Topology in physics

Aalto University

G. