N_0(E_F) A_{E_F}\}. \tag{4.38}$$

We see that the entropy tends toward the positive constant in the limit $T \to 0$,* and that the localization contribution to the specific heat is linked with a small ($\sim \partial A_{E_F}/\partial E_F$) correction to the thermodynamic state density, which was ignored above. Similarly, the small localization correction to the correlation contribution to the

*This contribution is connected with the entropy of "free" spins at the singly-occupied states and disappear when some kind of spin–spin interaction leading to magnetic order as $T \to 0$ is taken into account.

compressibility is (μ is the chemical potential)

$$\delta\kappa_{\text{loc}} = -\frac{\partial^2}{\partial\mu^2}\,\delta\Omega_{\text{loc}} = -v_0\int_{-\infty}^{\infty} dE A_E N_0(E)\frac{\partial^2}{\partial\mu^2} f^2(E)$$

$$= v_0\frac{\partial}{\partial E_F}\{A_{E_l} N_0(E_F)\}. \tag{4.39}$$

It is thus quite obvious that a singular contribution (4.21) to the "thermodynamic" density of states is fully consistent with the third law of thermodynamics and that it "signals" the appearance of a band of singly occupied states below the Fermi level.

Our analysis has so far been completely general in nature. In our further analysis we have to use a particular single-electron model of the Anderson transition. We will then be able to examine the "regular" contributions to the density of states which are associated with the nonsingular terms in (4.15). We will confine ourselves solely to the Fock contribution to (4.11), since the Hartree contribution, as was pointed out in Refs. 9, 129, and 146, is small if the interaction potential falls off at a distance greater than the inverse Fermi momentum. We will see below that such estimates also apply to a "regular" contribution to (4.11) in the localization region. In the case of a point-like interaction, however, the Hartree contribution, as we have already seen from (4.19), is twice as large (taking the account of spin) as the Fock contribution, so that the results which we will obtain below should be taken with a different sign (compare also with Ref. 150). As the single-electron model of the Anderson transition, we use the self-consistent theory of localization, since we can easily derive within its framework the explicit expressions for all the relevant quantities. Specifically, we find from (4.12) the "regular" contribution to $\tilde{\Sigma}_E^F$ at $T = 0$ to be

$$\tilde{\Sigma}^F_{E_{\text{reg}}} = -\frac{1}{\pi}\int_{-\infty}^{0} dE' \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q})\frac{D_{E_l}\mathbf{q}^2}{(E'-E)^2 + [\omega_0^2(E_F)\tau_{E_l} + D_{E_l}\mathbf{q}^2]^2} \tag{4.40}$$

where we have used the Berezinskiĭ–Gor'kov spectral density calculated from (2.74). Accordingly, the correction to the density of states is

$$\frac{\delta N(E)}{N_0(E)} \approx -\frac{d\tilde{\Sigma}^F_{E_{\text{reg}}}}{dE} = -\frac{1}{\pi}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q})\, \frac{D_{E_F}\mathbf{q}^2}{E^2 + [\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^2}. \tag{4.41}$$

Using, for simplicity, the point-like interaction model, and recalling that all the energies so far in this part of the article were reckoned from the Fermi level E_F, we find $(2 < d < 4)$ *

$$\frac{\delta N(E)}{N_0(E_F)} \approx \frac{v_0}{\pi}\frac{S_d}{d-2} D_{E_F}^{-d/2} \times \begin{cases} |E - E_F|^{(d-2)/2} - \tilde{E}^{(d-2)/2}; & |E - E_F| \gg \omega_0^2(E_F)\tau_{E_F} \\ \omega_0^{d-2}(E_F)\tau_{E_F}^{(d-2)/2} - \tilde{E}^{(d-2)/2}; & |E - E_F| \ll \omega_0^2(E_F)\tau_{E_F}, \end{cases} \tag{4.42}$$

where the characteristic energy $\tilde{E}$ is connected with the choice of the cutoff parameter at the upper limit of the integral over $\mathbf{q}$ in (4.41). This cutoff is necessary because the "diffusion" approximation for the integrand cannot be used at large values of the momentum. As we have done in the analysis of a similar cutoff in (2.77), we will set the cutoff momentum equal to the Fermi momentum, so that

$$\tilde{E} = D_{E_F} p_F^2, \tag{4.43}$$

An alternative approach would be to set the cutoff parameter equal to the reciprocal mean free path l^{-1}, but, as we have pointed out above, near the mobility edge we would have $l^{-1} \sim p_F$, so that both options would be equivalent in this region. According to the scaling argument advanced by Lee [136], the cutoff parameter should be set equal to about R_{loc}^{-1} near the mobility edge when $R_{\text{loc}}(E_F) \gg l, p_F^{-1}$; however, such a choice of this parmaeter in the main equation of the self-consistent theory (2.77) leads to some contradictory results. Since our further analysis is essentially based on the self-consistent theory, we will use Eq. (4.43). It is easy to see that the estimate (4.42) is valid if the following condition is satisfied:

$$|E - E_F|; \qquad \omega_0^2(E_F)\tau_{E_F} \ll \tilde{E}. \tag{4.44}$$

*Slight modifications of (4.42) associated with the frequency dependence of generalized diffusion coefficient [53, 60] can be found in Ref. 157.

In the special case of $d = 2$, instead of (4.42) we get:

$$\frac{\delta N(E)}{N_0(E_F)} = \frac{v_0}{4\pi^2 D_{E_F}} \begin{cases} \ln \dfrac{|E - E_F|}{\tilde{E}}; & |E - E_F| \gg \omega_0^2(E_F)\tau_{E_F} \\ \ln \dfrac{\omega_0^2(E_F)\tau_{E_F}}{\tilde{E}}; & |E - E_F| \ll \omega_0^2(E_F)\tau_{E_F}. \end{cases} \tag{4.45}$$

For $\omega_0^2(E_F) = 0$, i.e., in the metallic region, Eqs. (4.42) and (4.43) are consistent with the known results of Aronov and Al'tshuler [9, 128, 129, 136, 146], if the renormalized diffusion coefficient (2.75) is substituted for the classical diffusion coefficient. For $|E - E_F| \gg \omega_0^2(E_F)\tau_{E_F}$ this behavior also applies to the dielectric region; however, for $|E - E_F| \ll \omega_0^2(E_F)\tau_{E_F}$ a cusp in the density of states at the Fermi level of a metal flattens out, giving way to a smooth minimum.

Let us consider some specific dependences found in the self-consistent localization theory. Discarding unessential constants, we find from (2.93), (2.95), and (2.96), for $2 < d < 4$

$$\omega_0^2(E_F)\tau_{E_F} \sim \frac{\lambda}{4-d} \frac{E_F}{(p_F R_{\text{loc}}(E_F))^d} \sim \frac{\lambda E_F}{4-d} \left| \frac{E_F - E_c}{E_c} \right|^{d\nu}, \tag{4.46}$$

$$D_{E_F} \sim \frac{1}{4-d} \frac{1}{m} (p_F R_{\text{loc}}(E_F))^{2-d} \underset{E_F \leqslant E_c}{\sim} \frac{1}{m} \left| \frac{E_F - E_c}{E_c} \right|^{(d-2)\nu}, \tag{4.47}$$

$$\tilde{E} \sim \frac{dE_F}{4-d} \left(\frac{d\omega_0^2(E_F)}{4E_F^2} \right)^{(d-2)/2}$$

$$\sim E_F (p_F R_{\text{loc}}(E_F))^{2-d} \underset{E_F \leqslant E_c}{\sim} E_F \left| \frac{E_F - E_c}{E_c} \right|^{(d-2)\nu}, \tag{4.48}$$

where ν is the critical index of the localization length (2.97). We see that condition (4.44) can easily be satisfied. For the correction to the density of states at the Fermi level, or more precisely, for $|E - E_F| \ll \omega_0^2(E_F)\tau_{E_F}$ we find from (4.42) and (4.46)–(4.48)

$$\frac{\delta N(E_F)}{N_0(E_F)} \approx v_0 \frac{4-d}{d-2} m^{d/2} E_F^{(d/2)-1} \{1-(p_F R_{\text{loc}}(E_F))^{d-2}\}_{E_F \sim E_c}$$

$$\underset{E_F \sim E_c}{\sim} - v_0 N_0(E_F) \left| \frac{E_F - E_c}{E_c} \right|^{-(d-2)\nu}. \tag{4.49}$$

where $N_0(E_F)$ in the last equation denotes the density of states of free electrons at the Fermi level.* A divergence of the correction to the density of states in the limit of $E_F \to E_c$ (there is a similar divergence in the metallic region) shows that our analysis breaks down in the immediate vicinity of the mobility edge. The estimates which we have made, however, are valid so long as $\delta N(E) \ll N_0(E_F)$.

For $d = 2$, we find [see Eqs. (2.93) and (2.95)]

$$\omega_0^2(E_F)\tau_{E_F} \sim E_F \exp\left(-\frac{1}{\lambda}\right)$$

$$\tilde{E} \sim E_F \left\{1 - \exp\left(-\frac{1}{\lambda}\right)\right\}, \tag{4.50}$$

so that from (4.45) we get:

$$\frac{\delta N(E)}{N_0} \sim v_0 N_0 \begin{cases} \ln \dfrac{|E-E_F|}{E_F}: & |E-E_F| \gg E_F \exp\left\{-\dfrac{\pi E_F}{\gamma}\right\} \\ -\dfrac{\pi E_F}{\gamma}: & |E-E_F| \ll E_F \exp\left\{-\dfrac{\pi E_F}{\gamma}\right\}, \end{cases} \tag{4.51}$$

where $N_0 = m/2\pi$ is the density of states of free electrons in the two-dimensional space, and γ is the Born scattering frequency.

