Coexistence of superconductivity and magnetism at the nanoscale

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Kourovka, 2012
Outline I

- Recall on magnetism and superconductivity coexistence.
- Origin and the main peculiarities of the proximity effect in superconductor-ferromagnet systems.
- Josephson $\pi$-junction.
- Spin-valve effet. Long ranged triplet correlations. Domain wall superconductivity.
- $\varphi$-junction. Direct coupling between ferromagnetism and superconductivity.
- Perspectives and possible applications.
Outline II

1. Larkin-Ovchinnikov-Fulde-Ferrell state. LOFF or FFLO states. The origin of the non-uniform modulated superconducting state.

2. Experimental evidences of FFLO state.

3. Exactly solvable models of FFLO state.

4. Vortices in FFLO state. Role of the crystal structure.

5. Superfluid ultracold Fermi gases with imbalanced state populations: one more candidate for FFLO state?
The critical temperature variation versus the concentration \( n \) of the Gd atoms in \( \text{La}_{1-x}\text{Gd}_x\text{Al}_2 \) alloys (Maple, 1968). \( T_{c0} = 3.24 \text{ K} \) and \( n_{cr} = 0.590 \) atomic percent Gd.

The earlier experiments (Matthias et al., 1958) demonstrated that the presence of the magnetic atoms is very harmful for superconductivity.

\[
\frac{dT_c}{dx} \approx -\Theta_m x
\]

(Abrinkosov and Gorkov, 1960)
FIG. 3: Schematic mechanism of the indirect RKKY exchange (EX) interaction $H_{RKKY}$ in rare earth superconductors. The spin $S_i$ at $R_i$ polarizes spins of conduction electrons with the spin susceptibility $\chi(R_i - R_j)S_{R_i}\cdot S_j$. The polarization is transferred to the spin $S_j$ at $R_j$. 

$$H_{ex} = -J^2 \chi(R_i - R_j)S_i \cdot S_j$$
Antagonism of magnetism (ferromagnetism) and superconductivity

- Orbital effect (Lorentz force)

- Paramagnetic effect (singlet pair)

Electromagnetic mechanism (breakdown of Cooper pairs by magnetic field induced by magnetic moment)

Exchange interaction
No antagonism between antiferromagnetism and superconductivity

Usually $T_c > T_N$

<table>
<thead>
<tr>
<th>Compound</th>
<th>$T_c$ (K)</th>
<th>$T_N$ (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NdRh$_4$B$_4$</td>
<td>5.3</td>
<td>1.31</td>
</tr>
<tr>
<td>SmRh$_4$B$_4$</td>
<td>2.7</td>
<td>0.87</td>
</tr>
<tr>
<td>TmRh$_4$B$_4$</td>
<td>9.8</td>
<td>0.4</td>
</tr>
<tr>
<td>GdMo$_6$S$_8$</td>
<td>1.4</td>
<td>0.84</td>
</tr>
<tr>
<td>TbMo$_6$S$_8$</td>
<td>2.05</td>
<td>1.05</td>
</tr>
<tr>
<td>DyMo$_6$S$_8$</td>
<td>2.05</td>
<td>0.4</td>
</tr>
<tr>
<td>ErMo$_6$S$_8$</td>
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<td>0.2</td>
</tr>
<tr>
<td>GdMo$_6$Se$_8$</td>
<td>5.6</td>
<td>0.75</td>
</tr>
<tr>
<td>ErMo$_6$Se$_8$</td>
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<tr>
<td>DyNi$_2$B$_2$C</td>
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<td>11</td>
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<tr>
<td>ErNi$_2$B$_2$C</td>
<td>10.5</td>
<td>6.8</td>
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<tr>
<td>TmNi$_2$B$_2$C</td>
<td>11</td>
<td>1.5</td>
</tr>
<tr>
<td>HoNi$_2$B$_2$C</td>
<td>8</td>
<td>5</td>
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Characteristic energy for magnetism (per one atom/electron)

$$E_{mag} \sim \Theta \sim I^2/E_F$$

Characteristic energy for superconductivity (per one electron)

$$E_S \sim T_c(T_c/E_F) \ll T_c$$
\[ \mathbf{M}(r) = n_\text{eff} \mu S \Rightarrow B_{\text{ep}}(r) \Rightarrow j_{\text{sc}}(r) \Rightarrow B_{\text{sr}}(r) \]
\[ \Rightarrow B(r) = B_{\text{ep}}(r) + B_{\text{sr}}(r) = \nabla \times A(r) \]
FERROMAGNETIC CONVENTIONAL (SINGLET) SUPERCONDUCTORS

A. C. susceptibility and resistance versus temperature in ErRh\textsubscript{4}B\textsubscript{4} (Fertig \textit{et al.}, 1977).

RE-ENTRANT SUPERCONDUCTIVITY in ErRh\textsubscript{4}B\textsubscript{4} !
Electromagnetic interaction

\[ F_z = F_m + \int \left\{ a |\psi|^2 + \frac{b}{2} |\psi|^4 + \frac{\hbar^2}{4m} \left| \left( V - \frac{2ie}{\hbar c} A \right) \psi \right|^2 + \frac{B^2}{8\pi} - \frac{BH}{4\pi} \right\} dV \]

\[ \delta F_z (A) = \int \left\{ \frac{e^2 |\psi|^2}{mc^2} (A)^2 \right\} dV = \frac{e^2 |\psi|^2}{mc^2} \sum_q |A_q|^2 = \frac{1}{8\pi\lambda^2} \sum_q |A_q|^2 \]

\[ F_m = \int \left\{ AM^2 + \frac{B}{2} M^4 + a^2 (V M)^2 \right\} dV \]

\[ \delta F_z (A) + F_m = \sum_q (A + q^2 a^2) |M_q|^2 + \frac{1}{8\pi\lambda^2} \sum_q |A_q|^2 \]

if \( q\lambda \gg 1 \), then \( B = 4\pi M \), \( B = \text{rot} A \), \( |B_q|^2 = q^2 |A_q|^2 = (4\pi)^2 |M_q|^2 \)

\[ \delta F = (q^2 a^2) |M_q|^2 + \frac{2\pi}{\lambda^2 q^2} |M_q|^2 \]

\[ q_{a} = \frac{1}{\sqrt{a\lambda}} \]

\[ d^2 /q \quad a << d << \]
Coexistence phase

\[ E_m = -\sum_Q \frac{\chi(Q)}{2} |h_Q|^2 \]

h = I <S>

At T=0 and \( Q\xi_0 >> 1 \) following (Anderson and Suhl, 1959)

\[ \chi_s(Q) - \chi(Q) = \frac{\pi}{2Q\xi_0} \]

\[ d \sim (\xi a^2)^{1/3} \]

Intensity of the neutron Bragg scattering and resistance as a function of temperature in an ErRh\(_4\)B\(_4\) (Sinha et al., 1982). The satellite position corresponds to the wavelength of the modulated magnetic structure around 92 Å.

\[ d \sim (\xi a)^{1/2} \]

a \ll d \ll \xi
HoMo$_6$S$_8$

$T_c=1.8$ K

$T_m=0.74$ K

$T_c=0.7$ K

Figure 21. Measurements of small-angle scattering in the vicinity of the re-entrant transition in HoMo$_6$S$_8$. The peak at $|\mathbf{k}| = 0.027$ Å develops upon cooling and disappears below $0.64$ K [53].
Auto-waves in reentrant superconductors?

$T < T_{c2}$

current $I$
magnetic superconductors: re-entrant phase

Classical explanation: Jaccarino-Peter effect
- strong paramagnetic limitation in low field
- compensated by external field at higher fields
- $J<0$ (magnetic ions/conduction electrons)

INTRODUCTION: Superconductivity in heavy fermion systems

Unconventional pairing mechanism (role of correlations, magnetic «soft modes» close to the QCP)

Unconventional superconducting state (symmetry breaking: driven by the pairing mechanism)

Data: G. Knebel et al
FERROMAGNETIC UNCONVENTIONAL (TRIPLET) SUPERCONDUCTORS

**UGe$_2$** (Saxena *et al.*, 2000)
and **URhGe** (Aoki *et al.*, 2001)

*Triplett pairing*

Very recently (2007): **UCoGe** $\theta=3K$; $T_c=0.8K$

**URhGe**
(a) The total magnetic moment $M_{\text{total}}$ and the component $M_b$ measured for $H// \text{to the b axis.}$
In (b), variation of the resistance at 40 mK and 500 mK with the field re-entrance of SC between 8-12 T (Levy *et al.* 2005).
Heavy fermion ferromagnetic superconductors: re-entrant phase

- no paramagnetic limitation (p-wave)
- same electrons => $J>0$ (ferromagnetic)
- new phase is very robust:
  - higher $T_c$
  - higher orbital limit

**URhGe**

$T_{\text{Curie}} = 9.5$ K
$m_s = 0.4 \mu_B/\text{URhGe}$, $M_s \sim 500$G

Heavy fermion superconductors: ferromagnetic superconductors

- Spontaneous vortex state? need [1] \( M_s > B_{c1} \left( \frac{\lambda}{W} \right)^{\gamma_3} \)
  not met in URhGe, maybe in UCoGe [2]

