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SUPERCONDUCTIVITY IN STRONGLY DISORDERED SYSTEMS

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1 INTRODUCTION

The concept of electron localization [1] is basic for the understanding of electron properties of disordered systems [2,3]. In recent years a number of review papers had appeared, extensively discussing this problem [4,5,6,7]. According to this concept introduction of sufficiently strong disorder into a metallic system leads to spatial localization of electronic states near the Fermi level and thus to a transition to dielectric state (Anderson transition). After this transition dc conductivity (at zero temperature, $T = 0$) vanishes, despite the finite value of electronic density of states at the Fermi level (at least in one-electron approximation).

At the same time it is well-known that even the smallest attraction of electrons close to the Fermi level leads to formation of Cooper pairs and the system becomes superconducting at sufficiently low temperatures [8,9]. It is known that the introduction of disorder which does not break the time—reversal invariance (normal, nonmagnetic impurities etc.) does not seriously influence superconding transition temperature T_c and superconductivity in general (Anderson theorem) [10,11,12,13].

Thus a problem appears of the mutual interplay of these two possible electronic transitions in a disordered system which leads to quite different (even opposite) ground states (insulator or superconductor). This problem is very important both from theoretical and experimental points of view. Actually superconducting properties of many compounds depend strongly on structural disorder. In this respect we can mention amorphous systems (metallic glasses) and superconductors disordered by different forms of irradiation by high-energy particles (fast neutrons, electrons, heavy-ions etc.). It appears that in many of these systems superconductivity is realized when the system in normal state is quite close to the metal—insulator transition induced by disorder. In this case many anomalies of superconducting properties appear which cannot be satisfactorily explained within the standard theory of "dirty" superconductors [9,10,11,12,13].

The discovery of high-temperature superconductivity in metallic oxides [14,15] has lead to the entirely new opportunities in the studies of strong disorder effects in superconductors. Very soon it had been established that high—T_c superconductors are quite sensitive to structural disordering which leads to rather fast destruction of superconductivity and metal—insulator transition. However, the high values of initial T_c,

as well as a small size of Cooper pairs and quasi-two dimensional nature of electronic states in these systems are very appropriate for the studies of the mutual interplay of localization and superconductivity [16]. It may be stated with some confidence that in these systems superconductivity can be observed even in the region of localization (Anderson insulator).

This review is mainly concerned with theoretical aspects of localization and superconductivity close to Anderson transition. However, we shall pay some attention to experiments demonstrating the importance of localization phenomena for the correct analysis of superconductivity in strongly disordered systems. Special emphasis will be on the experiments with high—T_c superconductors. We shall limit ourselves with discussing only three-dimensional and quasi-two-dimensional (in case of HTSC) systems, excluding any discussion of purely two-dimensional systems, which are quite special both in respect to localization and superconductivity. In this case we refer a reader to recent review [17] which is specifically concerned with two-dimensional case.

The usual theory of "dirty superconductors" [9,10,11,12,13] is a cornerstone of our understanding of superconducting properties of disordered metals. It is based on the following main statements:

1. As impurity concentration (disorder) grows a transition takes place from the "pure" limit, when the electron mean-free path l is much larger than the superconducting coherence length ξ_0: $l \gg \xi_0 = \hbar v_F/\pi\Delta_0$ to a "dirty" superconductor with $\xi_0 \gg l \gg \hbar/p_F$ (Here v_F, p_F—are Fermi velocity and momentum, Δ_0—is the zero—temperature energy gap). Transition from the free electron motion to diffusive one does not change T_c at all (Anderson's theorem).

2. Superconducting coherence length ξ (at $T = 0$) determining the spatial scale of superconducting order-parameter (the size of a Cooper pair) diminishes with l so that $\xi \approx \sqrt{\xi_0 l}$ in the limit of $\hbar/p_F \ll l \ll \xi_0$.

Theory of "dirty" superconductors is the basis of our understanding of superconducting properties of many disordered alloys. However, the main results of this theory must be modified[18,19,20,21,22] for the mean-free path values l of the order of inverse Fermi momentum $\hbar/p_F$ (i.e. of the order of interatomic distance). In three—dimensional systems the growth of disorder leads to destruction of diffusive motion of electrons and transition from extended to localized states at critical disorder determined by $l_c \approx \hbar/p_F$, i.e. to transition to Anderson insulator. This metal—insulator transition is reflected in a continuous drop to zero of the static metallic conductivity (at $T = 0$) as $l \to l_c$. Transition from diffusion to localization is realized at the conductivity scale of the order of the so-called "minimal metallic conductivity" $\sigma_c \approx (e^2 p_F/\pi^3\hbar^2) \approx (2-5)10^2 Ohm^{-1}cm^{-1}$. The usual theory of "dirty" superconductors does not consider localization effects and is valid for conductivities in the interval $\sigma \gg \sigma_c$.

In our review we shall present an extensive discussion of theoretical problems of the interplay of superconductivity and localization. First of all we shall briefly describe the main principles of modern theory of electron localization and physics of metal—insulator transition in disordered systems, which will be necessary for clear understanding of the main problem under discussion. After that we shall give rather detailed description superconductivity close to the Anderson transition. Finally, we

shall describe the present experimental situation. We shall briefly describe some of the experiments with traditional superconductors, but our main emphasis will be on high—T_c oxides. We shall concentrate on the experiments with high—temperature—superconductors disordered by fast neutron irradiation which we consider one of the best methods to introduce disorder in a controlled fashion.

2 ANDERSON LOCALIZATION AND METAL-INSULATOR TRANSITION IN DISORDERED SYSTEMS

2.1 Basic Concepts of Localization

A number of detailed review papers exist dealing with basic theoretical aspects of Anderson localization [4,5,6,7,24,25.26]. Here we shall remind the main points of this theory and introduce the accepted terminology.

In 1958 Anderson [1] has shown for the first time that the wave function of a quantum particle in a random potential can qualitatively change its nature if randomness becomes large enough. Usually, when disorder is small, the particle (e.g. electron) is scattered randomly and the wave function changes at the scale of the order of mean free path l. However, the wave function remains extended plane—wave—like (Bloch wave—like) through the system. In case of strong enough disorder, the wave function becomes localized, so that its amplitude (envelope) drops exponentially with distance from the center of localization $\mathbf{r}_0$:

$$|\psi(\mathbf{r})| \sim exp(|\mathbf{r} - \mathbf{r}_0|/R_{loc}) \tag{1}$$

where R_{loc} is localization length. The physical meaning of Anderson localization is relatively simple: coherent tunneling of electrons is possible only between energy levels with the same energy (e.g. between equivalent sites in crystalline lattice). However, in case of strong randomness the states with the same energy are too far apart in space for tunneling to be effective.

At small disorder dc conductivity of a metal at $T = 0$ is determined by Drude expression:

$$\sigma_0 = \frac{ne^2}{m}\tau = \frac{ne^2}{p_F}l \tag{2}$$

where τ — is the mean free time (due to elastic disorder scattering), n — is electron density and e — its charge. Usual kinetic theory can be applied if

$$\frac{p_F l}{\hbar} \gg 1 \text{ or } \frac{E_F \tau}{\hbar} \gg 1 \tag{3}$$

which is a condition of weak scattering (disorder). From Eq. (2) and Eq. (3), taking into account $n = p_F^3/(3\pi^2\hbar^3)$, we can estimate the lower limit of conductivity for which Drude approximation is still valid:

$$\sigma_0 = \frac{e^2 p_F}{3\pi^2\hbar^2}\left(\frac{p_F l}{\hbar}\right) \gg \frac{e^2 p_F}{3\pi^2\hbar^2} \tag{4}$$

The conductivity value:

$$\sigma_c \approx \frac{e^2 p_F}{3\pi^2\hbar^2} \tag{5}$$

is usually called the "minimal metallic conductivity" [2,3]. As disorder grows the mean free path diminishes and becomes of the order of lattice spacing a. so that we reach $p_F l/\hbar \sim 1$, and the usual kinetic theory based upon Boltzmann equation becomes inapplicable. This was first noted by Ioffe and Regel [27], who observed that at such disorder the qualitative form of wave function must change, transforming from extended to localized accompanied by metal—insulator transition. From Eq.(5) it is clear that this transition takes place at the conductivity scale of the order of $\sigma_c \sim (2-5)10^2 Ohm^{-1}cm^{-1}$ for typical $\hbar/p_F \sim a \sim (2-3)10^{-8}cm$.

As Fermi energy moves towards the band—edge (or with the growth of disorder) the critical energy E_c (mobility edge) separating extended and localized states crosses the Fermi level. If E_F belongs to the region of localized states the system becomes insulating, conductivity is possible only for $T > 0$ or by exciting the carriers by alternating electric field. The appearance of these hopping mechanisms of conductivity signals Anderson transition[2,3].

One of the main problems is the qualitative behavior of conductivity when the Fermi level E_F crosses the mobility edge E_c (at $T = 0$). While Mott assumed the discontinuous drop of conductivity from σ_c to zero [2,3] modern approach[4,5,6.26] based mainly on the scaling theory to localization [28] demonstrates *continuous* transition. Experiments at low temperatures clearly confirm this type of behavior [6], and σ_c acts as a characteristic conductivity scale close to transition. Static conductivity of a metal at $T = 0$ close to Anderson transition within this approach is written as:

$$\sigma = A\frac{e^2}{\hbar\xi_{loc}} \approx \sigma_c \left|\frac{E_F - E_c}{E_c}\right|^{(d-2)\nu} \tag{6}$$

where A — is a numerical constant, d — is space dimension, and $\sigma_c \approx Ae^2/(\hbar a^{d-2})$. Here we introduced the correlation length of scaling theory diverging at the transition:

$$\xi_{loc} \approx \frac{\hbar}{p_F}\left|\frac{E_F - E_c}{E_c}\right|^{-\nu} \tag{7}$$

Critical exponent ν determines this divergence. In one—electron approximation and in the absence of magnetic scattering $\nu \sim 1$ [6,7,26,29]. In the region of localized states (i.e. for $E_F < E_c$) ξ_{loc} coincides with localization length of electrons R_{loc}. In metallic region ξ_{loc} determines the effective size of a sample at which "Ohmic" behavior appears. i.e. conductivity becomes independent of a sample size [6,30]. "Minimal metallic conductivity" σ_c determines, as we noted, the conductivity scale close to a transition.

In the vicinity of Anderson transition conductivity acquires an important frequency dependence[31]. For $E_F = E_c$ i.e. at the transition we have:

$$\sigma(\omega) \approx \sigma_c(i\omega\tau)^{\frac{d-2}{d}} \tag{8}$$

which is valid also close to the transition (from either side) for frequencies $\omega \gg \omega_c \sim [N(E_F)\xi_{loc}^d]^{-1}$. For $d = 3$ this reduces to the famous [32] Gotze's law $\omega^{1/3}$.

The spatial dimension $d = 2$ is the so called "lower critical dimensionality"[4,5,6,7] For $d = 2$ all electronic states are localized for infinitesimal disorder[28], and there is no Anderson transition.

Quasi—two—dimensional systems are especially interesting, mainly because most of high—T_c oxides demonstrate strongly anisotropic electronic properties. Here we shall make the simplest estimates for such systems on the line of Ioffe—Regel approach. Consider a system made of highly—conducting "planes" where the current carriers are "nearly—free", while the interplane tunneling is possible only due to some small transfer integral $w \ll E_F$ (E_F — is the Fermi energy of two—dimensional gas within the plane). Conductivity within the plane is determined for small disorder as:

$$\sigma_\parallel = e^2 D_\parallel N(E_F) \tag{9}$$

where $D_\parallel = v_F^2\tau/2$, $N(E_F) = m/(\pi a_\perp\hbar^2)$, $a_\perp$ — is interplane spacing, which is noticeably larger than interatomic distance within the plane. Interplane conductivity is given by:

$$\sigma_\perp = e^2 D_\perp N(E_F) \tag{10}$$

where $D_\perp = (wa_\perp)^2\tau/\hbar^2$. The appropriate mean free paths are $l_\parallel = v_F\tau$, $l_\perp = wa_\perp\tau/\hbar$. Ioffe-Regel criterion for a quasi—two—dimensional system can be written as:

$$l_\perp = wa_\perp\tau/\hbar \sim a_\perp \tag{11}$$

which is equivalent to $w\tau/\hbar \sim 1$ — the condition of breaking of coherent tunneling between the planes. Elementary estimate shows that this corresponds to:

$$\sqrt{\sigma_\parallel\sigma_\perp} \sim \frac{e^2}{\sqrt{2}\pi\hbar a} \sim \sigma_c \tag{12}$$

where a — is interatomic distance *within the planes*. In isotropic case this reduces to Eq.(5). For strongly anisotropic system when $\sigma_\parallel \gg \sigma_\perp$ it is clear that Eq.(12) can be satisfied even for $\sigma_\parallel \gg \sigma_c$, because of small values of $\sigma_\perp$. Formally, for $\sigma_\perp \to 0$, critical value of $\sigma_\parallel$ diverges, that reflects on this elementary level the tendency towards complete localization for purely two—dimensional case.

The important property of energy spectrum in the region of localized states is its local discretness. As we noted above, the physical meaning of localization itself leads to a picture of close energy levels being far apart in space, despite the continuous nature of average density of states. Due to exponential decay of the localized wave functions it leads to the absence of tunneling [1]. The energy spacing between levels of electrons localized within the sphere of the radius of the order of $R_{loc}(E)$ can be estimated [2,3] as:

$$\delta_{E_F} \approx [N(E_F)R_{loc}^d]^{-1} \tag{13}$$

As the metallic system moves toward Anderson transition, i.e. as the mean free path drops to interatomic distance and conductivity becomes less than $\sim 10^3 Ohm^{-1}cm^{-1}$ there appear the well known anomalies like the negative temperature coefficient of resistivity [27,33]. These anomalies are apparently closely connected with localization phenomena [6].

Up to now we discussed Anderson transition. neglecting electron interactions. Its importance for the problem of metal—insulator transition in disordered systems was known for a long time [2]. In recent years there was a serious progress in the general approach to a theory of "dirty" metals, based on the analysis of interference of impurity scattering and Coulomb interactions [34,35.36]. Later we shall review its implications for the general picture of Anderson transition. Apparently the continuous nature of metal—insulator transition is not changed though interaction lead to a number of specific effects. e.g. in the behavior of the density of states at the Fermi level, as well as to the growth of magnetic (spin) fluctuations. Here we shall briefly describe the concept of "soft" Coulomb gap appearing below the transition in the region of localized states [37,38,39,40]. Coulomb interaction between localized electrons can be estimated as $e^2/\epsilon R_{loc}$, and it is obviously important if this energy is comparable with the local level spacing $[N(E_F)R_{loc}^3]^{-1}$ (for three—dimensional system). As a result a Coulomb pseudogap appears at the Fermi level with the width:

$$\Delta_c \approx (e^3/\epsilon^{3/2})[N(E_F)]^{1/2} \tag{14}$$

where ϵ is the dielectric constant. We shall see later that close to the Anderson transition $\epsilon \approx 4\pi e^2 N(E_F) R_{loc}^2$ and accordingly:

$$\Delta_c \approx [N(E_F)R_{loc}^3]^{-1} \approx \delta_{E_F} \tag{15}$$

so that in case Coulomb effects are comparable with the effects of disc of energy spectrum in localized phase. At the moment there is no complete theory connecting the localization region with metallic phase within the general approaches of interaction theory.

2.2 Elementary Scaling Theory of Localization

The behavior of electronic system close to the Anderson transition can be described by a scaling theory similar to that used in the theory of critical phenomena [41.42,43]. The main physical idea of this approach is based upon a series of scale transformations from smaller to larger "cells" in coordinate space with appropriate description of a system by transformed parameters of initial Hamiltonian. These transformations are usually called renormalization group. In the theory of critical phenomena this approach is usually motivated by the growth of correlation length of order—parameter fluctuations near the critical point [41]. This is analogous to the growth of localization length on the approach of mobility edge from Anderson insulator.

The accepted scaling approach to localization problem was proposed by Abrahams, Anderson, Licciardello and Ramakrishnan [28]. In this theory localization is described in terms of *conductance* g as a function of a sample size L. For a small disorder ($p_F l/\hbar \gg 1$) the system is in a metallic state and conductivity σ is determined by Eq. (2) and is independent of a sample size if this size is much larger than the mean free path, $L \gg l$. Conductance is determined in this case just by Ohm law and for a d—dimensional hypercube we have:

$$g(L) = \sigma L^{d-2} \tag{16}$$

If electronic states near the Fermi level are localized, conductivity of an infinite system at $T = 0$ is zero and matrix elements for transitions between different electronic states

drop exponentially on distances of the order of R_{loc}. Then it can be expected that for $L \gg R_{loc}$. the effective conductance becomes exponentially small:

$$g(L) \sim exp(-L/R_{loc}) \tag{17}$$

Elementary scaling theory of localization assumes that in general case the conductance of a hypercube of a size L satisfies the simplest differential equation of a renormalization group:

$$\frac{dlng(L)}{dlnL} = \beta_d(g(L)) \tag{18}$$

Most important assumption here is the dependence fo $\beta_d(g)$ only on one variable g (one—parameter scaling). Then the qualitative behavior of β_g can be analyzed in a simplest possible way interpolating between limiting forms given by Eq. (16) and Eq. (17)[28]. From this analysis the following behavior of conductivity of an infinite system follows for metallic phase ($g_0 > g_c$):

$$\sigma \approx A\frac{e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(ln\frac{g_0}{g_c}\right)^{(d-2)\nu} \approx A\frac{e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(\frac{g_0-g_c}{g_c}\right)^{(d-2)\nu} \tag{19}$$

where g_0 is microscopic conductance determining the acyual state of the system and g_c is its critical value given by the zero of β—function in Eq. (18) which defines the metal—insulator transition[28].Here $A = const$ and we have explicitly introduced the conductivity scale of the order of σ_c. (Cf. Eq. (5)).

Let us define now correlation length of localization transition as:

$$\xi_{loc} \sim a\left|\frac{g_0-g_c}{g_c}\right|^{-\nu} \tag{20}$$

For $g_0 < g_c$ this length coincides with localization length R_{loc}. It is easy to see that Eq. (19) can be written as: [47]

$$\sigma \approx Ag_c\frac{e^2}{\hbar\xi_{loc}^{d-2}} \tag{21}$$

It follows that for $g > g_c$ correlation length ξ_{loc} determines behavior of conductivity close to the mobility edge, when this length becomes much larger than interatomic distance and mean free path.

Let us consider three—dimensional case in more details. Integrating Eq. (18) with $\beta_3(g) \approx 1 - g_c/g$ [28] gives $g(L) = (\hbar/e^2)\sigma_L L = (\hbar/e^2)\sigma + g_c$ so that for a finite sample close to the mobility edge ($\xi_{loc} \gg l$) we obtain:

$$\sigma_L = \sigma + \frac{e^2 g_c}{\hbar L} \tag{22}$$

where in correspondence with Eq. (21)

$$\sigma \approx Ag_c\frac{e^2}{\hbar\xi_{loc}} \tag{23}$$

It follows that for $L \gg \xi_{loc} \gg l$ conductivity $\sigma_L \to \sigma$ while for $l \ll L \ll \xi_{loc}$ conductivity σ_L and the appropriate diffusion coefficient, determined by Einstein relation $\sigma = e^2 D N(E_F)$ are equal to:

$$\sigma_L \approx \frac{e^2 g_c}{\hbar L} \tag{24}$$

$$D_L \approx \frac{g_c}{N(E_F)} \frac{1}{\hbar L} \tag{25}$$

where $N(E_F)$ is the electron density of states at the Fermi level. Thus in this latest case conductivity is not Ohmic, diffusion of electrons is "non—classical" [18,6]. From this discussion it is clear that the characteristic length ξ_{loc} in metallic region determines the scale on which conductivity becomes independent of sample size. Close to the mobility edge when $\xi_{loc} \to \infty$ only the samples with growing sizes $L \gg \xi_{loc}$ can be considered as macroscopic. These considerations allow to understand the physical meaning of diverging length ξ_{loc} of scaling theory in metallic region [30]. Close to the mobility ξ_{loc} is considered as the only relevant length in the problem (with an exception of a sample size L) and the scaling hypothesis is equivalent to the assumption of:

$$g(L) = f\left(\frac{L}{\xi_{loc}}\right) \tag{26}$$

where $f(x)$—is some universal (for a given dimensionality d) function. In metallic region for $L \gg \xi_{loc} \gg l$ it is obvious that $f(x) \sim x^{d-2}$ which reproduces Eq. (21).

For finite frequencies ω of an external electric field a new length appears in the system [31]:

$$\mathrm{L}_\omega = \left[\frac{D(\omega)}{\omega}\right]^{1/2} \tag{27}$$

where $D(\omega)$—is the frequency dependent diffusion coefficient. L_ω is a length of electron diffusion during one cycle of an external field. Close to the mobility edge ξ_{loc} is large and for $L_\omega < \xi_{loc}$, L and L_ω becomes the relevant length scale. In general, for finite ω localization transition is smeared, a sharp transition is realized only for $L^{-1} = L_\omega^{-1} = 0$. Thus for the finite frequency case the scaling hypothesis of Eq. (26) can be generalized as: [31]

$$g(L,\omega) = f\left(\frac{L}{\xi_{loc}}, \frac{L_\omega}{\xi_{loc}}\right) \tag{28}$$

where g denotes a real part of conductance. In metallic phase for $L \gg \xi_{loc}$ we have $g \sim L^{d-2}$ so that:

$$\sigma(\omega) = \frac{e^2}{\hbar} L^{2-d} f\left(\frac{L}{\xi_{loc}}, \frac{L_\omega}{\xi_{loc}}\right) \to \frac{e^2}{\hbar} \xi_{loc}^{2-d} f\left(\infty, \frac{L_\omega}{\xi_{loc}}\right) \equiv \frac{e^2}{\hbar \xi_{loc}^{d-2}} F\left(\frac{\xi_{loc}}{L_\omega}\right) \tag{29}$$

For small frequencies, when $L_\omega \gg \xi_{loc}$, the universal function $F(x) \approx Ag_c + Bx^{d-2}$ which reproduces Eq. (21) and the small frequency corrections found earlier in [45].

For $L_\omega \ll \xi_{loc}$ i.e. for high frequencies or close to mobility edge the relevant length is L_ω and frequency dependent part of conductivity is dominating. In particular at the mobility edge itself the length ξ_{loc} drops out and must cancel in Eq. (28) which leads to:

$$\sigma(\omega, E_F = E_c) \sim L_\omega^{2-d} \sim \left[\frac{\omega}{D(\omega)}\right]^{\frac{d-2}{2}} \tag{30}$$

On the other hand, according to Einstein relation we must have $\sigma(\omega) \sim D(\omega)$. Accordingly, from $[\omega/D(\omega)]^{(d-2)/2} \sim D(\omega)$ we get at the mobility edge:

$$\sigma(\omega, E_F = E_c) \sim D(\omega) \sim \omega^{\frac{d-2}{d}} \tag{31}$$

For $d = 3$ this leads [47,32] to $\sigma(\omega) \sim D(\omega) \sim \omega^{1/3}$. The crossover between different types of frequency dependence occurs for $L_\omega \sim \xi_{loc}$ which determines characteristic frequency: [31]

$$\omega_c \sim \frac{1}{\hbar \xi_{loc}^d N(E_F)} \tag{32}$$

The $\omega^{(d-2)/d}$—behavior is realized for $\omega \gg \omega_c$, while for $\omega \ll \omega_c$ we get small corrections of the order of $\sim \omega^{(d-2)/2}$ to Eq. (21).

Finally we must stress that for finite-temperatures there appear *inelastic* scattering processes which destroy the phase correlations of wave functions at distances greater than a characteristic length of the order of $L_\varphi = \sqrt{D\tau_\varphi}$, where D is the diffusion coefficient due to *elastic* scattering processes considered above and τ_φ is the "dephasing" time due to inelastic processes [35]. For $T > 0$ this length L_φ effectively replaces the sample size L in all expressions of scaling theory when $L \gg L_\varphi$, because on distances larger than L_φ all information on the nature of wave functions (e.g. whether they are localized or extended) is smeared out. Taking into account the usual low—temperature dependence like $\tau_\varphi \sim T^{-p}$ (where p is some integer, depending on the mechanism of inelastic scattering) this can lead to a non—trivial temperature dependence of conductivity, in particular to a possibility of a negative temperature coefficient of resistivity of "dirty" metals [30] which are close to localization transition. It is important to stress that similar expressions determine the temperature dependence of conductivity also for the localized phase until $L_\varphi < R_{loc}$. Only for $L_\varphi > R_{loc}$ the localized nature of wave functions starts to signal itself in temperature dependence of conductivity and the transition to exponentially activated hopping behavior takes place, which becomes complete for $T < [N(E_F) R_{loc}^d]^{-1}$.

2.3 Self—Consistent Theory of Localization

2.3.1 Isotropic Systems

It is obvious that qualitative scaling picture of Anderson transition described in the previous section requires microscopic justification. At the same time we need a practical method of explicit calculations for any physical characteristic of electronic system close to the mobility edge. Here we shall briefly describe the main principles of so called self—consistent theory of localization which while leaving aside some points of principle, leads to an effective scheme for analysis of the relevant physical characteristics which will

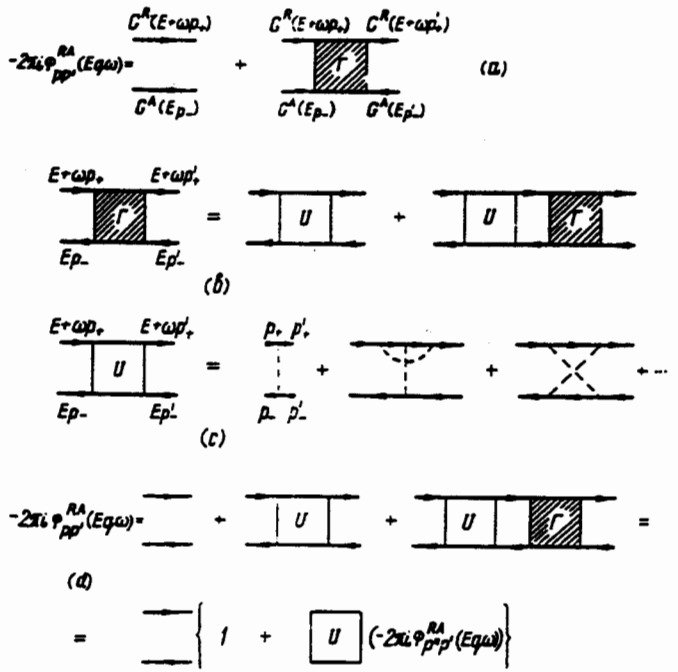

Figure 1: Graphical representation of: (a) — two—electron Green's function $\Phi^{RA}_{pp'}(Eq\omega)$; (b) — equation for full vertex part $\Gamma^{E}_{pp'}(q\omega)$; (c) — typical diagrams for irreducible vertex $U^{E}_{pp'}(q\omega)$ (d) — Bethe—Salpeter equation. Dashed line denotes "interaction" $U_0(p-p') = \rho|V(p-p')|^2$, where ρ — is density of scatterers, $V(p-p')$—is Fourier transform of a single scatterer potential.

be important for us. This approach. first formulated by Gotze [48,32] was later further developed by Vollhardt and Wolfle and other authors [49,50,29,51,52,7].

For the complete description of Anderson transition it is sufficient to know the two—particle Green's function [49]:

$$\Phi^{RA}_{E}(q\omega) = -\frac{1}{2\pi i}\sum_{pp'} < G^{R}(p_+ p'_+ E+\omega) G^{A}(p'_- p_- E) > \qquad (33)$$

where $p_{+-} = p \pm (1/2)q$. in most cases below E just coincides with the Fermi energy E_F. Angular brackets denote averaging over disorder.

Using Bethe—Salpeter equation shown diagrammatically in Fig. 1 and exact Ward identities we can obtain the approximate equation for $\Phi^{RA}_{E}(q\omega)$ [49.29,7]. For small ω and q the solution of this equation has a typical diffusion—pole form:

$$\Phi^{RA}_{E}(q\omega) = -N(E)\frac{1}{\omega + iD_E(q\omega)q^2} \qquad (34)$$

where $N(E)$—is electron density of states at energy E and the *generalized* diffusion coefficient $D_E(q\omega)$ is expressed through the so called relaxation kernel $M_E(q\omega)$:

$$D_E(q\omega) = i\frac{2E}{dm}\frac{1}{M_E(q\omega)} = \frac{v_F^2}{d}\frac{i}{M_E(q\omega)} \qquad (35)$$

where v_F is Fermi velocity of an electron. The retarded density—density response function at small ω and q is given by:

$$\chi^{R}(q\omega) = \omega\Phi^{RA}_{E}(q\omega) + N(E) + O(\omega, q^2) \qquad (36)$$

or from Eq. (34):

$$\chi^{R}(q\omega) = N(E)\frac{iD_E(q\omega)q^2}{\omega + iD_E(q\omega)q^2} \qquad (37)$$

For relaxation kernel $M_E(q\omega)$ (or for generalized diffusion coefficient) a self—consistency equation can be derived, which is actually the main equation of the theory [49,26.29]. If we take for the irreducible kernel in Bethe—Salpeter equation the so called approximation of "maximally crossed diagrams" (Cooperon), the equation for $M(q = 0\omega)$ takes the following form:

$$M_E(\omega) = 2i\gamma\left\{1 + \frac{1}{\pi N(E)}\sum_{|k|<k_0}\frac{i}{\omega + \frac{2E}{dm}\frac{k^2}{M_E(\omega)}}\right\} \qquad (38)$$

or for generalized diffusion coefficient itself:

$$\frac{D_0}{D_E(\omega)} = 1 + \frac{1}{\pi N(E)}\sum_{|k|<k_0}\frac{1}{-i\omega + D_E(\omega)k^2} \qquad (39)$$

where

$$D_0 = \frac{E}{md\gamma} = \frac{1}{d}v_F^2\tau \qquad (40)$$

is the classical (bare) diffusion coefficient determining Drude conductivity Eq. (2). For point scatterers randomly distributed with spatial density ρ (V is scattering amplitude) we have:

$$\gamma = \frac{1}{2\tau} = \pi\rho V^2 N(E_F) \qquad (41)$$

Cut—off in momentum space in Eq. (38) and Eq. (39) is determined by the limit of applicability of diffusion—pole approximation of Eq. (34) or [7]:

$$k_0 \approx Min\{p_F, l^{-1}\} \qquad (42)$$

Close to the mobility edge $p_F \sim l^{-1}$. Note, that from here on we are generally using natural units with Planck constant $\hbar = 1$, however in some of the final expressions we shall write $\hbar$ explicitly.

Conductivity can be expressed as:[49,29]

$$\sigma(\omega) = \frac{ne^2}{m}\frac{i}{\omega + M_E(\omega)} \to e^2 D_E(\omega)N(E) \text{ for } \omega \to 0 \qquad (43)$$

where we have used $n/N(E) = 2E/d$. It is clear that for metallic phase $M_E(\omega \to 0) = i/\tau_E$, where τ_E is generalized mean free time. Far from Anderson transition (for weak disorder) $\tau_E \approx \tau$ from Eq. (41) and Eq. (43) reduces to standard Drude expression.