The corrections to the density of states found above can be determined from the following heuristic considerations. Let us consider the interaction of an electron in the state ν with energy E with electrons in the states with energy E_F. The relative correction to its wave function in first-order of perturbation theory would be

$$\frac{\delta\varphi_\nu}{\varphi_\nu} \sim \int_0^\infty dt H_{\text{int}}(t), \tag{4.52}$$

*In case of cutoff scheme proposed by Lee [136] instead of power-law (4.49) we get the logarithmic divergence of the density of states correction at the mobility edge.

where at $t = 0$ the interaction is "switched" on and $H_{\text{int}}(t)$ is used in the interaction representation. The electron "diffuses" within the limits of the volume $(D_{E_F} t)^{d/2}$ during the time t. The matrix element for the short-range repulsive interaction can then be estimated as $\sim v_0 (D_{E_F} t)^{-d/2}$. Thus we find

$$\frac{\delta\varphi_\nu}{\varphi_\nu} \sim v_0 \int_{t_{\min}}^{t_{\max}} dt (D_{E_F} t)^{-d/2} \sim \frac{v_0}{D_{E_F}^{d/2}} \{ t_{\min}^{1-(d/2)} - t_{\max}^{1-(d/2)} \}. \quad (4.53)$$

Here $t_{\min}$ can be determined from the condition of applicability of "diffusion" approximation $(D_{E_F} t_{\min})^{1/2} \sim p_F^{-1}$, i.e., $t_{\min} \sim (D_{E_F} p_F^2)^{-1} \sim \bar{E}^{-1}$. The time $t_{\max}$ is determined by two factors. First, the interaction matrix element vanishes over the time intervals $t > |E - E_F|^{-1}$ because of the time-dependent oscillations of the wave functions. Secondly, the interacting electrons in the region of localized states cannot move apart to a distance greater than $R_{\text{loc}}(E_F)$ from each other and the "diffusion" approximation is valid as long as the time $t \leqslant R_{\text{loc}}^2(E_F)/D_{E_F} \sim [\omega_0^2(E_F)\tau_{E_F}]^{-1}$. Therefore, $t_{\max} \sim \min\{|E - E_F|^{-1};\ [\omega_0^2(E_F)\tau_{E_F}]^{-1}\}$. Setting $\delta N(E)/N_0(E_F) \sim \delta\varphi_\nu/\varphi_\nu$ [compare with (1.4)], we immediately find (4.42). This estimate is, of course, purely explanatory in nature.

These results represents a simple generalization of the analysis of Aronov and Al'tshuler [9, 128, 129] for the insulator side of the Anderson transition. There is no "Coulomb gap" in these approximations [130–133] principally because of the short-range nature of the interaction and possibly because of the crudeness of the model which is based solely on the first-order perturbation theory corrections in terms of the interaction. Although the advantage of the formalism which has been used is its clarity, it obviously cannot be used with any measure of success for a generalization in which higher-order corrections in terms of the interaction could be incorporated, because in this case we have to know the behavior of "higher-order" spectral densities.

4.2. *Electron–Electron Interaction in the Self-Consistent Theory of Localization*

We will consider below the role of the first-order corrections in the perturbation theory over interaction within the framework of the

standard diagram formalism of the self-consistent theory of localization. By using this approach we can improve the accuracy of the results obtained above and make some generalizations. Clearly, the diagram formalism, in principle, is not limited to the first-order perturbation theory in terms of the interaction.

Following the procedure of Al'tshuler and Aronov [128], we will examine a very simple correction to a single-electron Green's function, illustrated graphically in Fig. 15a, where the "triangular" vertex is given by the equation, illustrated graphically in Fig. 15b,

$$\gamma(\mathbf{q}\omega) = 1 + \int \frac{d^d\mathbf{p}'}{(2\pi)^d}\, \Gamma_{\mathbf{p}\mathbf{p}'}(\mathbf{q}\omega) G(\epsilon + \omega \mathbf{p}'_+) G(\epsilon \mathbf{p}_-). \qquad (4.54)$$

To avoid confusion we wish to point out that here and elsewhere in the text we have $\epsilon = E - E_F$; i.e., this expression denotes the energy reckoned from the Fermi level. The correction to a single-electron density of states will then be

$$\begin{aligned}
\frac{\delta N(\epsilon)}{N_0(E_F)} &= -\frac{1}{\pi N_0(E_F)} \int \frac{d^d\mathbf{p}}{(2\pi)^d}\, \mathrm{Im}\, \delta G^R(\epsilon \mathbf{p}) \\
&= -\frac{1}{\pi N_0(E_F)}\, \mathrm{Im} \int \frac{d^d\mathbf{p}}{(2\pi)^d}\, i[G^R(\epsilon\mathbf{p})]^2 \\
&\quad \times \int \frac{d^d\mathbf{q}}{(2\pi)^d} \int_\epsilon^\infty \frac{d\omega}{2\pi}\, \gamma^2(\mathbf{q}\omega) G^A(\epsilon + \omega \mathbf{p} + \mathbf{q}) \\
&= -\frac{1}{2\gamma^2(E_F)}\, \mathrm{Im} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \int_\epsilon^\infty \frac{d\omega}{2\pi}\, \gamma^2_{RA}(\mathbf{q}\omega) v(\mathbf{q}). \qquad (4.55)
\end{aligned}$$

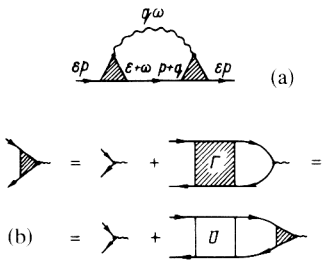

Figure 15. (a) A simplest correction to a single-electron Green's function. (b) An equation for a "triangular" vertex.

The single-electron Green's functions in (4.55) are taken in a simplest approximation (2.34). Clearly, for a "triangular" vertex with external legs we would have (see Fig. 3)

$$\mathcal{T}_{\mathbf{p}}^{RA}(\mathbf{q}\omega) \equiv G^{R}(\epsilon + \omega \mathbf{p}_{+}) G^{A}(\epsilon \mathbf{p}_{-}) \gamma^{RA}(\mathbf{q}\omega)$$

$$= -2\pi i \int \frac{d^{d}\mathbf{p}'}{(2\pi)^{d}} \phi_{\mathbf{pp}'}^{RA}(\mathbf{q}\omega). \tag{4.56}$$

Using (2.50) and (2.54)–(2.57), we then find an expression for the relevant vertex in the self-consistent theory of localization

$$\gamma^{RA}(\mathbf{q}\omega) = \left\{ \omega + \frac{1}{m} \mathbf{p} \cdot \mathbf{q} + 2i\gamma(E_F) \right\} \frac{\omega + M_{E_F}(\mathbf{q}\omega) - \dfrac{1}{m} \mathbf{p} \cdot \mathbf{q}}{\omega^{2} + \omega M_{E_F}(\mathbf{q}\omega) - \dfrac{2E_F}{dm} \mathbf{q}^{2}}$$

$$\approx \frac{2\gamma(E_F)}{-i\omega + D_{E_F}(\mathbf{q}\omega)\mathbf{q}^{2}}, \tag{4.57}$$

where the last expression is valid only for small values of ω and $\mathbf{q}$. Accordingly, the vertex $\gamma(\mathbf{q}\omega)$ in the self-consistent theory has the same form as in a "dirty" metal [9, 128, 129], although the classical diffusion coefficient is replaced by a generalized diffusion coefficient which was defined in (2.71). We see that in the localization region ($E_F < E_c$) Eqs. (2.73) and (2.74) imply in the limit of $\omega \to 0$

$$\gamma^{RA}(\mathbf{q}\omega) \underset{\omega \to 0}{\approx} \frac{2i\gamma(E_F)}{\omega + i\delta} A_{E_F}(\mathbf{q}). \tag{4.58}$$

This rather general result may be attributed to the localization criterion formulated in Sec. 1. The use of general relation (2.26) in (2.56) in fact shows that in the self-consistent theory we would have

$$\psi_{\mathbf{p}}^{\mathbf{q}}(E_F) \simeq \frac{\operatorname{Im} G^{R}(E_F \mathbf{p})}{\pi N_0(E_F)} \chi_{\mathbf{q}}(E_F), \tag{4.59}$$

where $\chi_{\mathbf{q}}(E_F)$ is given by (2.62). We then find from (2.26) and (4.56)

$$\mathcal{T}_{\mathbf{p}}^{RA} = 2\pi i \frac{\psi_{\mathbf{p}}^{\mathbf{q}}(E_F)\chi_{-\mathbf{q}}(E_F)}{\omega + i\delta} + \cdots. \tag{4.60}$$

Using (4.59) and (2.50), this yields (4.58) in the limit of $\omega \to 0$, taking into account the general relation (2.29).