- Survives when single domain!
  - superconducting if \( H_{c2}(T) > \alpha M_s \)
  - internal field:
    \[
    B_{\text{eff}} = \mu_0 H_{\text{int}} + \alpha M_s
    \]
    \[
    \mu_0 H_{\text{int}} = 0 \text{ for } M < M_s
    \]
    \[
    \mu_0 H_{\text{int}} = B_0 - \mu_0 N M_s \text{ for } M = M_s
    \]

**URhGe:** \( T_{\text{Curie}} = 9.5 \text{ K} \)
\( m_s = 0.4 \mu_B / \text{URhGe}, M_s \sim 500 \text{G} \)


Hardy & Huxley PRL94, 247006 (2005)
UCoGe: a newcomer


$T_{\text{curie}} \sim 2.5 \text{K}$, $T_c \sim 0.5 \text{K}$, $m_s \sim 0.07 \mu_B$

E. Hassinger, D. Aoki, G. Knebel and J. Flouquet, JPSJ 77 No. 7, July 2008, 073703

V. Mineev, JPSJ 77, October, 2008, 103702

**UCoGe: new crystals... and re-entrant phase uncovered**

- **UGe$_2$:** $T_{c_{\text{max}}} \approx 0.8\, \text{K}$, $T_{\text{curie}} \approx 35\, \text{K}$, at 12kbars $m_s \approx 1.4\, \mu_B$ at 0kbar.

- **UCoGe:** $T_c \approx 0.5\, \text{K}$, $T_{\text{curie}} \approx 2.5\, \text{K}$, $m_s \approx 0.07\, \mu_B$.

- **URhGe:** $T_c \approx 0.25\, \text{K}$, $T_{\text{curie}} \approx 9.5\, \text{K}$, $m_s \approx 0.4\, \mu_B$.
Superconducting order parameter behavior in ferromagnet

Standard Ginzburg-Landau functional:

\[ F = a|\Psi|^2 + \frac{1}{4m} |\nabla \Psi|^2 + \frac{b}{2} |\Psi|^4 \]

The minimum energy corresponds to \( \Psi = \text{const} \)

The coefficients of GL functional are functions of internal exchange field \( h \)!

Modified Ginzburg-Landau functional:

\[ F = a|\Psi|^2 - \gamma |\nabla \Psi|^2 + \eta |\nabla^2 \Psi|^2 + ... \]

The non-uniform state \( \Psi \sim \exp(\mathrm{i}qr) \) will correspond to minimum energy and higher transition temperature
Proximity effect in ferromagnet?

In the usual case (normal metal):

\[ a \Psi - \frac{1}{4m} \nabla^2 \Psi = 0, \text{ and solution for } T > T_c \text{ is } \Psi \propto e^{-qx}, \text{ where } q = \sqrt{4ma} \]

\[ F = (a - \gamma q^2 + \eta q^4) |\Psi_q|^2 \]

\( \Psi \sim \exp(iqr) \) - Fulde-Ferrell-Larkin-Ovchinnikov state (1964).

Only in pure superconductors and in the very narrow region.

Proximity effect in ferromagnet?
The total momentum of the Cooper pair is
\[-(k_F - \delta k_F) + (k_F - \delta k_F) = 2 \delta k_F\]
In ferromagnet (in presence of exchange field) the equation for superconducting order parameter is different

\[ a\Psi + \gamma \nabla^2 \Psi - \eta \nabla^4 \Psi = 0 \]

Its solution corresponds to the order parameter which decays with oscillations! \( \Psi \sim \exp[-(q_1 + i q_2) x] \)

Order parameter changes its sign!
Proximity effect as Andreev reflection

Classical Andreev reflection

Quantum Andreev reflection

$p_F^{\uparrow} \neq p_F^{\downarrow}$
Eilenberger equations

\[
\left( \omega + i \hbar(x) + \frac{1}{2\tau} G(x, \omega) \right) f(x, \theta, \omega) + \frac{1}{2} v_F \cos \theta \frac{\partial f(x, \theta, \omega)}{\partial x} \\
= \left( \Delta(x) + \frac{1}{2\tau} F(x, \omega) \right) g(x, \theta, \omega),
\]

\[
G(x, \omega) = \int \frac{d\Omega}{4\pi} g(x, \theta, \omega), \quad F(x, \omega) = \int \frac{d\Omega}{4\pi} f(x, \theta, \omega),
\]

\[
f(x, \theta, \omega) f^*(x, \theta, \omega) + g^2(x, \theta, \omega) = 1, \quad (A3)
\]
Theory of S-F systems in dirty limit

Analysis on the basis of the Usadel equations

\[ -\frac{D_f}{2} \vec{\nabla}^2 F_f \xi, \omega, h \xi + \xi + i \hbar \vec{F}_f \xi, \omega, h \xi = 0 \]

\[ G_f \xi, \omega, h + F_f \xi, \omega, h \vec{F}_f^* \xi, \omega, -h \xi = 1 \]

leads to the prediction of the oscillatory-like dependence of the critical current on the exchange field \( h \) and/or thickness of ferromagnetic layer.
Remarkable effects come from the possible shift of sign of the wave function in the ferromagnet, allowing the possibility of a «π-coupling» between the two superconductors (π-phase difference instead of the usual zero-phase difference).
The oscillations of the critical temperature as a function of the thickness of the ferromagnetic layer in S/F multilayers has been predicted in 1990 and later observed on experiment by Jiang et al. PRL, 1995, in Nb/Gd multilayers.
SF-bilayer $T_c$-oscillations

Ryazanov et al. JETP Lett. 77, 39 (2003) Nb-Cu$_{0.43}$Ni$_{0.57}$

V. Zdravkov, A. Sidorenko et al cond-mat/0602448 (2006) Nb-Cu$_{0.41}$Ni$_{0.59}$

$d_{F_{\text{min}}} = (1/4) \lambda_{ex}$ largest $T_c$-suppression
Josephson effect

\[ \Psi_1 = |\Psi_1| \exp(i\theta_1) \]
\[ \Psi_2 = |\Psi_2| \exp(i\theta_2) \]

\[ |\Psi_1| = |\Psi_2| \]

superconducting phase difference: \[ \varphi = \theta_1 - \theta_2 \]

Josephson relations

\[ I_s = I_c \sin \varphi \]

\[ V = \frac{\hbar}{2e} \frac{d\varphi}{dt} \]

Electromagnetic radiation at the frequency \( f \)

\[ f = \frac{V}{\Phi_0} \]
S-F-S Josephson junction in the clean/dirty limit

Damping oscillating dependence of the critical current $I_c$ as the function of the parameter $\alpha = \frac{h d_F}{v_F}$ has been predicted. (Buzdin, Bulaevskii and Panjukov, JETP Lett. 81)

$h$- exchange field in the ferromagnet, $d_F$ - its thickness

$J(\phi) = I_c \sin \phi$

$E(\phi) = - I_c \left( \frac{\Phi_0}{2 \pi c} \right) \cos \phi$

$\alpha$
The oscillations of the critical current as a function of temperature (for different thickness of the ferromagnet) in S/F/S trilayers have been observed on experiment by Ryazanov et al. 2000, PRL

and as a function of a ferromagnetic layer thickness by Kontos et al. 2002, PRL
Phase-sensitive experiments

\[ \pi \text{-junction in one-contact interferometer} \]

0-junction minimum energy at 0

\[ I = I_c \sin(\pi + \phi) = -I_c \sin \phi \]

\[ E = E_J [1 - \cos(\pi + \phi)] = E_J [1 + \cos \phi] \]

\[ 2\pi LI_c > \Phi_0/2 \]

\[ \phi = \pi = (2\pi/\Phi_0) \int_{A_{dl}} \]

\[ = 2\pi \Phi/\Phi_0 \]

Spontaneous circulating current in a closed superconducting loop when \( \beta_L > 1 \) with NO applied flux

\[ \beta_L = \Phi_0/(4\pi LI_c) \]

\[ \Phi = \Phi_0/2 \]

Bulaevsky, Kuzii, Sobyanin, JETP Lett. 1977

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Cluster Designs (Ryazanov et al.)