If the frequency behavior of relaxation kernel leads to the existence of a limit $lim_{\omega \to 0} \omega M_E(\mathbf{q}\omega)$ a singular contribution appears in Eq. (34) for $\omega \to 0$:[7]

$$\Phi_E^{RA}(\mathbf{q}\omega) \approx -\frac{N(E)}{\omega} \frac{1}{1 - \frac{2E}{md} \frac{q^2}{\omega M_E(\mathbf{q}\omega)}} \approx -\frac{N(E)}{\omega} \frac{1}{1 + R_{loc}^2 q^2} \tag{44}$$

where we have defined:

$$R_{loc}^2(E) = -\frac{2E}{md} lim_{\omega \to 0} \frac{1}{\omega M_E(\omega)} \tag{45}$$

According to the general criterion of localization [55,7] this behavior corresponds to the region of localized states.

Convenient formalism to consider general properties of disordered system is based upon exact eigenstate representation for an electron in a random field created by disorder. These eigenstates $\phi_\nu(\mathbf{r})$ are formally defined by the Schroedinger equation:

$$H\phi_\nu(\mathbf{r}) = \varepsilon_\nu \phi_\nu(\mathbf{r}) \tag{46}$$

where H is one—particle Hamiltonian of disordered system under consideration, ε_ν are exact eigenvalues of electron energy in a random potential. Obviously $\phi_\nu(\mathbf{r})$ and ε_ν are dependent on locations of scatterers $\mathbf{R_n}$ for a given realization of random field.

The general criterion for localization is conveniently expressed via the following two—particle spectral density [55]:

$$\ll \rho_E(\mathbf{r}) \rho_{E+\omega}(\mathbf{r}') \gg^F = \frac{1}{N(E)} < \sum_{\nu\nu'} \phi_\nu^*(\mathbf{r}) \phi_{\nu'}(\mathbf{r}) \phi_{\nu'}^*(\mathbf{r}') \phi_\nu(\mathbf{r}') \delta(E - \varepsilon_\nu) \delta(E + \omega - \varepsilon_{\nu'}) > \tag{47}$$

Localization leads leads to the appearance of $\delta(\omega)$—contribution to this spectral density:

$$\ll \rho_E(\mathbf{r}) \rho_{E+\omega}(\mathbf{r}') \gg^F = A_E(\mathbf{r} - \mathbf{r}') \delta(\omega) + \rho_E^F(\mathbf{r} - \mathbf{r}'\omega) \tag{48}$$

or in momentum representation:

$$\ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^F = A_E(\mathbf{q}) \delta(\omega) + \rho_E^F(\mathbf{q}\omega) \tag{49}$$

where the second terms are regular in ω. It is easy to show that

$$A_E(\mathbf{r} - \mathbf{r}') = \frac{1}{N(E)} < \sum_\nu \delta(E - \varepsilon_\nu) |\phi_\nu(\mathbf{r})|^2 |\phi_\nu(\mathbf{r}')|^2 > \tag{50}$$

$$A_E = A_E(\mathbf{r} - \mathbf{r}')|_{\mathbf{r}=\mathbf{r}'} \sim R_{loc}^{-d}$$

$A_F(\mathbf{r} - \mathbf{r}')$ represents the so called inverse participation ratio [24]. Roughly speaking its value at $\mathbf{r} = \mathbf{r}'$ is inversely proportional to the number of atomic orbitals which effectively form quantum state ν.

It is easily shown that the small ω and $\mathbf{q}$ behavior of Gorkov—Berezinskii spectral density can be obtained from[7]:

$$\ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^F = \frac{1}{\pi N(E)} Im \Phi_E^{RA}(\mathbf{q}\omega) \tag{51}$$

Now we immediately obtain from Eq. (44) the singular contribution to Gorkov—Berezinskii spectral density as[7]:

$$A_E(\mathbf{q}) = \frac{1}{1 + R_{loc}^2(E) q^2} \to 1 - R_{loc}^2(E) q^2 \text{ for } q \to 0 \tag{52}$$

Then $R_{loc}(E)$ as defined in Eq. (45) is actually the localization length[55,7]. It is useful to define a characteristic frequency[49]:

$$\omega_0^2(E) = -lim_{\omega \to 0} \omega M_E(\omega) > 0 \tag{53}$$

so that

$$R_{loc}(E) = \sqrt{\frac{2E}{md}} \frac{1}{\omega_0(E)} \tag{54}$$

Thus, the localization transition is signalled by the divergence of relaxation kernel for $\omega \to 0$ [49], so that two characteristic types of it behavior for $\mathbf{q} = 0$ and $\omega \to 0$ appear:

$$M_E(0\omega) \approx \begin{cases} \frac{i}{\tau_E} \text{ for } E \geq E_c \\ \frac{i}{\tau_E} - \frac{\omega_0^2(E)}{\omega} \text{ for } E \leq E_c \end{cases} \tag{55}$$

The frequency $\omega_0(E)$ is in some crude sense analogous to the order parameter in the usual theory of phase transitions. It appears in the localized phase signalling about Anderson transition.

In general case we can get the following explicit expression for Berezinskii—Gorkov spectral density which is valid for small ω and $\mathbf{q}$ [56,7]:

$$\ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^F = \begin{cases} \frac{1}{\pi} \frac{D_E q^2}{\omega^2 + (D_E q^2)^2} \text{ (Metal)} \\ A_E(\mathbf{q}) \delta(\omega) + \frac{1}{\pi} \frac{D_E q^2}{\omega^2 + [\omega_0^2(E)\tau_E + D_E q^2]^2} \text{ (Insulator)} \end{cases} \tag{56}$$

where we have introduced renormalized diffusion coefficient, determined by relaxation time τ_E:

$$D_E = \frac{2E}{dm} \tau_E = \frac{1}{d} v_F^2 \tau_E \tag{57}$$

Substituting Eq. (55) into self—consistency equation Eq. (38) we can obtain equations for τ_E and $\omega_0(E)$ [50,51,7] and thus determine all the relevant characteristics of the system. For $d > 2$ Eq. (38) and Eq. (39) do really describe metal—insulator transition [50,51,7,26]. For $d = 2$ all electronic states are localized [49].

Below we present some of the results of this analysis which will be important for the following. For $2 < d < 4$ a correlation length similar to that of Eq. (7) and Eq. (20) appears:

$$\xi_{loc}(E) \sim \frac{1}{p_F} \left| \frac{E - E_c}{E_c} \right|^{-\nu} \text{ for } E \sim E_c \tag{58}$$

where $\nu = 1/(d-2)$. The position of the mobility edge is determined by a condition:

$$\left.\frac{E}{\gamma}\right|_{E=E_c} = \frac{d}{\pi(d-2)} \tag{59}$$

which follows if we assume the cut—off $k_0 = p_F$ in Eq. (38) and Eq. (39). Static conductivity in metallic phase ($E > E_c$) is given by (Cf. Eq. (21):

$$\sigma = \frac{\sigma_0}{[p_F \xi_{loc}(E)]^{d-2}} \tag{60}$$

where $\sigma_0 = (ne^2/m)\tau$ is usual Drude conductivity. In particular, for $d = 3$:

$$\left.\frac{E}{\gamma}\right|_{E=E_c} = p_F l|_{E=E_c} = \frac{3}{\pi} \tag{61}$$

in complete accordance with Ioffe—Regel criterion . and

$$\sigma = \frac{\sigma_0}{p_F \xi_{loc}(E)} \tag{62}$$

Critical exponent $\nu = 1$. Mean free path which follows from Eq. (61) corresponds to Drude conductivity :

$$\sigma_c = \left.\frac{ne^2}{m}\tau\right|_{E=E_c} = \frac{e^2 p_F}{3\pi^2\hbar^2}\left(\frac{p_F l}{\hbar}\right)\Big|_{E=E_c} = \frac{e^2 p_F}{\pi^3\hbar^2} \tag{63}$$

which is equivalent to elementary estimate of Eq. (5).

Eq. (62) can also be rewritten as [20] :

$$\sigma = \sigma_0\left\{1 - \frac{\sigma_c}{\sigma_0}\right\} = \sigma_0 - \sigma_c \tag{64}$$

where Drude conductivity σ_0 is now the measure of disorder. It is obvious that for small disorder (large mean free path) $\sigma_0 \gg \sigma_c$ and Eq. (64) reduces to $\sigma \approx \sigma_0$. As disorder grows (mean free path drops) conductivity $\sigma \to 0$ for $\sigma_0 \to \sigma_c$.

In dielectric phase ($E < E_c$) we have $\xi_{loc}(E) = R_{loc}(E)$ and finite $\omega_0^2(E)$ from Eq. (53) which tends to zero as $E \to E_c$ from below. This frequency determines dielectric function of insulating phase [7] :

$$\epsilon(\omega \to 0) = 1 + \frac{\omega_p^2}{\omega_0^2(E)} = 1 + \kappa_D^2 R_{loc}^2(E) \sim \left|\frac{E - E_c}{E_c}\right|^{-2\nu} . \tag{65}$$

where $\omega_p^2 = 4\pi ne^2/m$ is the square of plasma frequency, $\kappa_D^2 = 4\pi e^2 N(E)$ is the square of inverse screening length of a metal.

Thus the main results of self—consistent theory of localization coincide with the main predictions of elementary scaling theory of localization. Vollhardt and Wolfle had shown [50.29] that equations of this theory and especially the main differential equation of renormalization group Eq. (18) for conductance may be explicitly derived

from self—consistency equations Eq. (38) and Eq. (39) reformulated for a finite system by introduction of low—momentum cut—off at $k \sim 1/L$, where L is the system size.

The results considered up to now are valid for $\omega \to 0$. Self—consistent theory of localization allows to study the frequency dependence of conductivity (generalized diffusion coefficient) [29]. At finite frequency the main Eq. (39) for the generalized diffusion coefficient for $d = 3$ can be rewritten as [29]:

$$\frac{D_E(\omega)}{D_0} = 1 - \left(\frac{E_c}{E}\right)^{1/2} + \frac{\pi}{2}\left(\frac{E_c}{E}\right)^{1/2}\left\{-\frac{i\omega}{2\gamma}\frac{D_0}{D_E(\omega)}\right\}^{1/2} \tag{66}$$

which can be solved explicitly. With sufficient for our aims accuracy this solution may be written as:

$$D_E(\omega) \approx \begin{cases} D_E & \omega \ll \omega_c \quad E \geq E_c \text{ (Metal)} \\ D_0\left(-\frac{i\omega}{2\gamma}\right)^{1/3} & \omega \gg \omega_c \text{ (Metal and Insulator)} \\ D_E \frac{-i\omega}{-i\omega + \frac{3D_E}{v_F^2}\omega_0^2(E)} & \omega \ll \omega_c \quad E < E_c \text{ (Insulator)} \end{cases} \tag{67}$$

where (Cf. Eq. (32)):

$$\omega_c \sim 2\gamma[p_F \xi_{loc}]^{-d} \sim \frac{1}{N(E)\xi_{loc}^d} \tag{68}$$

Here the renormalized diffusion coefficient:

$$D_E = \frac{D_0}{p_F \xi_{loc}(E)} \tag{69}$$

At the mobility edge itself $\xi_{loc}(E = E_c) = \infty$, so that $\omega_c = 0$ and we get the $\omega^{1/3}$—behavior (Cf. Eq. (31)):

$$D_E(\omega) = D_0\left(-\frac{i\omega}{2\gamma}\right)^{1/3} \tag{70}$$

Note that ω_c is in fact determined by $D_E(\omega_c) \sim D_E \sim D_0(\omega_c/2\gamma)^{1/3}$. The meaning of the limit $\omega \to 0$ used above (Cf. e.g. Eq. (55)) is just that $\omega \ll \omega_c$. In particular, the expression Eq. (56) for Gorkov—Berezinskii spectral density is valid only for $\omega \ll \omega_c$. For $\omega_c \leq \omega \leq 2\gamma$, using Eq. (70) in Eq. (34) we get from Eq. (51):

$$\ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^F = \frac{\sqrt{3}}{2\pi}\frac{\alpha^{2/3}\omega^{1/3}q^2}{\omega^2 + \alpha^{2/3}\omega^{4/3}q^2 + \alpha^{4/3}\omega^{2/3}q^4} \tag{71}$$

where $\alpha = D_0 v_F/2\gamma = D_0 l \sim [N(E)]^{-1}$, where the last estimate is for $l \sim p_F$. Eq. (71) is valid also at the mobility edge itself where $\omega_c = 0$. Obviously the correct estimate can be obtained from Eq. (56) by a simple replacement $D_E \to D_0(\omega/\gamma)^{1/3}$.

In the following analysis we will also need a correlator of *local* densities of states defined as:

$$\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^H = \frac{1}{N(E)} < \sum_{\nu\nu'} |\phi_\nu(\mathbf{r})|^2 |\phi_{\nu'}(\mathbf{r}')|^2 \delta(E - \varepsilon_\nu)\delta(E + \omega - \varepsilon_{\nu'}) > \tag{72}$$

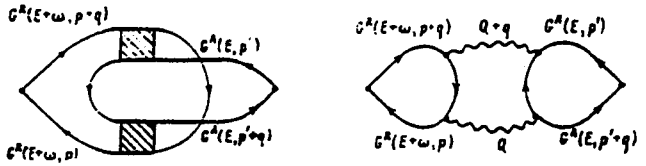

Figure 2: Two equivalent forms of diagram for the correlator of local density of states. Wavy lines denote diffusion propagator, i.e. the sum of ladder diagrams.

This correlator again can be expressed via two—particle Green's function[7]. Far from the Anderson transition (weak disorder) we can estimate the most important contribution to that correlator from the diagram shown in Fig. 2.[58] The same contribution comes from the diagram which differs from that in Fig. 2 by direction of electron lines in one of the loops. Direct calculation gives:

$$\ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^{H} \sim \frac{N(E)}{\gamma^2}(\rho V^2)^2 Re \int d^d\mathbf{Q} \frac{1}{-i\omega + D_0\mathbf{Q}^2} \frac{1}{-i\omega + D_0(\mathbf{Q}+\mathbf{q})^2}$$
$$\sim \frac{1}{N(E)} Re \frac{1}{D_0^{d/2}} \frac{1}{(-i\omega + D_0 q^2)^{2-d/2}} \qquad (73)$$

For the first time similar result for this correlator was found for some special model by Oppermann and Wegner [59]. For $d = 3$ from Eq. (73) we find:

$$\ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^{H} \sim \frac{1}{N(E) D_0^{3/2}} \left\{ \frac{D_0 q^2}{\omega^2 + (D_0 q^2)^2} + [\omega^2 + (D_0 q^2)^2]^{-1/2} \right\}^{1/2} \qquad (74)$$

It is obvious that for the estimates close to the mobility edge we can in the spirit of self—consistent theory of localization replace D_0 in Eq. (73) and Eq. (74) by the generalized diffusion coefficient $D(\omega)$. In particular, for system at the mobility edge ($\omega_c = 0$) $D_0 \to D_0(\omega/\gamma)^{1/3}$ in Eq. (74).

Surely. the self—consistent theory of localization is not free of some difficulties. Apparently the main is an uncontrollable nature of self—consistency procedure itself. In more details these are discussed in Refs.[7,26]. Here we shall concentrate only on some problems relevant for the future discussion. From the definition of generalized diffusion coefficient in Eq. (35) it is clear that it may be a function of both ω and q, i.e. it can also possess spatial dispersion. Self—consistent theory of localization deals only with the limit of $D_E(\mathbf{q} \to 0\omega)$. At present it is not clear whether we can in any way introduce spatial dispersion into equations of self—consistent theory. Using scaling considerations the q—dependence of $D_E(\mathbf{q}\omega \to 0)$ can be estimated as follows.[6,78] We have seen above that for the system of finite size of $L \ll \xi_{loc}$ elementary scaling theory of localization predicts the L—dependent diffusion coefficient $D_E \approx (g_c/N(E))/L^{d-2}$ (Cf. Eq. (25) for $d = 3$). From simple dimensional considerations we can try the replacement $L \to q^{-1}$ and get:

$$D_E(\omega \to 0\mathbf{q}) \approx \begin{cases} D_E \text{ for } q\xi_{loc} \ll 1 \\ \alpha q^{d-2} \text{ for } q\xi_{loc} \gg 1 \end{cases} \qquad (75)$$

where $\alpha \sim g_c/N(E) \sim D_0 l$ and $E \sim E_c$, $l^{-1} \sim p_F$. Obviously an attempt to incorporate such q—dependence into equations of self—consistent theory of localization (like Eq. (38) and Eq. (39)) will radically change its structure. At the same time the L—dependence like $D_E \sim \alpha L^{d-2}$ (for $L \ll \xi_{loc}$) can be directly derived from Eq. (39) as equations of elementary scaling theory are derived from it [50,29,26]. Thus the foundations for the simple replacement $L \to q^{-1}$ like in Eq. (75) are not completely clear.

Finally we should like to stress that self—consistent theory of localization can not be applied "deep" inside localization region. Its derivation is based on a kind of extrapolation of "metallic" expressions and it does not take into account local discreteness of energy spectrum in the region of localized states as discussed in previous section. This is reflected in the form of one—particle Green's function used in self—consistent theory [49,29,26,7]. It does not describe the effects of local level repulsion, though it does not contradict it.[61] Thus self—consistent theory of localization can be applied within localized region only until local energy spacing given by Eq. (13) is much smaller than other relevant energies of the problem under consideration. In fact this always leads to a condition of sufficiently large localization length R_{loc}, i.e. the system must be in some sense close to the mobility edge.

2.3.2 Quasi-Two-Dimensional Systems

Self—consistent theory of localization for quasi—two—dimensional systems was first analyzed by Prigodin and Firsov [62]. The electronic spectrum of a quasi—two—dimensional system can be modelled by nearly—free electrons within highly conducting planes and tight—binding approximation for interplane electron transfer:

$$E(\mathbf{p}) - E_F = v_F(|\mathbf{p}_\parallel| - p_F) - w\varphi(p_\perp) \qquad (76)$$

Here w is the interplane transfer integral and $\varphi(p_\perp) = cosp_\perp a_\perp$, where $-\pi/a\perp \leq p_\perp \leq \pi/a_\perp$. Then the equations of self—consistent theory of localization for anisotropic generalized diffusion coefficient take the following form [62]:

$$D_j(\omega) = D_j^0 - \frac{1}{\pi N(E)} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{D_j(\omega)}{-i\omega + D_\parallel(\omega)q_\parallel^2 + D_\perp(\omega)(1-\varphi(q_\perp))} \qquad (77)$$

where $j = \parallel, \perp$, and $D_\parallel^0 = v_F^2\tau/2$, $D_\perp^0 = (wa_\perp)^2\tau$ are inplane and interplane bare Drude diffusion coefficients, τ is the mean free time due to elastic scattering (disorder). It can be seen that the initial anisotropy of diffusion coefficient does not change as disorder grows up to the Anderson transition and in fact we have only to find one unknown ratio:

$$\alpha(\omega) = \frac{D_j(\omega)}{D_j^0} = \frac{\sigma_j(\omega)}{\sigma_j^0} \qquad (78)$$

which is determined by algebraic equation following from Eq. (77):

$$\alpha(\omega) = 1 - \frac{1}{2\pi E_F \tau} ln \frac{2}{[-i\omega\tau/\alpha(\omega)] + (w\tau)^2 + [(-i\omega\tau/\alpha(\omega))(-i\omega/\alpha(\omega) + 2w^2\tau^2)]^{1/2}} \qquad (79)$$

Due to a quasi—two—dimensional nature of the system there is no complete localization for any degree of disorder which is typical for purely two—dimensional system. However the tendency for a system to become localized at lower disorder than in isotropic case is clearly seen. All states at the Fermi level become localized only for $w < w_c$, where

$$w_c = \sqrt{2}\tau^{-1} exp(-\pi E_F \tau) \tag{80}$$

Thus the condition of localization is actually more stringent than given by the simplest Ioffe—Regel type estimate as in Eq. (11). For fixed w the mobility edge appears at:

$$E_F = E_c = \frac{1}{\pi\tau} ln\left(\frac{\sqrt{2}}{w\tau}\right) \tag{81}$$

Thus in case of strong anisotropy when $w\tau \ll 1$ localization can in principle take place even in case of $E_F \gg \tau^{-1}$, i.e. at relatively weak disorder. These estimates are in qualitative accordance with Eq.(11), which is valid in case of relatively strong disorder $E_F\tau \sim 1$.

In the metallic phase close to the Anderson transition:

$$\sigma_j = \sigma_j^0 \frac{E_F - E_c}{E_c} \tag{82}$$

For $w \to 0$ we have $E_c \to \infty$ which reflects complete localization in two dimensions. We can also define inplane Drude conductivity at $E_F = E_c$ as a kind of a "minimal metallic conductivity" in this case as a characteristic conductivity scale at the transition:

$$\sigma_\parallel^c = e^2 N(E_F) D_\parallel^0(E_F = E_c) = \frac{1}{\pi^2}\frac{e^2}{\hbar a_\perp} ln\left(\frac{\sqrt{2}\hbar}{w\tau}\right) \approx \frac{1}{\pi^2}\frac{e^2}{\hbar a_\perp} ln\left(\frac{E_F}{w}\right) \tag{83}$$

where we have used $N(E_F) = m/(\pi a_\perp \hbar^2)$, m is inplane effective mass, and the last equality is valid for $E_F\tau/\hbar \sim 1$, i.e. for a case of sufficiently strong disorder. For the time being we again use $\hbar$ explicitly. From these estimates it is clear that inplane "minimal conductivity" is logarithmically enhanced in comparison with usual estimates (Cf. Eq. (5)). This logarithmic enhancement grows as the interplane overlap of electronic wave functions diminishes. Accordingly in case of small overlap ($w\tau/\hbar \ll 1$) this conductivity scale may be significantly larger than $(3-5)10^2 Ohm^{-1}cm^{-1}$ which is characteristic for isotropic systems. Thus in quasi—two—dimensional case Anderson transition may take place at relatively high values of inplane conductivity. For a typical estimate in a high—T_c system we can take something like $E_F/w > 10$ so that the value of $\sigma_\parallel^c$ may exceed $10^3 Ohm^{-1}cm^{-1}$. Obviously these estimates are in qualitative accordance with elementary estimates based upon Ioffe—Regel criterion of Eq. (11) and Eq. (12). Similar conclusions can be deduced from the analysis presented in Ref.[63] where it was shown by a different method that in case of anisotropic Anderson model the growth of disorder leads to a significant drop of a critical disorder necessary to localize all states in a conduction band.

Now let us quote some results for the frequency dependence of generalized diffusion coefficient in quasi—two—dimensional case which follow from the solution of Eq.

(79)[62]. We shall limit ourselves only to the results valid close to the mobility edge in metallic phase:

$$\alpha(\omega) \approx \begin{cases} \frac{E_F - E_c}{E_c} & \omega \ll \omega_c \\ (2\pi E_F w\tau^2)^{-2/3}(-i\omega\tau)^{1/3} & \omega_c \ll \omega \ll w^2\tau \\ 1 - \frac{1}{2\pi E_F \tau} ln\left(\frac{1}{-i\omega\tau}\right) & w^2\tau \ll \omega \ll \tau^{-1} \end{cases} \tag{84}$$

where

$$\omega_c \approx [2\pi E_F w\tau^2]^2 \frac{1}{\tau}\left|\frac{E_F - E_c}{E_c}\right|^3 \tag{85}$$

From these expressions we can see the crossover from $\omega^{1/3}$—behavior typical for isotropic three—dimensional systems to logarithmic dependence on frequency which is characteristic for two—dimensional systems.

2.3.3 Self-Consistent Theory of Localization in Magnetic Field

Early version of self-consistent theory of localization as proposed by Vollhardt and Wolfle was essentially based upon time-reversal invariance [49,29]. This property is obviously absent in the presence of an external magnetic field. In this case in addition to Eq. (33) we have to consider two—particle Green's function in particle—particle (Cooper) channel:

$$\Psi_E(\mathbf{q},\omega) = -\frac{1}{2\pi i}\sum_{\mathbf{p}+\mathbf{p}'_-} < G(\mathbf{p}_+,\mathbf{p}'_+,-E+\omega)G(-\mathbf{p}'_-,-\mathbf{p}_-,-E> \tag{86}$$

which for small ω and $\mathbf{q}$ again has diffusion—pole form like that of Eq. (34), but with *different* diffusion coefficient. Appropriate generalization of self-consistent theory of localization was proposed by Yoshioka, Ono and Fukuyama [64]. This theory is based on the following system of coupled equations for relaxation kernels $M_j(\mathbf{q},\omega)$, corresponding to diffusion coefficients in particle—hole and particle—particle channels:

$$M_1 = 2i\gamma\left\{1 - \frac{1}{\pi N(E)}\sum_{n=0}^{N_0}\frac{2}{\pi L_H}\int_0^{\sqrt{q_0^2 - 4m\omega_H(n+1/2)}}\frac{dq_z}{2\pi}\frac{1}{\omega - \frac{D_0}{\tau M_2}[q_z^2 + 4m\omega_H(n+1/2)]}\right\} \tag{87}$$

$$M_2 = 2i\gamma\left\{1 - \frac{1}{\pi N(E)}\sum_{|\mathbf{q}|<q_0}\frac{1}{\omega - D_0 q^2/(\tau M_1)}\right\} \tag{88}$$

Here $\omega_H = eH/mc$ is cyclotron frequency, $L_H = (c/eH)^{1/2}$ is magnetic length and $N_0 = q_0^2/4m\omega_H$. These equations are the basis of self-consistent theory of localization in the absence of time-reversal invariance. These equations were extensively studied in Refs.[64,65.66.67,68]. Alternative formulations of self—consistent theory in magnetic field were given in Refs. [69,70,71,72,73]. All these approaches lead to qualitatively similar results. Here we shall concentrate on formulations given in Ref. [68].

Let us introduce the dimensionless parameter $\lambda = \gamma/\pi E$ as a measure of disorder and generalized diffusion coefficients in diffusion and Cooper channels D_1 and D_2 defined as in Eq. (35) with M replaced by M_1 and M_2 respectively. We shall use dimensionless $d_j = D_j/D_0$ $(j = 1,2)$ in the following.

We are mainly interested in diffusion coefficient in the Cooper channel, which as we shall see defines the upper critical field of a superconductor. Self—consistent equation for it after some algebra can be written as:

$$2mD_2 = \pm (\frac{\omega_c}{E})^{1/3} + (-\frac{i\omega}{E})^{1/2}(2mD_2)^{-1/2} + \frac{3\lambda}{1+3\lambda}\Delta_2 \qquad (89)$$

where + corresponds to metallic, and − to insulating phases, while characteristic frequency

$$\omega_c = \left(\frac{|1-3\lambda|}{(3/2)\pi\lambda}\right)^3 E \qquad (90)$$

can be considered as a measure of disorder and separate regions with different frequency dependencies of diffusion coefficient. In Eq. (89)

$$\Delta_2 = -(2\omega_H/E)^{1/2}\sum_{p=1}^{\infty}\frac{(-1)^p}{p^{1/2}}f(2\pi p\kappa) \qquad (91)$$

where

$$f(y) = \sqrt{2/\pi}\int_0^\infty \frac{cos(t)dt}{\sqrt{t+y}}; \qquad \kappa = \frac{-i\omega/E}{2\omega_H/E}\frac{1}{2mD_2} \qquad (92)$$

This gives:

$$\Delta_2 = \begin{cases} W(2\omega_H/E)^{1/2} & |\kappa| \ll 1 \\ \frac{1}{48}\left((-i\omega/E)\frac{1}{2mD_2}\right)^{-3/2}(2\omega_H/E)^2 & |\kappa| \gg 1 \end{cases} \qquad (93)$$

where $W = -\sum_{p=1}^{\infty}(-1)^p/p^{1/2} \approx 0.603$.

Solutions of Eq. (89) for different limiting cases can be found in Ref. [68]. The usual diffusion coefficient D_1 is given by the same expressions as D_2 with the replacement of the coefficient $3\lambda/(1+3\lambda)$ before the field—dependent correction by $1/(1+3\lambda)$. Here we only quote the results for D_2 in case of $\omega_c \ll (\omega_H/E)^{3/2}$:

$$D_2 = \frac{1}{2m}\left\{\pm(\omega_c/E)^{1/3} + \left[\frac{3\lambda}{1+3\lambda}\right]W(2\omega_H/E)^{1/2}\right\} \approx \frac{1}{4m}W(2\omega_H/E)^{1/2} \qquad \omega \ll \omega_c^* \qquad (94)$$

$$D_2 = \frac{1}{2m}\left\{(-i\omega/E)^{1/3} + \frac{2}{3}\left[\frac{3\lambda}{1+3\lambda}\right]\frac{1}{48}\frac{(2\omega_H/E)^2}{(-i\omega/E)}\right\} \qquad \omega \gg \omega_c^* \qquad (95)$$

where $\omega_c^* = (W/2)^3(2\omega_H/E)^{3/2}E$.

Note that for high frequencies larger than ω_c^* the correction term becomes quadratic in field which differs from usual square root behavior at low frequencies.

It is easy to see that in the absence of the external magnetic field these equations reduce to the usual self-consistency equation as derived by Vollhardt and Wolfle with a single relaxation kernel.

2.4 Interaction Effects and Anderson Transition

The main unsolved problem of the theory of metal—insulator transition in disordered systems is the role of electron—electron interactions. The importance of interactions

for this problem is known for a long time [2]. In recent years the decisive importance of interactions was revealed in the theory of "dirty metals" [34,35,36], as well as in the concept of Coulomb gap at the Fermi level of strongly localized electrons [37,38,39, 40]. We have already briefly discussed Coulomb gap. It appears for strongly localized states. In case of "dirty metals" diffusive nature of electronic transport leads to special interference effects between Coulomb interaction and disorder scattering [34,36]. Most important is an appearance of some kind of a precursor to Coulomb gap already in metallic state. It is connected with simple exchange correction to electron self—energy which leads to the following cusp—like correction to one—particle density of states in case of the screened Coulomb interaction in three—dimensional system [34]:

$$\delta N(E) = \frac{|E-E_F|^{1/2}}{2\sqrt{2}\pi^2 D_0^{3/2}} \qquad (96)$$

where D_0 is the usual Drude diffusion coefficient. In two—dimensional case this correction is logarithmic [36]. General belief is that this cusp somehow transforms into Coulomb gap as system moves from metal to insulator. However, up to now there is no complete solution for this problem.