Using (4.57) in (4.55), we find

$$\frac{\delta N(\epsilon)}{N_0(E_F)} = -\frac{1}{\pi} \operatorname{Im} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \times \int_{\epsilon}^{\infty} d\omega \frac{1}{[-i\omega + D_{E_F}(\mathbf{q}\omega)\mathbf{q}^2]^2}. \tag{4.61}$$

Using (2.73) ($\mathbf{q} \to 0$), we can write (4.61) in the form

$$\frac{\delta N(\epsilon)}{N_0(E_F)} = -\frac{2}{\pi} \int_{\epsilon}^{\infty} d\omega \int \frac{d^d\mathbf{q}}{(2\pi)^d} v(\mathbf{q}) D_{E_F}\mathbf{q}^2 \times \left\{ \frac{\omega}{[\omega^2 + (\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2)^2]^2} + \frac{\omega_0^2(E_F)\tau_{E_F}}{\omega} \frac{\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2}{[\omega^2 + (\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2)^2]^2} \right\}. \tag{4.62}$$

Calculating the integral over ω in (4.62), we find

$$\frac{\delta N(\epsilon)}{N_0(E_F)} = \left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{I}} + \left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{II}},$$

where

$$\left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{I}} = -\frac{1}{\pi} \int \frac{d^d\mathbf{q}}{(2\pi)^d} v(\mathbf{q}) \times \frac{D_{E_F}\mathbf{q}^2}{\epsilon^2 + [\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^2} \tag{4.63}$$

is in agreement with expression (4.41) derived above, and

$$\left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{II} = \frac{1}{\pi}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q}) D_{E_F}\mathbf{q}^2 \left\{\frac{1}{\epsilon^2 + [\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^2}\right.$$
$$\times \frac{\omega_0^2(E_F)\tau_{E_F}}{\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2} + \frac{\omega_0^2(E_F)\tau_{E_F}}{[\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^3}$$
$$\left.\times \ln \frac{\epsilon^2}{\epsilon^2 + [\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^2}\right\}$$
$$\underset{|\epsilon|\to 0}{\approx} \frac{2}{\pi}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q}) \frac{\omega_0^2(E_F)\tau_{E_F}}{[\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^3} D_{E_F}\mathbf{q}^2$$
$$\left.\times \ln \frac{|\epsilon|}{\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2}\right\}. \tag{4.64}$$

At $\omega_0^2(E_F) = 0$, i.e., in the metallic region, contribution (4.64) vanishes. The fact that it exists shows that, in general, the diagram formalism used here does not match uniquely the formalism of the exact eigenfunctions which was used to obtain the results in the preceding section. Such a contribution can, in principle, be linked with the second term in Eq. (2.18) which was dropped in the derivation of Eq. (2.74). An estimate of the integral in (4.64) on the basis of the point-like interaction model yields

$$\left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{II} \simeq \frac{v_0}{\pi}\frac{S_d}{4-d} D_{E_F}^{-d/2}[\omega_0^2(E_F)\tau_{E_F}]^{(d-2)/2} \ln \frac{|\epsilon|}{\omega_0^2(E_F)\tau_{E_F}}. \tag{4.65}$$

We see that Eq. (4.65) plays a dominant role only in an exponentially small region near the Fermi level (or the mobility edge), whereas contribution (4.42) accounts for the main correction in a broader energy region and, in this sense, is the principal contribution. However, the logarithmic singularity in (4.65) unequivocally shows that the higher-order corrections in terms of the interaction must be taken into account if we wish to describe correctly the immediate neighborhood of the Fermi level (mobility edge) in the localization region. Using (4.46) and (4.47), we find

$$\left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\text{II}} \sim v_0 m^{d/2} E_F^{(d/2)-1} \ln\left\{\frac{|\epsilon|}{\lambda E_F}[p_F R_{\text{loc}}(E_F)]^d\right\}$$

$$\sim v_0 N_0(E_F) \ln\left\{\frac{|\epsilon|}{\lambda E_F}\left|\frac{E_F - E_c}{E_c}\right|^{-d\nu}\right\}. \tag{4.66}$$

Using (4.50) for $d = 2$ we find in a similar way, that for $|\epsilon| \ll E_F \exp(-1/\lambda)$ the contribution (4.64) is always dominant

$$\frac{\delta N(\epsilon)}{N_0} \sim v_0 N_0 \frac{1}{\lambda} \ln\frac{|\epsilon|}{E_F}, \tag{4.67}$$

so that the second expression in (4.51) has, in fact, no region of applicability. The results obtained in the formalism of the exact eigenfunctions can thus be substantially modified by using the diagram formalism. This method can also be used to analyze the role of other diagrams, whose contributions are different from that of the simplest "Fock" diagram shown in Fig. 15a. There are, in fact, several other diagrams of the first-order perturbation theory in terms of the interaction. Some examples of these diagrams are shown in Fig. 16. Let us examine first of all the "Hartree" diagram of Fig. 16a. We can estimate its contribution within the context of a self-consistent theory of localization analogously to how it was done in Refs. 9, 129, and 151. In a self-consistent theory the full vertex

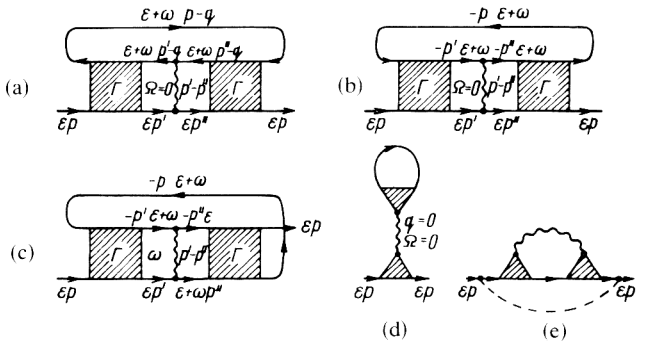

Figure 16. Examples of various first-order diagrams, whose contribution is compared with the contribution of the diagram in Fig. 15a.

$\Gamma_{\mathbf{pp}'}(\mathbf{q}\omega)$ (see Fig. 3) can be written in the following simple form, for small values of $\mathbf{q}$ and ω:

$$\Gamma^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega) \approx \frac{2\gamma(E_F)\rho V^2}{-i\omega + D_{E_l}(\mathbf{q}\omega)\mathbf{q}^2}. \tag{4.68}$$

This expression generalizes the "diffusion" expression (2.86) which is derived by taking the sum over the ladder diagrams in Fig. 5b. The result in (4.68) is basically self-evident. It is obvious, in particular, that its use in Eq. (4.54) yields (4.57). Equation (4.68) can also be derived formally. In this connection, let us examine an auxiliary vertex $\tilde{\Gamma}^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega)$, which is determined graphically in Fig. 17 (compare with Fig. 3). We find

$$\begin{aligned}\tilde{\Gamma}^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega) &= (\rho V^2)^2(-2\pi i)\int \frac{d^d\mathbf{p}''}{(2\pi)^d}\int \frac{d^d\mathbf{p}'''}{(2\pi)^d}\,\phi^{RA}_{\mathbf{p}''\mathbf{p}'''}(\mathbf{q}\omega) \\ &= -2\pi i(\rho V^2)^2\phi^{RA}(\mathbf{q}\omega) \approx \frac{2\gamma(E_F)\rho V^2}{-i\omega + D_{E_F}(\mathbf{q}\omega)\mathbf{q}^2},\end{aligned} \tag{4.69}$$

where in the last equality we have used (2.69). Let us assume that in the limit of $\mathbf{q}\to 0$ we have $\Gamma^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega) \approx \tilde{\Gamma}^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega)$. We can then reproduce (4.69) by using (4.68) in the second diagram on the right side of Fig. 17, whereas the first diagram is nonsingular in the limits of $\omega\to 0$ and $\mathbf{q}\to 0$. This demonstrates that our assumption is self-consistent. We can now compare directly the contributions to the electron self-energy from the diagrams in Figs. 16a and 15a:

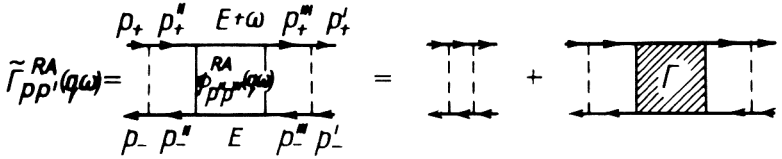

Figure 17. Graphic determination of the auxiliary vertex $\tilde{\Gamma}^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega)$.

$$\Sigma_H^R(\epsilon\mathbf{p}) = \int \frac{d^d\mathbf{p}'}{(2\pi)^d} \int \frac{d^d\mathbf{p}''}{(2\pi)^d} \int \frac{d^q\mathbf{q}}{(2\pi)^d} \int_\epsilon^\infty \frac{d\omega}{2\pi}\, v(\mathbf{p}'-\mathbf{p}'')$$
$$\times \frac{4(\rho V^2)\gamma^2(E_F)}{[-i\omega + D_{E_F}(\omega)\mathbf{q}^2]^2}\, G^R(\epsilon\mathbf{p}')G^R(\epsilon\mathbf{p}'')G^A(\mathbf{p}''-\mathbf{q}\epsilon+\omega)$$
$$\times G^A(\mathbf{p}'-\mathbf{q}\epsilon+\omega)G^A(\mathbf{p}-\mathbf{q}\epsilon+\omega) \tag{4.70}$$

$$\Sigma_F^R(\epsilon\mathbf{p}) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \int_\epsilon^\infty \frac{d\omega}{2\pi}\, v(\mathbf{q}) \frac{4(\rho V^2)^2\gamma^2(E_F)}{[-i\omega + D_{E_F}(\omega)\mathbf{q}^2]^2}\, G^A(\epsilon+\omega\mathbf{p}-\mathbf{q}). \tag{4.71}$$