2 x 2

unfrustrated

fully-frustrated

checkerboard-frustrated

6 x 6

fully-frustrated

checkerboard-frustrated

30\mu m
2 x 2 arrays: spontaneous vortices

Fully frustrated

Checkerboard frustrated
Scanning SQUID Microscope images
(Ryazanov et al., Nature Physics, 2008)

\[ I_c \]

\[ T = 1.7K \quad T = 2.75K \quad T = 4.2K \]
Critical current density vs. F-layer thickness (V.A. Oboznov et al., PRL, 2006)

\[ I_c = I_{c0} \exp\left(-\frac{d_F}{\xi_{F1}}\right) \left| \cos \left(\frac{d_F}{\xi_{F2}}\right) + \sin \left(\frac{d_F}{\xi_{F2}}\right) \right| \]

\[ d_F \gg \xi_{F1} \]

Spin-flip scattering decreases the decaying length and increases the oscillation period.

\[ \xi_{F2} > \xi_{F1} \]

Nb-Cu_{0.47}Ni_{0.53}-Nb

\[ I = I_c \sin \phi \]

\[ I = I_c \sin(\phi + \pi) = -I_c \sin(\phi) \]
Critical current vs. temperature

\[
\frac{1}{\xi_{F1}} = \frac{1}{\xi_F} \sqrt{1 + \left(\frac{1}{\hbar \tau_s}\right)^2},
\]

\[
\frac{1}{\xi_{F2}} = \frac{1}{\xi_F} \sqrt{1 + \left(\frac{1}{\hbar \tau_s}\right)^2 - \left(\frac{1}{\hbar \tau_s}\right)},
\]

\[\xi_{F2} > \xi_{F1}\]

\[\left(\omega + i E_{ex} + \frac{\hbar \cos \Theta}{\tau_s}\right) \sin \Theta - \frac{\hbar D}{2} \frac{\partial^2 \Theta}{\partial x^2} = 0.\]

Effective spin-flip rate
\[\Gamma(T) = \cos \Theta(T)/\tau_s;\]

\[G = \cos \Theta(T); \quad F = \sin \Theta(T)\]
Critical current vs. temperature (0-π- and π-0- transitions)

\[ J_c, \text{A/cm}^2 \]

\[ T, \text{K} \]

\( d_F = 9 \text{ nm} \)
\( d_F = 18 \text{ nm} \)
\( d_F = 11 \text{ nm} \)
\( d_F = 22 \text{ nm} \)

\( d_{F_1} = 10-11 \text{ nm} \)
\( d_{F_2} = 22 \text{ nm} \)

\( \text{Nb-Cu}_{0.47}\text{Ni}_{0.53}-\text{Nb} \)

(V.A.Oboznov et al., PRL, 2006)
Triplet correlations


\[
D \partial_x^2 \hat{f} - 2|\omega| \hat{f} + i \text{sgn}(\omega)(\hat{f}\hat{V}^* - \hat{V}\hat{f}) = 0
\]

\[
\hat{V} = J \begin{pmatrix} \cos \alpha & \pm i \sin \alpha \\ \mp i \sin \alpha & -\cos \alpha \end{pmatrix}
\]

Structure of the functions f:

\[
f = i \tau_2 (f_3(x) \sigma_3 + f_0(x)) + i \tau_1 \sigma_1 f_1(x)
\]

\[
f_3 \propto \langle \psi \psi \rangle - \langle \psi \psi \rangle
\]

- Singlet condensate

\[
f_0 \propto \langle \psi \psi \rangle + \langle \psi \psi \rangle
\]

- Triplet condensate (with projection 0 on z-axis)

\[
f_1 \propto \langle \psi \psi \rangle \propto \langle \psi \psi \rangle
\]

- Triplet condensate (with projection +1, -1)

\[
\sigma, \tau \quad \text{-Pauli matrices (spin, Nambu)}
\]

- positive chirality
Triplet proximity effect may substantially increase the decaying length in the dirty limit.

The same, but larger amplitude

No oscillations
Some source of triplet correlations?

Why difficult to observe? Magnetic scattering and spin-orbit scattering are harmful for long ranged triplet component.

\[ \xi_f = \sqrt{\frac{D_f}{\hbar}} \]
Supercurrent measured in NbTiN/CrO2/NbTiN junctions

Klapwijk’s group in Delft

Long junctions with « large » $I_c$

CrO$_2$ is half-metallic!
FIG. 1: Geometry of $S/F'/F/F''/S$ junction. The arrows indicate non-collinear orientations of magnetizations in each layer with thickness $d_L$, $d$, $d_R$, respectively ($L = d_L + d + d_R$).

\[ \xi_f \ll L \ll \xi_0 \]

\[ e R_F I_c = -\frac{2 \Delta(T)^2 h_0^2}{\pi^3 T_c^3} \sin \theta_R \sin \theta_L \]

(+ small term)
FIG. 2: Critical current induced by long range triplet proximity effect in S/F'/F/F''/S junction, in units of $(\pi G\Delta(T)^2/4eT_c)$, for varying length of F' and F'' layers, at $d_L = d_R \sim \xi_f \ll d \ll \xi_0$, and for different orientations of the magnetization in the layers.

Rather sharp maximum of the critical current at $d_L=d_R=\xi_f$
SFS junction with different types of domains
Clean SFS junctions with sharp domain walls

Domain walls with collinear and non-collinear magnetic moments

(a) and (b) diagrams showing magnetic moments with angles and distances labeled.
Clean SFS junctions, BdG equations

\[-i\hbar v_F \hat{T}_z \frac{d}{dt} \hat{g} + \hbar (\mathbf{R} + t\mathbf{n}_F) \tilde{\sigma} \hat{g} + \left( \begin{array}{cc} 0 & \Delta(\mathbf{R} + t\mathbf{n}_F) \\ \Delta^*(\mathbf{R} + t\mathbf{n}_F) & 0 \end{array} \right) \hat{g} = \epsilon \hat{g} \]

\[ g = (u,v), u \text{ and } v \text{ are the electronlike and holelike parts of the quasiparticle wave function } \]

\[
I_{2D} = \frac{k_F}{2\pi} \int ds \int d\mathbf{n}_F \left( j(\varphi + \gamma) + j(\varphi - \gamma) \right)(\mathbf{n}_F, \mathbf{n}_i) ,
\]

\[
I_{3D} = \frac{k_F^2}{4\pi^2} \int ds \int d\mathbf{n}_F \left( j(\varphi + \gamma) + j(\varphi - \gamma) \right)(\mathbf{n}_F, \mathbf{n}_i) ,
\]

\[ j(\varphi) = \frac{e \Delta_0}{2\hbar} \sin(\varphi/2) \tanh \left( \frac{\Delta_0 \cos(\varphi/2)}{2T} \right) \]

\[ 2D: \quad \delta I = \frac{e \Delta_0^2 k_F \xi_f}{2\pi \hbar T} \sin \varphi \left( \sin(d/\xi_f) - \frac{d^2}{\xi_f^2} \int_0^\infty \frac{\sin y}{y^3} dy \right) \quad l \sim \sqrt{\xi_f/d} \]

\[ 3D: \quad \delta I \simeq \frac{e \Delta_0^2 k_F^2 \xi_f L_y}{2\pi^2 \hbar T} \sqrt{\frac{\pi \xi_f}{2d}} \cos(d/\xi_f + \pi/4) \sin \varphi \quad l \sim \xi_f/d \]

\[ \delta I(\alpha) = \sin^2 \frac{\alpha}{2} \cdot \delta I(\pi) \]
Long–range triplet proximity effect in dirty SF systems

\[ f = i\sigma_y (f_s + i f_t \bar{\sigma}) \]

T. Champel et al. PRL (2008)

\[
\left( \frac{D}{2} \nabla^2 - \omega_n \right) f_s = -hf_t
\]

\[
\left( \frac{D}{2} \nabla^2 - \omega_n \right) f_t = hf_s
\]
Dirty SFS junctions with sharp domain walls. Usadel equations.

\[ f = i\sigma_y (f_s + if_t \bar{\sigma}) \left( \frac{D}{2} \nabla^2 - \omega_n \right) f_s = -hf_t \]

\[ \left( \frac{D}{2} \nabla^2 - \omega_n \right) f_t = hf_s \]

\[ \frac{\partial f_s}{\partial x} \bigg|_{\pm d/2} = F_\pm = \pm \frac{\pi \Delta}{\gamma b \sqrt{\Delta^2 + \omega_n^2}} e^{\pm i\varphi/2} , \]

\[ \frac{\partial f_t}{\partial x} = 0 , \]

\[ g_1 = ih \text{sign} z g_1 + \frac{D}{2} F \delta(x) , \]

\[ g_2 = -ih \text{sign} z g_2 + \frac{D}{2} F \delta(x) . \]

\[ I = \frac{2eDN_F T}{\pi} \int d^2 s \sum_n \text{Im} \left( f_s^* \frac{\partial}{\partial x} f_s - f_t^* \frac{\partial}{\partial x} f_t \right) \]
Dirty SFS junctions with sharp domain walls. Collinear magnetic moments

*No change in the length of decay of superconducting correlations*

*No long – range triplet component*

In accordance with the result of Crouzy et al. PRB 76 (2007)

Dirty SFS junctions with sharp domain walls. Non-collinear magnetic moments

\[ d < \xi_N \]

\[ \delta I = \frac{eN_FDL_y}{2} \frac{\xi_h^4}{\gamma_b d^2} \Delta \tanh \frac{\Delta}{2T} \sin \varphi \]
Clean SFS junctions with domain walls

Maximum contribution to the critical current:
Domain walls with antiparallel magnetic moments

Dirty SFS junctions with domain walls

Maximum contribution to the critical current:
Domain walls with non-collinear magnetic moments

Noncollinear domains in SFS junctions are sources of the long range triplet correlation.

The domain walls connecting superconducting leads can be considered as channels for triplet current.
F/S/F trilayers, spin-valve effect

If $d_s$ is of the order of magnitude of $\xi_s$, the critical temperature is controlled by the proximity effect.