Early attempt to describe electron—electron interactions in Anderson insulators in a Fermi—liquid like scheme was undertaken in Ref. [74]. Simple generalization of the theory of "dirty metals" [34,35,36] along the lines of self—consistent theory of localization was proposed in Refs.[75,56,7]. However the most general approach to this problem was introduced by McMillan [76] who proposed to describe the metal—insulator transition in a disordered system by a scaling scheme similar in spirit to elementary scaling theory of localization of noninteracting electrons discussed above. He formulated a simple system of coupled differential equations of renormalization group for two effective "charges": dimensionless conductance g and single—particle density of states $N(E)$. Later it was realized that this simple scheme can not be correct because it assumed for conductivity the relation like Eq. (43) with density of states while the correct Einstein relation for interacting system contains electron compressibility $dn/d\mu$ (μ is chemical potential)[77,78,79], which is not renormalized close to the metal—insulator transition as opposed to density of states. The most comprehensive approach to a scaling description of metal—insulator transition in disordered systems was formulated by Finkelstein [79,80,81]. Unfortunately more or less explicit solutions were only obtained neglecting the scattering and interaction processes in Cooper channel which are mainly responsible, as we have seen above, for localization itself. This approach is still under very active discussion [82,83,84,85,86,87,88,89] and demonstrate fundamental importance of interactions. However the problem is still unresolved and most of these works consider only the metallic side of transition.

Below we consider only some qualitative results of this approach, following mainly Refs.[83,84]. Fermi liquid theory survives the introduction of disorder [92], although with some important corrections [34,36], and is actually valid up to metal—insulator transition [79,80,83,84].

In the absence of translation invariance there is no momentum conservation and we have to use some unknown exact eigenstate $\phi_\nu(\mathbf{r})$ representation for electrons in random field to characterize quasiparticles with energies ε_ν (Cf. Ref.[93]). The free energy energy as a functional of quasi—particle distribution function $n_s(\varepsilon_\nu, \mathbf{r})$ (s—spin

variable) is written as in usual Fermi liquid theory:

$$F\{n_s(\varepsilon_\nu,\mathbf{r})\} = \sum_{s,\nu}\int d\mathbf{r} n_s(\varepsilon_\nu\mathbf{r})(\varepsilon_\nu-\mu) + \frac{1}{2}\sum_{ss'}\int d^d\mathbf{r}\delta N_s(\mathbf{r})\delta N_{s'}(\mathbf{r})f_{ss'} \quad (97)$$

where $N_s = \sum_\nu n_s(\varepsilon_\nu\mathbf{r})$ is the total density per spin and $f_{ss'} = f^a + ss'f^a$ is the quasi—particle interaction function. The angular dependence of f—function in dirty case can be neglected. because $n_s(\varepsilon_\nu\mathbf{r})$ is assumed to describe electrons on distances larger than mean free path there only s—wave scattering is important and Fermi—liquid interaction becomes point—like. In an external spin dependent field V_s the quasi—particle distribution function obeys a kinetic equation:

$$\frac{\partial}{\partial t}n_s - D\nabla^2 n_s + \left(\frac{\partial n_s}{\partial\varepsilon}\right)(-D\nabla^2)[V_s + \sum_{s'} f_{ss'}N_{s'}] = 0 \quad (98)$$

where D is quasi—particle diffusion coefficient. Eq. (98) is obtained from usual Fermi—liquid kinetic equation [93] by replacing $v_F\partial/\partial\mathbf{r}$ by $-D\nabla^2$ which reflects a crossover from ballistic to diffusive transport in disordered system. Solving Eq. (98) for density—density and spin—spin response functions one gets:[79.80,82]

$$\chi_\rho(\mathbf{q}\omega) = \frac{dn}{d\mu}\frac{D_\rho q^2}{D_\rho q^2 - i\omega} \quad (99)$$

$$\chi_s(\mathbf{q}\omega) = \frac{\chi D_s q^2}{D_s q^2 - i\omega} \quad (100)$$

where $dn/d\mu = N(E_F)/(1+F_0^s)$, $\chi = N(E_F)\mu_B^2/(1+F_0^a)$ (μ_B is Bohr's magneton) and

$$D_\rho = D(1+F_0^s) \quad (101)$$

$$D_s = D(1+F_0^a) \quad (102)$$

Landau parameters $F_0^{s,a}$ are defined by

$$N(E_F)f^s = F_0^s \qquad N(E_F)f^a = F_0^a \quad (103)$$

Here $N(E_F)$ is quasi-particle density of states at the Fermi level (for both spin directions). If we neglect Fermi—liquid renormalization effects Eq. (99) reduces to Eq. (37). Conductivity is given now by $\sigma = e^2D(dn/d\mu)$.

As system moves towards metal—insulator transition Hubbard—like interaction of electrons close to a given impurity site becomes more and more important. It is known for a long time [2,7] that this interaction leads to the appearance of a band of single—occupied states just below the Fermi level of a system on the dielectric side of Anderson transition. These states actually simulate paramagnetic centers and lead to Curie—like contribution (diverging as temperature $T \to 0$)[2,7]. Thus on the metallic side of transition static magnetic susceptibility χ is expected to diverge since it is infinite (at $T = 0$) on the insulating side. At the same time $dn/d\mu$ remains finite. Therefore $D_s/D_\rho = (dn/d\mu)/\chi$ goes to zero, i.e. *spin diffusion is much slower than charge diffusion* close to metal—insulator transition. This fact was first noted in Ref. [81] where it was assumed that it leads to a possibility of local magnetic appearing in

metallic phase before a transition. This idea was further elaborated in Refs. [87,88,89], where extensive discussion of this magnetic transition was given.

The idea of paramagnetic moments appearing already in metallic phase apparently can much simplify the analysis of metal—insulator transition and allow its description by equations of elementary scaling theory of localization [90,91,36]. In general case electron interactions in diffusion channel can be classified by total spin of an electron and hole j [36]. It can be shown that all interaction corrections with $j = 0$ do not depend on electron—electron coupling constant (charge) and are universal [36]. If paramagnetic scattering is operating in the system it dumps scattering processes in Cooper (localization) channel [94] as well as interaction processes in diffusion channel with $j = 1$ [36]. In this case only interaction processes with $j = 0$ determine corrections to classical (Drude) conductivity. Due to universal nature of these corrections (independence of electronic charge) their structure is actually coincide with that of localization corrections (Cooperon).[90,91] This means that renormalization group has only one effective "charge" — dimensionless conductance g. In this case differential equation for the conductance of a finite system is again given by Eq. (18) with the same asymptotic forms of $\beta_d(g)$. This approach is valid for systems with linear size $L < L_T = \sqrt{\hbar D/T}$. This length L_T replaces in the theory of interacting electrons characteristic length of phase coherence L_φ of noninteracting theory. The appearance of this new length is due to the fact that characteristic time of interaction processes [36] is $\sim \hbar/T$.

As in noninteracting case for $d = 3$ Eq. (18) again possess unstable fixed point responsible for the existence of mobility edge and absence of minimal metallic conductivity at the metal—insulator transition. However, in this case there are no special reasons to believe that the critical exponent ν of localization correlation length ξ_{loc} will coincide with its value for noninteracting theory. At finite temperatures as in usual scaling picture conductivity for $d = 3$ is given by:[90,91,36]

$$\sigma \approx \frac{e^2}{\hbar\xi_{loc}} f\left(\frac{\xi_{loc}}{L_T}\right) \quad (104)$$

As system approaches insulating phase $\xi_{loc} \to \infty$. For $\xi_{loc} \ll L_T$ we have $f(\xi_{loc}/L_T) = A + B(\xi_{loc}/L_T)$, where A and B are some numerical constants. Thus in this region conductivity corrections are proportional to $\sqrt{T}$ [34]. In case of $\xi_{loc} \gg L_T$, i.e. very close to transition:

$$\sigma \approx C\frac{e^2}{\hbar L_T} = C\frac{e^2}{\hbar}\sqrt{T/D\hbar} \quad (105)$$

where again $C \sim 1$. Using Einstein relation [77] $\sigma = e^2D(dn/d\mu)$ we immediately obtain:

$$D = \frac{C^{2/3}}{\hbar}T^{1/3}\left(\frac{dn}{d\mu}\right)^{-2/3} \quad (106)$$

and

$$\sigma = C^{2/3}\frac{e^2}{\hbar}\left(T\frac{dn}{d\mu}\right)^{1/3} \quad (107)$$

which is valid for $L_T < \xi_{loc}$, where $L_T = [C/(T dn/d\mu)]^{1/3}$.

In case of a system in alternating electric field with frequency $\omega \gg T/\hbar$ the relevant length becomes $L_\omega = [D/\omega]^{1/2}$ as in Eq. (27): Accordingly for $L_\omega \ll \xi_{loc}$ instead of Eq. (107) we get:

$$\sigma(\omega) \approx \frac{e^2}{\hbar}\left(\omega\frac{dn}{d\mu}\right)^{1/3} \tag{108}$$

which is analogous to Eq. (31) and Eq. (70).

The metal—insulator transition can be viewed as a gradual breakdown of the Fermi—Liquid state [84]. As we approach the transition different Fermi—liquid parameters, such as D, $N(E_F)$, χ etc. change continuously and at a critical point some of these may either diverge or go to zero. This behavior is related to the divergence of correlation length ξ_{loc} characterized by a critical exponent ν. On the insulating side of the transition this length can be also interpreted as the scale inside which a Fermi liquid description of the system still holds.

For high—T_c superconductors problems of interplay of localization and interactions become especially important because of unusual nature of normal state of these systems. In the absence of accepted theory of this normal state we shall limit ourselves only to few remarks on one specific model. The so called "marginal" Fermi—liquid theory [95,96] is a promising semi—phenomenological description of both normal and superconducting properties of these systems. We shall see that localization effects are apparently greatly enhanced in this case [97].

Basically the idea of "marginal" Fermi—liquid is expressed by the following form of one—particle Green's function [95]:

$$G(E\mathbf{p}) = \frac{Z_\mathbf{p}}{\varepsilon - \xi_\mathbf{p} - i\gamma_\mathbf{p}} + G_{incoh} \tag{109}$$

where $\xi_\mathbf{p}$ is renormalized quasi—particle energy, $\gamma_\mathbf{p} \approx Max[\varepsilon, T]$ is anomalous (linear) decay—rate for these quasiparticles which is quite different from quadratic in ε or T decay—rate of the usual Fermi—liquid theory [93]. The concept of "marginality" arises due to peculiar behavior of quasiparticle residue:

$$Z_\mathbf{p}^{-1} \approx ln\frac{\tilde{\omega}_c}{|\xi_\mathbf{p}|} \approx ln\frac{\tilde{\omega}_c}{|\varepsilon|} \tag{110}$$

where $\tilde{\omega}_c$ is characteristic frequency scale of some kind of electronic excitations, which is the phenomenological parameter of the theory. From Eq. (110) it is clear that quasiparticle contribution to Green's function Eq. (109) vanishes precisely at the Fermi level, while exists close to it though with logarithmically reduced weight. Note that in the case of usual Fermi—liquid $Z_\mathbf{p} \approx 1$[93].

For disordered system we can estimate the impurity contribution to the scattering rate of quasi—particles as [97]:

$$\gamma = 2\rho V^2 Z_\mathbf{p} Im \sum_\mathbf{p} \Lambda^2(\mathbf{p}+\mathbf{q},\mathbf{p}) G(\mathbf{p}+\mathbf{q}\varepsilon) \approx 2\pi\rho V^2 Z^2 \Lambda^2(\mathbf{q}\to 0) N(E_F) \approx Z\Lambda^2\gamma_0 \tag{111}$$

where Λ is the appropriate vertex—part renormalized by Fermi—liquid effects, ρ again is impurity concentration, V is impurity potential and $N(E_F) = Z^{-1}N_0(E_F)$ is the

renormalized density of states in Fermi—liquid. Here $N_0(E_F)$ is density of states for noninteracting electrons at the Fermi level, γ_0 is scattering rate for noninteracting case. To get the last relation in Eq. (111) a weak dependence of vertices and self—energy on momentum was assumed. Now we can use the Ward identity for $\Lambda(\mathbf{q}\to 0\omega = 0)$ vertex of disordered Fermi—liquid theory [92,83,84]:

$$\Lambda(\mathbf{q}\to 0\omega = 0) = (1+F_0^s)^{-1}Z^{-1} \tag{112}$$

where F_0^s is Landau parameter introduced above. As a result we can easily get a simple relation between the mean free paths of interacting and noninteracting quasiparticles [97]:

$$l = (p_F/m^*)\gamma^{-1} = (p_F/m)\gamma_0^{-1}/\Lambda^2(\mathbf{q}\to 0) = l_0(1+F_0^s)^2Z^2 \tag{113}$$

Here $m^* = Z^{-1}m$ is the effective mass of quasiparticle. Assuming $F_0^s \approx const < 1$ and using Eq. (110) we get at $T = 0$:

$$l = l_0/\left[ln\frac{\tilde{\omega}_c}{|\varepsilon|}\right]^2 \tag{114}$$

Then from usual Ioffe—Regel criterion for localization $p_F l \approx 1$ we obtain that all quasiparticle state within the region of the order of

$$|\varepsilon_c| \approx \tilde{\omega}_c exp(-\sqrt{p_F l}) \tag{115}$$

around the Fermi—level in high—T_c oxides are localized even for the case of weak impurity scattering $p_F l \gg 1$. For realistic estimates of $\tilde{\omega}_c \approx 0.1 - 0.2eV$ [95] and $p_F l < 5$ the width of this localized band may easily be of the order of hundreds of degrees K, while for $p_F l \approx 10$ and $\tilde{\omega}_c \approx 1000K$ we get $|\varepsilon_c| \approx 40K$. Obviously this band grows with disorder as the mean free path l_0 drops. We can safely neglect this localization for $T \gg |\varepsilon_c|$, but for low enough temperatures localization effects become important and all states are localized in the ground state. Unfortunately these ideas are still at this elementary level and we may quote only one paper attempting to put them on more sound basis of scaling theory of metal—insulator transition of interacting electrons [98].

3 SUPERCONDUCTIVITY AND LOCALIZATION: STATISTICAL MEAN—FIELD APPROACH

3.1 BCS Model and Anderson Theorem

We shall start our analysis of superconductivity in strongly disordered systems within the framework of simple BCS—model [8,9] which assumes the existence of some kind of effective electron—electron attraction within energy region of the order of $2 < \omega >$ around the Fermi level. In usual superconductors $< \omega > \sim \omega_D$, where ω_D is Debye frequency, because pairing is determined by electron—phonon mechanism, however we shall use some effective $< \omega >$ as an average frequency of some kind of Bose—like excitations responsible for pairing, e.g. in high—T_c superconductors. At the moment

we shall not discuss microscopic nature of this attraction which in general case is determined by the balance of attraction due to Boson—exchange and Coulomb repulsion. Here we just assume (as always is done in simple BCS—approach) that this effective attraction is described by some interaction constant g, which is considered just as a parameter. More detailed microscopic approach will be given in later sections.

Nontrivial results concerning superconductivity in disordered systems were obtained very soon since the discovery of BCS—theory [10,11,12,13]. The concept of "dirty" superconductor described the experimentally very important case of the mean free path l short in comparison with superconducting coherence length $\xi_0 \sim \hbar v_F/T_c$, i.e. the case when:

$$\xi_0 \gg l \gg \hbar/p_F \tag{116}$$

Already in this case of not so strongly disordered (in the sense of closeness to metal—insulator transition) system Cooper pairing takes place not between electrons with opposite momenta and spins as in regular case, but between time—reversed exact eigenstates of electrons in disordered system [13.9]:

$$(\mathbf{p}\uparrow, -\mathbf{p}\downarrow) \Longrightarrow (\phi_\nu(\mathbf{r})\uparrow, \phi_\nu^*(\mathbf{r})\downarrow) \tag{117}$$

In the following we consider only singlet pairing. The underlying physics is simple: in disordered systems such as e.g. an alloy the electron momentum becomes badly determined due to the lack of translational invariance. However, in random potential field we can always define exact eigenstates $\phi_\nu(\mathbf{r})$, which are just solutions of Schroedinger equation in this random field (for a given configuration of this field). We don't need to know the explicit form of these eigenstates at all, the pairing partner of $\phi_\nu(\mathbf{r})$ is being given by time—reversed $\phi_\nu^*(\mathbf{r})$. This leads to a relative stability of a superconducting state with respect to disordering in the absence of scattering mechanisms which break the time—reversal invariance such as e.g. magnetic impurities.

If we consider temperatures close to superconducting transition temperature T_c, when $\Delta(\mathbf{r})$ is small, and the linearized gap equation determining T_c takes the form:

$$\Delta(\mathbf{r}) = gT \int d\mathbf{r}' \sum_{\varepsilon_n} K(\mathbf{r}\mathbf{r}'\varepsilon_n)\Delta(\mathbf{r}') \tag{118}$$

where the kernel:

$$K(\mathbf{r}\mathbf{r}'\varepsilon_n) = G_\uparrow(\mathbf{r}\mathbf{r}'\varepsilon_n)G_\downarrow^*(\mathbf{r}'\mathbf{r}\varepsilon_n) \tag{119}$$

is formed by exact one—electron Green's functions of a normal metal. We can use now an exact eigenstate representation for an electron in a random field of a disordered system to write:

$$G_\uparrow(\mathbf{r}\mathbf{r}'\varepsilon_n) = \sum_\nu \frac{\phi_{\nu\uparrow}(\mathbf{r})\phi_{\nu\uparrow}^*(\mathbf{r}')}{i\varepsilon_n - \varepsilon_\nu} \tag{120}$$

where ε_ν are exact energy levels of an electron in disordered system. Then

$$K(\mathbf{r}\mathbf{r}'\varepsilon_n) = Tg \sum_{\mu\nu} \frac{\phi_{\nu\uparrow}(\mathbf{r})\phi_{\nu\uparrow}^*(\mathbf{r}')\phi_{\mu\downarrow}^*(\mathbf{r}')\phi_{\mu\downarrow}(\mathbf{r})}{(i\varepsilon_n - \epsilon_\nu)(-i\varepsilon_n + \epsilon_\mu)} \tag{121}$$

In the following for brevity we shall drop spin variables always assuming singlet pairing. In case of a system with time—reversal invariance (i.e. in the absence of an external

magnetic field. magnetic impurities etc.) Eq. (121) can be rewritten as:

$$K(\mathbf{r}\mathbf{r}'\varepsilon_n) = G(\mathbf{r}\mathbf{r}'\varepsilon_n)G(\mathbf{r}'\mathbf{r} - \varepsilon_n) = \sum_{\mu\nu} \frac{\phi_\nu(\mathbf{r})\phi_\nu^*(\mathbf{r}')\phi_\mu(\mathbf{r}')\phi_\mu^*(\mathbf{r})}{(i\varepsilon_n - \varepsilon_\nu)(-i\varepsilon_n - \varepsilon_\mu)} \tag{122}$$

Averaging over disorder we get:

$$< \Delta(\mathbf{r}) >= gT \int d\mathbf{r} \sum_{\varepsilon_n} < K(\mathbf{r}\mathbf{r}'\varepsilon_n)\Delta(\mathbf{r}') > \tag{123}$$

Practically in all papers on the superconductivity in disordered systems it is assumed that we can make simplest decoupling in Eq. (123) to get the following linearized equation for the average order—parameter:

$$< \Delta(\mathbf{r}) >= gT \int d\mathbf{r}' \sum_{\varepsilon_n} K(\mathbf{r} - \mathbf{r}'\varepsilon_n) < \Delta(\mathbf{r}') > \tag{124}$$

where the averaged kernel in case of time—invariance is given b;

$$\begin{aligned} K(\mathbf{r} - \mathbf{r}'\varepsilon_n) = K^*(\mathbf{r} - \mathbf{r}'\varepsilon_n) =< K(\mathbf{r}\mathbf{r}'\varepsilon_n) >= \\ =< \sum_{\mu\nu} \frac{\phi_\nu(\mathbf{r})\phi_\mu^*(\mathbf{r})\phi_\mu(\mathbf{r}')\phi_\nu^*(\mathbf{r}')}{(i\varepsilon_n - \varepsilon_\nu)(-i\varepsilon_n - \varepsilon_\mu)} >= \\ = \int_{-\infty}^{\infty} dE N(E) \int_{-\infty}^{\infty} d\omega \frac{\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F}{(i\varepsilon_n + E)(E + \omega - i\varepsilon_n)} \end{aligned} \tag{125}$$

where we have again introduced Gorkov—Berezinskii spectral density [55] defined in Eq. (47). Here $N(E)$ is an exact electron density of states per *one spin direction* as it always appears in superconductivity theory (above, while discussing localization we always used density of states for both spin directions).

Usually the decoupling procedure used in Eq. (123) to reduce it to Eq. (124) is justified by the assumption that the averaging of $\Delta(\mathbf{r})$ and of Green's functions in Eq. (123) forming the kernel can be performed independently because of essentially different spatial scales [12]: $\Delta(\mathbf{r})$ changes at a scale of the order of coherence length (Cooper pair size) ξ, while $G(\mathbf{r}\mathbf{r}'\varepsilon_n)$ are oscillating on the scale of interatomic distance $a \sim \hbar/p_F$, and we always have $\xi \gg a$. Actually it is clear that this decoupling is valid only if the order—parameter is *self—averaging* (i.e. in fact nonrandom) quantity: $\Delta(\mathbf{r}) =< \Delta(\mathbf{r}) >$, $< \Delta^2(\mathbf{r}) >=< \Delta(\mathbf{r}) >^2$. Below we shall see that for a system close to mobility edge the property of self—averageness of $\Delta(\mathbf{r})$ is absent and situation is actually highly nontrivial. In this case the so called *statistical* fluctuations [58] leading to inequality of $< \Delta^2(\mathbf{r}) >$ and $< \Delta(\mathbf{r}) >^2$ become quite important. However, we shall start with what we call statistical mean—field approach which completely neglects these fluctuations and allows the simple analysis using Eq. (124), as a necessary first step to understand superconductivity in strongly disordered systems, which will allow to find most of the important deviations from the usual theory of "dirty" superconductors. The role of statistical fluctuations will be analyzed later.

If we look for the solution of Eq. (124) $\Delta(\mathbf{r}) = const$ (homogeneous gap), we immediately obtain the following equation for transition temperature T_c:

$$\begin{aligned} 1 = gT_c \int d\mathbf{r} \sum_{\varepsilon_n} K(\mathbf{r} - \mathbf{r}'\varepsilon_n) = \\ = gT_c \int d\mathbf{r} \int_{-\infty}^{\infty} dE N(E) \int_{-\infty}^{\infty} d\omega \frac{\ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F}{(E + i\varepsilon_n)(E + \omega - i\varepsilon_n)} \end{aligned} \tag{126}$$

Using the general sum—rule [55]:

$$\int dr \ll \rho_E(r\rho_{E+\omega}(r') \gg^F = \delta(\omega) \tag{127}$$

we immediately reduce Eq. (127) to a standard BCS form:

$$1 = gT_c \int_{-\infty}^{\infty} dEN(E) \sum_{\varepsilon_n} \frac{1}{E^2 + \varepsilon_n^2} = g \int_0^{<\omega>} dEN(E) \frac{1}{E} th \frac{E}{2T_c} \tag{128}$$

where we introduced the usual cut—off at $\varepsilon_n \sim 2 < \omega >$. Note that $N(E)$ here is an exact one-particle density of states (per one spin direction) in a normal state of a disordered system. From Eq. (128) we get the usual result:

$$T_c = \frac{2\gamma}{\pi} < \omega > exp\left(-\frac{1}{\lambda_p}\right) \tag{129}$$

where $\lambda_p = gN(E_F)$ is dimensionless pairing constant, $ln\gamma = C = 0.577...$ is Euler constant. This is the notorious Anderson theorem: in the absence of scattering processes breaking time—reversal invariance disorder influence T_c only through the possible changes of the density of states $N(E_F)$ under disordering (which are usually relatively small).

Due to the sum—rule of Eq. (127) all singularities of Berezinskii—Gorkov spectral density reflecting possible localization transition do not appear in equation determining T_c: there is no explicit contribution from $\delta(\omega)$ term of Eq. (48) and Eq. (128) has the same form both in metallic and localized phases (Cf. Ref.[100]).

The only limitation here which appears on the physical grounds is connected with the local discreteness of electronic spectrum in localized phase discussed above. It is clear that Cooper pairing is possible in localized phase only between electrons with centers of localization within the distance of the order of $\sim R_{loc}(E)$, because only in that case their wave functions overlap [19,20]. However, these states are splitted in energy by δ_E defined in Eq. (13). Obviously, we have to demand that superconducting gap Δ (at $T = 0$, $\Delta \sim T_c$) be much larger than this δ_E:

$$\Delta \sim T_c \gg \delta_E \sim \frac{1}{N(E)R_{loc}^3(E)} \tag{130}$$

i.e on the energy interval of the order of $\Delta \sim T_c$ there must be many discrete levels, with centers of localization within distance $\sim R_{loc}(E)$ from each other. In this case the problem of Cooper pairs formation within $\sim R_{loc}(E)$ is qualitatively the same as in metallic state, e.g. we can replace summation over discrete levels ε_ν by integration. Analogous problem was considered previously in case of Cooper pairing of nucleons in finite nuclei [93] and also of Cooper pairing of electrons in small metallic particles (granular metals) [101,102]. For strongly anisotropic high—T_c systems we must similarly have [16]:

$$\Delta \sim T_c \gg [N(E)R_{loc}^a R_{loc}^b R_{loc}^c]^{-1} \tag{131}$$

where we have introduced the appropriate values of localization lengths along the axes of an orthorhombic lattice.

Obviously Eq. (130) is equivalent to a condition of large enough localization length:

$$R_{loc}(E) \gg [N(E)\Delta]^{-1/3} \sim (\xi_0/p_F^2)^{1/3} \sim (\xi_0 l^2)^{1/3} \tag{132}$$

i.e. the system must be close enough to mobility edge or just slightly localized. Here we used the usual estimate of mean free path close to Anderson transition $l \sim p_F^{-1}$. Below we shall see that Eq. (132) is just a condition that Cooper pairs must be much smaller than localization length, only in this case Cooper pairing is possible in localized phase [19,20].

3.2 T_c Degradation

In usual BCS model discussed above pairing interaction g is assumed to be a given constant in the vicinity of the Fermi level. In more realistic approach this interaction is determined by the balance of interelectron attraction, due e.g. to electron—phonon coupling (as in traditional superconductors) or some other Boson—exchange mechanism (as is apparently the case in high—T_c superconductors), and Coulomb repulsion. It is clear that in strongly disordered system all these interactions can, in principle, be strongly renormalized in comparison with "pure" case. The aim of this section is to discuss these effects on the approach to metal—insulator transition induced by disorder.

Usually the Coulomb repulsion within Cooper pair is strongly reduced in comparison with electron—phonon attraction due to a retarded nature of electron—phonon coupling [9]. Characteristic time of electron—phonon interaction is of the order of ω_D^{-1}, while for Coulomb interaction in "pure" metal it is determined by $\sim \hbar/E_F$—the time during which electrons "pass" each other in the pair. Due to metallic screening both interactions are more or less point—like. However, in a disordered metal ballistic transport changes to diffusion and as disorder grows electron motion becomes slower effectively leading to the growth of Coulomb repulsion within Cooper pair and the appropriate drop of T_c as was first claimed by Anderson, Muttalib and Ramakrishnan [18]. Actually electron—phonon interaction can also change under disordering but a common belief is that these changes are less significant than in case of Coulomb interaction [103,104]. This problem is still under active discussion and some alternative points of view were expressed [105,106,107]. However, the general agreement is that some kind of diffusion renormalization of effective interaction of electrons within Cooper pair provides effective mechanism of T_c degradation under disordering. Below we shall mainly use the approach of Ref. [20], with the main aim to study a possibility of superconductivity surviving up to Anderson transition.

Later in this section we shall also consider the possible mechanisms of T_c degradation under disordering due to magnetic fluctuations (or local moments) which appear close to metal—insulator transition. Possible relation of these mechanisms to enhanced Coulomb effects will also be discussed.

The general problem of T_c degradation under disordering becomes much more complicated in case of high—temperature superconductors because of unknown nature of pairing in these systems. However, we believe that the mechanism based upon the growth of Coulomb repulsion within Cooper pair is also operational here, while of course it is difficult to say anything about disorder effects upon attractive interactions leading to Cooper pair formation in these systems.

We shall always assume some kind of Boson—exchange (phonons, excitons, spin fluctuations etc.) model of pairing interaction. so that T_c can be obtained from some generalized form of Eliashberg equations and thus be given by the famous Allen—Dynes expression [108]:

$$T_c = \frac{f_1 f_2}{1.20}\omega_{log} exp\left\{-\frac{1.04(1+\lambda)}{\lambda - \mu^*(1+0.62\lambda)}\right\} \tag{133}$$

where

$$f_1 = [1+(\lambda/\lambda_1)^{3/2}]^{1/3}; \qquad f_2 = 1 + \frac{[<\omega^2>^{1/2}/\omega_{log} - 1]\lambda^2}{\lambda^2 + \lambda_2^2}$$

$$\lambda_1 = 2.46(1+3.8\mu^*); \qquad \lambda_2 = 1.82(1+6.3\mu^*)\frac{<\omega^2>^{1/2}}{\omega_{log}} \tag{134}$$

Here ω_{log} is the mean logarithmic frequency and $<\omega>^2$ is the mean square frequency of Bosons responsible for pairing (the averaging is over the spectrum of these Bosons), μ^* is the Coulomb pseudopotential, λ is the dimensionless pairing constant due to Boson—exchange. At relatively weak coupling $\lambda \leq 1.5$ Allen—Dynes expression effectively reduces to McMillan formula:

$$T_c = \frac{\omega_{log}}{1.20} exp\left\{-\frac{1.04(1+\lambda)}{\lambda - \mu^*(1+0.62\lambda)}\right\} \tag{135}$$

which for the weak coupling limit gives the usual BCS result $T_c \sim <\omega> exp(-1/\lambda - \mu^*)$. For very strong pairing interaction Eq. (133) gives the asymptotic behavior $T_c \approx 0.18\sqrt{\lambda <\omega^2>}$. In most parts of this review we shall limit ourselves to weak coupling approximation. Coulomb pseudopotential μ^* in the "pure" system is given by:

$$\mu^* = \frac{\mu}{1 + \mu ln\frac{E_F}{<\omega>}} \tag{136}$$

where μ is the dimensionless Coulomb constant. The mechanism of T_c degradation under disordering due to the growth of Coulomb repulsion is reflected in the appropriate growth of μ^* [18,20].

3.2.1 Coulomb Kernel

Let us use again the exact eigenstate $\phi_\nu(\mathbf{r})$ representation for an electron in random system, with exact energy levels ε_ν. These functions and energies may correspond either to extended or to localized states. Consider the one—electron Green's function in this representation and take its diagonal element $G_{\nu\nu}(\varepsilon)$. The influence of interaction is described by the appropriate irreducible self—energy $\Sigma_\nu(\varepsilon)$ [93,110]:

$$G_{\nu\nu}(\varepsilon) = \frac{1}{\varepsilon - \varepsilon_\nu - \Sigma_\nu(\varepsilon)} \tag{137}$$

Here energy zero is at the Fermi level. Let us introduce a "self—energy" $\Sigma_E(\varepsilon)$ averaged over some surface of constant energy $E = \varepsilon_\nu$ and over random field configurations [110]:

$$\Sigma_E(\varepsilon) = \frac{1}{N(E)} < \sum_\nu \delta(E - \varepsilon_\nu)\Sigma_\nu(\varepsilon) > \tag{138}$$

Consider model with short—range static interelectron interaction $v(\mathbf{r}-\mathbf{r}')$. Then for the simplest Fock correction we find:

$$\Sigma_\mu^F = -\int d\mathbf{r}\int d\mathbf{r}' v(\mathbf{r}-\mathbf{r}')\sum_\nu f_\nu \phi_\mu^*(\mathbf{r}')\phi_\nu^*(\mathbf{r})\phi_\mu(\mathbf{r})\phi_\nu(\mathbf{r}') \tag{139}$$

where $f_\nu = f(\varepsilon_\nu)$ is Fermi distribution function. Accordingly from Eq. (138) we get [56]:

$$\Sigma_E^F = -\int_{-\infty}^{\infty} d\omega f(E+\omega)\int d\mathbf{r}\int d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F \tag{140}$$

where we again introduced Berezinskii—Gorkov spectral density as defined in Eq. (47).