An important point here is that in contrast with the "Fock" diagram, the momentum transferred along the interaction line in the "Hartree" diagram is not small, and the frequency transfer is zero. We can then easily estimate the ratio

$$F = \frac{\Sigma_H^R(\epsilon\mathbf{p})}{\Sigma_F^R(\epsilon\mathbf{p})} \approx \left\{\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(0)\right\}^{-1} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \int \frac{d^d\mathbf{p}'}{(2\pi)^d}$$
$$\times \int \frac{d^d\mathbf{p}''}{(2\pi)^d} (\rho V^2)^2 G^R(\epsilon\mathbf{p}')G^A(\epsilon\mathbf{p}')v(\mathbf{p}'-\mathbf{p}'')G^R(\epsilon\mathbf{p}'')G^A(\epsilon\mathbf{p}'')$$
$$\approx \left\{\int d\Omega v(0)\right\}^{-1} \int d\Omega \frac{1}{N_0^2} \int \frac{d^d\mathbf{p}'}{(2\pi)^d} \int \frac{d^d\mathbf{p}''}{(2\pi)^d}$$
$$\times \delta\left(E_F - \frac{\mathbf{p}'^2}{2m}\right) v(\mathbf{p}'-\mathbf{p}'')\delta\left(E_F - \frac{\mathbf{p}''^2}{2m}\right). \tag{4.72}$$

and finally find the same result as that for a metal [9, 129, 151]:

$$F = \frac{\int d\Omega v\left(2p_F \sin\frac{\theta}{2}\right)}{\int d\Omega v(0)}, \tag{4.73}$$

where the integration is over the solid angle at the Fermi surface (θ is the angle between two momenta at the Fermi sphere). We easily see that $F < 1$ if the interaction potential falls off at a distance greater than the inverse Fermi momentum. If, for example, we would use a

screened Coulomb interaction as a model potential,* we would find, for $d = 3$ (Ref. 9),

$$F = \frac{1}{2}\int_0^{\pi} d\theta \sin\theta \frac{1}{1 + \frac{4p_F^2}{\kappa_D^2}\sin^2\frac{\theta}{2}} = \frac{\kappa_D^2}{4p_F^2}\ln\left\{1 + \frac{4p_F^2}{\kappa_D^2}\right\}, \tag{4.74}$$

where $\kappa_D^2 = 4\pi e^2 N_0(E_F)$ is the inverse square of the screening length. We can infer from this that $F \ll 1$ if $\kappa_D^2 \ll p_F^2$, but $F \to 1$ if $p_F^2 \ll \kappa_D^2$.

Since $\gamma(E_F) \sim \rho V^2 N_0(E_F)$ (2.35), we find $F \sim \kappa_D^2/p_F^2 \sim e^2/m\rho V^2\gamma(E_F)/E_F \sim e^2/m\rho V^2\lambda(E_F)$, so that $F \sim e^2/m\rho V^2$ near the mobility edge ($E_F \sim E_c \sim E_{sc}$). For a point-like interaction we would have $F = 1$ and, after incorporating the spin, the Hartree contribution in Fig. 16a would be twice as large as the Fock contribution. In this case, all the corrections to the density of states which we have considered would simply change sign, as we have mentioned above. The contribution of the diagram in Fig. 16b, where the interaction occurs in the Cooper channel, in the absence of interactions which break down the time-reversal invariance (magnetic field, magnetic impurities, etc.), is equal to the contribution of the diagram in Fig. 16a (Ref. 152), so that it is also small over the parameter F. A similar small parameter is also found for the Fock diagram in Fig. 16c, where the interaction occurs in the Cooper channel. The difference between this diagram and the one in Fig. 15a is again linked to the appreciable momentum transfer along the interaction line. The diagram in Fig. 16d is canceled out because of the total electrical neutrality of the system, and for the contribution of the diagram in Fig. 16a we easily find

$$\operatorname{Im}\Sigma^R(\epsilon\mathbf{p}) = -\pi\rho V^2 \delta N(\epsilon), \tag{4.75}$$

where $\delta N(\epsilon)$ is the correction to the density of states due to the diagram in Fig. 15a. Diagrams of this type therefore have an additional smallness over the parameter ρV^2.

*We will see below that the static (zero frequency-transfer) interaction in a Coulomb system in the localization region is given by a screened Coulomb potential. This example is therefore quite realistic.

The problem of the relevance of higher-order interaction corrections remains essentially unresolved. The solution of this problem, in addition to the analysis of many new diagrams, should also consider how the interaction can be incorporated into the determination of the current-relaxation kernel of the self-consistent theory $M_{E_f}(\mathbf{q}\omega)$. This can be done, for example, by introducing the interaction lines and vertices into the calculation of the irreducible kernel $U^{E_f}_{\mathbf{pp}'}(\mathbf{q}\omega)$ in (2.58). Clearly, such effects could be ignored only in the first-order corrections over interaction, a circumstance which was exploited above. The higher-order corrections in terms of the interaction are clearly important in the immediate vicinity of the mobility edge and the Fermi level, but the first-order corrections are presumably adequate in the remaining energies. The diagram formalism creates the opportunity, at least in principle, to analyze these problems systematically.

4.3. Polarization Operator, Screening, and Coulomb Interaction

Let us consider in general terms and within the self-consistent theory how the localization affects the behavior of the polarization operator; i.e., let us consider, in fact, the screening of the electric field in Fermi glass.

Working again in the representation of exact eigenfunctions of a single-electron problem, we find the following expression for the Fourier transform of the polarization operator of noninteracting electrons:

$$\Pi(\mathbf{q}\omega) = \left\langle \sum_{\mu\nu\mathbf{pp}'} \frac{f_\mu - f_\nu}{\epsilon_\nu - \epsilon_\mu + \omega + i\delta \operatorname{sign} \omega} \varphi_\nu(\mathbf{p}_+)\varphi^*_\nu(\mathbf{p}'_+)\varphi_\mu(\mathbf{p}'_-)\varphi^*_\mu(\mathbf{p}_-) \right\rangle$$

$$= \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\Omega \frac{f(E) - f(E+\Omega)}{\Omega + \omega + i\delta \operatorname{sign} \omega} N_0(E) \langle\langle \rho_E \rho_{E+\Omega} \rangle\rangle^E_{\mathbf{q}} \qquad (4.76)$$

in the zero-temperature formalism, or

$$\Pi(\mathbf{q}\omega_m) = \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\Omega \frac{f(E) - f(E+\Omega)}{\Omega + i\omega_m} N_0(E) \langle\langle \rho_E \rho_{E+\Omega} \rangle\rangle^E_{\mathbf{q}} \qquad (4.77)$$

in the Matsubara technique ($\omega_m = 2\pi m T$). Substituting into this

expression the singular part of (4.15), at $T = 0$ we find

$$\Pi_{\text{loc}}(\mathbf{q}\omega) = 0 \tag{4.78}$$

and the nonvanishing contribution arises only from the regular part of (4.15),

$$\Pi(\mathbf{q}\omega) = \Pi_{\text{reg}}(\mathbf{q}\omega) = \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\Omega \frac{f(E) - f(E+\Omega)}{\Omega + \omega + i\delta \operatorname{sign} \omega} N_0(E) \rho_E^E(\mathbf{q}\Omega). \tag{4.79}$$

The situation is quite different in the Matsubara technique ($T \neq 0$):

$$\Pi_{\text{loc}}(\mathbf{q}\omega_m \neq 0) = 0 \tag{4.80}$$

$$\Pi_{\text{loc}}(\mathbf{q}\omega_m = 0) = \int_{-\infty}^{\infty} dE \left(-\frac{\partial f(E)}{\partial E}\right) N_0(E) A_E(\mathbf{q}) \approx N_0(E_F) A_{E_F}(\mathbf{q}).$$

We thus find

$$\Pi(\mathbf{q}\omega_m) = \frac{1}{T} \delta_{\omega_m 0} \int_{-\infty}^{\infty} dE f(E)[1 - f(E)] N_0(E) A_E(\mathbf{q}) + \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\Omega \frac{f(E) - f(E+\Omega)}{\Omega + i\omega_m} N_0(E) \rho_E^E(\mathbf{q}\Omega). \tag{4.81}$$

Taking into account the explicit form of the "regular" part of the Berezinskiĭ–Gor'kov spectral density which arises in the self-consistent theory of localization [see Eq. (2.73)], we find from (4.79), after some straightforward calculations the following expression:

$$\Pi(\mathbf{q}\omega) = \Pi_{\text{reg}}(\mathbf{q}\omega) = N_0(E_F) \frac{D_{E_F} \mathbf{q}^2}{D_{E_F} \mathbf{q}^2 + \omega_0^2(E_F) \tau_{E_F} - i(\omega + i\delta \operatorname{sign} \omega)}. \tag{4.82}$$