Firstly the FI/S/FI trilayers has been studied experimentally in 1968 by Deutscher et Meunier. In this special case, we see that the critical temperature of the superconducting layers is reduced when the ferromagnets are polarized in the same direction.
In the dirty limit, we used the quasiclassical Usadel equations to find the new critical temperature $T^*_c$. We solved it self-consistently supposing that the order parameter can be taken as:

$$\Delta = \Delta_0 \left(1 - \frac{x^2}{L^2}\right)$$

with $L \gg d_s$


In the case of a perfect transparency of both interfaces

$$d^* = \gamma \sqrt{\frac{h}{D_n}} \frac{D_s}{4\pi T_c}$$

$$\ln\left(\frac{T_{c\uparrow\downarrow}^*}{T_c}\right) = \Psi\left(\frac{1}{2}\right) - \text{Re} \Psi\left(\frac{1}{2} + \frac{d^* T_c}{d_s T_{c\uparrow\downarrow}^*}\right)$$

$$\ln\left(\frac{T_{c\uparrow\uparrow}^*}{T_c}\right) = \Psi\left(\frac{1}{2}\right) - \Psi\left(\frac{1}{2} + \frac{d^* T_c}{d_s T_{c\uparrow\downarrow}^*}\right)$$
Recent experimental verifications

CuNi/Nb/CuNi
Gu, You, Jiang, Pearson, Bazaliy, Bader, 2002

Ni/Nb/Ni
Moraru, Pratt Jr, Birge, 2006
Evolution of the difference between the critical temperatures as a function of interfaces’ transparency

\[ \gamma_B = 0 \]

Infinite transparency

\[ \gamma_B \approx 5 \]

Finite transparency

\[
\ln \left( \frac{T_{c\uparrow\uparrow}^*}{T_c} \right) = \Psi \left( \frac{1}{2} \right) - \text{Re} \left[ \Psi \left( \frac{1}{2} + \frac{\tilde{d} T_c}{d_s T_{c\uparrow\uparrow}^*} \right) + i \right]
\]

\[
\ln \left( \frac{T_{c\uparrow\downarrow}^*}{T_c} \right) = \Psi \left( \frac{1}{2} \right) - \Psi \left( \frac{1}{2} + \frac{\tilde{d} T_c}{d_s T_{c\uparrow\downarrow}^*} \right)
\]

\[
\tilde{d} = \frac{D_s}{4\pi T_c} \gamma \sqrt{\frac{\hbar}{D_n}} \ln \left( \frac{\hbar}{D_n} \right) \left( 1 + \left( \frac{\gamma}{\hbar} \right) \sqrt{\frac{\hbar}{D_n}} \right)
\]
Similar physics in F/S bilayers

In practice, magnetic domains appear quite easily in ferromagnets

\( w: \) width of the domain wall

Localized (domain wall) superconducting phase.


\[ \text{Ni}_{0.80}\text{Fe}_{0.20}/\text{Nb} \ (20\text{nm}) \]

Thin films: Néel domains

Rusanov et al., PRL, 2004
Inverse effect: appearance of the dense domain structure under the influence of superconductivity. Not observed yet.
Domain wall superconductivity in purely electromagnetic model

\[- \left( \nabla + \frac{2\pi i}{\Phi_0} \vec{A}(\vec{r}) \right)^2 \Psi = \frac{1}{\xi^2(T)} \Psi \]

Ferromagnetic layer

Superconducting film

E.B. Sonin (1988)

Pb-Co/Pt

\( w \gg D \)

\( w \ll D \)

\( H_{c3} \)
Superconductivity nucleation at a single domain wall

Thin domains

Step-like magnetic field profile

Wavefunction $\eta_+$
Domain Wall S/F bilayer with domain structure

\[ B_0 = 4\pi M \sim 1 - 10kOe \frac{dH_{c2}}{dT} \sim 0.5kOe/K \]

\[ T_c \sim 9K \]

\[ \delta T_c \sim 1 - 3K \]
Thick domains

$B_0$-maximum field induced by the domain wall

Local approximation: Particle in a linear B profile
Superconducting nucleus in a periodic domain structure in an external field

$H \neq 0$

\[
\frac{\pi B_0 w^2}{\Phi_0} = 5
\]

\[
\frac{\pi B_0 w^2}{\Phi_0} = 1
\]

Domain wall superconductivity
Domain wall superconductivity

\( H > 0 \)

\( H = 0 \)

\( H < 0 \)

Nb/BaFe$_{12}$O$_{19}$

Z. YANG et al, Nature Materials, 2004
Atomic layered S-F systems


Magnetic layered superconductors like RuSr$_2$GdCu$_2$O$_8$

Also even for the quite small exchange field ($h>T_c$) the $\pi$-phase must appear.
Crystal structure of layered HTS

- $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+x}$ (Bi2212)
- $\text{Bi}_2\text{Sr}_2\text{CuO}_{6+x}$ (Bi2201)

Also Tl2212, Tl2201 etc.
It is possible to obtain the exact solution of this model and to find all Green functions.
The limit $t \ll T_{co}$
Atomic thickness F/S/F heterostructure. Inversion of the proximity effect at low temperature.

Also the exact solution is possible. \( T_{cP} < T_{cAP} \)  

\[
\delta T_P - \delta T_{AP} \approx -\frac{1397\zeta(7)}{512\pi^6} \frac{t^4h^2}{T_{c0}^6} \approx -0.0029 \frac{t^4h^2}{T_{c0}^6};
\]  

(Tollis et al., PRB, 2005)
The inversion of the proximity effect occurs at the temperature

\[ T^* = 0.47T_{c0} \]

for the weak field \( h \).
Superconducting multilayered systems

(Buzdin, Cayssol and Tollis, PRL 2005,)

layered superconductors with a structure like high-$T_c$

Zeeman effect, i.e. the exchange field $\mu_B H$, or ferromagnetic superconductor

At low temperature the paramagnetic limit may be strongly exceed $\mu_B H \sim t_1$. $\pi$-phase with FFLO modulation in plane.
Superconducting bilayer

$S \ll 0$  BCS coupling

$S \ll \pi$

Energy spectrum

$\pi$-phase
Compensation of the Zeeman splitting at $\mu_B H = h \sim t$
At low temperature the paramagnetic limit may be strongly exceeded by $\mu_B H \sim t_1$. \( \pi \)-phase with FFLO modulation in plane. Field-induced superconductivity!
More complicated phase diagramm
Very special situation is possible when the weak link in Josephson junction is a non-centrosymmetric magnetic metal with broken inversion symmetry! Suitable candidates: MnSi, FeGe.

Josephson junctions with time reversal symmetry: \( j(-\varphi) = -j(\varphi) \); i.e. higher harmonics can be observed \( \sim j_n \sin(n\varphi) \) – the case of the \( \pi \) junctions.

Without this restriction a more general dependence is possible

\[
j(\varphi) = j_0 \sin(\varphi + \varphi_0).
\]

**Rashba-type** spin-orbit coupling

\[
\vec{\alpha} \vec{\sigma} \times \vec{\sigma} \cdot \vec{n}
\]

\( \vec{n} \) is the unit vector along the asymmetric potential gradient.
Geometry of the junction with BIS magnetic metal
\[ F = a |\Psi|^2 + \gamma |\vec{D}\Psi|^2 + \frac{b}{2} |\Psi|^4 - \varepsilon \hbar \left[ \int \left( \psi \delta \psi^* + \psi^* \delta \psi \right) \right], \]

\[ D_i = -i \partial_i - 2eA_i \]

\[ a\Psi - \gamma \frac{\partial^2 \Psi}{\partial x^2} + 2i \varepsilon \hbar \frac{\partial \Psi}{\partial x} = 0, \]

\[ \Psi \propto \exp(i\tilde{\varepsilon}x) \exp(-x \sqrt{\frac{a - a_c}{\gamma}}), \quad \text{where} \quad \tilde{\varepsilon} = \frac{\varepsilon \hbar}{\gamma} \]

\[ \varphi_0 - \text{Josephson junction} \quad (A. \text{Buzdin, PRL, 2008}). \]
$\Psi \propto \exp(i\tilde{c}x) \exp(-x\sqrt{\frac{a-a_c}{\gamma}}), \quad \text{where } \tilde{c} = \frac{\varepsilon h}{\gamma}$

In contrast with a Π junction it is not possible to choose a real $\Psi$ function!
\( \varphi_0 \) Josephson junction

\[
 j \varphi = j_c \sin \varphi + \varphi_o
\]

where

\[
 \varphi_o = \frac{2e h L}{\gamma}
\]

The phase shift \( \varphi_0 \) is proportional to the length and the strength of the BIS magnetic interaction.

The \( \varphi_0 \) Junction is a natural phase shifter.