Let us define the Coulomb kernel by the following functional derivative:

$$K_c(E-E') = -\frac{\delta\Sigma_E^F}{\delta f(E')} \tag{141}$$

which characterize the change of electron energy due to a variation of its distribution function. It is easy to see that:

$$K_c(\omega) = \frac{1}{N(E)} < \sum_{\mu\nu} < \mu\nu|v(\mathbf{r}-\mathbf{r}')|\nu\mu > \delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_\mu) >=$$

$$= \int d\mathbf{r}\int d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \ll \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}') \gg^F \tag{142}$$

is actually Fock—type matrix element of interaction averaged over two surfaces of constant energy E and $E' = E+\omega$ and over disorder. We can use $K_c(\omega)$ as a kernel in the linearized gap equation determining T_c which is a reasonable generalization of a Coulomb kernel used in the theory of ordered superconductors [111]. In momentum representation:

$$K_c(\omega) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} v(\mathbf{q}) \ll \rho_E\rho_{E+\omega} \gg_{\mathbf{q}}^F \tag{143}$$

In the weak coupling approximation over pairing interaction it is the only relevant Coulomb contribution in the gap equation, in case of strong coupling there are additional contributions, e.g. connected with diffusional renormalization of the density of states Eq. (96)[105,106,107,112,113].

In the following we assume point—like interaction: $v(\mathbf{q}) = v_0$. During our discussion of localization we have discovered that for small $\omega \ll \gamma$ and $q \ll l^{-1}$ Gorkov—Berezinskii spectral density acquires a diffusional contribution:

$$\ll \rho_E\rho_{E+\omega} \gg_{\mathbf{q}}^{Fdiff} = \frac{1}{\pi N(E)} Im\Phi_E^{RA}(\mathbf{q}\omega) \tag{144}$$

where

$$\Phi_E^{RA}(\mathbf{q}\omega) = -\frac{N(E)}{\omega + iD_E(\omega)q^2} \tag{145}$$

and the generalized diffusion coefficient in metallic phase is given by:

$$D_E(\omega) \approx \begin{cases} D_E & |\omega| \ll \omega_c \approx 2\gamma(\sigma/\sigma_c)^3 \\ D_0\left(-\frac{i\omega}{2\gamma}\right)^{1/3} & |\omega| \gg \omega_c \end{cases} \tag{146}$$

In the absence of disorder this diffusional contribution disappears and the kernel $K_c(\omega)$ for $|\omega| < E_F$ reduces to usual Coulomb potential $\mu = N(E)v_o$.[9,111] Accordingly we can use the following approximation [20]:

$$K_c(\omega) \approx \mu\theta(E_F - |\omega|) + K_c^{diff}(\omega) \tag{147}$$

where

$$K_c^{diff}(\omega) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} v_0 \ll \rho_E \rho_{E+\omega} \gg_{\mathbf{q}}^{Fdiff} \tag{148}$$

This form of the Coulomb kernel gives correct interpolation between the strong disorder limit and "pure" case. Note, that in case of disordered system besides diffusional contribution which contains singularities associated with Anderson transition there also appear "regular" contributions to $K_c(\omega)$ which may be modelled by μ, making it different from its value in "pure" system.

3.2.2 Electron—Phonon Interaction

The case of electron—phonon interaction is different. Diffusion renormalization of electron—phonon vertex does not appear because the relevant corrections compensate each other if we take into account impurity vibrations [104]. Surely the value of electron—phonon contribution to pairing interaction do change in a disordered system in comparison with "pure" case [103]. However, these changes are relatively insignificant in the sense of absence of drastic changes at the Anderson transition. We shall demonstrate the absence of diffusion renormalization of electron—phonon vertex using the lowest order diagrams of perturbation theory following the approach of Ref.[104].

Let us limit our analysis to homogeneous continuous medium. The appearance of deformation u leads to the variation of density of the medium given by $\delta\rho = -\rho div\mathbf{u}$. Accordingly, taking into account the electroneutrality condition we get the variation of electron density as $\delta n = -ndiv\mathbf{u}$. This leads to the following change of the free electron Green's function:

$$\delta G^{-1}(E\mathbf{p}) = -ndiv\mathbf{u}\frac{d}{dn}[E - v_F(|\mathbf{p}| - p_F)] = \\ = -nv_F div\mathbf{u}\frac{dp_F}{dn} = -\frac{1}{3}v_F p_F div\mathbf{u} \tag{149}$$

where we have used $n = p_F^3/(3\pi^2)$. Let us define electron—phonon vertex $\mathbf{\Lambda}$ by:

$$\frac{\delta G}{\delta \mathbf{u}} = G\mathbf{\Lambda}G = -G\frac{\delta G^{-1}}{\delta \mathbf{u}}G; \qquad \mathbf{\Lambda} = -\frac{\delta G^{-1}}{\delta \mathbf{u}} \tag{150}$$

For $\mathbf{u}(\mathbf{r},t) = \mathbf{u}exp(i\mathbf{qr} - i\omega t)$ we get from Eq. (149):

$$\delta G^{-1}(E\mathbf{p}) = -i\mathbf{qu}\frac{v_F p_F}{3} \tag{151}$$

so that the "bare" electron—phonon vertex (i is vector index):

$$\Lambda_{1i}^{(0)} = iq_i\frac{v_F p_F}{3} \tag{152}$$

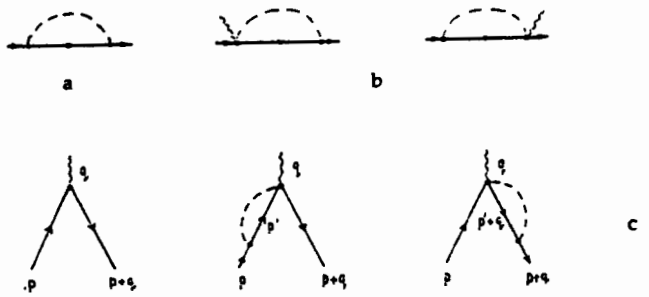

Figure 3: Electron—phonon interaction and impurity scattering: (a) Self—energy due to impurity scattering; (b) Diagrams representing changes of (a) due to impurity vibrations; (c) Diagrams for "bare" electron—phonon vertex in case of vibrating impurities.

Consider the system with impurities randomly placed at points $\mathbf{R}_n$ which create the potential:

$$U(\mathbf{r}) = \sum_n V(\mathbf{r} - \mathbf{R}_n) \tag{153}$$

Vibrations of the medium lead to vibrations of impurity atoms, so that $\mathbf{R}_n \to \mathbf{R}_{0n} + \mathbf{u}_n(t)$ with $\mathbf{u}_n(t) = \mathbf{u}exp(i\mathbf{qR}_{0n} - i\omega t)$. Random field of static impurities leads to a simplest self—energy correction given by Fig. 3 (a) [53,54]. Impurity vibrations can be accounted for by the additional interaction term:

$$\delta V(\mathbf{r} - \mathbf{R}_n) = \frac{\partial V(\mathbf{r} - \mathbf{R}_{n0})}{\partial \mathbf{R}_{n0}}\mathbf{u}exp(i\mathbf{qR}_{n0} - i\omega t)$$

so that

$$\Lambda_{2i}u_i = \frac{\delta\Sigma}{\delta u_i}u_i = \\ =< \sum_n \left\{ \frac{\partial V(\mathbf{r} - \mathbf{R}_{n0})}{\partial R_{n0}^i} G(\mathbf{r}t, \mathbf{r}'t')V(\mathbf{r}' - \mathbf{R}_{n0})u_{in} + \right. \\ \left. + V(\mathbf{r} - \mathbf{R}_{n0})G(\mathbf{r}t, \mathbf{r}'t')\frac{\partial V(\mathbf{r}' - \mathbf{R}_{n0})}{\partial R_{n0}^i}u_{in} \right\} > \tag{154}$$

where the angular brackets define as usual the averaging over random impurity positions. In momentum representation and for point—like impurities we get in lowest order over ω/E_F and q/p_F:

$$\Lambda_{2i}(\mathbf{p},\mathbf{q}) = \rho V^2 \int \frac{d^3\mathbf{p}'}{(2\pi)^3}[-i(p_i - p_i')G(E\mathbf{p}') + i(p_i' - p_i)G(E\mathbf{p}')] = \\ = 2\rho V^2 \int \frac{d^3\mathbf{p}'}{(2\pi)^3}[-i(p_i - p_i')G(E\mathbf{p}')] = \\ = 2\pi\rho V^2 N(E)p_i = 2\gamma p_i \tag{155}$$

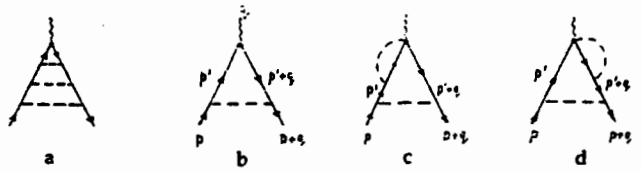

Figure 4: Electron—phonon vertex renormalization: (a) Impurity "ladder" (diffusion) renormalization; (b), (c), (d) Simplest corrections due to impurity vibrations.

The relevant diagrams are shown in Fig. 3 (b) [114]. "Bare" electron—phonon vertex is thus given by the sum of three diagrams shown in Fig. 3 (b) and reduces to:

$$\Lambda_i^{(0)} = \Lambda_{1i}^{(0)} + \Lambda_{2i}^{(0)} = iq_i\frac{v_F p_F}{3} + 2\gamma p_i \tag{156}$$

Diffusion renormalization of electron—phonon vertex can appear due to impurity scattering ladder corrections as shown in Fig. 4 (a). However, let us consider simplest corrections shown in Fig. 4 (b,c,d). For the contribution of graph of Fig. 4 (b) we have:

$$\Lambda_{1i}^{(1)} = \rho V^2 iq_i\frac{v_F p_F}{3}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}G(E\mathbf{p}')G(E+\omega\mathbf{p}'+\mathbf{q}) \approx$$
$$\approx iq_i\frac{v_F p_F}{3}[1+i\omega/2\gamma - D_0q^2/2\gamma] \approx iq_i\frac{v_F p_F}{3} \qquad \omega, q \to 0 \tag{157}$$

and for the sum of graphs of Fig. 4 (c,d):

$$\Lambda_{2i}^{(1)} = 2\rho V^2\gamma\int\frac{d^3\mathbf{p}'}{(2\pi)^3}G(E\mathbf{p}')G(E+\omega\mathbf{p}'+\mathbf{q})p_i' \approx$$
$$\approx 2\rho V^2\gamma q_i\int\frac{d^3\mathbf{p}'}{(2\pi)^3}p_i'G(E\mathbf{p}')\frac{\partial}{\partial p_i}G(E+\omega\mathbf{p}') \approx$$
$$\approx 2\gamma\rho V^2 q_i p_F\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{v_F}{3}G(E\mathbf{p}')G^2(E\mathbf{p}') = -iq_i\frac{v_F p_F}{3} \tag{158}$$

Thus for $\omega \to 0$, $q \to 0$ we obtain:

$$\Lambda_{1i}^{(1)} + \Lambda_{2i}^{(1)} = 0 \tag{159}$$

and we have total cancellation of initial diagrams contributing to diffusion ladder. Accordingly there is no diffusion renormalization of electron—phonon vertex (for $\omega, q \to 0$): this cancellation is valid for any graph obtained from the simplest corrections by adding further impurity lines to the ladder. Similar cancellation takes place in case of adding to diagrams of Fig. 4 (b,c,d) corrections due to maximally crossed impurity lines (Cooper channel). Thus there is no significant change of electron—phonon vertex due to Cooperon. In this latest conclusion we disagree with Ref.[105]. Thus the only relevant contribution to electron—phonon vertex in impure system is defined by the sum of diagrams of Fig. 3 (b) leading to Eq. (156) which does not contain diffusion

type renormalization. Localization appears via generalized diffusion coefficient which replaces the Drude one. Thus localization singularities does not appear in electron—phonon vertex, though surely this interaction is really changed by disorder scattering in comparison with "pure" case.

In the following we shall model pairing interaction due to phonon exchange by some constant λ as in BCS model. Of course we must stress that this constant is different from that in regular metal. It is constant in a sense that it does not contain singularities due to metal—insulator transition. Electron—phonon kernel in the linearized gap equation can be taken in the simplest form:

$$K_{ph}(E,E') = \begin{cases} -\lambda & |E|,|E'| < \omega_D \\ 0 & |E|,|E'| > \omega_D \end{cases} \tag{160}$$

and consider λ as relatively weakly dependent on disordering. More detailed discussion of electron—phonon pairing in disordered systems can be found in Refs.[103,105,106].

As we mentioned above it is quite difficult to speculate on disorder dependence of pairing interaction in high—temperature superconductors. In case of the "marginal" Fermi—liquid approach [95,96] pairing interaction can be modelled as in Eq. (160) with the replacement of Debye frequency ω_D by some phenomenological *electronic* frequency $\tilde{\omega}_c$ which we briefly mentioned above while discussing localization in "marginal" Fermi—liquid. In the following we shall just assume that this pairing interaction is weakly dependent on disorder as in the case of phonon mechanism of pairing.

3.2.3 Metallic Region

In metallic region we can use Eqs. (143—145) and Eq. (147) and find the diffusional contribution to Coulomb kernel:

$$K_c^{diff}(\omega) = -\int\frac{d^3\mathbf{q}}{(2\pi)^3}v_0 Im\frac{1}{\omega + iD_E q^2} \approx$$
$$\approx \frac{v_0}{2\pi^3}\left[\frac{1}{|D_E(\omega)l|} - \frac{|\omega|^{1/2}}{|D_E^{3/2}(\omega)|}\right] \approx$$
$$\approx \frac{v_0}{2\pi^3}\begin{cases} \frac{1}{D_E l} - \frac{|\omega|^{1/2}}{D_E^{3/2}} & |\omega| \ll \omega_c \\ \frac{1}{D_0 l}\left(\frac{\omega}{2\gamma}\right)^{-1/3} & |\omega| \gg \omega_c \end{cases} \tag{161}$$

Accordingly for the Coulomb kernel defined by Eq. (147) we get [20]:

$$K_c(\omega) = \mu\theta(E_F - |\omega|) + \frac{\mu}{p_F l}\begin{cases} \frac{\varrho_c}{\varrho} & |\omega| < \omega_c \\ \frac{1}{p_F l}\left(\frac{\omega}{2\gamma}\right)^{-1/3} & \omega_c < \omega < \gamma \sim E_F \end{cases} \tag{162}$$

Upper limit cut—off in the integral in Eq. (161) was taken $\sim l^{-1}$. Rough estimate of contribution of higher momenta can be achieved introducing cut—off $\sim p_F$ (Cf. Ref.[112]). This will cancel $(p_F l)^{-1}$ in Eq. (162). Close to Anderson transition $l^{-1} \sim p_F$ and this correction is irrelevant. We shall assume that far from transition these higher momenta corrections can be included in the definition of μ. From Eq. (162) we can see that diffusion renormalization of Coulomb kernel leads to substantial growth of

Coulomb repulsion close to Anderson transition (i.e. when conductivity drops below σ_c—"minimal metallic conductivity").

Superconducting transition temperature T_c is determined by the linearized gap equation [111] which in the weak coupling approximation can be written as[115,116]:

$$\Delta(\omega) = \lambda\theta(<\omega> - \omega)\int_0^{<\omega>}\frac{d\omega'}{\omega'}\Delta(\omega')th\frac{\omega'}{2T_c} - \\ -\theta(E_F-\omega)\int_0^{E_F}\frac{d\omega'}{\omega'}K_c(\omega-\omega')\Delta(\omega')th\frac{\omega'}{2T_c} \tag{163}$$

Consider metallic region and take $\omega_c \gg <\omega>$ which in accordance with ω_c estimate given in Eq. (146) roughly corresponds to $\sigma \geq \sigma_c$ for typical $E_F/<\omega> \sim 10^2$, so that the system is not very close to Anderson transition. The change of T_c due to diffusion contribution in Coulomb kernel Eq. (162) can be determined by perturbation theory over $K_c^{diff}(\omega)$ in gap equation. First iteration of Eq. (163) gives:

$$\frac{\delta T_c}{T_{c0}} \approx \frac{\int_0^\infty\frac{d\omega}{\omega}\int_0^\infty\frac{d\omega'}{\omega'}\Delta_0(\omega)th\frac{\omega}{2T_{c0}}K_c^{diff}(\omega-\omega')\Delta_0(\omega')th\frac{\omega'}{2T_{c0}}}{\frac{1}{2T_{c0}}\int_0^\infty d\omega[\Delta_0(\omega)]^2[ch\frac{\omega}{2T_{c0}}]^{-2}} \tag{164}$$

where $\Delta(\omega)$ is the usual "two—step" solution of Eg. (163)[9,111] which is valid for standard form of Coulomb kernel $K_c(\omega) = \mu\theta(E_F - |\omega|)$,

$$T_{c0} = 1.13 <\omega> exp\left(-\frac{1}{\lambda-\mu_0^*}\right) \tag{165}$$

is a critical temperature in regular superconductor when the Coulomb pseudopotential is given by :

$$\mu_0^* = \frac{\mu}{1+\mu ln\frac{E_F}{<\omega>}} \tag{166}$$

Using the first relation in Eq. (162) we get from Eq. (164):

$$\frac{\delta T_c}{T_{c0}} \approx -\frac{\mu}{(\lambda-\mu_0^*)^2}\frac{1}{p_F l}\frac{\sigma_c}{\sigma} \tag{167}$$

This change of T_c is equivalent to the following change of Coulomb pseudopotential [20]:

$$\delta\mu^* \approx \mu\frac{\sigma_c^2}{\sigma(\sigma+\sigma_c)} \tag{168}$$

where we have used Eq. (64) and $p_F l \approx \sigma_0/\sigma_c = (\sigma+\sigma_c)/\sigma_c$ to replace $p_F l$ in Eq. (167). As we noted above this later factor disappears from Eq. (167) if we use cut—off at $q \sim p_F$ in Eq. (161). According to Eq. (168) Coulomb pseudopotential μ^* grows as σ drops and this dependence is more strong than a similar one obtained in Ref.[18], which is connected with our use of the results of self—consistent theory of localization. Method of Ref.[18] is based upon the use of q—dependence of diffusion coefficient as given by Eq. (75) . Our expression for $\delta\mu^*$ leads to a significant growth of μ^* for conductivities $\sigma \leq 10^3 Ohm^{-1}cm^{-1}$. This growth can easily explain the typical

T_c degradation in "very dirty" superconductors as their conductivity in normal state drops approaching the Ioffe—Regel limit [27]. At the same time expressions for μ^* proposed in Ref.[18] can explain experimental data only under the assumption that a characteristic conductivity scale determining μ^* is an order of magnitude larger than Ioffe—Regel limit., for which we see no serious grounds. More extensive discussion can be found in Ref.[105].

Let us consider now the situation at the mobility edge itself, when $\sigma = 0$ and $\omega_c = 0$ so that $K_c(\omega)$ is determined by the second expression in Eq. (162) for all frequencies below $\gamma \sim E_F$. In this case we can show [20] that the influence of Coulomb repulsion on T_c is again described by effective pseudopotential μ^* which can be estimated as:

$$\mu^* \sim \alpha\mu\left(\frac{<\omega>}{2\gamma}\right)^{-1/3} \qquad \alpha \sim 1 \tag{169}$$

In this case T_c may remain finite at the mobility edge only under very strict conditions: both $E_F \sim \gamma$ and μ must be very small, while λ must be at least close to unity. As a crude estimate we can demand something like $\lambda \sim 1$, $\mu \leq 0.2$ and $E_F \leq 10^3 T_{c0}$. Obviously only some narrow band superconductors like Chevrel phases can satisfy these conditions among traditional systems. High—T_c superconductors are especially promising. Experimental situation will be discussed later.

Using Eq. (168) and Eq. (169) we can write down a simple interpolation formula for the conductivity dependence of μ^*:[20]

$$\mu^* \approx \mu_0^* + \frac{\alpha\mu(<\omega>/2\gamma)^{-1/3} - \mu_0^*}{1+(<\omega>/2\gamma)^{-1/3}\sigma(\sigma+\sigma_c)/\sigma_c^2} \tag{170}$$

To get an expression via observable parameters take into account $<\omega>/\gamma \approx (<\omega>/E_F)(1+\sigma/\sigma_c)$. These expressions describe continuous crossover from the region of weak localization corrections to the vicinity of Anderson transition where its influence upon T_c becomes very strong. This crossover takes place at $\omega_c \sim <\omega>$.

3.2.4 Localization Region

Let us now consider Anderson insulator. According to Eq. (143) and Eq. (48) Coulomb kernel acquires in this case $\delta(\omega)$—contribution:

$$K_c^{loc}(\omega) = v_0 A_E\delta(\omega) = v_0\frac{1}{N(E)} < \sum_\nu \delta(E-\varepsilon_\nu)|\phi(\mathbf{r})|^2|\phi(\mathbf{r})|^2 > \tag{171}$$

$$A_E = A_E(\mathbf{r}-\mathbf{r}')|_{\mathbf{r}=\mathbf{r}'} \sim R_{loc}^{-3}$$

which is actually connected with "Hubbard—like" repulsion of electrons in a single quantum state becoming nonzero in localization region [117,75,7]. This mechanism acts in addition to diffusion contributions in Coulomb pseudopotential μ^* considered above, which are due to "regular" part of Gorkov—Berezinskii spectral density. Using Eq. (172) as a full Coulomb vertex in linearized gap equation (163) we can solve it exactly [20] and find:

$$\Delta(\omega) = \frac{\theta(<\omega> - |\omega|)\Delta_1}{1+\frac{\mu A_E}{2N(E)}\frac{1}{\omega}th\frac{\omega}{2T_c}} \tag{172}$$

where

$$\Delta_1 = \lambda \int_0^{<\omega>} d\omega \Delta(\omega) \frac{1}{\omega} th \frac{\omega}{2T_c} \tag{173}$$

and equation for T_c takes the form:

$$1 = \lambda \int_0^{<\omega>} d\omega \frac{th\frac{\omega}{2T_c}}{\omega + \frac{\mu A_E}{2N(E)} th\frac{\omega}{2T_c}} \tag{174}$$

To account for "regular" diffusion contributions to μ^* we can just replace here $\lambda \to \lambda^* = \lambda - \mu^*$, where μ^* is given by Eq. (169). Then our equation for T_c can be approximately represented by[20]:

$$ln\frac{T^*}{T_c} \approx \psi\left(\frac{1}{2} + \frac{\mu A_E}{4T_c N(E)}\right) - \psi\left(\frac{1}{2}\right) \tag{175}$$

where $\psi(x)$ is digamma function, and T^* is taken to be equal to T_c of the system at the mobility edge which is given by Eq. (165) with μ_0^* replaced by μ^* from Eq. (169). Here we slightly overestimate the role of Coulomb repulsion in localization region. We can see that this additional "Hubbard—like" repulsion acts upon T_c as magnetic impurities [9,111] with effective spin—flip scattering rate:

$$\frac{1}{\tau_{sf}} = \pi \frac{\mu A_E}{N(E)} \sim \frac{\mu}{N(E) R_{loc}^3(E)} \tag{176}$$

Obviously this result is connected with the appearance below the mobility edge of the "band" of singly occupied electron states of the width $v_0 R_{loc}^{-3}$ [117,75,7,118]. Superconductivity persists until $\tau_{sf}^{-1} < 0.57 T_c^*$, i.e. until

$$R_{loc}(E) > \left[\frac{\mu}{N(E)T_c^*}\right]^{1/3} \sim (\xi_0 p_F^{-2})^{1/3} \sim (\xi_0 l^2)^{1/3} \tag{177}$$

where the last estimates are valid for typical values of parameters and correspond to the simple estimate of Eq. (130). Thus the Coulomb repulsion in a single (localized) quantum state leads to a sharp reduction of T_c below the mobility edge even if superconductivity survived up to the Anderson transition. Another interpretation of this effect is the influence of "free" spins of Mott's band of singly occupied states below the Fermi level of Anderson insulator.

Coulomb gap [37,38,39,39,40] effects can be neglected here [20] because according to the estimates given in Eq. (14) and Eq. (15) the Coulomb gap width:

$$\Delta_c \sim [N(E) R_{loc}^3(E)]^{-1} \ll T_c \sim \Delta \tag{178}$$

i.e. is small in comparison to superconducting gap Δ (or T_c) under conditions given by Eq. (130) which is necessary for the observation of superconductivity in localization region.

3.2.5 Spin Fluctuations

As we mentioned during our discussion of interaction effects upon Anderson transition the role of magnetic fluctuations (spin effects) in general becomes stronger as

we approach metal—insulator transition. The band of single—occupied states is being formed below the Fermi level of Anderson insulator, which is equivalent to the appearance of localized moments [117,7,118]. These effects actually may become important already before metal—insulator transition[81,87,88,89,90,91,36], and lead to additional mechanism of T_c degradation. Unfortunately there is no complete theoretical understanding of these effects and accordingly only few estimates can be done concerning superconductivity. Here we shall mention only some of these crude estimates following Refs.[119,120,121].

In the framework of Hubbard model with weak disorder it can be shown [119] that the spin susceptibility is represented by:

$$\chi_s = \frac{\chi_0}{1 - UN(E) + \gamma_0 - \gamma'} = \frac{\chi_0}{\eta_0 - \gamma'} \equiv \frac{\chi_0}{\eta} \tag{179}$$

where χ_0 is spin susceptibility of free electrons, $\eta_0 = 1 - UN(E) + \gamma_0$ is enhancement factor for the ordered case (U is Hubbard interaction, γ_0 is correlation correction to RPA approximation), γ' is the correction due to the interference of Hubbard interaction and disorder scattering:

$$\gamma' = B\lambda^2 \qquad B = 6\sqrt{3}\pi^2 [N(E)U]^2 \left\{1 - \frac{1}{2} UN(E)\right\} \tag{180}$$

Here $\lambda = 1/(2\pi E\tau) = 1/(p_F l)$ is the usual perturbation theory parameter for disorder scattering. As $\gamma' > 0$ we can see from Eq. (179) that disordering leads to diminishing denominator $\eta = \eta_0 - \gamma'$. If we reach a critical disorder defined by:

$$\lambda_c = \sqrt{\frac{\eta_0}{B}} \qquad p_F l|_c = 2\sqrt{\frac{B}{\eta_0}} \tag{181}$$

we get $\chi_s \to \infty$. It should be stressed that this divergence of χ_s in a disordered system must not be identified with any kind of ferromagnetic instability but may signify something like the appearance of a spin—glass state or just of localized moments. In any case it means the growth of spin dependent effects under disordering.

If the initial enhancement of spin susceptibility is strong enough (e.g. due to a large U), i.e. $\eta_0 \ll 1$, the critical disorder defined by Eq. (181) may be lower than the critical disorder for Anderson localization, appearing at $p_F l \sim 1$. Then these spin dependent effects may become important well before Anderson transition. In the opposite case these effects will appear only very close to metal—insulator transition. In general case the relation between these two transitions depends on parameters.

If spin fluctuations are strong enough ($\eta \ll 1$) a strong mechanism for T_c degradation in superconducting state appears[120] analogous to similar effect due to magnetic impurities [9,111]:

$$ln\frac{T_{c0}}{T_c} = \psi\left(\frac{1}{2} + \rho\right) - \psi\left(\frac{1}{2}\right) \tag{182}$$

where [120]:

$$\rho = \frac{9\sqrt{3}\pi}{2} \lambda^2 \frac{UN(E)}{\eta} = \frac{9\sqrt{3}}{2} \left[\frac{UN(E)}{B}\right] \frac{\lambda^2}{\lambda_c^2 - \lambda^2} \tag{183}$$

As ρ from Eq. (183) diverges as $(\lambda_c - \lambda)^{-1}$ for $\lambda \to \lambda_c$ superconducting transition temperature T_c drops to zero.

If $\lambda_c \ll 1$, which is possible for $\eta_0 \ll 1$, superconductivity will be destroyed long before metal—insulator transition. In the opposite case this mechanism may lead to its destruction on the either side of metal—insulator transition depending on the parameters of the system, such as U. In general we need a more accurate analysis which must include the mutual interplay of magnetic fluctuations and disorder scattering leading to metal—insulator transition. In any case magnetic mechanisms of T_c degradation close to metal—insulator transition may be as important as Coulomb effects considered above.

3.3 Ginzburg—Landau Theory and Anderson Transition

3.3.1 General Analysis

The main result of the previous analysis may be formulated as follows. Despite many mechanisms leading to T_c degradation and destruction of superconductivity in strongly disordered systems there seems to be no general rule prohibiting a possibility of a superconducting state in Anderson insulator. Of course we must meet very rigid conditions if we hope to observe this rather exotic state. There is almost no chance to observe it in traditional superconductors but high—T_c systems seem promising. The following analysis will be based on the general assumption that T_c survives in a strongly disordered system or even in Anderson insulator, i.e. that these strict conditions are met. Our aim is to study superconducting properties of such a strongly disordered system to determine specific characteristics which will make this case different from the usual case of "dirty" superconductors. We shall see that even before Anderson transition there are significant deviations from the predictions of standard theory which make strongly disordered system different. So on the practical side our aim is simply to generalize the usual theory of "dirty" superconductors for the case of strong disorder in a sense of the mean free path becoming of the order of interatomic spacing or $l \sim p_F^{-1}$.

To claim that superconductivity is possible close to disorder—induced metal—insulator transition it is not sufficient just to demonstrate the finite values of T_c. Even more important is to show the existence of superconducting response to an external electromagnetic potential $\mathbf{A}$. In general case the analysis of response functions of a superconductor with strong disorder seems to be a difficult task. However, close to T_c significant simplifications take place and actually we only have to show that the free—energy density of the system can be expressed in the standard Ginzburg—Landau form [122,12,9]:

$$F = F_0 + A|\Delta|^2 + \frac{1}{2}B|\Delta|^4 + C|(\nabla - \frac{2ie}{\hbar c}\mathbf{A})\Delta|^2 \qquad (184)$$

where F_n is free energy density of the normal state. Our problem is thus reduced to a microscopic derivation of expressions for the coefficients A, B, and C of Ginzburg—Landau expansion Eq. (184) taking into account the possibility of electron localization. This will be the generalization of the famous Gorkov's derivation [12] of similar expressions for the case of "dirty" superconductors. Such analysis was first done by Bulaevskii and Sadovskii [19,20] and later by Kotliar and Kapitulnik [21,22]. Recently the same results were obtained by Kravtsov [124].