In the metal region we would have $\omega_0^2(E_F) = 0$, and (4.82) reduces to the familiar expression for the polarization operator of a "dirty"

metal [9, 128, 135]. In the localization region, after incorporating $\omega_0^2(E_F)\tau_{E_F} = D_{E_F}R_{\text{loc}}^{-2}(E_F)$ [compare with (2.74)], we find

$$\Pi_{\text{reg}}(\mathbf{q}0) = N_0(E_F)\frac{\mathbf{q}^2}{\mathbf{q}^2 + R_{\text{loc}}^{-2}(E_F)}. \tag{4.83}$$

For the Matsubara polarization operator some analogous calculations yield

$$\Pi(\mathbf{q}\omega_m) = N_0(E_F)\left\{ A_{E_F}(\mathbf{q})\delta_{\omega_m 0} + \frac{D_{E_F}(\omega_m)\mathbf{q}^2}{\omega_m + D_{E_F}(\omega_m)\mathbf{q}^2}\,\theta(\omega_m) + \frac{D_{E_F}(-\omega_m)\mathbf{q}^2}{-\omega_m + D_{E_F}(-\omega_m)\mathbf{q}^2}\,\theta(-\omega_m)\right\}; \quad \theta(\omega_m) = \begin{cases} 1 & m \geq 0 \\ 0 & m < 0 \end{cases}, \tag{4.84}$$

where $A_{E_F}(\mathbf{q})$ is given by expression (2.74′), and the generalized diffusion coefficient in the Matsubara technique is

$$D_{E_F}(\omega_m) = \frac{2E_F}{dm}\frac{i}{M_{E_F}(\omega_m)}; \qquad M_{E_F}(\omega_m) = \frac{i}{\tau_{E_F}} - \frac{\omega_0^2(E_F)}{i\omega_m}. \tag{4.85}$$

The result in (4.84) can also be obtained by a direct calculation from the diagram in Fig. 18a if we take into account that the triangular vertex of the self-consistent theory in the Matsubara technique is

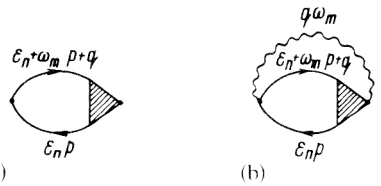

Figure 18. (a) A diagram for the polarization operator of noninteracting electrons. (b) "Fock" contribution to the thermodynamic potential.

given by

$$\gamma(\mathbf{q}\omega_m\epsilon_n) = \theta(\epsilon_n)\theta(\epsilon_n + \omega_m) + \theta(-\epsilon_n)\theta(-\epsilon_n - \omega_m) + 2\gamma(E_F)\left\{\frac{\theta(\epsilon_n)\theta(-\epsilon_n - \omega_m)}{-\omega_m + D_{E_F}(-\omega_m)\mathbf{q}^2} + \frac{\theta(-\epsilon_n)\theta(\epsilon_n + \omega_m)}{\omega_m + D_{E_I}(\omega_m)\mathbf{q}^2}\right\}; \quad \epsilon_n = (2n+1)\pi T. \tag{4.86}$$

This expression is the well-known Aronov–Al'tshuler result [128], in which the generalized diffusion coefficient (4.85) is substituted for the classical diffusion coefficient.

We see that the "localization" contribution

$$\Pi_{\text{loc}}(\mathbf{q}\omega_m) = \int_{-\infty}^{\infty} dE\left(-\frac{\partial f(E)}{\partial E}\right) N_0(A) A_E(\mathbf{q})\delta_{\omega_m 0} = \frac{1}{T}\delta_{\omega_m 0}\int_{-\infty}^{\infty} dE f(E)[1 - f(E)]N_0(E)A_E(\mathbf{q}) \underset{T\to 0}{\approx} N_0(E_F)A_{E_I}(\mathbf{q})\delta_{\omega_m 0} \tag{4.87}$$

arises in polarization operators (4.81) and (4.84). Let us consider the simplest (Fock) contribution to the thermodynamic potential, shown schematically in Fig. 18b. The localization contribution in (4.87) would then correspond to

$$\langle \delta\Omega_{\text{loc}}\rangle = \frac{1}{2}T\sum_m \int \frac{d^d\mathbf{q}}{(2\pi)^d} v(\mathbf{q})\Pi_{\text{loc}}(\mathbf{q}\omega_m) = \frac{1}{2}\int_{-\infty}^{\infty} dE f(E)[1 - f(E)]N_0(E)\int \frac{d^d\mathbf{q}}{(2\pi)^d} v(\mathbf{q})A_E(\mathbf{q}) = -\frac{1}{2}\int_{-\infty}^{\infty} dE[1 - f(E)]N_0(E)\tilde{\Sigma}^E_{E_{\text{loc}}}. \tag{4.88}$$

In the last equality in (4.88) we have used (4.18). We have found the exact analog of the localization contribution in (4.28). The term

with unity, as was noted in (4.29), goes into the renormalization of the chemical potential (taking the account of the regular contribution) and the factor $\frac{1}{2}$ is canceled by summing over the spin. Equation (4.88) then reduces to the localization part of (4.30). Consequently, we find the diagram "recipe" for determining the localization contribution to the self-energy $\tilde{\Sigma}_E$ which, as we can see from its definition (4.6), has no rigorous diagrammatic meaning.

The difference in the behavior of the polarization operator at $T = 0$ and in the Matsubara technique, which is seen only at zero frequency (static-field screening), is a consequence of the known difference between the adiabatic (Kubo) static response and the isothermal static response in systems with nonergodic behavior [39] manifested, as was noted in Sec. 1, in a $\delta(\omega)$ contribution to the Berezinskiĭ–Gor'kov spectral density. As we can see from a general analysis carried out by Kwok and Schulz [40], the Matsubara response is sensitive to nonergodic behavior, whereas the response determined by the commutator Green's functions (Kubo) does not "feel" it. At $\omega_m = 0$ the corresponding Matsubara Green's function determines the *isothermal* response, whereas the limit $\omega \to 0$ of the corresponding commutator Green's function (which can be obtained by continuing the Matsubara function analytically $i\omega_m \to \omega + i\delta$) determines the *adiabatic* response. The physical difference between these two responses is that the adiabatic (Kubo) response describes a system which is isolated from the heat bath after the external perturbation is applied, whereas the isothermal response corresponds to a system inside the heat bath during the time this perturbation is active [39]. The polarization operator, as we know, is associated with the electronic compressibility and the dielectric constant. For an isothermal static compressibility we would have

$$\kappa^T(\mathbf{q}0) = \Pi(\mathbf{q}\omega_m = 0), \tag{4.89}$$

whereas the adiabatic compressibility can be determined by means of

$$\kappa^A(\mathbf{q}0) = \Pi_{\text{reg}}(\mathbf{q}\omega \to 0). \tag{4.90}$$

From (4.79)–(4.81) we then find

$$\kappa^T(\mathbf{q}0) - \kappa^A(\mathbf{q}0) = N_0(E_F) A_{E_F}(\mathbf{q}) = \frac{N_0(E_F)}{1 + R_{\text{loc}}^2(E_F)\mathbf{q}^2}. \quad (4.91)$$

The last equation in (4.91) is valid in the self-consistent theory of localization. Götze [50] and Prelovšek [63] have pointed out for the first time that the quantity $A_{E_F}(\mathbf{q})$ determines the difference between the isothermal static compressibility and the adiabatic static compressibility.

For an isothermal static polarization operator from (4.80) and (4.83) we find

$$\begin{aligned}\Pi^T(\mathbf{q}0) &= \Pi(\mathbf{q}\omega_m = 0) = \Pi_{\text{loc}}(\mathbf{q}0) + \Pi_{\text{reg}}(\mathbf{q}0) \\ &= N_0(E_F)\left\{\frac{1}{1 + R_{\text{loc}}^2(E_F)\mathbf{q}^2} + \frac{\mathbf{q}^2}{\mathbf{q}^2 + R_{\text{loc}}^{-2}(E_F)}\right\} = N_0(E_F).\end{aligned} \quad (4.92)$$

Correspondingly, for an *adiabatic* static dielectric constant we find

$$\begin{aligned}\epsilon^A(\mathbf{q}\omega \to 0) &= 1 + \frac{4\pi e^2}{\mathbf{q}^2}\Pi_{\text{reg}}(\mathbf{q}\omega \to 0) \\ &= \begin{cases} 1 + \dfrac{\kappa_D^2}{\mathbf{q}^2}; & \mathbf{q} \gtrsim R_{\text{loc}}^{-1}(E_F) \\ 1 + \kappa_D^2 R_{\text{loc}}^2(E_F); & \mathbf{q} \lesssim R_{\text{loc}}^{-1}(E_F), \end{cases}\end{aligned} \quad (4.93)$$

where $\kappa_D^2 = 4\pi e^2 N_0(E_F)$, consistent with (2.100), whereas the *isothermal* static dielectric constant is

$$\epsilon^T(\mathbf{q}0) = 1 + \frac{4\pi e^2}{\mathbf{q}^2}\Pi^T(\mathbf{q}0) = 1 + \frac{\kappa_D^2}{\mathbf{q}^2}, \quad (4.94)$$

consistent with the standard Thomas–Fermi result for a *metal*. This last dielectric constant corresponds to a real experiment on screening of the external static field. Rice [153] and Jäckle [154] have obtained this important result for the first time at the qualitative level, and we have shown here the formal "mechanism" of this