Energy \( E_J(\varphi) \sim -j_c \cos(\varphi + \varphi_0) \)
$E_J(\varphi) \sim -j_c \cos(\varphi + \varphi_0)$

$E(\varphi = \varphi_0) \neq E(\varphi = -\varphi_0)$
Spontaneous flux (current) in the superconducting ring with $\Phi_0$-junction.

$$E(\varphi) = \frac{j_c}{2e} \left( - \cos(\varphi + \varphi_0) + \frac{k \varphi^2}{2} \right)$$

$$k = \frac{c \Phi_0}{2\pi L j_c}$$

In the $k<<1$ limit the junction generates the flux $\Phi = \Phi_0(\varphi_0/2\pi)$

$$\varphi_0 = \frac{2\varepsilon \hbar L}{\gamma}$$

**Very important**: The $\Phi_0$ junction provides a mechanism of a direct coupling between supercurrent (superconducting phase) and magnetic moment (z component).
Let us consider the following geometry:

\[ \varphi_0 = x \frac{v_{so}}{v_F} \frac{M_y}{M_0} \]

\[ \sin \theta = \frac{I}{I_c} \Gamma \quad \text{with} \quad \Gamma = \frac{E_J}{K \gamma} x \frac{v_{so}}{v_F} \]

\[ \varphi(t) = \omega_J t \]

voltage-biased Josephson junction
Magnetic anisotropy (easy z-axis) energy:

\[ E_M = -\frac{KV}{2} \left( \frac{M_z}{M_0} \right)^2. \]

Coupling parameter:

\[ \Gamma = \frac{E_J}{KVx} \frac{v_{so}}{v_F}. \]

Weak coupling regime: \( \Gamma < 1 \).
Strong coupling regime: \( \Gamma > 1 \).

Let us consider first the \( \phi_0 \) - junction when a constant current \( I < I_c \) is applied:

Minimum energy condition:

\[ \partial_\varphi E_{tot} = \partial_{\phi_0} E_{tot.} = 0. \]
The current provokes rotation of the magnetic moment:

\[ \sin \theta = \frac{I}{I_c} \Gamma \]

\[ M_y = M_0 \sin \theta \]

For the case \( \Gamma > 1 \) when \( I > I_c / \Gamma \) the moment will be oriented along the y-axis.

Applying to the \( \varphi_0 \) junction a current (phase difference) we can generate the magnetic moment rotation.

a.c. current -> moment’s precession!
What happens if the SO gradient is along $y$-axis?

The total energy has two minima $\theta = (0, \pi)$.

The current pulses would provoke the switches of $\mathbf{M}$ between $\theta = 0$ and $\theta = \pi$ orientations. This corresponds to the transition of the junction from $+\varphi_0$ to $-\varphi_0$ state.

$$\varphi_0 = \frac{x (v_{SO}/v_F)}{\cos \theta}$$
Magnetic moment precession – voltage-biased $\varphi_0$- junction.

\[ \dot{\varphi}(t) = \omega_J t \]

\[ \frac{dM}{dt} = \gamma M \wedge H_{\text{eff}} + \frac{\alpha}{M_0} \left( M \wedge \frac{dM}{dt} \right), \]

where $H_{\text{eff}} = -\delta F/\mathcal{V} \delta M$ is the effective magnetic field.

\[ H_{\text{eff}} = \frac{K}{M_0} \left[ \Gamma \sin \left( \omega_J t - r \frac{M_y}{M_0} \right) \hat{y} + \frac{M_z}{M_0} \hat{z} \right] \]

\[ r = x v_{so}/v_F \]
Landau-Lifshitz equation:

\[
\frac{dM}{dt} = \gamma M \wedge H_{\text{eff}} + \frac{\alpha}{M_0} \left( M \wedge \frac{dM}{dt} \right)
\]

Magnetic moment precession:

\[
H_{\text{eff}} = \frac{K}{M_0} \left[ \Gamma \sin \left( \omega_J t - r \frac{M_y}{M_0} \right) \hat{y} + \frac{M_z}{M_0} \hat{z} \right]
\]

\[
\frac{I}{I_c} = \sin \omega_J t + \frac{\Gamma r}{2} \frac{1}{\omega^2 - 1} \sin 2\omega_J t + \ldots
\]
Weak coupling regime: $\Gamma \ll 1$.

Without damping

$$m_x(t) = \frac{\Gamma \omega \cos \omega_J t}{1 - \omega^2} \quad \text{and} \quad m_y(t) = -\frac{\Gamma \sin \omega_J t}{1 - \omega^2}.\]

$$\frac{I}{I_c} = \sin \omega_J t + \frac{\Gamma r}{2} \frac{1}{\omega^2 - 1} \sin 2\omega_J t + \ldots,$$

With damping

$$m_y(t) = \frac{\omega_+ - \omega_-}{\Gamma r} \sin \omega_J t + \frac{\alpha_- - \alpha_+}{\Gamma r} \cos \omega_J t,$$

$$\omega_\pm = \frac{\Gamma r}{2} \frac{\omega \pm 1}{\Omega_\pm} \quad \text{and} \quad \alpha_\pm = \frac{\Gamma r}{2} \frac{\alpha}{\Omega_\pm} \quad \text{with} \quad \Omega_\pm = (\omega \pm 1)^2 + \alpha^2.$$

$$I(t) \approx I_c \sin \omega_J t + I_c \frac{\omega_+ - \omega_-}{2} \sin 2\omega_J t +$$

$$+ I_c \frac{\alpha_- - \alpha_+}{2} \cos 2\omega_J t + I_0(\alpha)$$

$$I_0(\alpha) = \frac{\alpha \Gamma r}{4} \left( \frac{1}{\Omega_-} - \frac{1}{\Omega_+} \right)$$

The current acquires a d..c. component!
Strong coupling regime: $\Gamma >> 1$.

If $r << 1$ then $m_y \approx 0$ and without damping we have:

\[
\begin{align*}
    m_x(t) &= \sin \left[ \frac{\Gamma}{\omega} \left( 1 - \cos \omega_j t \right) \right] \\
    m_z(t) &= \cos \left[ \frac{\Gamma}{\omega} \left( 1 - \cos \omega_j t \right) \right]
\end{align*}
\]

Complete reversal being induced by $\Gamma / \omega > \pi / 2$.

Comparison between analytic results (dashed line) and numerical computation.
Complicated regime of the magnetic dynamics:

For more details – see (F. Konschelle and A. Buzdin, PRL, 2009).
Complementary Josephson logic

**RSFQ-logic using \( \pi \)-shifters**


**RSFQ-logic: Rapid Single Quantum logic**

**Conventional RSFQ-cell**

\[ LI_c > \Phi_0 \]

**Fluxon memorizing cell**

\[ L \rightarrow 0 \]

**RSFQ-** \( \pi \)-**cell**

\[ L_j = \Phi_0 / (2\pi I_c) \]

\[ \tau \sim 1/(I_c R) \]

**To operate at 20 GHz clock rate**

\[ I_c R \sim 50 \ \mu V \text{ has to be} \]

**We have** \[ I_c R > 0.1 \ \mu V \text{ for the present} \]

\( \pi \)-**RSFQ – Toggle Flip-Flop**
Implementation of superconductor/ferromagnet/superconductor π-shifters in superconducting digital and quantum circuits

A. K. Feofanov¹, V. A. Oboznov², V. V. Bol'ginov², J. Lisenfeld¹, S. Poletto¹, V. V. Ryazanov², A. N. Rossolenko², M. Khabipov³, D. Balashov³, A. B. Zorin³, P. N. Dmitriev⁴, V. P. Koshelets⁴ and A. V. Ustinov¹*

Figure 1 | Complementary d.c.-SQUIDs. a, Schematic diagram of a complementary d.c.-SQUID employing two conventional Josephson...
Superconducting phase qubit

Digital bit

Quantum bit

or π-shift due to π-junction

|0⟩

|1⟩

Φ_{ext} = Φ_0 / 2

or π-shift due to π-junction

qubit operation
Implementation of superconductor/ferromagnet/superconductor \( \pi \)-shifters in superconducting digital and quantum circuits

A. K. Feofanov\(^1\), V. A. Oboznov\(^2\), V. V. Bol'ginov\(^2\), J. Lisenfeld\(^1\), S. Poletto\(^1\), V. V. Ryazanov\(^2\), A. N. Rossolenko\(^2\), M. Khabipov\(^3\), D. Balashov\(^3\), A. B. Zorin\(^3\), P. N. Dmitriev\(^4\), V. P. Koshelets\(^4\) and A. V. Ustinov\(^1\*)

\[\text{Figure 3 | Self-biased phase qubit. a, Schematic diagram of a phase qubit circuit used to test the decoherence properties of the \( \pi \)-junction. The qubit is realized by the central loop with embedded conventional and \( \pi \)-Josephson junctions. The larger loop to its left is a d.c.-SQUID for qubit readout. To the right of the qubit is a coupled weakly flux bias coil. b, Scanning electron microscope picture of the realized phase qubit employing a \( \pi \)-junction in the qubit loop. The flux bias coil is not shown.}\]

\[\text{Figure 4 | Rabi oscillations between the ground and the excited qubit states resulted from resonant microwave driving. a,b, Rabi oscillations observed in the phase qubit with an embedded \( \pi \)-junction (a) and a conventional phase qubit made on the same wafer as a reference (b). Each}\]
I. conclusions

• Unusual non-uniform magnetic and superconducting phases.