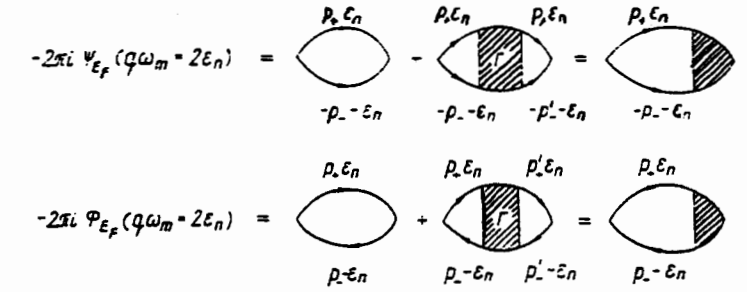

Figure 5: Graphic representation of two—particle Green's functions $\Psi_E(q\omega_m)$ and $\Phi_E(q\omega_m)$ (for $\omega_m = 2\varepsilon_n$). There is no summation over ε_n in the loops.

Within the BCS model coefficients A and B which determine the transition temperature and the equilibrium value of the order—parameter Δ do not change in comparison with their values found in the theory of "dirty" superconductors, even if the system is close to Anderson transition. This corresponds to the main statement of Anderson theorem. Less trivial is the behavior of the coefficient C, which in fact defines the superconducting response. In the usual theory of "dirty" superconductors [12] this coefficient is proportional to diffusion coefficient of electrons, i.e. to conductivity (at $T = 0$). As the Fermi level approaches the mobility edge conductivity drops to zero. However, we shall see that the coefficient C remains finite in the vicinity of Anderson transition, even in the region of localized states.

To derive Ginzburg—Landau coefficients we must know the two—electron Green's function in the normal state [12]. Let us introduce the following two-particle Matsubara Green's functions in momentum representation [20]:

$$\Psi_E(\mathbf{q}, \omega_m, \varepsilon_n) = -\frac{1}{2\pi i}\sum_{\mathbf{p}_+\mathbf{p}'_-} < G(\mathbf{p}_+, \mathbf{p}'_+, -\varepsilon_n + \omega_m)G(-\mathbf{p}'_-, -\mathbf{p}_-, -\varepsilon_n) > \qquad (185)$$

$$\Phi_E(\mathbf{q}, \omega_m, \varepsilon_n) = -\frac{1}{2\pi i}\sum_{\mathbf{p}_+\mathbf{p}'_-} < G(\mathbf{p}_+, \mathbf{p}'_+, -\varepsilon_n + \omega_m)G(\mathbf{p}'_-, \mathbf{p}_-, -\varepsilon_n) > \qquad (186)$$

where $\mathbf{p}_{+-} = \mathbf{p} \pm \mathbf{q}/2$ and $\omega_m = 2\pi m T$. Graphically these functions are represented in Fig. 5. Then Ginzburg—Landau coefficients are defined by [12,123]:

$$A = \frac{1}{g} + 2\pi i T \sum_{\varepsilon_n} \Psi_E(\mathbf{q} = 0\omega_m = 2\varepsilon_n) \qquad (187)$$

$$C = i\pi T \sum_{\varepsilon_n} \frac{\partial^2}{\partial q^2}\Psi_E(\mathbf{q}\omega_m = 2\varepsilon_n)|_{q=0} \qquad (188)$$

Thus the superconducting properties are determined by the Green's function Ψ_E describing the propagation of electronic (Cooper) pair. At the same time we have seen that the Green's function Φ_E determines transport properties of a normal metal

and Anderson transition. In case of time—invariance (i.e. in the absence of external magnetic field or magnetic impurities) we have [64]:

$$\Psi_E(\mathbf{q}\omega_m\varepsilon_n) = \Phi_E(\mathbf{q}\omega_m\varepsilon_n) \tag{189}$$

and it is sufficient to know only $\Phi_E(\mathbf{q}\omega_m = 2\varepsilon_n)$ to determine Ginzburg—Landau coefficients.

As a one—electron model of Anderson transition we can take the self—consistent theory of localization which will allow us to perform all calculations explicitly. We only have to formulate the main equations of this theory in Matsubara formalism (finite T) [20]. For small q and ω_m, analogously to Eq. (34), we have:

$$\Phi_E(\mathbf{q}\omega_m) = -\frac{N(E)}{i|\omega_m| + D_E(|\omega_m|)q^2} \qquad \omega_m = 2\pi mT \tag{190}$$

where the generalized diffusion coefficient $D_E(\omega_m)$ is determined by the self—consistency equation analogous to Eq. (39):

$$\frac{D_0}{D_E(\omega_m)} = 1 - \frac{i}{\pi N^2(E)} \sum_{|\mathbf{q}|<k_0} \Phi_E(\mathbf{q}\omega_m) \tag{191}$$

In three-dimensional case Eq. (191) reduces to (Cf. Eq. (66)):

$$\frac{D_E(\omega_m)}{D_0} = 1 - \frac{\lambda}{\lambda_c} + \frac{\pi}{2}\frac{\lambda}{\lambda_c}\left[\frac{D_0}{D_E(\omega_m)}\frac{\omega_m}{2\gamma}\right]^{1/2} \tag{192}$$

where we have used the same notations as in our discussion of self—consistent theory of localization. Analogously to Eq. (67) and with accuracy sufficient for our aims we can write down the solution of Eq. (192) as:

$$D_E(\omega_m) \approx Max\left\{D_E\frac{\omega_m}{\omega_m + 3D_E\omega_0^2(E)/v_F^2};\quad D_0\left(\frac{\omega_m}{2\gamma}\right)^{1/3}\right\} \tag{193}$$

where D_E is the renormalized diffusion coefficient defined in Eq. (69) and ω_0 is the fundamental frequency defined by Eq. (53), which signals a transition to insulator.

As we have already noted Ginzburg—Landau coefficients A and B are given by the usual expressions valid also for "dirty" superconductors [12,20]:

$$A = N(E_F)ln\frac{T}{T_c} \approx N(E_F)\frac{T - T_c}{T_c} \tag{194}$$

where T_c is given by the usual BCS relation of Eq. (129), and

$$B = \frac{7\zeta(3)}{8\pi^2T_c^2}N(E) \tag{195}$$

where $\zeta(x)$ is Riemann zeta—function ($\zeta(3) = 1.202...$). These coefficients depend on disorder only through the appropriate disorder dependence of $N(E_F)$ and are valid even in localized phase. This is equivalent to the main statement of Anderson theorem.

Significant changes appear in the gradient term coefficient C. Using Eqs. (188)—(190) with Eq. (193) we can find that in different limiting cases this coefficient can be expressed as [19,20]:

$$C \equiv N(E_F)\xi^2 \approx N(E_F)\begin{cases} \frac{\pi}{8T_c}D_{E_F} & \xi_{loc}(E_F) < (\xi_0 l^2)^{1/3};\quad E_F > E_c \\ \left(\frac{D_0 l}{T_c}\right)^{2/3} \approx (\xi_0 l^2)^{2/3} & \xi_{loc}(E_F) > (\xi_0 l^2)^{1/3}\quad E_F \sim E_c \\ R_{loc}^2(E_F)ln\frac{1.78D_{E_F}}{\pi T_c R_{loc}^2(E_F)} & R_{loc}(E_F) < (\xi_0 l^2)^{1/3};\quad E_F < E_c \end{cases} \tag{196}$$

where we have defined the coherence length ξ, and $\xi_0 = 0.18v_F/T_c$ is BCS coherence length, l as usual is the mean free path. Practically the same results were obtained in Refs.[21,22]using the approach based upon elementary scaling theory of localization, which is as we already noted is equivalent to our use of self—consistent theory of localization. In Ref.[124] the same results were confirmed using the σ—model approach to localization.

In metallic state, as Fermi level E_F moves towards the mobility edge E_c localization correlation length ξ_{loc} grows and the coefficient C initially drops as the generalized diffusion coefficient D_{E_F}, i.e. as conductivity of a system in the normal state. However, in the vicinity of Anderson transition while $\sigma \to 0$ the drop of C coefficient saturates and it remains finite even for $E_F < E_c$, i.e. in Anderson insulator. With further lowering of E_F into localization region (or with E_c growth with disorder) the C coefficient is being determined by localization radius R_{loc} which diminishes as E_F moves deep into insulating state. However, remembering Eq. (130) and Eq. (132) we recognize that our analysis is valid only for large enough values of localization length, which satisfy Eq. (132). In this sense the last asymptotics in Eq. (196) is actually outside these limits of applicability.

The finite value of the coefficient C in Ginzburg—Landau expansion in the vicinity of Anderson transition signifies the existence of superconducting (Meissner) response to an external magnetic field. Accordingly, for $T < T_c$ the system can undergo a transition from Anderson insulator to superconductor. The physical meaning of this result can be understood from the following qualitative picture (Cf. Ref.[101] where the similar estimates were used for the granular metal). In Anderson insulator all electrons with energies E close to Fermi level are localized in spatial regions of the size of $\sim R_{loc}(E)$. Nearby regions are connected by some tunneling amplitude $\mathcal{V}$ which determines the probability of electron transition between such regions as:

$$P_T \approx 2\pi|\mathcal{V}|^2N(E)R_{loc}^3(E) \tag{197}$$

However, Anderson localization means that

$$|\mathcal{V}| < \frac{1}{N(E)R_{loc}^3(E)} \tag{198}$$

and coherent tunneling between states localized in these regions is impossible, and we have $P_T < 2\pi N^{-1}(E)R_{loc}^{-3}$. At the same time if conditions given by Eq. (130) or Eq. (132) are satisfied inside each region $\sim R_{loc}$ Cooper pairs may form and superconducting gap Δ appears in the spectrum. Then a kind of "Josephson" coupling appears between regions of localized states which determines the possibility of *pairs* tunneling:

$$E_J \approx \pi^2[N(E)R_{loc}^3(E)]^2|\mathcal{V}|^2\Delta \tag{199}$$

It is easy to see that for

$$\Delta > \frac{2}{\pi}\frac{1}{N(E)R^3_{loc}(E)} \tag{200}$$

we have $E_J > P_T$, so that if Eq. (130) is satisfied we can get $E_J \gg N^{-1}(E)R^{-3}(E)$ despite of Eq. (198) and tunneling of pairs between nearby regions of localized states is possible, even in the absence of single—particle tunneling.

It is convenient to rewrite Eq. (196) using the the relation between generalized diffusion coefficient and conductivity like Eq. (43) as well as Eqs. (62), (64). Then using the Ginzburg—Landau expansion and the expressions for its coefficients we can easily find the temperature dependent coherence length $\xi(T)$ [9,19,20]:

$$\xi^2(T) = \xi^2\frac{T_c}{T_c - T}\begin{cases} \xi_0 l\frac{\sigma_c}{\sigma+\sigma_c} & \sigma > \sigma^* \quad (E_F > E_c) \\ (\xi_0 l^2)^{2/3} & \sigma < \sigma^* \quad (E_F \sim E_c) \end{cases} \tag{201}$$

where $\sigma_c = e^2 p_F/(\pi^3\hbar^2)$ and characteristic conductivity scale σ^* is given by

$$\sigma^* \approx \sigma_c(p_F\xi_0)^{-1/3} \approx \sigma_c\left(\frac{T_c}{E_F}\right)^{1/3} \tag{202}$$

Thus in the region of very small conductivities $\sigma < \sigma^*$ the scale of $\xi(T)$ is defined not by $\xi \sim \sqrt{\xi_0 l}$ as in the usual theory of "dirty" superconductors [12,9] but by the new length $\xi \sim (\xi_0 l^2)^{1/3} \sim (\xi_0/p_F^2)^{1/3}$, which now is the characteristic size of Cooper pair close to Anderson transition.

In a case if $\omega^{1/3}$—law for a diffusion coefficient at the mobility edge is invalid and we have ω^δ—behavior, with some unknown critical exponent δ (which is possible because the modern theory actually cannot guarantee precise values of critical exponents at Anderson transition [47,7]) we can easily show in a similar way that for conductivities $\sigma < \sigma^* \approx \sigma_c(p_F\xi_0)^{-\delta}$ the coherence length is defined by $\xi \sim \xi_0^{\frac{1-\delta}{2}} l^{\frac{1+\delta}{2}}$. Qualitatively this leads to the same type of behavior as above.

From Eq. (201) we can see that $\xi^2(T)$ initially diminishes as we approach metal—insulator transition proportionally to σ as in the case of a "dirty" superconductor. However, already in metallic region for $\sigma < \sigma^*$ it diminishes much slower remaining finite both at the transition itself and below.

The superconducting electron density n_s can be defined as [9]:

$$n_s(T) = 8mC\Delta^2(T) = 8mC(-A)/B \tag{203}$$

Close to Anderson transition we can estimate:

$$n_s \sim mN(E_F)\xi^2\Delta^2 \sim mp_F(\xi_0/p_F^2)^{2/3}\Delta^2 \sim n(T_c^{1/2}/E_F^2)^{2/3}(T_c - T) \tag{204}$$

where $n \sim p_F^3$ is total electron density. If we take here $T \sim 0.5T_c$ i.e. more or less low temperatures we get a simpler estimate:

$$n_s \sim n\left(\frac{T_c}{E_F}\right)^{4/3} \tag{205}$$

which is actually valid up to $T = 0$, as we shall see below. From these estimates we can see that only a small fraction of electrons are superconducting in a strongly

disordered case. However this confirms a possibility of superconducting response of Anderson insulator.

Characteristic conductivity σ^* defined in Eq. (202) gives an important conductivity scale at which significant influence of localization effects upon superconducting properties appear [20]. While σ_c is of the order of Mott's "minimal metallic conductivity" [2,3] σ^* is in general even lower. However, for small enough Cooper pairs (i.e. small ξ_0 which is characteristic of strong coupling and high—T_c superconductors) it is more or less of the order of σ_c. Experimentally it can be defined as a conductivity scale at which significant deviations from predictions of the standard theory of "dirty" superconductors appear under disordering.

We must stress that these results show the possibility of Cooper pairs being delocalized in Anderson insulator, while single—particle excitations of such superconductor are apparently localized, which may lead to some peculiar transport properties of "normal" electrons for $T < T_c$. First attempts to explore this peculiar situation were undertaken in Refs.[125,152,127,128].

These results are easily generalized for the case of strongly anisotropic quasi—two—dimensional systems such as high—T_c superconducting oxides. Using the analysis of such systems within the self—consistent theory of localization [62] we can write down the following Matsubara generalization of Eq. (84):

$$\frac{D_j(\omega_m)}{D_j^0} \approx \begin{cases} Max\left[\frac{E_F-E_c}{E_c}; (2\pi E_F w\tau^2)^{-2/3}(\omega_m\tau)^{1/3}\right] & \omega_m \ll w^2\tau \\ 1 - \frac{1}{2\pi E_F\tau}ln\left(\frac{1}{\omega_m\tau}\right) & \omega_m \gg w^2\tau \end{cases} \tag{206}$$

where $j = \|, \perp$. Now carrying out calculations similar to that of Ref.[20] we obtain for the coefficients of gradient terms in Ginzburg—Landau expansion [16,129]:

$$C_{\|,\perp} = N(E_F)\xi^2_{\|,\perp} \tag{207}$$

where for the coherence lengths $\xi_{\|,\perp}$ we obtain a number of different expressions, depending on the value of the ratio $w^2\tau/2\pi T_c\hbar$ which determines as we shall see the "degree of two—dimensionality" of the problem under study. For the case of $w^2\tau/2\pi T_c\hbar \gg 1$, corresponding to an anisotropic but three—dimensional system, we have:

$$\xi^2_{\|,\perp} = \frac{\pi}{8T_c}D^0_{\|,\perp}\left(\frac{E_F - E_c}{E_c}\right) \approx \xi^0_{\|,\perp}l_{\|,\perp}\left(\frac{E_F - E_c}{E_c}\right) \tag{208}$$

where $\xi^0_\| \sim \hbar v_F/T_c$, $\xi^0_\perp \sim wa_\perp/T_c$, $l_\| = v_F\tau$ and $l_\perp = wa_\perp\tau/\hbar$ are the longitudinal and transverse BCS coherence lengths and mean free paths. The above expressions are valid in the conductivity region $\sigma_\| > \sigma^*$, where

$$\sigma^* \sim \sigma^c_\|\frac{\xi^0_\|}{l_\|}\left(\frac{T_c^2}{E_F w}\right)^{2/3} \tag{209}$$

Here $\sigma^c_\|$ was defined in Eq. (83). The condition of $w^2\tau/2\pi T_c\hbar \gg 1$ is equivalent to the requirement:

$$\xi_\perp \sim \sqrt{\xi^0_\perp l_\perp} \gg a_\perp \tag{210}$$

which clarifies its physical meaning: the transverse size of a Cooper pair must be much greater than interplane lattice spacing. In this case we have just anisotropic three-dimensional superconductivity.

In the immediate vicinity of the Anderson transition, for $\sigma_{\|} < \sigma^*$ we have:

$$\xi^2_{\|,\perp} \approx (1-2^{-5/3})(16\pi^4)^{-1/3}\zeta(5/3)\frac{D_{\|,\perp}}{(E_F T_c w)^{2/3}\tau} \approx (\xi^0_{\|,\perp})^2\left(\frac{T_c^2}{E_F w}\right)^{2/3} \tag{211}$$

It is easy to see that for $w \sim E_F$ all these expressions naturally go over to those derived above for the three—dimensional case.

For the case of $w^2\tau/2\pi T_c\hbar < 1$ which corresponds to "almost two—dimensional" case of

$$\xi_{\perp} \sim \sqrt{\xi^0_{\perp} l_{\perp}} \leq a_{\perp} \tag{212}$$

i.e. of transverse size of Cooper pairs smaller than interplane spacing, we have

$$\xi^2_{\|,\perp} \approx \left\{ \begin{array}{ll} \frac{D^0_{\|,\perp}}{\pi T_c}\frac{E_F - E_c}{E_c} & (\sigma_{\|} > \sigma^*) \\ \frac{D^0_{\|,\perp}}{(4\pi^2 E_F T_c w)^{2/3}\tau} & (\sigma_{\|} < \sigma^*) \end{array} \right\} + (\pi^2/8-1)\frac{D^0_{\|,\perp}}{\pi T_c}\left(1 - \frac{1}{2\pi E_F\tau} ln \frac{1}{2\pi T_c \tau}\right) \tag{213}$$

Essential difference from just anisotropic case of Eq. (208) and Eq.(211) is the appearance here of a second term of "two-dimensional" type. In purely two—dimensional problem ($w = 0$) we have [123]:

$$\xi^2_{\|} = \frac{\pi D^0_{\|}}{8T_c}\left(1 - \frac{1}{2\pi E_F\tau} ln \frac{1}{2\pi T_c\tau}\right) \tag{214}$$

For high—Tc oxides it is reasonable to estimate $\xi^0_{\|} \sim l_{\|}$, $T_c \sim w$, $T_c \sim 0.1E_F$, so that $\sigma^* \sim \sigma^c_{\|}$, i.e. these systems are always more or less close to the Anderson transition. For $T_c \sim w$ and $\hbar/\tau \sim E_F$ which is characteristic of rather strongly disordered case, we have $w^2\tau/2\pi T_c\hbar < 1$, so that for these systems we can realize almost two—dimensional behavior, though in general high—T_c oxides are apparently an intermediate case between strongly anisotropic three—dimensional and nearly two—dimensional superconductors.

The significant change of Ginzburg—Landau coefficients and the new scale of coherence length close to the Anderson transition lead to an increased width of critical region of thermodynamic fluctuations near T_c[21,22]. These are well known to be important for any second—order phase transition. The width of the critical region is defined by the so called Ginzburg criterion [41,43] which may be expressed via the coefficients of Landau expansion. Mean—field approximation for the order parameter in Landau theory is valid (for $d = 3$) for[41,43]

$$1 \gg \left|\frac{T - T_c}{T_c}\right| \gg \frac{B^2 T_c^2}{\alpha C^3} \equiv \tau_G \tag{215}$$

where α is defined by $A = \alpha(T - T_c)/T_c$. In case of superconducting transition we have: $\alpha = N(E_F)$, $B \sim N(E_F)/T_c^2$ and $C = N(E_F)\xi^2$. Accordingly, from Eq. (215) we get the following estimate for the critical region:

$$\tau_G \sim \frac{1}{N^2(E_F)\xi^6 T_c^2} \sim \left(\frac{E_F}{T_c}\right)^2 \frac{1}{\xi^6 p_F^6} \tag{216}$$

In the "pure" limit $\xi = \xi_0 \sim v_F/T_c$ and we get $\tau_G \sim (T_c/E_F)^4$, so that critical region is practically unobservable. In a "dirty" superconductor $\xi \sim \sqrt{\xi_0 l}$ and

$$\tau_G \sim \left(\frac{T_c}{E_F}\right)\frac{1}{(p_F l)^3} \tag{217}$$

and again we have $\tau_G \ll 1$. However, for a superconductor close to mobility edge $\xi \sim (\xi_0/p_F^2)^{1/3}$ and from Eq. (216) we get:[21,22]

$$\tau_G \sim 1 \tag{218}$$

Note that in fact τ_G may still be small because of numerical constants which we have dropped in our estimates. Anyhow, the critical region in this case becomes unusually wide and superconducting transition becomes similar in this respect to superfluid transition in Helium. Fluctuation effects may thus become observable even in bulk three—dimensional superconductor. Note that in localized phase $\xi \sim R_{loc}$ and $\tau_G \sim [N^2(E_F)R^6_{loc}T_c^2]^{-1} > 1$ if the condition given by Eq. (130) is violated.

Finally we should like to mention that thermodynamic fluctuations lead [21,22] to an additional mechanism of T_c degradation for a system which is close to Anderson transition. This follows from the general result on thr reduction of mean—field transition temperature due to critical fluctuations. If these fluctuations are small (and we can use the so called one—loop approximation) for a three-dimensional system it can be shown that [21.22]:

$$T_c = T_{c0} - \frac{7\zeta(3)}{16\pi^4\xi^3 N(E_F)} \tag{219}$$

where T_{c0} is the mean—field transition temperature. If we use here our expressions for ξ valid close to metal—insulator transition we easily find for $\sigma > \sigma^*$[22]:

$$T_c \approx T_{c0}\left[1 - 0.5\left(\frac{\sigma_c}{\sigma}\right)^{3/2}\left(\frac{T_{c0}}{E_F}\right)^{1/2}\right] \tag{220}$$

For $\sigma < \sigma^*$ this fluctuation correction saturates as the further drop of coherence length stops there. Obviously higher—order corrections are important here, but unfortunately little is known on the importance of this mechanism of T_c degradation outside the limits of one—loop approximation.

3.3.2 Upper critical field

Direct information on the value of $\xi^2(T)$ can be obtained from the measurements of the upper critical field H_{c2} [9]:

$$H_{c2} = \frac{\phi_0}{2\pi\xi^2(T)} \tag{221}$$

where $\phi_0 = \pi c\hbar/e$ is superconducting magnetic flux quantum. Using Eq. (201) we obtain the following relation between normal state conductivity σ, the slope of the upper critical field at $T = T_c$ given by $(dH_{c2}/dT)_{T_c}$ and the value of electronic density of states at the Fermi level (per one spin direction) $N(E_F)$[19,20]:

$$-\frac{\sigma}{N(E_F)}\left(\frac{dH_{c2}}{dT}\right)_{T_c} \approx \left\{ \begin{array}{ll} \frac{8e^2}{\pi^2\hbar}\phi_0 & \sigma > \sigma^* \\ \phi_0\frac{\sigma}{(\xi_0 l^2)^{2/3}T_c} \approx \phi_0\frac{\sigma}{[N(E_F)T_c]^{1/3}} & \sigma < \sigma^* \end{array} \right. \tag{222}$$

For $\sigma > \sigma^*$ the r.h.s. of Eq. (222) contains only the fundamental constants. This so called Gorkov's relation [12] is often used to interpret experimental data in "dirty" superconductors. Using it we may find $N(E_F)$ for different degrees of disorder from measurements of $(dH_{c2}/dT)_{T_c}$ and conductivity σ. On the other hand $N(E_F)$ can in principle be determined from independent measurements e.g. of electronic contribution to specific heat. However, our expression for $\sigma < \sigma^*$ which is valid close to metal—insulator transition shows that in this region Gorkov's relation becomes invalid and its use can "simulate" the drop of $N(E_F)$ with the growth of resistivity (disorder). Roughly speaking Eq. (222) shows that under the assumption pf relatively smooth change of $N(E_F)$ and T_c with disorder the usual growth of $(dH_{c2}/dT)_{T_c}$ with disorder saturates in conductivity region of $\sigma < \sigma^*$ close to the Anderson transition and the slope of the upper critical field becomes independent of resistivity. This stresses the importance of independent measurements of $N(E_F)$.

Note that the qualitative behavior given by Eq. (222) is retained also in the case when ω^δ—dependence of diffusion coefficient at the mobility edge (with some arbitrary critical exponent δ), only the expression for σ^* is changed as noted above. Thus this behavior is not related to any specific approximations of self—consistent theory of localization, except the general concept of continuous transition.

For an anisotropic (quasi—two—dimensional) system we have similar relations:

$$\left(\frac{dH_{c2}^{\perp}}{dT}\right)_{T_c} = -\frac{\phi_0}{2\pi\xi_{\parallel}^2 T_c} \tag{223}$$

$$\left(\frac{dH_{c2}^{\parallel}}{dT}\right)_{T_c} = -\frac{\phi_0}{2\pi\xi_{\parallel}\xi_{\perp} T_c} \tag{224}$$

with $\xi_{\parallel\perp}$ given above during our discussion after Eq. (207). This leads to relations and qualitative behavior similar to Eq. (222). However, we should like to note an especially interesting relation for the anisotropy of the slopes of the upper critical field [16]:

$$\frac{(dH_{c2}^{\parallel}/dT)_{Tc}}{(dH_{c2}^{\perp}/dT)_{Tc}} = \frac{\xi_{\parallel}}{\xi_{\perp}} = \frac{v_F}{wa/\hbar} \tag{225}$$

We see that the anisotropy of $(dH_{c2}/dT)_{T_c}$ is actually determined by the anisotropy of the Fermi velocity irrespective of the regime of superconductivity: from the "pure" limit, through the usual "dirty" case, up to the vicinity of the Anderson transition.

The above derivation of C coefficient of Ginzburg—Landau expansion explicitly used the time— reversal invariance expressed by Eq. (189). This is valid in the absence of the external magnetic field and magnetic impurities. Accordingly the previous results for the upper critical field are formally valid in the limit of infinitesimal external field and this is sufficient for the demonstration of superconducting (Meissner) response and for the determination of $(dH_{c2}/dT)_{T_c}$, because $H_{c2} \to 0$ as $T \to T_c$. In a finite external field we must take into account its influence upon localization. The appropriate analysis was performed in Refs.[68,130] and with a slightly different method in Ref.[73]. The results are essentially similar and below we shall follow Ref.[68]. The standard scheme for the analysis of superconducting transition in an external magnetic field

[12,9,131,132] gives the following equation determining the temperature dependence of $H_{c2}(T)$:

$$ln\frac{T}{T_c} = 2\pi T\sum_{\varepsilon_n}\left\{\frac{1}{2|\varepsilon_n| + 2\pi D_2(2|\varepsilon_n|)H/\phi_0} - \frac{1}{2|\varepsilon_n|}\right\} \tag{226}$$

where $D_2(2|\varepsilon_n|)$ is the generalized diffusion coefficient in the Cooper channel as defined after Eqs. (87) and (88). Eq. (226) is valid [9] for

$$R_H = \frac{mcv_F}{eH} \gg \xi \tag{227}$$

R_H is Larmor radius of an electron in magnetic field, ξ is the coherence length. Note that Eq. (226) describes only the orbital motion contribution to H_{c2}. In fact H_{c2} is also limited by the paramagnetic limit [9,132]:

$$\frac{1}{2}g_0\mu_B H < \Delta \tag{228}$$

where g_0 is the usual g—factor of an electron, μ_B is Bohr magneton.

Standard approach of the theory of "dirty" superconductors is based upon the replacement of $D_2(2|\varepsilon_n|)$ in Eq. (226) by Drude diffusion coefficient D_0 which is valid for a metal with $l \gg p_F^{-1}$. For a system which is close to the Anderson transition we must take into account both the frequency dependence of diffusion coefficient and the fact that in magnetic field D_2 is not equal to D_1 — the usual diffusion coefficient determining electronic transport. Actually we shall see that the external magnetic field influence upon localization leads to rather small corrections to $H_{c2}(T)$ practically everywhere except the region of localized states [68]. Thus we may really neglect this influence as a first approximation as that was done in Refs.[19,20] and start with the replacement of D_2 in Eq. (226) by $D_1 = D_E$, where D_E is the frequency dependent generalized diffusion coefficient in the absence of magnetic field. Detailed analysis of Eq. (226) can be found in Ref.[68].

Summation over Matsubara frequencies in Eq. (226) must be cut—off at some frequency of the order of $< \omega >$ — the characteristic frequency of Bose excitations responsible for pairing interaction. It is convenient here to measure the distance from Anderson transition (degree of disorder) via frequency ω_c defined in Eqs. (32),(68) or Eq. (90) . If a system is far from Anderson transition, so that $\omega_c \gg < \omega >$ we can completely neglect the frequency dependence of diffusion coefficient and find the usual results of the theory of "dirty" superconductors:

$$H_{c2}(T) = \frac{4}{\pi^2}\frac{\phi_0 T_c}{D_0} ln\frac{T_c}{T} \qquad T \sim T_c \tag{229}$$

$$H_{c2} = \frac{1}{2\gamma}\frac{\phi_0 T_c}{D_0}\left[1 - \frac{1}{24}\left(\frac{4\gamma T}{T_c}\right)^2\right] \qquad T \ll T_c \tag{230}$$

where $\gamma = 1.781....$ For the H_{c2} derivative at $T = T_c$ we find from here the first relation of Eq. (222), and $H_{c2}(T = 0)$ is conveniently expressed as[12,131]:

$$-\frac{H_{c2}(0)}{T_c(dH_{c2}/dT)_{T_c}} = \frac{\pi^2}{8\gamma} \approx 0.69 \tag{231}$$

In this case $H_{c2}(T)$ curve is convex at all temperatures below T_c.[12,131,9,132] Very close to the Anderson transition, when $\omega_c \ll 2\pi T$, only $\omega^{1/3}$ behavior of diffusion coefficient is important in Eq. (226) and we find[68]:

$$H_{c2}(T) = m\frac{\phi_0}{\pi}\frac{(4\pi)^{2/3}}{c_1}T^{2/3}E^{1/3}ln\frac{T_c}{T} \qquad T \sim T_c \tag{232}$$

$$H_{c2}(T) = m\frac{\phi_0}{\pi}(\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}\left[1 - \frac{2}{3}c_2\left(\frac{4\gamma T}{T_c}\right)^{2/3}\right] \qquad T \ll T_c \tag{233}$$

where $c_1 = \sum_{n=0}^{\infty}(n+1/2)^{-5/3} \approx 4.615$ and $c_2 \approx 0.259$. From these expressions we get:

$$-\frac{1}{N(E)}\left(\frac{dH_{c2}}{dT}\right)_{T_c} = \frac{(4\pi)^{2/3}}{\pi c_1}m\phi_0(E/T_c)^{1/3} = \frac{2\pi}{c_1}\frac{\phi_0}{[N(E)T_c]^{1/3}} \tag{234}$$

which makes precise the second relation in Eq. (222), while for $H_{c2}(T=0)$ we obtain:

$$-\frac{H_{c2}(0)}{T_c(dH_{c2}/dT)_{T_c}} = \frac{c_1}{(4\gamma)^{2/3}} \approx 1.24 \tag{235}$$

As was first noted in Refs.[19,20] this ratio for the system at the mobility edge is significantly larger than its classical value 0.69. In this case $H_{c2}(T)$ curve is concave for all temperatures below T_c.[20]Detailed expressions for the intermediate disorder when $2\pi T \ll \omega_c \ll < \omega >$ can be found in Ref.[68].