behavior. Using the exact eigenfunction formalism, we can demonstrate directly that the first-order interaction corrections to $\Pi(\mathbf{q}0)$ cancel each other. These corrections account for the density of states behavior of the type shown in (4.42). There are no such features in $\Pi(\mathbf{q}0)$, in complete agreement with the important circumstance pointed out by Lee [136] and Finkel'shtein [137]: The screening range is determined by the quantity $dn/d\mu = \Pi(\mathbf{q} \to 00)$, rather than by the density of states. McMillan [134] did not take this fact into account. The physical mechanism of screening in a Fermi glass is connected with the electrons adjusting themselves in such a manner as to ensure total screening at any temperature, however small, as a result of hopping conductivity along the localized states in an Anderson dielectric. The typical time for such an adjustment is determined by the reciprocal frequency $\omega^* \sim D_{\text{hop}}\mathbf{q}^2$, where D_{hop} is the diffusion coefficient determined by the hopping conductivity [153, 154]. In the formalism described above, which ignores the hopping conductivity, the static nature of the field (and of the response) should be understood in the context of the condition $\omega \ll \omega^*$. In a real experiment, we would have $q \sim L^{-1}$, where L is the typical length-scale of the nonuniformity of the external field (which is determined by the size of the sample, etc.). The divergence of the dielectric constant, observed near the insulator-metal transition in certain experiments on Si doped with P [155, 156], is probably attributable to the divergence of the localization length $R_{\text{loc}}(E_F \to E_c)$ in Eqs. (2.100) and (4.93). These experiments were carried out at the external-field frequencies in the range from several hundred MHz to the infrared region of the optical spectrum, i.e., they measured a response of the type in (4.93). It would be clearly worthwhile to measure experimentally the dielectric constant of this system in a static field.

We can now turn from the analysis of the short-range repulsive potential to the case of real Coulomb interaction. In the following, we will carry out all the calculations for the three-dimensional case, using the Matsubara technique. We will consider, as in Ref. 128, only the contribution from the diagram in Fig. 15a, where the wavy line corresponds to the dynamically screened Coulomb interaction

$$V(\mathbf{q}\omega_m) = \frac{4\pi e^2}{\mathbf{q}^2 \epsilon(\mathbf{q}\omega_m)}. \tag{4.95}$$

The dielectric constant in (4.95) is

$$\epsilon(\mathbf{q}\omega_m) = 1 + \frac{4\pi e^2}{\mathbf{q}^2}\,\Pi(\mathbf{q}\omega_m) \tag{4.96}$$

and the polarization operator is taken from (4.84) (Fig. 18a). The "localization" contribution discussed above is now important, since in the calculation of the diagram in Fig. 15a it reduces to the contribution which vanishes in the limit of $T \to 0$. We can therefore confine ourselves below solely to the regular contribution. From Fig. 15a we find

$$\delta G(\epsilon_n \mathbf{p}) = T \sum_m \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, V(\mathbf{q}\omega_m)\gamma^2(\mathbf{q}\omega_m) G^2(\epsilon_n \mathbf{p}) G(\epsilon_n + \omega_m \mathbf{p} + \mathbf{q}), \tag{4.97}$$

where the screened interaction for small values of $\mathbf{q}$ and ω_m is

$$V(\mathbf{q}\omega_m \to 0) \approx \frac{-\omega_m + D_{E_F}(-\omega_m)\mathbf{q}^2}{N_0(E_F) D_{E_F}(-\omega_m)\mathbf{q}^2}\,\theta(-\omega_m) + \frac{\omega_m + D_{E_F}(\omega_m)\mathbf{q}^2}{N_0(E_F) D_{E_F}(\omega_m)\mathbf{q}^2}\,\theta(\omega_m). \tag{4.98}$$

Introducing, by definition,

$$\delta N(\epsilon_n) = -\frac{1}{\pi}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\delta G(\epsilon_n \mathbf{p}), \tag{4.99}$$

we find

$$\begin{aligned}\delta N(\epsilon_n) &= -\frac{1}{\pi}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\, T\sum_m V(\mathbf{q}\omega_m)\gamma^2(\mathbf{q}\omega_m\epsilon_n) \\ &\quad \times \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, G^2(\epsilon_n \mathbf{p}) G(\epsilon_n + \omega_m \mathbf{p} + \mathbf{q}) \\ &\approx -iN_0(E_F)\, T\sum_m \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, V(\mathbf{q}\omega_m) \\ &\quad \times \left\{ \frac{\theta(\epsilon_n)\theta(-\epsilon_n - \omega_m)}{[-\omega_m + D_{E_F}(-\omega_m)\mathbf{q}^2]^2} - \frac{\theta(-\epsilon_n)\theta(\epsilon_n + \omega_m)}{[\omega_m + D_{E_F}(\omega_m)\mathbf{q}^2]^2} \right\}. \end{aligned} \tag{4.100}$$

Using (4.98), for $\epsilon_n > 0$, we find

$$\delta N(\epsilon_n > 0) \approx -\frac{iT}{2\pi^2} \sum_{\omega_m=-\infty}^{-\epsilon_n} D_{E_F}^{-1}(\omega_m) \int_0^\infty d\mathbf{q} \frac{1}{D_{E_F}(-\omega_m)\mathbf{q}^2 - \omega_m}$$

$$= -\frac{iT}{2\pi^2 D_{E_F}^{3/2}} \sum_{\omega_m=-\infty}^{-\epsilon_n} \omega_m^{-2}\{\omega_m - \omega_0^2(E_F)\tau_{E_F}\}^2$$

$$\times \int_{-\infty}^{\infty} \frac{dx}{x^2 - \omega_m + \omega_0^2(E_F)\tau_{E_F}}. \tag{4.101}$$

The finite sum over the Bose frequencies appearing here can be calculated from the relation

$$T \sum_{m=-\infty}^{-n} \Phi(i\omega_m) = \int_{-i\epsilon_n+\infty}^{-i\epsilon_n-\infty} \frac{dz}{2\pi i} f_B(z)\Phi(z)$$

$$= \int_{-\infty}^{\infty} \frac{dz}{2\pi i} f_B(z - i\epsilon_n)\Phi(z - i\epsilon_n)$$

$$= \int_{-\infty}^{\infty} \frac{dz}{2\pi i} f(z)\Phi(z - i\epsilon_n), \tag{4.102}$$

where $f_B(z) = \{\exp(z/T) - 1\}^{-1}$ is the Bose function, and $f(z) = \{\exp(z/T) + 1\}^{-1}$ is the Fermi function. In our case, we would have

$$\Phi(z) = \{iz + \omega_0^2(E_F)\tau_{E_F}\} \frac{1}{2z^2} \int_0^\infty dx\{x^2 + iz + \omega_0^2(E_F)\tau_{E_F}\}^{-1}. \tag{4.103}$$

Carrying out in (4.102) and (4.103) an analytic continuation $i\epsilon_n \to E + i\delta$ and calculating the imaginary part, taking the account of $\operatorname{Im}\Phi(z) = -\operatorname{Im}\Phi(-z)$, we find the correction to the density of states corresponding to (4.101),

$$\delta N(\epsilon) = \frac{1}{4\pi^3 D_{E_F}^{3/2}} \int_0^{\tilde{E}} dz\{f(z + \epsilon) + f(z - \epsilon) - 1\} \operatorname{Im}\Phi(z). \tag{4.104}$$

Expression (4.102), strictly speaking, is valid for $\Phi(z)$ which falls off rapidly enough in the limit $|z| \to \infty$. Since the function $\Phi(z)$ in

(4.103) generally does not satisfy this condition, we have introduced a cutoff at the upper limit of $\tilde{E}$, where $\tilde{E}$ has the same meaning as in Eq. (4.43), in order to take into account the "constant" that arises in (4.104). Using the asymptotic expression

$$\operatorname{Im}\Phi(z) \approx \begin{cases} \dfrac{\pi}{4\sqrt{2z}}; & z \gg \omega_0^2(E_F)\tau_{E_F} \\ \dfrac{3\pi}{4\sqrt{2}}\dfrac{1}{z}[\omega_0^2(E_F)\tau_{E_F}]^{1/2}; & z \ll \omega_0^2(E_F)\tau_{E_F}, \end{cases} \tag{4.105}$$

we find

$$\delta N(\epsilon) \approx \begin{cases} \dfrac{1}{2^{5/2}\pi^2 D_{E_F}^{3/2}}\left\{ T^{1/2}\varphi\left(\dfrac{|\epsilon|}{2T}\right) - \tilde{E}^{1/2}\right\}; & |\epsilon|,\ T \gg \omega_0^2(E_F)\tau_{E_F} \\ \dfrac{1}{2^{5/2}\pi^2 D_{E_F}^{3/2}}\{(\omega_0^2(E_F)\tau_{E_F})^{1/2} - \tilde{E}^{1/2}\} & \\ + \dfrac{3(\omega_0^2(E_F)\tau_{E_F})^{1/2}}{2^{5/2}\pi^2 D_{E_F}^{3/2}} \ln\dfrac{\max\{|\epsilon|,\ T\}}{\omega_0^2(E_F)\tau_{E_F}}; & |\epsilon|,\ T \ll \omega_0^2(E_F)\tau_{E_F}, \end{cases} \tag{4.106}$$

where (Ref. 128)

$$\varphi(x) = \frac{1}{\sqrt{2}}\int_0^\infty dy\sqrt{y}\left\{\frac{1}{ch^2(x-y)} + \frac{1}{ch^2(x+y)}\right\} \approx \begin{cases} 1.07; & x \ll 1 \\ \sqrt{2x}; & x \gg 1. \end{cases} \tag{4.107}$$

The results which we have obtained extend directly the corresponding expressions, derived by Aronov and Al'tshuler [128], to the insulator side of the Anderson transition. It is easy to see that our correction to the density of states, due to dynamically screened Coulomb interaction, is the same within a constant as the results given in (4.42) and (4.65). Consequently, there is no "Coulomb gap" even in the Coulomb-interaction model, which probably may be attributed to the crudeness of our model. We should emphasize, however, that a standard analysis [130–133] of the Coulomb gap is carried out without regard for the screening and quantum effects. The problems which arise here can in our opinion be solved only

through a systematic analysis of the higher-order corrections in terms of the electron–electron interaction.