• Ferromagnetic superconductors – unconventional types of superconductivity. Fundamental properties of this superconductivity?

• Superconductor-ferromagnet heterostructures permit to study superconductivity under huge exchange field ($h>>T_c$).

• The $\pi$-junction realization in S/F/S structures is quite a general phenomenon.

• The BIS magnets provide a mechanism of the realization of the novel $\phi_0$ - junctions. In these $\phi_0$ - junctions a direct (linear) coupling between superconductivity and magnetism is realized. Seems to be ideal for superconducting spintronics.


Coexistence of Superconductivity and Magnetism –Bulaevskii et al., ADV. IN PHYSICS, 1985, VOL. 34, 175.
Non-uniform (FFLO) states and quantum oscillations in superconductors and superfluid ultracold Fermi gases

1. Larkin-Ovchinnikov-Fulde-Ferrell state. LOFF or FFLO states. The origin of the non-uniform modulated superconducting state.

2. Experimental evidences of FFLO state.

3. Exactly solvable models of FFLO state.

4. Vortices in FFLO state. Role of the crystal structure.

5. Supefluid ultracold Fermi gases with imbalanced state populations: one more candidate for FFLO state?
FFLO inventors

Fulde and Ferrell

Larkin and Ovchinnikov
The total momentum of the Cooper pair is
\(- (k_F - \delta k_F) + (k_F - \delta k_F) = 2 \delta k_F\)
Pairing of electrons with opposite spins and momenta unfavourable:

\[ [\varepsilon(k) - \mu_B H^{\text{eff}}] \neq [\varepsilon(-k) + \mu_B H^{\text{eff}}] \]

But:

\[ [\varepsilon(k + q) - \mu_B H^{\text{eff}}] \approx [\varepsilon(-k + q) + \mu_B H^{\text{eff}}] \text{ if } q \approx \frac{\mu_B H^{\text{eff}}}{v_F} \]

\[ \rightarrow \Delta(r) = \Delta \exp(iq \cdot r) \]

At \( T = 0 \), Zeeman energy compensation is exact in 1d, partial in 2d and 3d.

- the upper critical field is increased
- Sensivity to the disorder and to the orbital effect:

\[ q(T) \gg \frac{1}{\ell_{\text{imp}}}, \frac{1}{L_H} \] (clean limit)
The SC order parameter performs one-dimensional spatial modulations along $H$, forming planar nodes.
Modified Ginzburg-Landau functional:

\[ F = a|\Psi|^2 - \gamma |\nabla \Psi|^2 + \eta |\nabla^2 \Psi|^2 - \gamma' |\Psi|^4 + \beta |\nabla \Psi|^2 |\Psi|^2 + \beta' \left[ |\Psi^*|^2 \nabla^2 \Psi^2 + \Psi^2 \nabla (\nabla \Psi^*)^2 \right] + \delta |\Psi|^6 + ... \]

\[ \tilde{\nabla} = \nabla - \frac{2ie}{\hbar c} A \]

May be 1st order transition at \( T < T^* \approx 0.56T_c \)
2. Experimental evidences of FFLO state.

- Unusual form of $H_{c2}(T)$ dependence
- Change of the form of the NMR spectrum
- Anomalies in ultrasound absorption
- Unusual behaviour of magnetization
- Change of anisotropy ....
Organic superconductor

$\kappa$-(BEDT-TTF)$_2$Cu(NCS)$_2$ ($T_c=10.4\text{K}$)

Layered structure

Suppression of orbital effects in $H$ parallel to the planes

Cu[N(CN)$_2$]Br layer

BEDT-TTF layer

$\sim 15\text{Å}$

BEDT-TTF (donor molecule)
The Fulde-Ferrell-Larkin-Ovchinnikov State in the Organic Superconductor \(\kappa-(BEDT-TTF)_{2}\text{Cu(NCS)}_{2}\) as Observed in Magnetic Torque Experiments

B. Bergk\textsuperscript{a}, A. Demuer\textsuperscript{b}, I. Sheikin\textsuperscript{b}, Y. Wang\textsuperscript{c}, J. Wosnitza\textsuperscript{d}, Y. Nakazawa\textsuperscript{e}, and R. Lortz\textsuperscript{f}
Anomalous in-plane anisotropy of the onset of SC in \((\text{TMTSF})_2\text{ClO}_4\)

Field induced superconductivity (FISC) in an organic compound

\[ \lambda-(\text{BETS})_2\text{FeCl}_4 \]

\[ T_c^\text{max} \approx 4.2\text{K} \]

\[ B_0 \approx 33\text{Tesla} \]

Metal

Insulator AF

FISC

FISC for \( 18T < B < 45T \)

S. Uji et al., Nature \textbf{410} 908 (2001)

L. Balicas et al., PRL \textbf{87} 067002 (2001)

c-axis (in-plane) resistivity

\[ B \]

\[ R(\Omega \text{mm}) \]

\[ (a) T=5.4\text{K} \]

\[ \theta=0.5^\circ \]

\[ T=0.8\text{K} \]
Pauli paramagnetically limited superconducting state

$H-T$ phase diagram of CeCoIn$_5$

- $H \parallel ab$
  - $H_{c2}^{\text{orb}}$
  - $\frac{dH_{c2}}{dT} \bigg|_{T_c} = 24$ T/K

- $H \parallel c$
  - $H_{c2}^{\text{orb}}$
  - $\frac{dH_{c2}}{dT} \bigg|_{T_c} = 11$ T/K

- 1st order transition
- 2nd order transition
New high field phase of the flux line lattice in CeCoIn$_5$

This 2nd order phase transition is characterized by a structural transition of the flux line lattice.

Ultrasound and NMR results are consistent with the FFLO state which predicts a segmentation of the flux line lattice.
FFLO State in Neutron Star
Color superconductivity
R. Casalbuoni and G. Nardulli

Bose-Einstein-Condensate

Vortices

Superfluid ultracold Fermi gases with imbalanced state populations: one more candidate for FFLO state?

Massachusetts Institute of Technology:

Rice University, Houston:
3. Exactly solvable models of FFLO state.
The FFLO phase is the soliton lattice, first proposed by Brazovskii, Gordyunin and Kirova in 1980 for polyacetylene.

$$\Delta(x) = \Delta_0 sn(x / \xi, k)$$

At $T = 0$, $\mu_B H = \frac{2}{\pi} \Delta$

**Magnetic moment**
Spin-Peierls transitions - e.g. CuGeO$_3$

$u_n = (-1)^n \Delta(x = na)$

$T_{SP} = 14.2$ K
In 2D superconductors

Y. Matsuda and H. Shimahara
In 3D superconductors

The transition to the FFLO state is 1st order. The sequence of phases is similar to 2D case. **Houzet et al. 1999; Mora et al. 2002**
4. Vortices in FFLO state. Role of the crystal structure.
FFLO phase in the case of paramagnetic and orbital effect (3D BCS limit) – upper critical field

Note: The system with elliptic Fermi surface can be transformed by scaling transformation to the isotropic one. Sure the direction of the magnetic field will be changed.

\[ \Delta(r) \sim \exp(iQz) \exp(-\rho^2 eH/2\hbar c) \]

FFLO exists for Maki parameter \( \alpha > 1.8 \).

For Maki parameter \( \alpha > 9 \) the highest Landau level solutions are realized – Buzdin and Brison, 1996.

\[ \alpha \equiv \sqrt{2} \frac{H_{c2}^{orb}}{H_{c2}^P} \]

Lowest \( m=0 \) Landau level solution, Gruenberg and Gunter, 1966
FFLO phase in 2D superconductors in the tilted magnetic field - upper critical field

Highest Landau level solutions are realized – Bulaevskii, 1974; Buzdin and Brison, 1996; Houzet and Buzdin, 2000.

\[ \Delta(r) \sim e^{-i m \varphi^m} \exp(iQz) \exp(-\rho^2 eH / 2 \hbar c), \]
Exotic vortex lattice structures in tilted magnetic field

Generalized Ginzburg-Landau functional

Near the tricritical point, the characteristic length is

$$q(T)^{-1} \xrightarrow{T \to T^*} \infty$$

Microscopic derivation of the Ginzburg-Landau functional:

$$\frac{\mathcal{F}}{N(0)T_c^2} = 0.86 \frac{B - H_{\text{eff}}(T)}{H_{\text{eff},*}} |\Delta|^2 + 3.0 \frac{T - T^*}{T^*} \xi_0^2 |\nabla \Delta|^2 + 0.15 \frac{T - T^*}{T^*} |\Delta|^4$$

Instability toward FFLO state

Validity:

- large scale for spatial variation of $\Delta$ : vicinity of $T^*$
  - small orbital effect, introduced with
  $$\tilde{\nabla} = \nabla - \frac{2ie}{\hbar c} A$$
  
- we neglect diamagnetic screening currents (high-$\kappa$ limit)