On Fig. 6 we present the results of numerical solution of Eq. (226) for the different values of characteristic frequency ω_c, i.e. for the different disorder. A smooth crossover from the classical behavior of the theory of "dirty" superconductors [131,9,132] to anomalous temperature dependence close to the Anderson transition [20] is clearly seen.

Below the mobility edge (i.e. in Anderson insulator) and for $\omega_c = 1/(2\pi^2 N(E)R_{loc}^3) \ll 2\pi T$, i.e. very close to mobility edge we can again use $\omega^{1/3}$—behavior of diffusion coefficient and find the same temperature dependence of H_{c2} as at the mobility edge itself or just above it. For $2\pi T \ll \omega_c \ll 2\pi T_c$ Eq. (226) takes the form [68]:

$$ln(T/T_c) = xln(\gamma\omega_c/\pi T_c) + \frac{3}{2}(1+x)ln(1+x) \tag{236}$$

where we have defined $x = \omega_H/\omega_c^{2/3}E^{1/3}$. This equation implicitly defines $H_{c2}(T)$ and shows [68] that now $H_{c2}(T) \to \infty$ for $T \to 0$ (logarithmic divergence). Numerical solution of Eq. (226) is shown at the insert in Fig. 6. Below we shall see however, that this divergence of H_{c2} is lifted by the inverse influence of magnetic field upon diffusion.

Let us now turn to the problem of magnetic field influence upon diffusion and its consequences for H_{c2} temperature behavior. If we are far from the Anderson transition magnetic field influence is small on parameter $\sim \sqrt{\omega_H/E}$ and its influence upon H_{c2} is insignificant. Close to the transition magnetic field correction may overcome the value of $D(H=0)$ and we have to consider its influence in detail [68]. Accordingly we shall limit ourselves with the case of $\omega_c/E \ll (\omega_H/E)^{2/3}$ for which we have already discussed the magnetic field behavior of generalized diffusion coefficient in Cooper channel. It was

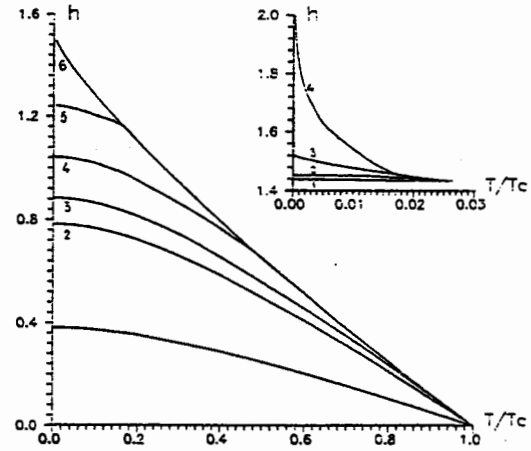

Figure 6: Temperature dependence of the upper critical field H_{c2}: Numerical solution of Eq. (226) for the dependence of $h = \omega_H/T_c^{2/3}E^{1/3}$ on T/T_c for different values of $\theta = \omega_c/T_c$: 1. $\theta = 100$; 2. $\theta = 10$; 3. $\theta = 2\pi$; 4. $\theta = 3$; 5. $\theta = 1$; 6. $\theta = 0$ (Mobility edge). No magnetic field influence on diffusion. At the insert:Low temperature part of h on T/T_c close to the Anderson transition: 1. Mobility edge ($\theta = 0$) with magnetic field influence on diffusion; 2. Metallic phase ($\theta = 0.1$). no magnetic field influence; 3. Mobility edge ($\theta = 0$), no magnetic field influence; 4. Insulating phase ($\theta = 0.1$), no magnetic field influence. Numerical cut—off was taken at $< \omega >= 100T_c$.

given in Eq. (94) and Eq. (95). In this case we have seen that characteristic frequency ω_c is replaced by:

$$\omega_c' = (\varphi\omega_H/E)^{3/2} \tag{237}$$

where $\varphi = W^2/2 \approx 0.18$.($W$ was defined during our discussion of localization in magnetic field). For $T \sim T_c$ there is no change in the slope of H_{c2} given by Eq. (234) as was noted already in Ref.[20]. Here we shall consider the case of $T \ll T_c$.

For $2\pi T > \omega_c'$ in all sums over Matsubara frequencies we can take $D(\omega) \sim \omega^{1/3}$ and actually we can neglect magnetic field influence upon diffusion. In this case $H_{c2}(T)$ behaves like in Eq. (233) i.e. as at the mobility edge in the absence of magnetic field effects. For $2\pi T < \omega_c'$ we find

$$H_{c2}(T) = m\frac{\phi_0}{\pi}(1+\varphi)^{-1/3}(\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}\left[1 - \frac{4\gamma}{3\varphi^{1/3}(1+\varphi)}\frac{T}{T_c}\right] \tag{238}$$

Accordingly we have

$$-\frac{H_{c2}(0)}{T_c(dH_{c2}/dT)_{T_c}} = (1+\varphi)^{-1/3}\frac{c_1}{(4\gamma)^{2/3}} \approx 1.18 \tag{239}$$

and the change in comparison with Eq. (235) is actually small. However, for $2\pi T < \omega_c'$ the $H_{c2}(T)$ curve becomes convex. The inflexion point can be estimated as $T^* = \omega_c'/2\pi \approx 0.02T_c$. This behavior is shown in the insert on Fig. 6.

Consider now insulating region. We shall see that the magnetic field effects on diffusion lead to the effective cut—off of the weak divergence of H_{c2} as $T \to 0$ noted above. It can be shown[68] that the external field defined by

$$\frac{W}{2}\sqrt{2\omega_H/E} > (\omega_c/E)^{1/3} \tag{240}$$

transfers the system from insulating to metallic state. If the system remains close to mobility edge we can estimate the upper critical field as above by $\omega_H \approx (\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}$ and Eq. (240) reduces to:

$$\omega_c \approx \frac{1}{2\pi^2 N(E)R_{loc}^3} < \frac{\pi}{\gamma}(W/\sqrt{2})^3 T_c \approx 0.14T_c \tag{241}$$

and practically in all the interval of localization lengths where according to our main criterion of Eq. (130) we can have superconductivity in Anderson insulator *the upper critical field in fact destroys localization* and the system becomes metallic. Accordingly there is no way to observe the divergence of the upper critical field as $T \to 0$ and the $H_{c2}(T)$ curves in "insulating" phase all belong to the region between the curves of $H_{c2}(T)$ at the mobility edge defined in the absence of magnetic field (curve 3 in the insert on Fig. 6) and at the mobility edge defined in magnetic field (curve 1 in the insert). This result actually shows that it may be difficult to confirm the insulating ground state of strongly disordered superconducting system just applying strong enough magnetic field to destroy superconductivity and perform usual transport measurements at low temperatures.

Note another mechanism for the change of $H_{c2}(T)$ at low temperatures was proposed by Coffey, Levin and Muttalib [133]. They have found the enhancement of H_{c2} at low temperatures due to the magnetic field dependence of the Coulomb pseudopotential μ^* which appears via the magnetic field dependence of diffusion coefficient. Magnetic field suppression of localization effects leads to the reduction of Coulomb pseudopotential enhancement due to these effects [18]. Accordingly we get the enhancement of H_{c2} at low temperatures. Unfortunately the apparently more important effects of the frequency dependence of generalized diffusion coefficient were dropped.

Returning to general criteria of validity of Eq. (226) we note that the condition of $R_H \gg \xi$ is reduced to $\omega_H \ll T_c^{1/3}E_F^{2/3}$ which is obviously satisfied in any practical case. Note, however, that our estimates for H_{c2} at low temperatures lead to $\omega_H = \Delta_0(E_F/\Delta_0)^{1/3} > \Delta_0$ which can easily overcome paramagnetic limit. In this case the experimentally observed H_{c2} of course will be determined by paramagnetic limit and anomalous behavior due to localization will be unobservable at low temperatures. At the same time in case of H_{c2} being determined by paramagnetic limit it may become possible to obtain insulating ground state of the system applying the strong enough magnetic field. Note that the effective masses entering to cyclotron frequency and paramagnetic splitting may be actually very different and there may be realistic cases when orbital critical field may dominate at low T. For $T \sim T_c$ H_{c2} is always determined by orbital contribution.

3.4 Superconductivity in Anderson Insulator at $T = 0$

We have already considered the superconducting response of a system which is close to Anderson transition within Ginzburg—Landau approximation, i.e. for temperatures $T \sim T_c$. In fact it is not difficult to obtain similar results also for $T = 0$.[23]

Superconducting current density at $T = 0$ is given by [9]:

$$\mathbf{j}_s = -\frac{n_s e^2}{mc}\mathbf{A} \tag{242}$$

where n_s is superconducting electron density, $\mathbf{A}$ is vector potential of an external magnetic field. On the other hand, using exact eigenstates representation DeGennes has obtained the following beautiful relation between superconducting response at $T = 0$ and conductivity of a system in the normal state [9,23]:

$$\mathbf{j}_s = \left\{\frac{1}{2\pi c}\int d\xi \int d\xi' L(\xi,\xi') Re\sigma(\xi - \xi') - \frac{ne^2}{mc}\right\}\mathbf{A} \tag{243}$$

All characteristics of a superconducting state are contained here in the kernel:

$$L(\xi,\xi') = \frac{1}{2}\frac{EE' - \xi\xi' - \Delta_0^2}{EE'(E+E')} \tag{244}$$

where $E = \sqrt{\xi^2 + \Delta_0^2}$ and Δ_0 is superconducting gap at $T = 0$. Note that in normal state $\mathbf{j}_s = 0$ and we can rewrite Eq. (243) as:

$$\mathbf{j}_s = \frac{1}{2\pi c}\int d\xi \int d\xi' [L(\xi,\xi')|_{\Delta=\Delta_0} - L(\xi,\xi')|_{\Delta=0}]\mathbf{A} \tag{245}$$

Taking into account that $L(\xi,\xi')|_{\Delta=\Delta_0} - L(\xi.\xi')|_{\Delta=0}$ for large $|\xi - \xi'|$ drops as $|\xi - \xi'|^{-3}$ it is sufficient to know only the low—frequency response of a system in normal state. In particular, for "pure" system (with no scattering) we have $Re\sigma(\omega) = (ne^2/m)\pi^{-1}\delta(\omega)$ and comparing Eq. (242) with Eq. (245) it is immediately clear that at $T = 0$ we have $n_s = n$, i.e. in an ideal system all electrons are superconducting.

Close to the Anderson transition we can use the results of elementary scaling theory of localization, e.g. Eq. (21) and Eq. (23) to write

$$\sigma(\omega) \approx \begin{cases} A\frac{g_c}{\xi_{loc}} & \omega < \omega_c \\ A\frac{g_c}{\xi_{loc}}\left(\frac{\omega}{\omega_c}\right)^{1/3} & \omega > \omega_c \end{cases} \tag{246}$$

where $\omega_c \sim [N(E)\xi_{loc}^3]^{-1}$ is defined in Eq. (32), g_c is the critical conductance of scaling theory ($g_c \sim 1$), $A \sim 1$. From Eq. (244) and Eq. (245) it is clear that the main contribution into integral in Eq. (245) comes from $|\xi - \xi'| \sim \Delta_0$, so that the value of n_s depends on the relation between Δ_0 and ω_c. For $\Delta_0 < \omega_c$ we have $\sigma(\Delta_0) = Ag_c/\xi_{loc}$ and

$$n_s = A\frac{m}{e^2}\Delta_0\frac{g_c}{\xi_{loc}} \tag{247}$$

For $\Delta_0 > \omega_c$ we have $\sigma(\Delta_0) = Ag_c[N(E)\Delta_0]^{1/3}$ and it becomes independent on the further growth of ξ_{loc} in the region of $\xi_{loc} > [N(E)\Delta_0]^{1/3}$. Accordingly n_s does not vanish at the mobility edge but saturate at

$$n_s = A\frac{m}{e^2}g_c[N(E)\Delta_0]^{1/3} \tag{248}$$

In localization region we can write instead of Eq. (246)

$$\sigma(\omega) \approx \begin{cases} 0 & \omega < \omega_c \\ Ag_c[N(E)\omega]^{1/3} & \omega > \omega_c \end{cases} \tag{249}$$

which again leads to $\sigma(\Delta_0) \approx Ag_c[N(E)\Delta_0]^{-1/3}$ and Eq. (248) remains valid until $R_{loc} > [N(E)\Delta_0]^{-1/3}$. Thus the density of superconducting electrons n_s remains finite close to Anderson transition both in metallic and insulating states.

However, from Eq. (248) it is easy to see that close to Anderson transition

$$\frac{n_s}{n} \sim \left(\frac{\Delta_0}{E_F}\right)^{4/3} \tag{250}$$

This coincide with an estimate of Eq. (205) based upon Ginzburg—Landau expansion. For typical Δ_0 and E_F only small part ($\sim 10^{-4}$ in traditional superconductors) of conduction electrons form Cooper pairs. The condition of $R_{loc} > [N(E)\Delta_0]^{-1/3} \sim a(E_F/\Delta_0)^{1/3}$ as discussed above defines the size of possible superconducting region in Anderson insulator. This region is of course quite small, e.g. if metal—insulator transition takes place with a change of some external parameter x (impurity concentration, pressure, fluence of fast neutrons etc.), so that $R_{loc} \sim a|(x - x_c)/x_c|^{-\nu}$, then for $\nu \approx 1$ and typical $E_F/\Delta_0 \sim 10^4$ we get $|x - x_c| < 0.1x_c$.

These estimates are in complete accordance with the results of our discussion of Ginzburg—Landau approximation [19,20] and we can obtain the qualitative picture of superconductivity in Anderson insulator both for $T \sim T_c$ and $T \to 0$, i.e. in the ground state.

4 STATISTICAL FLUCTUATIONS OF SUPERCONDUCTING ORDER PARAMETER

4.1 Statistical Critical Region

Thermodynamic fluctuations within Ginzburg critical region has been briefly considered above, with the conclusion of their increased importance in systems which are close to Anderson transition. Fluctuation conductivity of Cooper pairs (above T_c) is especially interesting in strongly disordered system because the usual single—particle contribution to conductivity drops to zero as the system moves towards Anderson transition. It can be analyzed within the standard approach [134,135] which takes into account fluctuational Cooper pairs formation above T_c. From these estimates we can see that as the system approaches the Anderson transition a temperature interval where the fluctuation contribution to conductivity is important widens. Fluctuation Cooper pair conductivity becomes comparable with a single—particle one for $\sigma < \sigma^* \approx \sigma_c(p_F\xi_0)^{-1/3} \approx \sigma_c(T_c/E_F)^{1/3}$, i.e. close enough to mobility edge. In fact this confirms the above picture of Cooper pairs remaining delocalized while single—particle excitations localize as the system undergoes metal—insulator transition.

Here we shall demostrate the appearance of the new type of fluctuations which are at least of the same importance as the usual critical fluctuations of superconducting

order—parameter. We call them statistical fluctuations [58] and their nature is closely connected to the problem of self—averaging properties of this order parameter (i.e. with a possibility of decoupling transforming Eq. (123) into Eq. (124)). We shall more or less follow Ref.[58], equivalent resuts were recently obtained in Ref.[136].

Let us return to the Eq. (118) and analyze the situation in more details. We shall use a simple iteration procedure assuming that fluctuations of the kernel $K(\mathbf{rr}')$ due to disorder are small. Similar approach was first used in Ref.[137]. In this case we can represent $K(\mathbf{rr}')$ and $\Delta(\mathbf{r})$ as

$$K(\mathbf{rr}') = K_0(\mathbf{r}-\mathbf{r}') + K_1(\mathbf{rr}'); \qquad K_0(\mathbf{r}-\mathbf{r}') = < K(\mathbf{rr}') >$$
$$\Delta(\mathbf{r}) = < \Delta > + \Delta_1(\mathbf{r}) \tag{251}$$

where $< \Delta >$ is the solution of linearized gap equation with averaged kernel $K_0(\mathbf{r}-\mathbf{r}')$ while $\Delta_1(\mathbf{r})$ is the first order correction over the perturbation defined by $K_1(\mathbf{rr}')$. We have seen that the linearized gap equation Eq. (124) with the averaged kernel $K_0(\mathbf{r}-\mathbf{r}')$ determines the standard transition temperature of BCS theory given by Eq. (129) which we shall now denote as T_{c0}. In the first order over K_1 there is no correction to T_{c0}: $< K_1 >=0$. In the second order of this perturbation theory we obtain the following change of transition temperature, defined as the temperature of appearance of homogeneous order—parameter:

$$\frac{T - T_{c0}}{T_{c0}} = \frac{1}{\lambda_p}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{K_1(\mathbf{q}0)K_1(0\mathbf{q})}{1 - K_0(\mathbf{q}, T_c)} \tag{252}$$
$$K_0 = \int d\mathbf{r} e^{i\mathbf{qr}} K(\mathbf{r}, T_c)$$

where

$$K_1(0\mathbf{q}) = K_1(-\mathbf{q}0) = \int d\mathbf{r}\int d\mathbf{r}' \epsilon^{i\mathbf{qr}}[K(\mathbf{rr}') - K_0(\mathbf{r}-\mathbf{r}')] =$$
$$= \lambda_p \int_0^{<\omega>} \frac{dE}{E} th\frac{E}{2T_c}\int d\mathbf{r} e^{i\mathbf{qr}}\left[\frac{1}{N(E)}\sum_\mu |\phi_\mu(\mathbf{r})|^2\delta(E - \varepsilon_\mu) - 1\right] \tag{253}$$

Here $\lambda_p = gN(E_F)$ and we have used the completenes and orthonormality of exact eigenfunctions $\phi_\mu(\mathbf{r})$. It is obvious that correction to T_{c0} given by Eq. (253) is always positive. After averaging Eq. (253) over disorder we get the relative change of transition temperature due to fluctuations as

$$\frac{\delta T_c}{T_{c0}} = \left\langle \frac{T_c - T_{c0}}{T_{c0}} \right\rangle = \lambda_p \int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{\varphi(\mathbf{q})}{1 - K_0(\mathbf{q}, T_c)} \tag{254}$$
$$\varphi(\mathbf{q}) = \int d\mathbf{r} e^{i\mathbf{qr}}\varphi(\mathbf{r})$$

where

$$\varphi(\mathbf{r}) = \int_0^{<\omega>} \frac{dE}{E} th\frac{E}{2T_c}\int_0^{<\omega>} \frac{dE'}{E'} th\frac{E'}{2T_c}\left\{\frac{1}{N(E)} \ll \rho_E(\mathbf{r})\rho_{E'}(0) \gg^H - 1\right\} \tag{255}$$

and we have introduced the spectral density of Eq. (72), which is actually a correlation function of local densities of states.

Remember now that in a "dirty" system [132]:

$$1 - K_0(\mathbf{q},T) = 1 - 2\pi T\lambda_p \sum_{\varepsilon_n} \frac{1}{2|\varepsilon_n| + D_E(2|\varepsilon_n|)q^2} \approx$$

$$\approx \lambda_p \left[\frac{T - T_{c0}}{T_{c0}} + \xi^2 q^2\right] \qquad \varepsilon_n = (2n+1)\pi T \tag{256}$$

where ξ is the coherence length defined previously e.g. in Eq. (196). The approximate equality here is valid for $|T - T_{c0}|/T_c \ll 1$, $\xi^2 q^2 \ll 1$. From Eq. (255) and Eq. (256) we get the change of transition temperature in the following form:

$$\frac{\delta T_c}{T_{c0}} = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{\varphi(\mathbf{q})}{\xi^2 q^2} \tag{257}$$

Here we must cut—off integration at $q \sim \xi^{-1}$ in accordance with limits of applicability of the last expression in Eq. (256). However, the contribution of short—wave fluctuations here may be also important.

The Ginzburg—Landau functional expressed via non—averaged order parameter $\Delta(\mathbf{r})$ has the following form [9]:

$$F\{\Delta\} = \int d\mathbf{r} \left\{ \frac{N(E_F)}{\lambda_p} |\Delta(\mathbf{r})|^2 - N(E_F) \int d\mathbf{r}' K(\mathbf{r}\mathbf{r}') \Delta(\mathbf{r}') \Delta(\mathbf{r}) + \frac{1}{2} B |\Delta(\mathbf{r})|^4 \right\} \tag{258}$$

where we have neglected the fluctuations of pairing interaction λ_p and of the coefficient B, which is defined by the standard expression given in Eq. (195). Using Eqs. (251)—(253) we can find Ginzburg—Landau equations which describe the slow changes of $\Delta(\mathbf{r})$:

$$\left\{ N(E_F) \frac{T_{c0} - T}{T_{c0}} + \delta A(\mathbf{r}) - B|\Delta(\mathbf{r})|^2 + C \frac{\partial^2}{\partial r^2} \right\} \Delta(\mathbf{r}) = 0 \tag{259}$$

where

$$\delta A(\mathbf{r}) = N(E_F) \int_0^{<\omega>} \frac{dE}{E} th \frac{E}{2T_{c0}} \left\{ \frac{1}{N(E_F)} \sum_\nu |\phi_\nu(\mathbf{r})|^2 \delta(E - \varepsilon_\nu) - 1 \right\} \tag{260}$$

describes the fluctuations of the coefficient A of Ginzburg—Landau expansion and we have neglected the fluctuations of the C coefficient .

Ginzburg—Landau equations with fluctuating coefficients were analyzed for the first time by Larkin and Ovchinnikov[138]. It was shown that $\delta A(\mathbf{r})$—fluctuations lead to a shift of transition temperature given by Eq. (257) and the solution of Eq. (259) for the order parameter in the first order over fluctuations has the form of Eq. (251) with:

$$\Delta_1(\mathbf{r}) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \Delta_1(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}}$$

$$\Delta_1(\mathbf{q}) = -\frac{<\Delta>}{N(E_F)} \frac{\delta A(\mathbf{q})}{\xi^2 q^2 + 2\tau} \tag{261}$$

where $\tau = (T_c - T)/T_c$ is temperature measured relative to the new transition temperature. The mean—square fluctuation of the order—parameter itself is determined from Eq. (261) by:

$$\frac{<\Delta^2>}{<\Delta>^2} - 1 = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{\varphi(\mathbf{q})}{[\xi^2 q^2 + 2\tau]^2} \tag{262}$$

where $\varphi(\mathbf{q})$ was introduced in Eqs. (255), (255). It is important to note that fluctuations of $\Delta(\mathbf{r})$ as opposed to T_c—shift are determined by small $\mathbf{q}$ behavior of $\varphi(\mathbf{q})$.

We can see now that all the physics of statistical fluctuations is described by the correlation function of local densities of states. This function was determined above in Eqs. (73). Using Eq. (74) for the metallic state not very close to the mobility edge we can get from Eq. (255):

$$\varphi(\mathbf{q} = 0) \sim \frac{\xi}{N^2(E_F) D_0^2} \tag{263}$$

where $\xi = \sqrt{\xi_0 l}$ and D_0 is the Drude diffusion coefficient. Estimating the T_c—shift from Eq. (257) we get:

$$\frac{\delta T_c}{T_{c0}} \sim \frac{1}{N^2(E_F) D_0^2 \xi^2} \sim \frac{T_c}{E_F} \frac{1}{(p_F l)^3} \sim \tau_G \tag{264}$$

where τ_G is the size of Ginzburg critical region defined by Eq. (215). We have seen that in the usual "dirty" superconductor $\tau_G \ll 1$. For the order—parameter fluctuations from Eq. (262) we obtain:

$$\frac{<\Delta^2>}{<\Delta>^2} - 1 \approx \frac{1}{8\pi} \frac{\varphi(\mathbf{q} = 0)}{\xi^3 \sqrt{2|\tau|}} \approx \left(\frac{\tau_D}{|\tau|}\right)^{1/2} \tag{265}$$

From here we can see that the width of the temperature region where statistical fluctuations are important is given by:

$$\tau_D \sim \frac{\varphi^2(0)}{\xi^6} \sim \frac{1}{N^4(E_F) D_0^4 \xi^4} \sim \left(\frac{T_c}{E_F}\right)^2 \frac{1}{(p_F l)^6} \sim \tau_G^2 \tag{266}$$

It is obvious that in a"dirty" superconductor we have $\tau_D \ll \tau_G \ll 1$ and statistical fluctuations are absolutely unimportant.

Situation change for a system which is close to the mobility edge. Using Eq. (74) with D_0 replaced by $D_0(\omega/\gamma)^{1/3}$ we obtain:

$$\varphi(\mathbf{q}) \approx \frac{\gamma^{1/2}}{N^2(E_F) D_0^3 T_c} \int_0^{T_c} \frac{d\omega}{\omega^{1/2}} [\omega^2 + D_0 \gamma^{-2/3} \omega^{2/3} q^4]^{-1/4} \sim \xi^3 ln \frac{1}{\xi q} \tag{267}$$

where $\xi \sim (\xi_0 p_F^{-2})^{1/3}$. Similarly we get:

$$\frac{<\Delta^2>}{<\Delta>^2} - 1 \approx \int_0^{\xi^{-1}} \frac{\xi^3 q^2 dq}{(\xi^2 q^2 + 2\tau)^2} ln \frac{1}{q\xi} \sim \frac{1}{\sqrt{|\tau|}} ln \frac{1}{|\tau|} \tag{268}$$

From Eq. (268) it follows that close to the mobility edge statistical fluctuations become important and even overcome thermodynamic fluctuations due to the logarithmic factor in $\varphi(\mathbf{q})$. Thus in this region we have $\tau_D > \tau_G \sim 1$.

The crossover from the regime of weak statistical fluctuations ($\tau_D \ll \tau_G$) to the strong fluctuation regime occurs at the conductivity scale $\sigma \sim \sigma^* \approx \sigma_c(p_F\xi_0)^{-1/3}$ which was extensively discussed above. Thus close to the mobility edge the superconducting order—parameter is no more a self averaging quantity. Here the mean—field theory approach becomes formally invalid due to thermodynamic and also because of statistical fluctuations. Below we shall analyze this situation in more details.

Finally we shal briefly discuss the region of localized localized states. Here we again encounter a singular $\delta(\omega)$—contribution to the correlator of local densities of states, the same as for Gorkov—Berezinskii spectral density, which leads to the additional contribution to $\varphi(\mathbf{q})$:

$$\varphi(\mathbf{q}) = \int_0^{<\omega>} \frac{dE}{E^2}\left(th\frac{E}{2T_c}\right)\frac{A_E(\mathbf{q})}{N(E_F)} + \cdots \sim \frac{A_{E_F}}{N(E_F)T_{c0}} + \cdots = \\ = \frac{1}{N(E_F)T_c(1+R_{loc}^2q^2)} + \cdots \qquad (269)$$

Accordingly a new contribution to $\Delta(\mathbf{r})$ fluctuations is given by:

$$\frac{<\Delta^2>}{<\Delta>^2} - 1 \approx \frac{1}{N(E_F)T_c}\int_0^{R_{loc}^{-1}} \frac{q^2dq}{(\xi^2q^2+2|\tau|)^2(1+R_{loc}^2q^2)} \sim \frac{1}{N(E_F)T_cR_{loc}^3\tau^2} \qquad (270)$$

and it grows fast as the localization length R_{loc} diminishes. Using our main criterion of superconductivity in localized phase given by Eq. (130) we can see that in all region of possible superconductivity statistical fluctuations of $\Delta(\mathbf{r})$ remain of the order of unity and are important in rather wide temperature interval around T_c.

4.2 Superconducting Transition at Strong Disorder

We consider now superconductivity in systems with strong statistical fluctuations of the "local transition temperature" as described by Eq. (259) and Eq. (260). In this analysis we shall follow Refs.[139,140]. For simplicity we assume Gaussian nature of these fluctuations. Note, however, that close to the mobility edge the fluctuations of local density of states become strongly non—Gaussian [141] and this can complicate the situation. Unfortunately the importance of this non—Gaussian behavior for superconductivity has not been studied up to now.

We shall see that as statistical fluctuations become strong enough the superconducting state can appear in inhomogeneous fashion even if the correlation length of disorder induced fluctuations of local transition temperature is small compared with the superconducting correlation length ξ (microscopic disorder). This case was first analyzed by Ioffe and Larkin[145]. Investigating the case of extremely strong disorder they have shown that as the temperature is lowered the normal phase acquires localized superconducting regions (drops) with characteristic size determined by $\xi(T)$. Far from T_c their density is low, but with further cooling the density and dimensions of the drops

increase and they begin to overlap leading to a kind of percolative superconducting transition.

Our treatment of superconductors with large statistical fluctuations will be based on the Ginzburg—Landau functional:

$$F\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\} = \int d\mathbf{r}\left\{\frac{\mathbf{B}^2(\mathbf{r})}{8\pi} + N(E_F)\left[(\tau + t(\mathbf{r}))|\Delta(\mathbf{r})|^2 + \right.\right. \\ \left.\left. + \xi^2\left|\left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}(\mathbf{r})\right)\Delta(\mathbf{r})\right|^2 + \frac{1}{2}\lambda|\Delta(\mathbf{r})|^4\right]\right\} \qquad (271)$$

where $\mathbf{B} = rot\mathbf{A}$ is magnetic field and we have redefined the coefficient of quartic term as $B = N(E_F)\lambda$. Here $t(\mathbf{r})$ is defined by Eq. (260) as $\delta A(\mathbf{r}) = N(E_F)t(\mathbf{r})$ and plays the role of the fluctuation of local "critical temperature", which appears due to fluctuations of local density of states. In general case it also can have contributions from local fluctuations of pairing interaction or other types of microscopic inhomogeneities. As noted above we assume Gaussian statistics of these fluctuations. though real situation close to the mobility edge may be more complicated [141]. Given the distribution of $t(\mathbf{r})$, the free energy of the system and the order—parameter correlator are equal to:

$$\mathcal{F}\{t(\mathbf{r})\} = -TlnZ, \quad Z = \int D\{\mathbf{A},\Delta\}exp[-F\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\}/T] \qquad (272)$$

$$<\Delta(\mathbf{r})\Delta(\mathbf{r}')> = Z^{-1}\int D\{\mathbf{A},\Delta\}\Delta(\mathbf{r})\Delta(\mathbf{r}')exp[-F\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\}/T] \qquad (273)$$

and must be averaged over the Gaussian distribution of $t(\mathbf{r})$. From our definition of $t(\mathbf{r})$ and using the approach of the previous section, assuming the short—range of fluctuations of local density of states(on the scale of ξ), it is easy to estimate the correlator of $t(\mathbf{r})$ as:

$$<t(\mathbf{r})t(\mathbf{r}')> = \gamma\delta(\mathbf{r}-\mathbf{r}'), \qquad \gamma \approx \tau_D^{1/2}\xi^3 \qquad (274)$$

Then the probability of a configuration with a given $t(\mathbf{r})$ is given by

$$\mathcal{P}\{t(\mathbf{r})\} = exp\left[-\frac{1}{2\gamma}\int dr t^2(\mathbf{r})\right] \qquad (275)$$

The problem reduces thus to calculation of the functions $\mathcal{F}\{t(\mathbf{r})\}$ and $<\Delta(\mathbf{r})\Delta(\mathbf{r}')>$ and their subsequent averaging over $\mathcal{P}\{t(\mathbf{r})\}$.