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References

1. N. F. Mott and E. A. Davis, *Electronic Processes in Non-Crystalline Materials*, Clarendon Press, Oxford, 1979.
2. N. F. Mott, *Metal-Insulator Transitions*, Taylor & Francis, London, 1974.
3. P. W. Anderson, *Phys. Rev.* **109**, 1492 (1958).
4. D. J. Thouless, *Phys. Reports* **13**, 93 (1974).
5. A. L. Efros, *Usp. Fiz. Nauk* **126**, 41 (1978).
6. B. I. Shklovskiĭ and A. L. Efros, *Electronic Properties of Doped Semiconductors*, Nauka, Moscow, 1979.
7. M. V. Sadovskiĭ, *Usp. Fiz. Nauk* **133**, 223 (1981).
8. I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems*, Nauka, Moscow, 1982.
9. B. L. Al'tshuler, A. G. Aronov, D. E. Khmel'nitskiĭ, and A. I. Larkin, in: *Quantum Theory of Solids*, Ed. by I. M. Lifshitz, Mir Publishers, Moscow, 1982.
10. *Anderson Localization*, Ed. by Y. Nogaoka and H. Fukuyama, Springer Series in Solid State Sciences, Vol. 39, Springer Verlag, Berlin, Heidelberg, New York, 1982.
11. S. F. Edwards, *Phil. Mag.* **3**, 1020 (1958).
12. N. F. Mott and W. D. Twose, *Adv. Phys.* **10**, 107 (1960).
13. B. I. Halperin, *Adv. Chem. Phys.* **13**, 123 (1967).
14. V. L. Berezinskiĭ, *Zh. Eksp. Teor. Fiz.* **65**, 1251 (1973).
15. A. A. Gogolin, *Phys. Reports* **86**, 1 (1982).
16. M. H. Cohen, *J. Non-Cryst. Sol.* **4**, 391 (1970).
17. T. F. Rosenbaum, K. Andres, G. A. Thomas, and R. N. Bhatt, *Phys. Rev. Lett.* **45**, 1723 (1980).

18. M. A. Paalanen, T. F. Rosenbaum, G. A. Thomas, and R. N. Bhatt, *Phys. Rev. Lett.* **48**, 1284 (1982).
19. G. A. Thomas, M. A. Paalanen, and T. F. Rosenbaum, *Phys. Rev.* **B27**, 3897 (1983).
20. A. F. Ioffe and A. R. Regel, *Progr. Semicond.* **4**, 237 (1960).
21. K. G. Wilson and J. Kogut, *Phys. Reports* **12**, 75 (1974).
22. Shang-keng Ma, *Modern Theory of Critical Phenomena*, W. A. Benjamin, Reading, Mass., 1976.
23. A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, *Fluctuation Theory of Phase Transitions*, Nauka, Moscow, 1982.
24. M. V. Sadovskiĭ, *Zh. Eksp. Teor. Fiz.* **70**, 1936 (1976).
25. F. J. Wegner, *Zs. Phys.* **25**, 327 (1976).
26. E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, *Phys. Rev. Lett.* **42**, 673 (1979).
27. J. T. Edwards and D. J. Thouless, *J. Phys.* **C5**, 807 (1972).
28. B. J. Last and D. J. Thouless, *J. Phys.* **C7**, 699 (1974).
29. D. C. Licciardello and D. J. Thouless, *Phys. Rev. Lett.* **35**, 1475 (1975).
30. D. C. Licciardello and D. J. Thouless, *J. Phys.* **C8**, 4157 (1975), **C11**, 925 (1978).
31. E. N. Economou and M. H. Cohen, *Phys. Rev.* **B5**, 2931 (1972).
32. M. V. Sadovskiĭ, *Zh. Eksp. Teor. Fiz.* **80**, 1135 (1981).
33. A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, *Methods of Quantum Field Theory in Statistical Physics*, Fizmatgiz, Moscow, 1982.
34. V. L. Berezinskiĭ and L. P. Gor'kov, *Zh. Eksp. Teor. Fiz.* **77**, 2498 (1979).
35. M. V. Sadovskiĭ, *Zh. Eksp. Teor. Fiz.* **83**, 1418 (1982).
36. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Chapter 18, Nauka, Moscow, 1974.
37. F. J. Wegner, *Zs. Phys.* **B36**, 209 (1980).
38. D. N. Zubarev, *Nonequilibrium Statistical Thermodynamics*, Chapters 16 and 17, Nauka, Moscow, 1971.
39. R. Kubo, *J. Phys. Soc. Jpn.* **12**, 570 (1957).
40. P. C. Kwok and T. D. Schulz, *J. Phys.* **C2**, 1196 (1969).
41. I. D. Vollhardt and P. Wölfle, *Phys. Rev.* **B22**, 4666 (1980).
42. S. V. Maleev and B. P. Toperverg, *Zh. Eksp. Teor. Fiz.* **69**, 1440 (1975).
43. L. P. Gor'kov, A. I. Larkin, and D. E. Khmel'nitskiĭ, *Pisma Zh. Eksp. Teor. Fiz.* **30**, 248 (1979).
44. J. S. Langer and T. Neal, *Phys. Rev. Lett.* **16**, 984 (1966).
45. G. C. Wick, *Phys. Rev.* **96**, 1124 (1954).
46. R. E. Cutkosky, *Phys. Rev.* **96**, 1135 (1954).
47. N. Nakanishi, *Suppl. Progr. Theor. Phys.* **43**, 1 (1969).
48. V. S. Vladimirov, *Equations of Mathematical Physics*, Nauka, Moscow, 1968.
49. W. Götze, *Sol. State Comm.* **27**, 1393 (1978); *J. Phys.* **C12**, 1279 (1979).
50. W. Götze, *Phil. Mag.* **B43**, 219 (1981).
51. W. Götze, in: *Modern Problems in Solid State Physics*, Vol. I—*Two-Dimensional Electron Systems*, Ed. by Yu. E. Lozovik and A. A. Maradudin, North-Holland Publishing Company.
52. D. Vollhardt and P. Wölfle, *Phys. Rev. Lett.* **45**, 842 (1980).
53. P. Wölfle and D. Vollhardt, in: Ref. 10, p. 26; *Phys. Rev. Lett.* **48**, 699 (1982).
54. A. V. Myasnikov and M. V. Sadovskiĭ, *Fiz. Tverd. Tela* **24**, 3569 (1982).
55. A. L. Efros, *Zh. Eksp. Teor. Fiz.* **59**, 880 (1970).