Next orders are important:

$$\frac{\mathcal{F}}{N(0)T_c^2} = 0.86 \frac{B - H_{\text{eff}}(T)}{H_{\text{eff},*}} |\Delta|^2 + 3.0 \frac{T - T^*}{T^*} \xi_0^2 |\tilde{\nabla} \Delta|^2 + 3.1 \xi_0^4 |\tilde{\nabla}^2 \Delta|^2 + 0.15 \frac{T - T^*}{T^*} |\Delta|^4$$

$$+ 0.85 \xi_0^2 \left\{|\Delta|^2 |\tilde{\nabla} \Delta|^2 + \frac{1}{8} \left[(\Delta^* \tilde{\nabla} \Delta)^2 + (\Delta \tilde{\nabla} \Delta^*)^2\right]\right\} + 0.011 |\Delta|^6$$

Next orders are important:
• 2nd order phase transition at

\[ 0.86 \frac{B - H_{\text{eff}}(T)}{H_{\text{eff}}} \Delta - 3.0 \frac{T - T^*}{T^*} \xi_0^2 \vec{\nabla}^2 \Delta + 3.1 \xi_0^4 \vec{\nabla}^4 \Delta = 0 \]

→ higher Landau levels

\[ \vec{\nabla}^2 \Delta_N = -\frac{4eH_\perp}{\hbar c} (N + \frac{1}{2}) \Delta_N \]

• Near the transition: minimization of the free energy with solutions in the form

\[
\psi_{\zeta = \rho + i\sigma}(x, y) = \frac{(2\sigma)^{\frac{1}{4}}}{(2^N N!)^\frac{1}{2}} e^{-\frac{\pi y^2 B_\perp}{\phi_0}} \sum_p \mathcal{H}_N \left( y \sqrt{\frac{2\pi B_\perp}{\phi_0}} + p \sqrt{2\pi \sigma} \right) e^{2i\pi p(x+iy)\sqrt{\frac{\sigma B_\perp}{\phi_0}} + i\pi p \zeta^2} \]

\[
\text{gauge } \quad A = (0, -y B_\perp, 0) \]

\[ \zeta \text{ Parametrizes all vortex lattice structures at a given Landau level } N \]

\[ (r_1, r_2) = \left( \sqrt{\frac{\phi_0}{\sigma B_\perp}}, \zeta \sqrt{\frac{\phi_0}{\sigma B_\perp}} \right) \] is the unit cell

All of them are described in the subset:

\[
\left[ |\zeta| > 1; 0 < \rho < \frac{1}{2} \right] \]
Analysis of phase diagram:
- cascade of 2nd and 1st order transitions between S and N phases
- 1st order transitions within the S phase
- exotic vortex lattice structures

At Landau levels $n > 0$, we find structures with several points of vanishment of the order parameter in the unit cell and with different winding numbers $w = \pm 1, \pm 2 \ldots$
Order parameter distribution for the asymmetric and square lattices with Landau level $n=1$. The dark zones correspond to the maximum of the order parameter and the white zones to its minimum.
Intrinsic vortex pinning in LOFF phase for parallel field orientation

\[ \Delta_n = \Delta_0 \cos(q r + \alpha_n) \exp(i \varphi_n(r)) \]

Josephson coupling between layers is modulated

\[ F_{n,n+1} = [-l_0 \cos(\alpha_n - \alpha_{n+1}) + l_2 \cos(q r) \cos(\alpha_n + \alpha_{n+1})] \cos(\varphi_n - \varphi_{n+1}) \]

\[ \varphi_n - \varphi_{n+1} = 2\pi x H_s / \Phi_0 + \pi n \]

s-interlayer distance, x-coordinate along \( q \)
Quasi-2D heavy fermion ($T_c=2.3$K)

Strong antiferromagnetic fluctuation

d-wave symmetry

Tetragonal symmetry

Modified Ginzburg-Landau functional:

$$\mathcal{F} = \alpha |\Psi|^2 - g \sum_{i=1}^{3} |\Pi_i \Psi|^2 + \gamma \sum_{i=1}^{3} \Pi_i^2 \Psi^2 + \varepsilon_z |\Pi_z^2 \Psi|^2 + \frac{\varepsilon_x}{2} (|\Pi_x \Pi_y \Psi|^2 + |\Pi_y \Pi_x \Psi|^2)$$

$$+ \frac{\tilde{\varepsilon}}{2} \left( |\Pi_z \Pi_x \Psi|^2 + |\Pi_x \Pi_z \Psi|^2 + |\Pi_z \Pi_y \Psi|^2 + |\Pi_y \Pi_z \Psi|^2 \right).$$

$$\Pi_j = \nabla_j - \frac{2e}{\hbar c} iA_j$$
No orbital effect

Modulation $\tilde{\epsilon}, \epsilon_z$ diagram in the case of the absence of the orbital effect (pure paramagnetic limit).

Areas with different patterns correspond to different orientation of the wave-vector modulation. The phase diagram does not depend on the $\epsilon_x$ value.
Modulation diagram in the case when the magnetic field applied along z axis. There are 3 areas on the diagram corresponding to 3 types of the solution for modulation vector $q_z$ and Landau level $n$. Modulation direction is always parallel to the applied field and $\varepsilon_x$ here is treated as a constant.
Magnetic field along z axis

\( z = \text{max}, n = 0 \)
\( q_z = 0, n > 0 \)
\( q_z = 0, n = \text{max} \)

\( z = \frac{x}{2} \)

xy-plane modulation

\( \theta = 0, \frac{\pi}{2} \)

xy-plane modulation

z-axis modulation

intermediate
Modulation diagram ($\xi$, $\varepsilon_z$) in the case when the magnetic field is applied along x axis. There are three areas on the diagram corresponding to different types of the solution for modulation vector $q_x$ and Landau level $n$. Modulation direction is always parallel to the applied field. The choice of the intersection point is determined by the coefficient $\varepsilon_x$. 
Magnetic field along x axis

$q_x^{\approx 0}, n = \text{max}$

$\varepsilon = -3\varepsilon_z$

$\varepsilon = -\varepsilon_x$

$\varepsilon = 3\varepsilon_z - 2\varepsilon_x$

$\varepsilon = 2\varepsilon_z$

$\theta = 0$

$\theta = \pi/2$

$\text{xy-plane modulation}$

$\text{z-axis modulation}$

$\text{intermediate}$

$xy$

intermediate
Small angle neutron scattering from the vortex lattice for $H \parallel c$

Neutron form factor seems to be consistent with FFLO state

The crystal structure effects influence on the FFLO state is very important.

The FFLO states with higher Landau level solutions could naturally exist in real 3D compounds (without any restrictions to the value of Maki parameter).

Wave vector of FFLO modulation along the magnetic field could be zero.

In the presence of the orbital effect the system tries in some way to reproduce optimal directions of the FFLO modulation by varying the Landau level index $n$ and wave-vector of the modulation along the field.
5. Supefluid ultracold Fermi gases with imbalanced state populations: one more candidate for FFLO state?
Massachusetts Institute of Technology:

Rice University, Houston:

**Experimental system:** Fermionic $^6$Li atoms cooled in magnetic and optical traps (mixture of the two lowest hyperfine states with different populations)

**Experimental result:** phase separation

rf transitions
76 MHz

Supefluid core
Normal gas
Rotating superfluid ultracold Fermi gases in a trap

Coils generating magnetic field

Fermion condensate

Laser beams

Vortex as a test for superfluidity

Images of vortex lattices

Questions:

1. What is the effect of confinement (finite system size) on FFLO states?

2. Effect of rotation on FFLO states in a trap (effect of magnetic field on FFLO state in a small superconducting sample).

3. Possible quantum oscillation effects.
Examples of quantum oscillation effects.

**Little-Parks effect. Switching between the vortex states.** Multiply-connected systems

Superconducting thin-wall cylinder

\[ v_s = \frac{\hbar}{mR} \left( L - \frac{\Phi}{\Phi_0} \right) \]

\[ \frac{\Delta T_c(H)}{T_{c0}} \sim \left( \frac{mv_s}{\hbar} \right)^2 \]

T\(_c\) (H) oscillations

Superconductor with a columnar defect or hole

Multiquantum vortices

A. Bezryadin, A.I. Buzdin, B. Pannetier (1994)

\[ \phi_0 = \frac{\pi \hbar c}{e} \]
Examples of quantum oscillation effects.

**Little-Parks effect.**

Simply-connected systems

Mesoscopic samples
dimensions $\sim$ several coherence lengths

Origin of $T_c$ oscillations:
Transitions between the states with different vorticity $L$

$$
\Delta = \left\{ e^{iL\theta} \right\}
$$

$L$ - Vorticity
(orbital momentum)

Multiquantum vortices
Examples of quantum oscillation effects.