We shall limit ourselves to consideration of noninteracting drops and no vortices. Then we can consider the phase of the order—parameter $\Delta(\mathbf{r})$ as nonsingular. After the gauge transformation

$$\mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + (c\hbar/2e)\nabla\phi(\mathbf{r}) \\ \Delta(\mathbf{r}) \to \Delta(\mathbf{r})exp[-i\phi(\mathbf{r})] \qquad (276)$$

where $\phi(\mathbf{r})$ is the *phase of the order parameter* we can use real $\Delta(\mathbf{r})$ and Ginzburg—Landau functional of Eq. (271) becomes:

$$F\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\} = \int d\mathbf{r}\left\{\frac{\mathbf{B}^2(\mathbf{r})}{8\pi} + N(E_F)\left[(\tau + t(\mathbf{r}))\Delta^2(\mathbf{r}) + \frac{4e^2\xi^2}{c^2\hbar^2}\mathbf{A}^2(\mathbf{r})\Delta^2(\mathbf{r}) + \right.\right. \\ \left.\left. + \xi^2(\nabla\Delta^2(\mathbf{r})) + \frac{1}{2}\lambda\Delta^4(\mathbf{r})\right]\right\} \qquad (277)$$

Integration over phase in Eq. (272) gives an inessential constant factor to the partition function which we disregard.

To average the logarithm of the partition function Eq. (272) over $t(\mathbf{r})$ we can use the replica trick[142] which permits the averaging to be carried out in explicit form. We express the average free energy Eq. (272) of the system in the form:

$$< \mathcal{F} >= -T \lim_{n\to 0} \frac{1}{n}[< Z^n > -1] \tag{278}$$

To calculate $< Z^n >$ in accordance with the idea of the replica method, we first assume n to be an arbitrary integer. Expressing Z^n in terms of an n—fold functional integral over the fields of the replicas $\mathbf{A}_\alpha$, $\Delta_\alpha(\mathbf{r})$, $\alpha = 1, ..., n$ and carrying out exact Gaussian averaging over $t(\mathbf{r})$, we get

$$< Z^n >= \int D\{\mathbf{A}, \Delta\} exp[-S_n\{\mathbf{A}_\alpha, \Delta_\alpha\}] \tag{279}$$

$$S\{\mathbf{A}_\alpha, \Delta_\alpha\} = \int d\mathbf{r} \left\{ \sum_\alpha^n \frac{\mathbf{B}^2(\mathbf{r})}{8\pi T} + \frac{N(E_F)}{T} \sum_\alpha^n \left[(\tau + t(\mathbf{r}))\Delta_\alpha^2(\mathbf{r}) + \frac{4e^2\xi^2}{c^2\hbar^2}\mathbf{A}_\alpha^2(\mathbf{r})\Delta_\alpha^2(\mathbf{r}) + \right.\right.$$
$$\left.\left. + \xi^2(\nabla\Delta_\alpha^2(\mathbf{r})) + \frac{1}{2}\lambda\Delta_\alpha^4(\mathbf{r})\right] - \frac{1}{2}\frac{N(E_F)}{T}\tilde{\gamma}\left[\sum_{\alpha=1}^n \Delta_\alpha^2(\mathbf{r})\right]^2 \right\}$$

The last expression here represents the "effective action" and $\tilde{\gamma} = \gamma N(E_F)/T_c \approx \tau_D^{1/2} N(E_F)/T_c$ grows with disorder . Note that the random quantities $t(\mathbf{r})$ have already dropped out of these expressions, and that the action $S\{\mathbf{A}_\alpha, \Delta_\alpha\}$ is translationally invariant. For the correlator of Eq. (273) we obtain:

$$< \Delta(\mathbf{r})\Delta(\mathbf{r}') >= \lim_{n\to 0} \frac{1}{n} \int D\{\mathbf{A}, \Delta\} exp[-S_n\{\mathbf{A}_\alpha, \Delta_\alpha\}] \sum_{\alpha=1}^n \Delta_\alpha(\mathbf{r})\Delta_\alpha(\mathbf{r}') \tag{280}$$

where we have symmetrized over the replica indices.

Far from the region of strong fluctuations of the order parameter $|\tau| \gg \tau_D, \tau_G$ the functional integrals in Eq. (279) and Eq. (273) can be calculated by the saddle—point method. The extrema of the action are determined by classical equations:

$$\left[\tau - \xi^2\nabla^2 + \lambda\Delta_\alpha^2 - \tilde{\gamma}\sum_{\beta=1}^n \Delta_\beta^2(\mathbf{r})\right]\Delta_\alpha(\mathbf{r}) = 0 \qquad \mathbf{A}_\alpha = 0 \tag{281}$$

The nontrivial conclusion is that these equations for $\Delta_\alpha(\mathbf{r})$ besides spatially homogeneous solutions do have localized solutions with finite action (*instantons*). These correspond at $\tau > 0$ to superconducting drops. We shall limit ourselves to a picture of noninteracting drops and consider only instanton solutions above T_c (at $\tau > 0$). We shall be interested only in those solutions that admit analytic continuation as $n \to 0$. We designate them $\Delta_\alpha^{(i)}(\mathbf{r})$, where the superscript i labels the type of solution. To find their contribution we must expand the action of Eq. (279) up to the terms quadratic in deviations $\varphi_\alpha(\mathbf{r}) = \Delta_\alpha(\mathbf{r}) - \Delta_\alpha^{(i)}(\mathbf{r})$. It can be shown that fluctuations of the fields $\mathbf{A}_\alpha(\mathbf{r})$ can be neglected if we consider noninteracting drops [139,140].

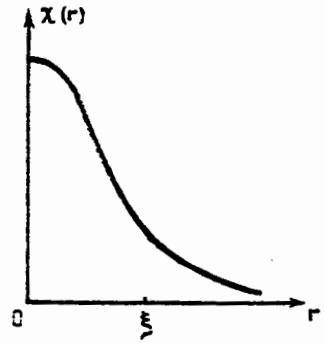

Figure 7: Qualitative form of instanton solution.

For $\tau > 0$ and for $\tilde{\gamma} > \lambda$ Eq. (281) possess (besides the trivial solution $\Delta_\alpha = 0$) the following nontrivial solution with finite action (instanton)(Cf. Refs.[143,144,7]):

$$\begin{aligned} &\Delta_\alpha^{(i)}(\mathbf{r}) = \Delta_0(\mathbf{r})\delta_{\alpha i}, \qquad i = 1, ..., n \\ &\Delta_0(\mathbf{r}) = \sqrt{\tfrac{\tau}{\tilde{\gamma}-\lambda}}\chi\left[\tfrac{r}{\xi(T)}\right]. \qquad \xi(T) = \tfrac{\xi}{\sqrt{\tau}} \end{aligned} \tag{282}$$

where the dimensionless function $\chi(x)$ satisfies the condition $d\chi(x)/dx|_{x=0} = 0$ and its asymptotic form: $\chi(x) \sim x^{-1}exp(-x)$ for $x \gg 1$ (for spatial dimension $d = 3$). The qualitative form of this solution is shown in Fig. 7.

From Eq. (282) it is seen that instantons are oriented along axes of replica space (there are n types of instanton solutions) which is due to the "cubic anisotropy" term $\lambda\Delta_\alpha^4$ in the effective action of Eq. (279). Index i characterizes the direction in replica space along which the symmetry breaking takes place. For $\lambda \to 0$ the action becomes $O(n)$—symmetric and instantons take the form :

$$\Delta_\alpha(\mathbf{r}) = \Delta_0(\mathbf{r})e_\alpha, \qquad \sum_{\alpha=1}^n e_\alpha^2 = 1 \tag{283}$$

i.e. are oriented along arbitrary unit vector $\vec{e}$ in replica space. Such instantons earlier were studied in the theory of localization [143,144,7].

The quadratic expansion of the effective action near instanton solution takes the form (Cf. analogous treatment in Refs.[143,144,7]):

$$S\{\Delta_\alpha\} = S\{\Delta_\alpha^{(i)}\} + \frac{1}{2}\int d\mathbf{r} \sum_{\alpha,\beta}(\varphi_\alpha \hat{M}_{\alpha\beta}^{(i)}\varphi_\beta) \tag{284}$$

where the operator $\hat{M}_{\alpha\beta}^{(i)}$ on instanton solutions is equal to:

$$\hat{M}_{\alpha\beta}^{(i)} = [\hat{M}_L\delta_{\alpha i} + \hat{M}_T(1 - \delta_{\alpha i})]\delta_{\alpha\beta} \tag{285}$$

with

$$M_{L,T} = \frac{2N(E_F)}{T}[-\xi^2\nabla^2 + \tau U_{L,T}(\mathbf{r})] \tag{286}$$

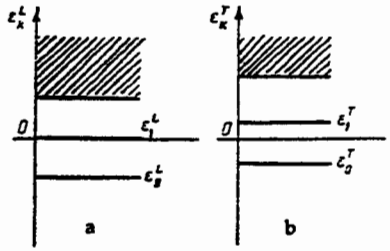

Figure 8: Qualitative structure of eigenvalues of M_L (a) and M_T (b) operators: $\varepsilon_1^L = 0$ — translation zero—mode; $\varepsilon_0^T \to 0$ for $\lambda \to 0$—transforms to "rotation" zero—mode. The continuous part of the spectrum is shaded.

where

$$U_L(\mathbf{r}) = 1 - 3\chi^2[r/\xi(T)]$$
$$U_T(\mathbf{r}) = 1 - (1 - \lambda/\tilde{\gamma})^{-1}\chi^2[r/\xi(T)] \qquad (287)$$

The value of Gaussian functional integral is determined by the spectra of eigenstates of operators M_L and M_T . Detailed analysis can be found in Refs.[139,140]. The qualitative form of these spectra is shown in Fig. 8.

Operator M_L always possess an eigenvalue $\varepsilon_1^L = 0$—the so called translation zero—mode, connected with translation symmetry: instanton center may be placed anywhere in space, the action does not change . However, this is not a lowest eigenvalue of M_L, there is always a negative eigenvalue $\varepsilon_0^L < \varepsilon_1^L = 0$. It can be shown rigorously that it is the only negative eigenstate of M_L[146]. Operator M_T possess also a single negative eigenvalue $\varepsilon_0^T < 0$[139,140], however this eigenvalue tends to zero for $\lambda \to 0$ becoming the "rotation" zero—mode, reflecting the arbitrary "direction" of instanton in replica space in the absence of cubic anisotropy in the action [143,144,7]. For $\lambda = \lambda^* = 2/3\tilde{\gamma}$ we have $M_L = M_T$ and the spectra of both operators coincide.

Including the contributions of instantons oriented along all the axes in replica space we obtain the following one—instanton contribution to the partition function entering Eq. (278)[139,140]:

$$< Z^n >= n\Omega \left(\frac{J_L}{2\pi}\right)^{d/2} [Det' M_L]^{-\frac{1}{2}} [Det M_T]^{\frac{1-n}{2}} exp\{-S_0(\tau)\} \qquad (288)$$

where Ω is the system volume,

$$J_L = \frac{1}{d}\int dr \left(\frac{\partial \Delta_0}{\partial r}\right)^2 = \frac{T}{2N(E_F)}\frac{S_0(\tau)}{\xi^2} \qquad (289)$$

and the action at the instanton is given by:

$$S_0(\tau) = \mathcal{A}\frac{\xi^3 T^{1/2}}{\gamma - \lambda T/N(E_F)} \qquad (290)$$

where $\mathcal{A} \approx 37.8$ is a numerical constant[147]. The prime on $Det M_L$ means that we must exclude the zero—eigenvalue $\varepsilon_1^L = 0$ from the product of eigenvalues determining

this determinant. The condition of applicability of the saddle—point approximation looks like $S_0(\tau) \gg 1$, and in fact all our analysis is valid outside the critical regions both for thermodynamic and statistical fluctuations.

In the limit of $n \to 0$ the total cancellation of imaginary contributions appearing due to negative eigenvalues takes place in Eq. (288) and using Eq. (278) we get for $\tilde{\gamma} > 3/2\lambda$ the following *real* contribution to the free energy:

$$\mathcal{F} = -\rho_s(\tau) T\Omega \qquad (291)$$

where the density of superconducting "drops"

$$\rho_s(\tau) = \left[\frac{T}{4\pi N(E_F)} S_0(\tau)\right]^{3/2} \xi^{-3} \left[\frac{Det M_T}{Det' M_L}\right]^{1/2} exp\{-S_0(\tau)\} \qquad (292)$$

Thus for $\tilde{\gamma} > 3/2\lambda$ even for $T > T_c$ the superconducting "drops" (instantons) appear in the system which directly contribute to the equilibrium free energy. This contribution given by Eqs. (291)— (292) exists along the usual thermodynamic fluctuations. The condition of $\tilde{\gamma} > 3/2\lambda$ defines critical disorder $\tau_D > \tau_D^* > \tau_G$, and this inhomogeneous picture of superconducting transition appears only for the case of sufficiently strong statistical fluctuations. The knowledge of qualitative structure of spectra of eigenvalues of M_L and M_T allows to analyze different asymptotics of Eq. (291)[139,140]. For $\tilde{\gamma} S_0(\tau) \ll \lambda \ll \lambda^*$ we get:

$$\rho_s(\tau) \approx \xi^{-3}(T) \left(\frac{\lambda}{\tilde{\gamma}}\right)^{1/2} S_0^{3/2}(\tau) exp[-S_0(\tau)] \qquad (293)$$

For $\lambda \to \lambda^*$ we obtain:

$$\rho_s(\tau) \approx \xi^{-3}(T) \left(\frac{\lambda^*}{\lambda} - 1\right)^{3/2} S_0^{3/2}(\tau) exp[-S_0(\tau)] \qquad (294)$$

Thus the density of superconducting "drops" $\rho_s(\tau)$ vanishes as $\lambda \to \lambda^*$. they are destroyed by thermodynamic fluctuations.

For the order—parameter correlator of Eq. (273) we get the following result:

$$< \Delta(\mathbf{r})\Delta(\mathbf{r}') >\approx \rho_s(\tau) \int d\mathbf{R_0} \Delta_0(\mathbf{r} + \mathbf{R_0})\Delta_0(\mathbf{r}' + \mathbf{R_0}) \qquad (295)$$

The integration over instanton center $\mathbf{R_0}$ here means in fact averaging over different positions of "drops". Note that over large distances this correlator decreases like $exp[-|\mathbf{r} - \mathbf{r}'|/\xi(T)]$ and does not contain the usual Ornstein—Zernike factor $|\mathbf{r} - \mathbf{r}'|^{-1}$.

We have found the free—energy of inhomogeneous superconducting state in the temperature region $\tau \gg \tau_D$, where the "drop" concentration is exponentially small and the picture of noninteracting "drops" is valid. They give exponentially small contribution to the specific heat and diamagnetic susceptibility. The characteristic size of "drops" is determined by $\xi(T)$ and as $T \to T_c$ the "drops" grow and begin to overlap leading to a percolative superconducting transition. Thus for $\tau_D > \tau_D^* > \tau_G$ superconductivity first appears in isolated "drops". This is similar to the picture of decay of a metastable state in case of the first—order phase transitions[148]. However, in this

latter case instantons give imaginary contribution to the free energy determining the decay rate of a "false" equilibrium state (critical bubble formation). Here instanton contributions lead as was noted above to real free energy and "drops" appear in the true equilibrium state.

It is more or less obvious that between isolated "drops" a kind of Josephson coupling may appear and lead to rather complicated phase diagram of the system in external magnetic field, e.g. including the "superconducting glass" phase[149,150]. Note the completely different appraoch to these problems proposed by Oppermann[151,152,153,154]. The existence of inhomogeneous regime of superconductivity will obviously lead to the rounding of BCS—like singularities of the density of states and superconductivity may become gapless. (Cf. Ref.[153]). Note that diffusion—enhanced Coulomb interactions can also lead to the gaplessness of strongly disordered superconductors via Coulomb—induced inelastic scattering [155]. Fluctuation conductivity in a similar inhomogeneous superconducting state was studied in Ref.[156].

Note the closely related problem of strongly disordered superfluids [157,158,159]. Some results here may be quite useful for the case of strongly disordered superconductors, though the limitations of this analogy are also important.

These and other possible manifestations of this new regime of superconductivity in strongly disordered systems define rather wide range of problems to be studied in future.

5 SUPERCONDUCTIVITY IN STRONGLY DISORDERED METALS: EXPERIMENT

Our review of experiments on strongly disordered superconductors will be in no sense exhaustive. This is mainly a theoretical review and the author is in no way an expert on experiment. However, we shall try to illustrate the situation with the interplay of Anderson localization and superconductivity in *bulk* (three—dimensional) superconductors, both traditional and high—temperature. Here we shall confine ourselves to a limited number of the experiments , which we consider most interesting from the point of view of illustration of some of the ideas expressed above. More than anywhere else in this review our choice of material is based on personal interest of the author, or our direct involvement in the discussion of experiment. We shall not deal with the general problem of disorder influence upon superconductivity, but shall consider only the systems which remain superconducting close to the disorder—induced metal—insulator transition. Complete review of the experiments on highly—disordered superconductors is left to somebody more competent.

5.1 Traditional Superconductors

There exists a number of strongly disordered systems which remain superconducting close to the metal—insulator transition induced by disorder.

The drop of T_c with conductivity decrease from the value of the order of $10^4 Ohm^{-1}cm^{-1}$ was observed in amorphous alloys of $GeAl$[160], $SiAu$[161] and $MoRe$[162], in Chevrel phase superconductors disordered by fast neutron irradiation, such as $Pb_{1-x}U_xMo_6S_8$[163],

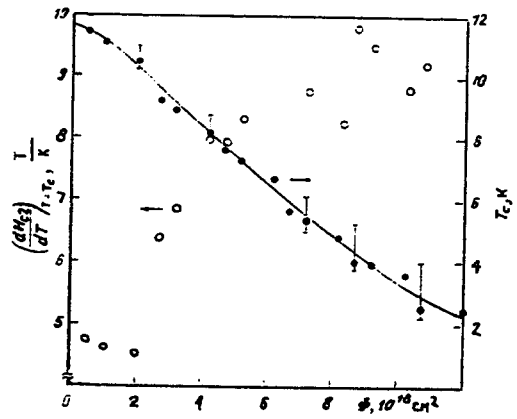

Figure 9: Fluence dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in $SnMo_5S_6$ (Ref.[164]).

$SnMo_6S_8$[164], Mo_6Se_8[165], in amorphous InO_x[166], in $BaPb_{1-x}Bi_xO_3$ in the concentration interval $0.25 < x < 0.30$[167] and in metallic glass $Zr_{0.7}Ir_{0.3}$[169]. In all of these systems superconducting transition is observed apparently not very far from the metal—insulator transition. For a number of these systems, such as $Pb_{1-x}U_xMo_6S_8$, $SnMo_6S_8$, Mo_6Se_8, $Zr_{0.7}Ir_{0.3}$ and $BaPb_{0.75}Bi_{0.25}O_3$[167] and some others a characteristic strongly negative temperature resistivity coefficient has been observed. The drop of T_c close to the mobility edge apparently was also observed in As_2Te_3[170]. However, in all of these systems T_c apparently vanishes before metal—insulator transition. Below we present some of the data on these and other similar systems.

On Fig. 9 we show the dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in $SnMo_5S_6$ (Chevrel phase superconductor) on the fluence of fast neutron irradiation (the number of neutrons which passed through a crossection of a sample during irradiation)[164].

In the region of large fluences (large disorder), when the system becomes amorphous, characteristic values of conductivity in the normal state are of the order of $\sim 10^3 Ohm^{-1}cm^{-1}$, which is not far from the values of "minimal metallic conductivity" $\sigma_c \sim 510^2 Ohm^{-1}cm^{-1}$, which define the conductivity scale of disorder induced metal—insulator transition. The negative temperature coefficient of resistivity was observed in this conductivity range. The experimental data on T_c decrease with the growth of resistivity in this system were rather well fitted in Ref.[20] using the μ^* dependence on resistivity given by Eq. (170). A clear tendency for $|dH_{c2}/dT|_{T_c}$ saturation with disorde is also observed. Analogous dependence of T_c and $|dH_{c2}/dT|_{T_c}$ on the resistivity in the normal state for Mo_6Se_8 disordered by fast neutrons is shown in Fig. 10[165].

Here superconductivity exists up to conductivities $\sigma \sim 250 Ohm^{-1}cm^{-1}$. Further disordering (irradiation) leads to the destruction of superconducting state and metal—insulator transition (an unlimited growth of resistivity with decrease of T, with variable—range hopping conduction[2,3] is observed). The slope of the upper critical field $|dH_{c2}/dT|_{T_c}$ also has a tendency to saturate with the growth of resistivity. Standard interpretation

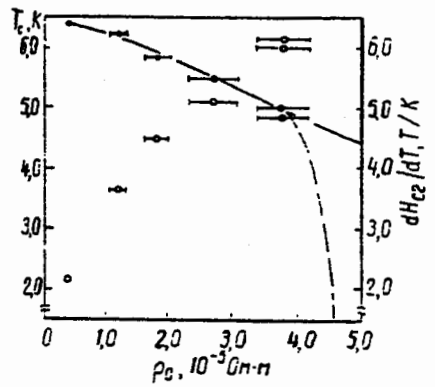

Figure 10: Resistivity dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in Mo_6Se_8 (Ref.[165]).

of such behavior of $|dH_{c2}/dT|_{T_c}$ was based upon the use of Gorkov's relation (Cf. first relation in Eq. (222)) and lead to the conclusion of $N(E_F)$ decrease under disordering. In fact, we have seen that no such conclusion can be reached for systems with conductivities $\sigma < 10^3 Ohm^{-1}cm^{-1}$, because such saturation behavior may be a natural manifestation of the approaching metal—insulator transition. Similar dependences were observed in other Chevrel phase superconductors [163,171,172].

In Fig. 11 we show the dependence of conductivity and T_c on the parameter $p_F l/\hbar$ in amorphous InO_x alloy[166].

In Fig. 12 from Ref.[173] the data on the temperature dependence of H_{c2} in amorphous In/InO_x (bulk) films are presented for different degrees of disorder. We can see that in the low temperature region $H_{c2}(T)$ deviates from the standard temperature dependence, but apparently confirm the qualitative form predicted above for systems which are close to Anderson transition.

Probably the most impressive are the data for amorphous $Si_{1-x}Au_x$ alloy [160, 161,174]. In Fig. 13 [161] the data on T_c and conductivity dependence on the gold concentration x are shown. In Fig. 14 $H_{c2}(T)$ dependence for this system is shown for different alloy compositions [161].

From these data it is clearly seen that T_c vanishes *before* metal—insulator transition. The metal—insulator transition itself is *continuous*, conductivity vanishes linearly with the decrease of gold concentration and the values of conductivity significantly less than the estimated "minimal metallic conductivity" are definitely observed. The system remains superconducting even for such low conductivity values. The slope of $H_{c2}(T)$ at $T = T_c$ is practically constant , despite the change of conductivity (disorder) in rather wide range. This behavior apparently cannot be explained only by the appearance of correlation pseudogap in the density of states observed in Ref.[174], which becomes significant only very close to metal—insulator transition. Low temperature deviation from standard convex dependence on T is also clearly seen. In Fig. 15 from Ref.[174] we show the temperature dependences of resistivity and superconducting energy gap (determined by tunneling) of a sample with $x = 0.21$. It nicely

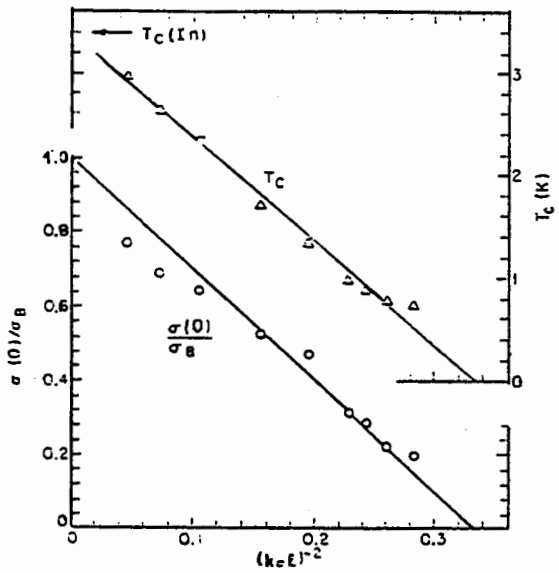

Figure 11: Conductivity σ and T_c dependence on the parameter $p_F l/\hbar$ in amorphous InO_x. σ_B is estimated Drude conductivity. (Ref.[166]).

demonstrates superconducting transition in the system which is very close to disorder induced metal—insulator transition.

Note, that according to Ref.[174] the superconducting energy gap in this sample is substantially broadened which may indicate the growth of statistical gap fluctuations due to the same fluctuations of the local density of states. These data are in obvious qualitative correspondence with the general theoretical picture described throughout this review.

These data show that in systems which are superconducting close to the disorder induced metal—insulator (Anderson) transition T_c decreases rather fast and in all reliable cases vanishes before the transition to insulating state. At the same time the temperature dependence of H_{c2} is not described by the standard theory of "dirty" superconductors both with respect to $(dH_{c2}/dT)_{T_c}$ behavior and at low temperatures, where the upward deviations from the standard dependence are readily observed. This confirms most of our theoretical conclusions.

Some indications of a possible superconducting state in the insulating phase of granular Al and $Al - Ge$ were observed in Refs.[175,176]. Obviously, the granular systems are more or less outside the scope of our review. However, we should like to mention that the strong smearing of BCS—like density of states and the gapless regime of superconductivity was observed (via tunneling measurements) in Refs.[177,178], close to the metal—insulator transition in these systems. This may confirm our picture of

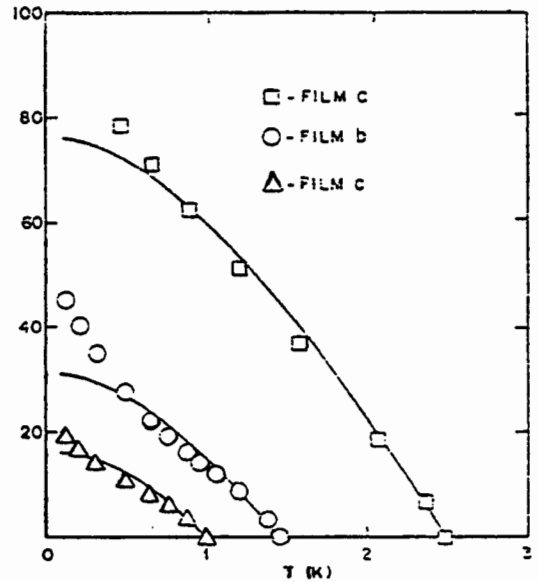

Figure 12: $H_{c2}(T)$ in amorphous films of In/InO_x. Dashed lines show standard theoretical dependences. (Ref.[173]).

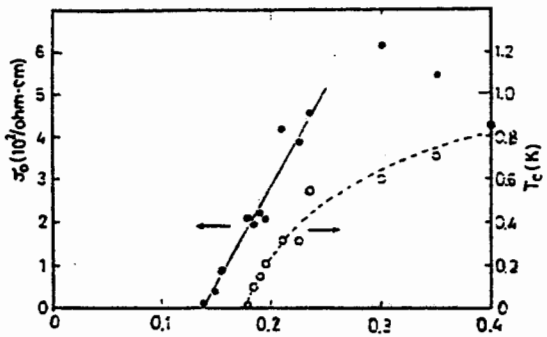

Figure 13: Conductivity σ and T_c dependence on gold concentration in amorphous $Si_{1-x}Au_x$ alloy. (Ref.[161]).

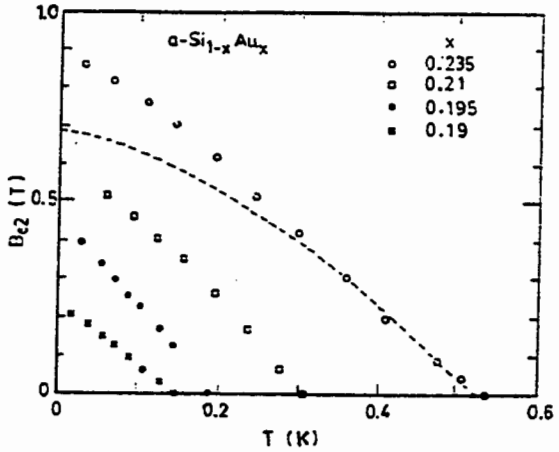

Figure 14: $H_{c2}(T)$ in amorphous $Si_{1-x}Au_x$ alloy. (Ref.[161]).

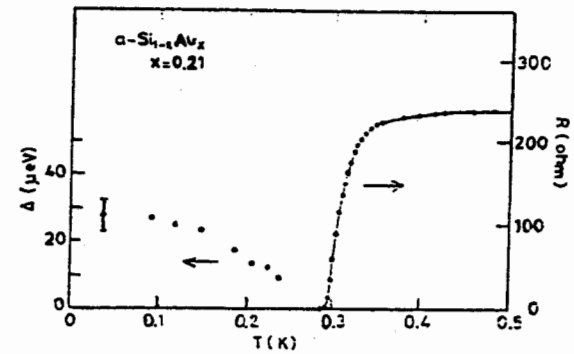

Figure 15: Temperature dependences of superconducting energy gap Δ and of the resistance R for amorphous $Si_{0.79}Au_{0.21}$. (Ref.[174]).

statistical fluctuation smearing of the density of states. Note, that more recent work on granular Al [179] apparently exclude the possibility of superconductivity in the insulating phase. Similar conclusions on superconductivity vanishing at the point of metal—insulator transition were reached for amorphous Al_xGe_{1-x}[180] and amorphous $Ga - Ar$ mixtures[181].

These results are not surprising since we have seen the existence of strong mechanisms of T_c degradation close to disorder induced metal—insulator transition.