56. S. L. Ginzburg, *Zh. Eksp. Teor. Fiz.* **63**, 2264 (1972).
57. M. V. Sadovskiĭ, *Zh. Eksp. Teor. Fiz.* **66**, 1720 (1974).
58. B. Shapiro and E. Abrahams, *Phys. Rev.* **B24**, 4025 (1981).
59. D. Pines and P. Nozieres, *The Theory of Quantum Liquids*, W. A. Benjamin, New York, 1966.
60. B. Sharpiro, *Phys. Rev.* **B25**, 4266 (1982).
61. D. J. Thouless, *J. Phys.* **C9**, L603 (1976).
62. W. Götze, in: *Recent Developments in Condensed Matter Physics*, Vol. I, p. 133, ed. by J. T. Devreese, Plenum Press, New York, 1981.
63. P. Prelovšek, *Phys. Rev.* **B23**, 1304 (1981).
64. E. A. Kotov and M. V. Sadovskiĭ, *Zs. Phys.* **B51**, 17 (1983).
65. B. Kramer, A. MacKinnon, and D. Weaire, *Phys. Rev.* **B23**, 6357 (1981).
66. J. Pichard and G. Sarma, *J. Phys.* **C14**, L127, L617 (1981).
67. A. MacKinnon and B. Kramer, *Phys. Rev. Lett.* **47**, 1546 (1981).
68. D. C. Licciardello and E. N. Economou, *Phys. Rev.* **B11**, 3697 (1975).
69. E. Domany and S. Sarker, *Phys. Rev.* **B23**, 6018 (1981).
70. G. Toulouse and P. Pfeuty, *Comptes Rendus* **B280**, 33 (1975).
71. D. J. Thouless, *J. Phys.* **C8**, 1803 (1975).
72. M. V. Sadovskiĭ, *Fiz. Tverd. Tela* **19**, 2334 (1977).
73. S. F. Edwards, M. B. Green, and G. Srinivasan, *Phil. Mag.* **35**, 1421 (1977).
74. A. Nitzan, K. F. Freed, and M. H. Cohen, *Phys. Rev.* **B15**, 4476 (1977).
75. K. F. Freed, *Phys. Rev.* **B5**, 4802 (1972).
76. P. G. de Gennes, *Phys. Lett.* **38A**, 339 (1972).
77. J. des Cloizeaux, *Phys. Rev.* **A10**, 1665 (1976).
78. D. J. Amit, *J. Phys.* **C7**, 3369 (1974).
79. E. Brezin and G. Parisi, *J. Phys.* **C13**, L307 (1980).
80. I. Ya. Pomeranchuk, V. V. Sudakov and E. A. Ter-Martirosyan, *Phys. Rev.* **103**, 784 (1956).
81. A. M. Polyakov, *Zh. Eksp. Teor. Fiz.* **57**, 271 (1969).
82. A. I. Larkin and D. E. Khmel'nitskiĭ, *Zh. Eksp. Teor. Fiz.* **56**, 2087 (1969).
83. N. N. Bogolyubov and D. V. Shirkov, *Introduction to Quantum Field Theory*, Nauka, Moscow, 1976.
84. A. A. Vladimirov and D. V. Shirkov, *Usp. Fiz. Nauk* **42**, 673 (1979).
85. I. F. Lyuksyutov, V. L. Pokrovskiĭ, and D. E. Khmel'nitskiĭ, *Zh. Eksp. Teor. Fiz.* **69**, 1817 (1975).
86. J. Zittartz and J. S. Langer, *Phys. Rev.* **148**, 741 (1966).
87. B. I. Halperin and M. Lax, *Phys. Rev.* **148**, 722 (1966).
88. B. I. Halperin and M. Lax, *Phys. Rev.* **153**, 802 (1967).
89. I. M. Lifshitz, *Zh. Eksp. Theor. Fiz.* **53**, 743 (1967).
90. J. S. Langer, *Ann. Phys.* **41**, 108 (1967).
91. J. L. Cardy, *J. Phys.* **C11**, L321 (1978).
92. M. V. Sadovskiĭ, *Fiz. Tverd. Tela* **21**, 743 (1979).
93. J. Collins and D. Soper, *Ann. Phys.* **112**, 209 (1978).
94. D. I. Kazakov and D. V. Shirkov, *Fortschr. Physik* **28**, 465 (1980).
95. J. Zinn-Justin, *Phys. Reports* **70**, 109 (1981).
96. F. J. Dyson, *Phys. Rev.* **85**, 631 (1952).
97. L. N. Lipatov, *Zh. Eksp. Teor. Fiz.* **72**, 411 (1977).
98. C. M. Bender and T. Wu, *Phys. Rev.* **D7**, 1620 (1973).
99. E. B. Bogomolny, *Phys. Lett.* **67B**, 193 (1977).
100. B. D. Dorfel, D. I. Kazakov, and D. V. Shirkov, *Preprint JINR, E2-10720*, 1977.

101. S. Coleman, *Phys. Rev.* **D15**, 2929 (1977).
102. C. G. Callan and S. Coleman, *Phys. Rev.* **D16**, 1762 (1977).
103. S. Coleman, V. Glaser, and A. Martin, *Comm. Math. Phys.* **58**, 211 (1978).
104. V. G. Makhankov, *Phys. Lett.* **61A**, 431 (1977).
105. R. Finkelstein, R. Le Levier, and M. Ruderman, *Phys. Rev.* **83**, 326 (1951).
106. A. B. Harris and T. C. Lubensky, *Phys. Rev.* **B23**, 2640 (1981).
107. E. Brezin and G. Parisi, *J. Stat. Phys.* **19**, 269 (1978).
108. B. I. Halperin, *Phys. Rev.* **139**, A104 (1965).
109. L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Part I, Chapt. XIV, Nauka, Moscow, 1976.
110. G. Parisi, *J. Phys.* **A14**, 735 (1981).
111. A. Houghton and L. Schäfer, *J. Phys.* **A12**, 1309 (1979).
112. A. Houghton, L. Schäfer, and F. J. Wegner, *Phys. Rev.* **B22**, 3598 (1980).
113. J. M. Cornwall, R. Jackiw, and E. Tomboulis, *Phys. Rev.* **D10**, 2428 (1974).
114. T. D. Lee and C. N. Yang, *Phys. Rev.* **117**, 22 (1960).
115. J. M. Luttinger and J. C. Ward, *Phys. Rev.* **118**, 1417 (1960).
116. P. Martin and D. DeDominicis, *J. Math. Phys.* **5**, 14 (1964).
117. G. Morandi, *J. Phys.* **A2**, 487 (1969).
118. F. J. Wegner, *Zs. Phys.* **B35**, 207 (1979).
119. L. Schäfer and F. J. Wegner, *Zs. Phys.* **B38**, 113 (1980).
120. A. Houghton, A. Jevicky, R. D. Kenway, and A. M. M. Pruisken, *Phys. Rev. Lett.* **45**, 394 (1980).
121. K. B. Efetov, A. I. Larkin, and D. E. Khmel'nitskiĭ, *Zh. Eksp. Teor. Fiz.* **79**, 1120 (1980).
122. A. J. McKane and M. Stone, *Ann. Phys.* **131**, 36 (1981).
123. S. Hikami, *Phys. Rev.* **B24**, 2671 (1981).
124. A. M. M. Pruisken and L. Schäfer, *Phys. Rev. Lett.* **46**, 490 (1981).
125. A. M. M. Pruisken and L. Schäfer, *Nucl. Phys.* **B200(FS4)**, 20 (1982).
126. K. B. Efetov, *Zh. Eksp. Teor. Fiz.* **82**, 872 (1982).
127. A. Aharony and Y. Imry, *J. Phys.* **C10**, L487 (1977).
128. B. L. Al'tshuler and A. G. Aronov, *Zh. Eksp. Teor. Fiz.* **77**, 2028 (1979).
129. B. L. Al'tshuler, A. G. Aronov, and P. A. Lee, *Phys. Rev. Lett.* **44**, 1288 (1980).
130. A. L. Efros and B. I. Shklovskiĭ, *J. Phys.* **C8**, L49 (1975).
131. A. L. Efros, *J. Phys.* **C9**, 2021 (1976).
132. S. D. Baranovskiĭ, A. L. Efros, B. L. Gelmont, and B. I. Shklovskiĭ, *J. Phys.* **C12**, 1023 (1979).
133. J. H. Davies, P. A. Lee, and T. M. Rice, *Phys. Rev. Lett.* **49**, 758 (1982).
134. W. L. McMillan, *Phys. Rev.* **B24**, 2739 (1981).
135. Y. Imry, Y. Gefen, and D. J. Bergmann, *Phys. Rev.* **B26**, 3436 (1982).
136. P. A. Lee, *Phys. Rev.* **B26**, 5882 (1982).
137. A. M. Finkel'shtein, *Zh. Eksp. Teor. Fiz.* **84**, 168 (1983).
138. G. S. Grest and P. A. Lee, *Phys. Rev. Lett.* **50**, 693 (1983).
139. R. Oppermann, *Zs. Phys.* **B49**, 273 (1983).
140. R. Freedman and J. A. Hertz, *Phys. Rev.* **B15**, 2384 (1977).
141. I. L. Fleishman, D. C. Licciardello and P. W. Anderson, *Phys. Rev. Lett.* **40**, 1340 (1978).
142. I. L. Fleishman, D. C. Licciardello, and P. W. Anderson, *Phys. Rev.* **B21**, 2366 (1980).
143. M. I. Katsnel'son and M. V. Sadovskiĭ, *Zh. Eksp. Teor. Fiz.* **87**, No. 8 (1984).
144. M. I. Katsnel'son and M. V. Sadovskiĭ, *Fiz. Tverd. Tela* **25**, 3372 (1983).

145. A. B. Migdal, Theory of Finite Fermi Systems and the Structure of the Nucleus, Chapt. II.4, Nauka, Moscow, 1965.
146. E. Abrahams, P. W. Anderson, P. A. Lee, and T. V. Ramakrishnan, *Phys. Rev.* **B24**, 6783 (1981).
147. N. F. Mott, *Phil. Mag.* **24**, 935 (1971).
148. H. Kamimura, *Phil. Mag.* **42**, 763 (1980).
149. V. L. Berezinskiĭ and L. P. Gor'kov, *Zh. Eksp. Teor. Fiz.* **78**, 813 (1980).
150. P. F. Maldague, *Phys. Rev.* **B23**, 1719 (1981).
151. B. L. Al'tshuler, D. E. Khmel'nitskiĭ, A. I. Larkin, and P. A. Lee, *Phys. Rev.* **B22**, 5142 (1980).
152. D. Yoshioka, Y. Ono, and H. Fukuyama, *J. Phys. Soc. Jpn.* **50**, 3419 (1981).
153. T. M. Rice, in: *Disordered Systems and Localization*, Lecture Notes in Physics, Vol. 149, p. 220, Springer Verlag, Berlin, Heidelberg, 1981.
154. J. Jäckle, *Phil. Mag.* **B46**, 313 (1982).
155. M. Capizzi, G. A. Thomas, F. DeRosa, R. N. Bhatt, and T. M. Rice, *Phys. Rev. Lett.* **44**, 1019 (1980).
156. H. F. Hess, K. DeKonde, T. F. Rosenbaum, and G. A. Thomas, *Phys. Rev.* **B25**, 5578 (1982).
157. L. N. Bulaevskiĭ, M. V. Sadovskiĭ, *J. Low Temp. Phys.* (to be published) 59, 89 (1985)