**FFLO states and Tc(H) oscillations in infinite 2D superconductors**

A.I. Buzdin, M.L. Kulic
(1984)

\[ \phi_a = \frac{\pi H_z}{\phi_0 k_0^2} \]

\( \tau \sim T_c - T_{c0} \)
Model: Modified Ginzburg-Landau functional (2D)

\[ F = \int ((a + V(\vec{r}))|\Psi|^2 - \beta |\nabla \Psi|^2 + \gamma |D^2 \Psi|^2) \, dx \, dy \]

\[ a = \alpha (T - T_{c_0}) \]

Trapping potential

\[ \vec{D} = \nabla - 2iM[\vec{\Omega}, \vec{r}] / \hbar \]

FFLO instability

\[ \vec{D} = \nabla - 2ie \vec{A} / \hbar c \]

Range of validity: vicinity of tricritical point

Confinement mechanisms:

1. Zero trapping potential.
   Boundary condition at the system edge
   \[ \vec{n} \vec{D} \Psi = 0 \]

2. Nonzero trapping potential
   \[ V(\vec{r}) = M \omega^2 r^2 / 2 \]
FFLO states in a 2D mesoscopic superconducting disk

Interplay between the system size, magnetic length, and FFLO length scale
Perpendicular magnetic field component $H_z = 0$

$$G = a|\Psi|^2 - \beta|\nabla \Psi|^2 + \gamma|\Delta \Psi|^2,$$

Eigenvalue problem:

$$-\Delta \Psi = k_0^2 q^2 \Psi, \quad \frac{\partial \Psi}{\partial r} |_{r=R} = 0.$$  

$k_0 = \beta/2\gamma$  Wave number of FFLO instability

$$\Psi = e^{i\theta} J_L(q\rho) \quad \frac{dJ_L(x)}{dx} |_{qk_0R} = 0$$

$x_{Ln}$ a set of zeros of the derivative of the Bessel function of the $L$-th order

The critical temperature:

$$T_c = T_{c0} + \frac{\beta^2}{4\gamma\alpha} f(q) \quad f(q) = 2q^2 - q^4$$

$$f \left( \frac{x_{Ln}}{k_0 R} \right) = \frac{2x_{Ln}^2}{(k_0 R)^2} - \frac{x_{Ln}^4}{(k_0 R)^4}. $$

![Graph showing the critical temperature as a function of $k_0 R$ with various curves for different $L$ values.](image)
$H - T$ Phase diagram: $H_z = 0$

$T_c = T_{c0} + \frac{\beta^2}{4\gamma\alpha} f(q),$

$T_c = \max_L \{T_L\}.$

$a = \alpha(T - T_{c0}), \beta = \beta(H/T), T_{c0} = T_{c0}(H)$

$\beta/2\gamma = k_0(H, T)$

FFLO state

$H \uparrow \rightarrow L \downarrow$
Tilted magnetic field: $H_z \neq 0$

\[ H = H_{\parallel} + H_z z_0 \]

\[ G = a|\Psi|^2 - \beta |D\Psi|^2 + \gamma |D^2\Psi|^2 , \]

\[ D = \nabla + \frac{2\pi i}{\Phi_0} A_{\parallel} \quad A_\theta = H_z r/2. \]

\[ \beta = \beta(H_{\parallel}, T) \]

Eigenvalue problem:

\[ -D^2\Psi = k_0^2 q^2 \Psi \quad \frac{\partial \Psi}{\partial r} \bigg|_{r=R} = 0. \]

\[ \psi(\rho) = e^{-\phi/2} \phi^{L/2} F(\alpha_L, b_L, \phi) , \]

\[ \phi_a = \frac{\pi R^2 H_z}{\Phi_0} \]

\[ T_c = T_{c0} + \frac{\beta^2}{4\gamma \alpha_0} f(q) , \quad f(q) = 2q^2 - q^4 . \]

Field induced superconductivity

\[ T_c = \max_L \{ T_L \} . \]
Tilted magnetic field: $H_z \neq 0$

Transitions with large jumps in vorticity

$T_c = \max_L \{T_L\}$.
Vortex solutions beyond the range of FFLO instability. Critical field of the vortex entry.

We focus on the limit $\xi_1 \ll \xi_2$

$$\Psi \approx e^{iL\theta}$$

Condition of the first vortex entry:

$$F(L = 1) - F(L = 0) = 0$$
**Condition of the first vortex entry:**

\[
\ln \frac{R}{\xi_m} - \Phi + \frac{\alpha}{2} + \frac{\alpha \xi_m^2}{R^2} \left( 3\Phi^2 - 4\Phi \ln \frac{R}{\xi_m} - \Phi^3 \right) = 0
\]

\[\xi_m = \max(\xi_1, \xi_2)\]

\[
\Phi = \frac{\pi H_z R^2}{\phi_0}
\]

\[
\alpha = \frac{\xi_2^4}{\xi_m^2 \xi_1^2}
\]

\[
\frac{R}{\xi_m} \gg 1
\]

**Limiting cases:**

\[\xi_2 \ll \xi_1 \quad \Phi = \ln \frac{R}{\xi_1} \quad H_z \propto \frac{1}{R^2} \ln \frac{R}{\xi_1}\]

\[\xi_2 \gg \xi_1 \quad \Phi = \left( \frac{R}{\xi_2} \right)^{2/3} \quad H_z \propto \left( \frac{1}{R} \right)^{4/3}\]

**Change in the scaling law**
FFLO states in a trapping potential

Interplay between the rotation effect, confinement, and FFLO instability
FFLO states in a 2D system in a parabolic trapping potential (no rotation)

\[ \Delta^4 \Psi + 2 \Delta^2 \Psi + (\tau + \nu_0 \rho^2) \Psi = 0 \]

\[ k_0 = \beta / 2\gamma \quad \text{FFLO length scale} \]

\[ \tau = a / \gamma k_0^4 \quad \text{Temperature shift} \]

\[ \nu_0 = M \omega^2 / 2 \gamma k_0^6 \quad \text{(Trapping frequency)}^2 \]

\[ \vec{\rho} = k_0 \vec{r} \quad \text{Dimensionless coordinate} \]

\[ q = 1 + x \]

\[ U(q) = q^4 - 2q^2 \approx -1 + 4x^2 \]

\[ \psi_{\vec{q}} = e^{-\lambda x^2} \quad \lambda = \frac{1}{\sqrt{\nu_0}} \]

Fourier transform:

\[ \Psi = \int e^{i\vec{q} \cdot \vec{\rho}} \psi_{\vec{q}} d^2 \vec{q} \]

\[ -\nu_0 \frac{\partial^2}{\partial q^2} \Psi + \left( q^4 - 2q^2 \right) \tilde{\Psi} = -\tau \Psi \]

\[ L = 0 \]
FFLO states in a 2D system in a parabolic trapping potential (no rotation)

**Phase diagram**

\[ \tau = 1 - 2\sqrt{\nu_0} \]

**Condensate wave function**

\[ \Psi \propto \sqrt{\frac{\pi}{k_0 r}} \cos(k_0 r - \pi/4)e^{-k_0^2 r^2 \sqrt{\nu_0}/4} \]

Suppression of wave function oscillations by the increase in the trapping frequency

\[ \frac{2}{\nu_0^{1/4}} \propto \frac{1}{\sqrt{\omega}} \]

= Number of observable oscillations
FFLO states in a rotating 2D gas in a parabolic trapping potential.

\[ \Psi(\vec{r}) = f_L(r)e^{iL\theta} \]

\[ D^4 f_L + 2D^2 f_L + (\tau + \nu_0 \rho^2) f_L = 0 \]

\[ D^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) - \left( \frac{L}{\rho} + \phi_a \rho \right)^2 \]

Expansion in eigenfunctions of the problem without a trap:

\[ f_L(\rho) = \sum_{n=0}^{\infty} c_n u_{nL}(\rho) \]

\[ -D^2 u_{nL} = q_{nL}^2 u_{nL} = 2\phi_a (2n+|L|+L+1) u_{nL} \]

\[ (2q_{nL}^2 - q_{nL}^4) c_n - \sum_{m} \nu_{nm} c_m = \pi c_n \]

\[ \nu_{nm} = \nu_0 \int_{0}^{\infty} u_{mL} u_{nL} \rho^3 d\rho \]

\[ k_0 = \beta / 2\gamma \quad \text{FFLO length scale} \]

\[ \tau = a / \gamma k_0^4 \quad \text{Temperature shift} \]

\[ \nu_0 = M\omega^2 / 2\gamma k_0^6 \quad \text{(Trapping frequency)}^2 \]

\[ \phi_a = 2M\Omega / \hbar k_0^2 \quad \text{rotation frequency} \]

\[ \vec{\rho} = k_0 \vec{r} \quad \text{Dimensionless coordinate} \]
FFLO states in a rotating 2D gas in a parabolic trapping potential.

Suppression of quantum oscillations by the increase in the trapping frequency.

First-order perturbation theory:

\[
\tau = \max_{L \geq 0} \left\{ 4\phi_a - v_0/\phi_a (2L + 1) + v_0 L/\phi_a - 4\phi_a^2 (2L + 1)^2 \right\}
\]

rotation induced superfluid phase
II. Conclusions

• There are strong experimental evidences of the existence of the FFLO state in organic layered superconductors and in heavy fermion superconductor CeCoIn$_5$.

• The interplay between FFLO modulation and orbital effect results in new type of the vortex structures, non-monotonic critical field behavior in layered superconductor in tilted field.

• Special behavior of fluctuations near the FFLO transition – vanishing stiffness.

• FFLO phase in ultracold Fermi gases with imbalanced state populations?