5.2 High—T_c Superconductors

Very soon after the discovery of high—temperature oxide superconductors [14,15] it was recognized that localization effects has an important role to play in these systems. There are many sources of disorder in these systems and the low level of conductivity indicate from the very beginning their closeness to Anderson transition. In the field where there are hundreds of papers published on the subject it is impossible to review or even to quote all of them. More or less complete impression of the status of high—T_c research can be obtained from Conference Proceedings[182]. Here we shall concentrate almost only on papers which deal with disordering by fast neutron irradiation which we consider probably the "purest" method to introduce disorder into the system (allowing to neglect the complicated problems associated with chemical substitutions). Also historically it was apparently the earliest method used to study disorder effect in high—T_c superconductors in a controllable way [183,184].

There are several reasons for localization effects to be important in high—T_c oxides:

- *Two—Dimensionality.* All the known high—T_c systems (with $T_c > 30K$) are strongly anisotropic or quasi-two-dimensional conductors. We have seen above that for such systems it is natural to expect the strong enhancement of localization effects due to the special role of spatial dimensionality $d = 2$: in purely two-dimensional case localization appears for infinitely small disorder[28,4,6,7]. The inplane conductivity scale for metal—insulator transition in such systems as given by Eq. (12) or Eq. (83) is larger than in isotropic case. Reasonable estimates show that the values of inplane "minimal metallic conductivity" may exceed $10^3 Ohm^{-1}cm^{-1}$. While due to continuous nature of Anderson transition there is no rigorous meaning of minimal metallic conductivity, these estimates actually define the scale of conductivity near the metal—insulator transition caused by disorder. Then it is clear that most of the real samples of high—T_c superconductors are quite close to Anderson transition and even the very slight disordering is sufficient to transform them into Anderson insulators[129].

- *"Marginal" Fermi Liquid.* During our discussion of interaction effects we have seen that there are serious reasons to believe that importance of localization effects in high—T_c oxides may be actually due to more fundamental reasons connected with anomalous electronic structure and interactions in these materials. The concept of "marginal" Fermi liquid [95] leads to extreme sensitivity of such system to disordering and the appearance of localized states around the Fermi level at infinitesimally weak disorder[97].

On the other hand high—T_c systems are especially promising from the point of view of the search for superconductivity in the Anderson insulator:

- High transition temperature T_c itself may guarantee the survival of superconductivity at relatively high disorder.

- Due to small size of Cooper pairs in high—T_c systems in combination with high—T_c (large gap !) we can easily satisfy the main criterion for superconductivity in localized phase as given by Eq. (130).

- Being narrow band systems as most of the conducting oxides high—T_c systems are promising due to low values of the Fermi energy E_F which leads to less effective T_c degradation due to localization enhancement of Coulomb pseudopotential μ^*. (Cf. Eq. (169)).

Anomalous transport properties of high—T_c oxides are well known[185]. Experimentally there are two types of resistivity behavior of good single-crystals of these systems. In highly conducting ab plane of $YBa_2Cu_3O_{7-\delta}$ and other oxides resistivity of a high quality single-crystal always shows the notorious linear—T behavior (by "good" we mean the samples with resistivity $\rho_{ab} < 10^3 Ohmcm$). However, along orthogonal c direction the situation is rather curious: most samples produce semiconductor-like behavior $\rho_c \sim 1/T$, though some relatively rare samples (apparently more pure) show metallic-like $\rho_c \sim T$ (with strong anisotropy $\rho_c/\rho_{ab} \approx 10^2$ remaining).[185] Metallic behavior in c direction was apparently observed only in the best samples of $YBa_2Cu_3O_{7-\delta}$ and almost in no other high—T_c oxide. In Fig. 16 taken from Ref.[186] we show the temperature dependence of ρ_c in a number of high—T_c systems. It is seen that $\rho_c(T)$ changes between metallic and semiconducting behavior depending on whether the resistivity is below or above the Ioffe—Regel limit defined for quasi-two-dimensional case by Eq. (12). Rather strange is the absence of any obvious correlation between the behavior of ρ_c and T_c.

This unusual behavior leads us to the idea that most of the samples of high—T_c systems which are studied in the experiment are actually already in localized phase due to internal disorder which is always present. This may be due to their quasi-two-dimensional nature or because of marginal Fermi liquid effects. In this case a simple conjecture on the temperature behavior of resistivity of single-crystals can be made qualitatively explaining the observations [187,188]. In case of localized states at the Fermi level and for finite temperatures it is important to compare localization length R_{loc} with diffusion length due to inelastic scattering $L_\varphi \approx \sqrt{D\tau_\varphi}$, where D is diffusion coefficient due to elastic scattering on disorder, while τ_φ is phase coherence time determined by inelastic processes. For $T > 0$ this length L_φ effectively replaces the sample size L in all expressions of scaling theory of localization when $L \gg L_\varphi$, because on distances larger than L_φ all information on the nature of wave functions (e.g. whether they are localized or extended) is smeared out. Taking into account the usual low—temperature dependence like $\tau_\varphi \sim T^{-p}$ (where p is some integer, depending on the mechanism of inelastic scattering) this can lead to a non—trivial temperature dependence of conductivity, in particular to a possibility of a negative temperature coefficient of resistivity [30]. Similar expressions determine the temperature dependence of conductivity also for the localized phase until $L_\varphi < R_{loc}$. In this case electrons do

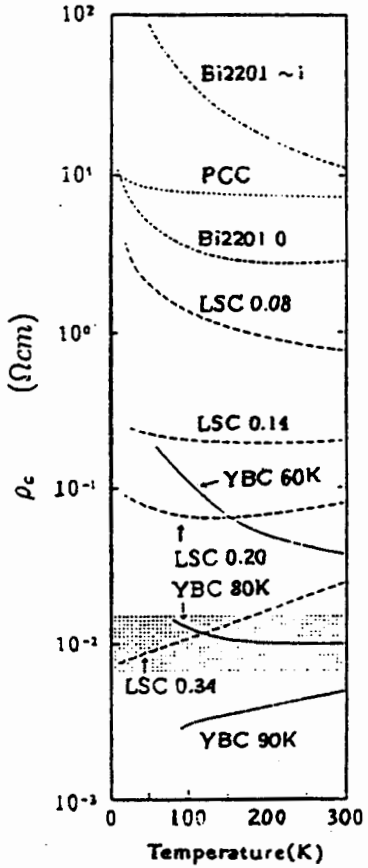

Figure 16: Temperature dependence of ρ_c for different high—T_c cuprates. The dashed region indicates the resistivity range corresponding to Ioffe—Regel limit of Eq. (12). (Ref.[186]).

not "feel" being localized and conductivity at high enough T will show metallic like behavior. For localization to be important we must go to low enough temperatures, so that L_φ becomes greater than R_{loc}. If high—T_c superconductors are Anderson insulators with very anisotropic localization length, $R^{ab}_{loc} \gg R^{c}_{loc}$ and both localization lengths diminish as disorder grows, L_φ is also anisotropic and we can have three different types of temperature behavior of resistivity[187]:

1. Low T or strong disorder, when we have

$$L^{ab}_\varphi \approx \sqrt{D_{ab}\tau_\varphi} \gg R^{ab}_{loc} \qquad L^{c}_\varphi \approx \sqrt{D_c\tau_\varphi} \gg R^{c}_{loc} \tag{296}$$

This gives semiconductor-like behavior for both directions.

2. Medium T or medium disorder, when

$$L^{ab}_\varphi < R^{ab}_{loc} \qquad L^{c}_\varphi > R^{c}_{loc} \tag{297}$$

and metallic behavior is observed in ab plane, while semiconducting temperature dependence of resistivity is observed along c axis.

3. High T or low disorder, when

$$L^{ab}_\varphi < R^{ab}_{loc} \qquad L^{c}_\varphi < R^{c}_{loc} \tag{298}$$

and metallic behavior is observed in both directions.

Here we do not speculate on the inelastic scattering mechanisms leading to the concrete temperature behavior in high—T_c oxides, in particular on linear T behavior in ab plane or $1/T$ behavior in c direction. Unfortunately too little is known on these mechanisms[185] to be able to make quantitative estimates on the different types of behavior predicted above. However, all experimental data as we shall see below at least do not contradict the idea of possibility of Anderson localization in high—T_c cuprates.

Now let us consider the experiments on controllable disordering of high—temperature superconductors. Already the first experiments on low temperature ($T = 80K$) fast neutron irradiation of ceramic samples of high—T_c systems [189,190,191,192,193,195] has shown that the growth of structural disorder leads to a number of drastic changes in their physical properties:

- continuous metal—insulator transition at very slight disordering,
- rapid degradation of T_c,
- apparent coexistence of hopping conductivity and superconductivity at intermediate disorder,
- approximate independence of the slope of H_{c2} at $T \sim T_c$ on the degree of disorder,
- anomalous *exponential* growth of resistivity with defect concentration.

These anomalies were later confirmed on single-crystals and epitaxial films[194,196,197, 198], and were interpreted[129,16] using the ideas of possible coexistence of Anderson localization and superconductivity.

In Fig. 17 we show data[129] on the dependence of the superconducting transition temperature and resistivity (at $T = 100K$, i.e. just before superconducting transition) on fast neutron fluence for $YBa_2Cu_3O_{6.95}$.

In all high—T_c compounds introduction of defects leads to strong broadening of superconducting transition. The derivative $(dH_{c2}/dT)_{T_c}$ in ceramic samples measured at the midpoint of the superconducting transition does not change as ρ_{100K} grows by an order magnitude. In Fig. 18[129] we show the temperature dependence of resistivity for samples of $YBa_2Cu_3O_{6.95}$ and $La_{1.83}Sr_{0.17}CuO_4$ for different degrees of disorder.

In all these materials the $\rho(T)$ curves vary in the same way. In the fluence range $\Phi > 10^{19}cm^{-2}$, where superconductivity is absent, $\rho(T)$ follows a dependence which is characteristic of conductivity via localized states[2,3]:

$$\rho(T) = \rho_0 exp(Q/T^{1/4}) \qquad Q = 2.1[N(E_F)R_{loc}^3]^{-1/4} \tag{299}$$

as shown in Fig. 19. (Mott's variable-range hopping conduction).

The most striking anomaly of resistivity behavior of all high—T_c systems under disordering is nonlinear, practically *exponential* growth of resistivity at fixed temperature (e.g. $\rho(T = 100K)$) with fluence, starting from the low fluences $\Phi < 710^{18}cm^{-2}$, including superconducting samples[189,129,190,191,192,193]. These data are shown in Fig. 20[129] for the dependence of $\rho(T = 80K)$ on Φ obtained from measurements made directly during the process of irradiation. For comparison the similar data for $SnMo_6S_8$ are shown which do not demonstrate such an anomalous behavior, its resistivity is just proportional to Φ and saturates at large fluences. We relate this exponential growth of ρ with the increase of Φ (i.e. of defect concentration) in all high—T_c systems to localization, which already appears for very small degrees of disorder in samples with high values of T_c. As we have seen in samples with much reduced or vanishing T_c localization is observed directly via Mott's hopping in the temperature behavior of resistivity given by Eq. (299).

From these results it follows that the electronic system of high—T_c superconductors is very close to the Anderson transition. The observed variation of ρ as a function both of fluence and of temperature can be described by the following empirical formula[189]:

$$\rho(T, \Phi) = (a + cT)exp(b\Phi/T^{1/4}) \tag{300}$$

Identifying the exponential factors in Eq. (299) and Eq. (300) it is possible to obtain a fluence dependence of localization length (Cf. Ref.[129]) and below).

Detailed neutron diffraction studies of structural changes in irradiated samples were also performed [189,129,199]. These investigations has shown definitely that there are no oxygen loss in $YBa_2Cu_3O_{6.95}$ during low temperature irradiation. Only some partial rearrangement of oxygens between positions $O(4)$ and $O(5)$ in the elementary cell occur as radiation-induced defects are introduced. In addition, in all high—T_c compounds the Debye— Waller factors grow and the lattice parameters a, b, c increase slightly[129,199]. The growth of Debye—Waller factors reflect significant atomic shifts, both static and dynamic, from their regular positions, which induce a random potential. This disorder is pretty small from the structural point of view, the lattice is only slightly distorted. However, we have seen that this small disorder is sufficient to induce metal—insulator transition and complete degradation of superconductivity. The

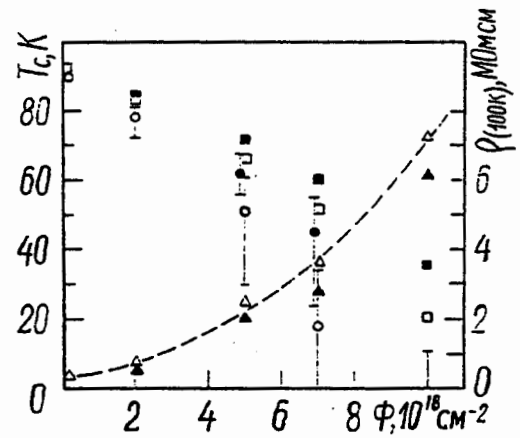

Figure 17: Dependence of the superconducting transition temperature and resistivity (at $T = 100K$) on neutron fluence for ceramic $YBa_2Cu_3O_{6.95}$. Different notations correspond to different methods of measurement and also evolution after annealing at $300K$. (Details see in Ref.[129]).

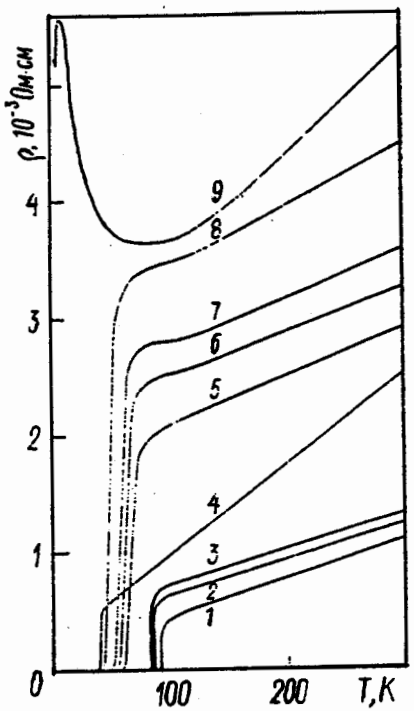

Figure 18: Temperature dependence of resistivity ρ for ceramic samples of $YBa_2Cu_3O_{6.95}$ (curves 1—3 and 5—8) and $La_{1.83}Sr_{0.17}CuO_4$ (curves 4, 9) irradiated at $T = 80K$ with different fluences: 1—$\Phi = 0$; 3, 6, 8 — $\Phi = 2.5$ and$710^{18} cm^{-2}$ plus annealing for 2 hours at $300K$; 2, 5, 7 — irradiated with $\Phi = 2.5$ and$710^{18} cm^{-2}$ plus annealing for 2 weeks at $300K$; 4 — $\Phi = 0$; 9 — $\Phi = 510^{18} cm^{-2}$ plus annealing for 2 hours at $300K$. (Ref.[129])

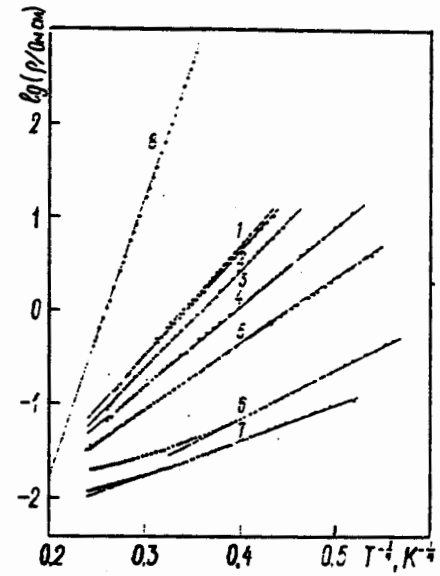

Figure 19: Dependence of $ln\rho$ on $T^{-1/4}$ for $YBa_2Cu_3O_{6.95}$ irradiated with a fluence of $\Phi = 1.210^{19} cm^{-2}$ at $T = 80K$ (curve 1), and after 20-minute annealing at $T = 150K(2); 200K(3); 250K(4); 300K(5)$ and two weeks annealing at $T = 300K(7)$. Similar dependences for $La_{1.83}Sr_{0.17}CuO_4$ for $\Phi = 210^{19} cm^{-2}$ annealed for 2 hours at $300K(6)$ and for La_2CuO_4 for $\Phi = 210^{19} cm^{-2}$ annealed for 2 hours at $300K(8)$. (Ref.[129]).

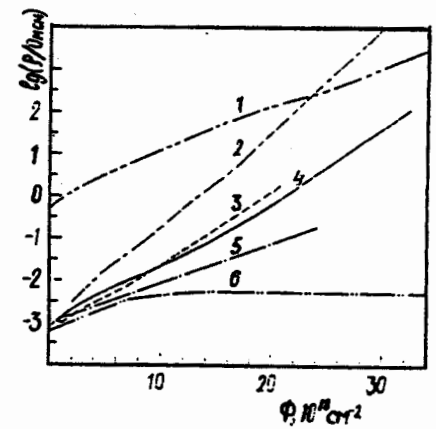

Figure 20: Dependence of $ln\rho$ on fluence Φ during irradiation at $T = 80K$: 1—La_2CuO_4; 2—$YBa_2Cu_3O_{6.95}$; 3—single crystaline ρ_{ab} in $YBa_2Cu_3O_{6.95}$; 4—$La_{1.83}Sr_{0.17}CuO_4$; 5—$Bi - Sr - Ca - Cu - O$; 6— $SnMo_6Se_8$. (Ref.[129]).

absence of oxygen loss implies that there is no significant change in concentration of carriers and we have really disorder—induced metal—insulator transition. This is also confirmed by other methods[195,200]. In Fig. 21 we show the data[195,196] on temperature dependence of the Hall concentration of ceramic samples of irradiated and oxygen deficient $YBa_2Cu_3O_{7-\delta}$.

It is seen that disordering weakens the anomalous temperature dependence of Hall effect, but Hall concentration n_H at low T practically does not change in striking difference with data on oxygen deficient samples, where n_H drops several times. This also confirms the picture of disorder—induced metal—insulator transition in radiation disordering experiments. Similar Hall data were obtained on epitaxial films[197] and single-crystals[198].

Qualitatively same resistivity behavior was also obtained in the experiments on radiation disordering of single-crystals[194,196] and epitaxial films[197]. Electrical resistivities of $YBa_2Cu_3O_{7-\delta}$ single crystals were measured at $T = 80K$ directly during irradiation by fast neutrons. The data are shown in Fig. 22.

We can see that ρ_{ab} increases exponentially with Φ (defect concentration) starting from the smallest doses, while ρ_c grows slower and only for $\Phi > 10^{19} cm^{-2}$ they grow with the same rate. At large fluences both ρ_{ab} and ρ_c demonstrate Mott's hopping $ln\rho_{ab,c} \sim T^{-1/4}$ [201]. Similar data of Ref.[197] show $ln\rho \sim T^{-1/2}$ characteristic of Coulomb gap. We do not know the reasons for this discrepancy between single-crystalline and epitaxial films data (note that another method of disordering by 1MeV Ne^+ ions was used in Ref.[197]). Anisotropy ρ_c/ρ_{ab} at $T = 80K$ drops rapidly (to the values ~ 30 for $\Phi = 10^{19} cm^{-2}$) and then practically does not change and "residual" anisotropy of the order of its room—temperature value in initial samples remains. This means that temperature dependence of anisotropy weakens in the disordered samples. Note, that unfortunately only the single-crystals with "semiconducting" temperature dependence of resistivity along c axis were investigated up to now.

The upper critical fields of $YBa_2Cu_3O_{7-\delta}$ single-crystals (determined from standard resistivity measurements) for different degrees of disorder are shown in Fig. 23[196].

Temperature dependence of H_{c2} in disordered samples is essentially nonlinear, especially for samples with low T_c. The estimated from high-field regions temperature derivative of $H_{c2}^{\perp}$ (field along the c axis) increases with disorder. However, similar derivative of $H_{c2}^{\parallel}$ (field along ab plane) drops in the beginning and then does not change. Anisotropy of H_{c2} decreases with disorder and in samples with $T_c \sim 10K$ the ratio of $(H_{c2}^{\parallel})'/(H_{c2}^{\perp})'$ is close to unity. According to Eq. (225) this means the complete isotropisation of the Cooper pairs. This is illustrated by Fig. 24[202].

The remaining anisotropy of resistivity may be connected with some kind of planar defects in the system.

Under irradiation localized moment contribution appears in the magnetic susceptibility of high—T_c oxides[189,129]. In the temperature range from T_c to $300K$ $\chi(T)$ is satisfactorily described by Curie—Weiss type dependence: $\chi(T) = \chi_0 + C/(T - \Theta)$. The value of χ_0 and the Curie constant C as a function of fluence Φ are given in Fig. 25. The value of C is proportional to fluence.

Note that the threefold larger slope of $C(\Phi)$ in $YBa_2Cu_3O_{6.95}$ as compared with $La_{1.83}Sr_{0.17}CuO_4$ is an evidence that this Curie-law temperature dependence is associated with localized moments forming on Cu (there are three times more coppers in

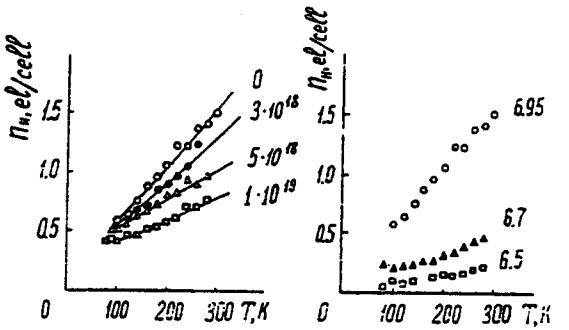

Figure 21: Temperature dependence of Hall concentration for the irradiated (left) and oxygen deficient (right) ceramic samples of $YBa_2Cu_3O_{7-\delta}$. (Ref.[195,196]).

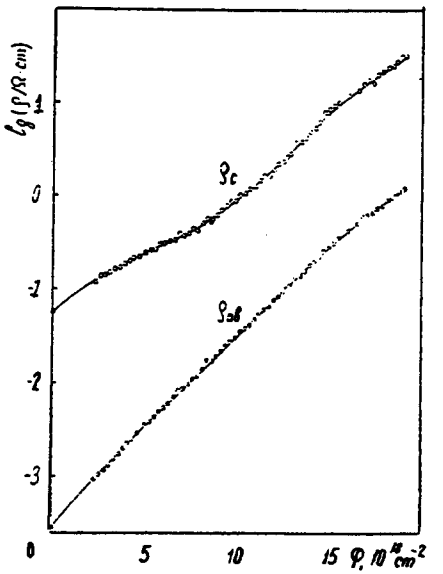

Figure 22: Fluence dependence of ρ_{ab} and ρ_c at $T = 80K$ during fast neutron irradiation. (Ref.[194,196]).

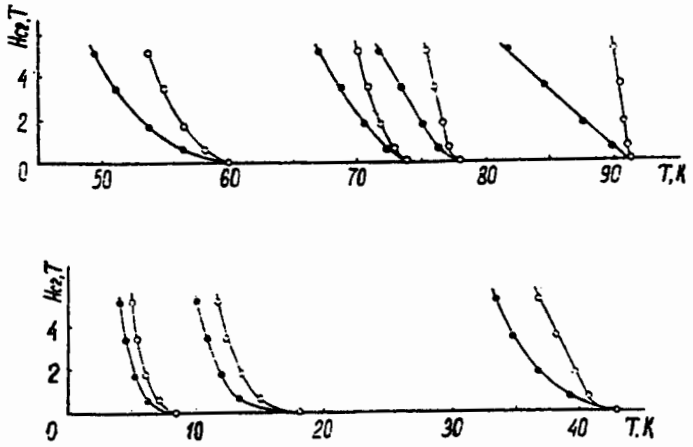

Figure 23: Temperature dependence of $H_{c2}^{\parallel}$ (upper curves) and $H_{c2}^{\perp}$ (lower curves) for the single-crystals of $YBa_2Cu_3O_{7-\delta}$ with different degrees of disorder. (Ref.[196]).

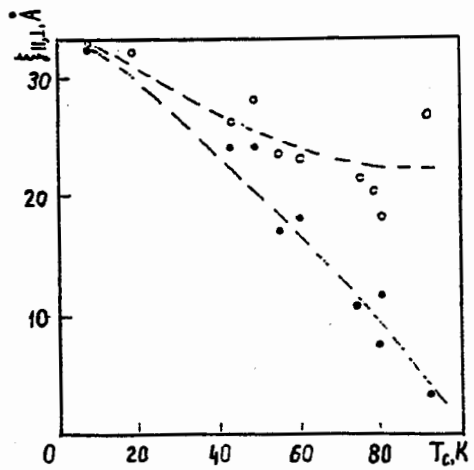

Figure 24: The dependence of coherence lengths determined from H_{c2} behavior under disordering on the critical temperature T_c: $\xi_{\parallel}$—circles; $\xi_{\perp}$—black circles. (Ref.[202]).

the elementary cell of Y compound than in La compound).

These data show that electronic properties of high—T_c systems are quite different under disordering from that of traditional superconductors [171,172] or even some closely related metallic oxides[198,203]. We associate these anomalies with the closeness of the Anderson transition and believe that real samples of high—T_c systems which always possess some noticeable disorder may well be already in the state of the Anderson insulator.

Using the data on electrical resistivity of disordered samples of $YBa_2Cu_3O_{6.95}$ and the relations given by Eq. (299) and Eq. (300) (assuming that exponentials there are identical) we can calculate the change of localization length R_{loc} as a function of fluence[129,190,191,192,193].

This dependence is shown on Fig. 26 along with fluence dependence of T_c. It is clearly seen that superconductivity is destroyed when localization length R_{loc} becomes smaller than $\sim 30\mathring{A}$, i.e. it becomes of the order or smaller than a typical size of the Cooper pair in this system (Cf. Fig. 24) in complete accordance with our basic criterion of Eq. (130). We can estimate the minimal value of R_{loc} for which superconductivity can still exist in a system of localized electrons via Eq. (130)[129] taking the free-electron value of $N(E_F) \approx 510^{33}(ergcm^3)^{-1}$ (for carrier concentration of $\sim 610^{21}cm^{-3}$) and the gap value $\Delta \sim 5T_c$, corresponding to very strong coupling[184]. We obtain the result shown in Fig. 26. In any case we can see that criterion of Eq. (130) ceases to be fulfilled for $\Phi \sim (5-7)10^{18}cm^{-2}$ in remarkably good agreement with the experiment.

In the absence of accepted pairing mechanism for high temperature superconductors it is very difficult to speculate on the reasons for T_c degradation in these systems. If we assume that the main mechanism of T_c degradation is connected with the growth of Coulomb effects during disordering, as discussed above in this review, we can try to use appropriate expressions to describe the experimental data. Assuming superconductivity in the localized phase we can use Eq. (175), estimating R_{loc} as above from empirical relation Eq. (300) and Eq. (299) (or directly expressing the parameters entering Eq. (175) via experimental dependence of resistivity on fluence as described by Eq. (300)[129]). The results of such a fit (with the assumption of $\mu \approx 1$) are also shown in Fig. 26. The agreement is also rather satisfactory, the more rapid degradation of T_c for small degrees of disorder can be related to additional contributions to Coulomb repulsion within Cooper pairs neglected in the derivation of Eq. (175). Surely we do not claim that this is a real explanation of T_c degradation in disordered high temperature superconductors. However, note its relation to localized moment formation under disordering.

According to Mott[117] (Cf. also Refs.[118,7]) the appearance of localized moments may be related to the presence of localized states (single occupied states below the Fermi level as briefly discussed above). We can then estimate the value of the effective magnetic moment (in Bohr magnetons) in unit cell as[129]:

$$\mu R_{loc}^{-3}\Omega_0 = p_{theor}^2 \tag{301}$$

where Ω_0 is the volume of a unit cell. For large degrees of disorder ($\Phi = 210^{19}cm^{-2}$) and $R_{loc} \approx 8\mathring{A}$ with $\mu \approx 1$ we obtain $p_{theor}^2 = 0.66$ for $YBa_2Cu_3O_{6.95}$ in full agreement with experiment. However, for smaller fluences p_{theor} is considerably smaller than the experimental value. Note, though , that the estimate of Eq. (301) is valid only for small

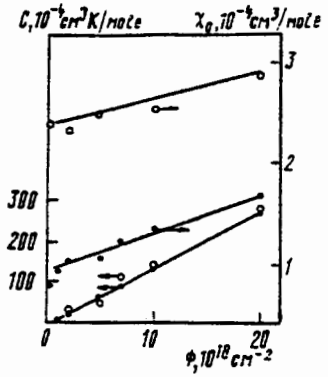

Figure 25: Dependence of the Curie constant C and the temperature-independent part χ_0 of magnetic susceptibility on neutron fluence Φ for $La_{1.83}Sr_{0.17}CuO_4$ (black circles) and $YBa_2Cu_3O_{6.95}$ (circles). (Ref.[129]).

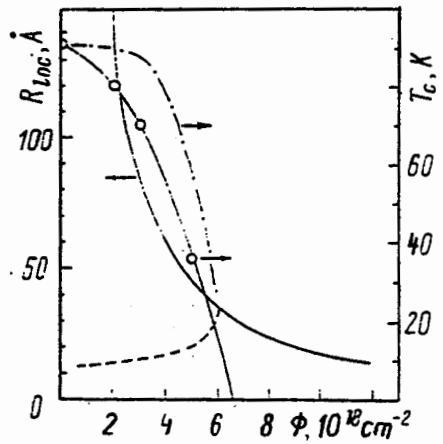

Figure 26: Dependence of T_c on fluence for $YBa_2Cu_3O_{6.95}$ (circles). The solid curve is the localization length calculated using Eq. (299) and Eq. (300). Dashed curve defines the minimum localiztion length at which superconductivity can exist at given T_c according to Eq. (130). Dashed-dotted curve is theoretical fit using Eq. (175). (Ref.[129]).

enough values of R_{loc}, i.e. when the Fermi level is well inside the localized region. On the other hand, the accuracy with which the Curie constant is determined in weakly disordered samples is considerably less than in strongly disordered case. Of course, the other mechanisms of local moment formation, which were discussed above and can become operational even before the metal—insulator transition can be important here.

We shall limit ourselves to this discussion of localization effects in high temperature superconductors. Our conclusion is that these effects are extremely important in these systems and some of the anomalies can be successfully described by theoretical ideas formulated in this review.

6 CONCLUSION

We conclude our review trying to formulate the basic unsolved problems. From the theoretical point of view probably the main problem is to formulate the theory of superconducting pairing in strongly disordered system along the lines of the general theory of interacting Fermi systems. This problem is obviously connected with the general theory of metal—insulator transition in such approach, which as we mentioned during our brief discussion above is rather far from its final form. These problems become even more complicated if we address ourselves to the case of high temperature superconductors, where we even do not know precisely the nature of pairing interaction in regular system.

Concerning the semiphenomenological approach to the theory of superconductivity close to the Anderson transition we must stress the necessity of further investigation of the region of strong statistical fluctuations with the aim of more detailed study of their influence upon different physical properties, like e.g. the upper critical field, density of states, nuclear relaxation etc. Obviously, all of them may be significantly changed in comparison with predictions of what we called the statistical mean—field theory.

As to experiment, certainly too much is still to be done for unambigous demonstration of exotic possibility of superconductivity of Anderson insulators.

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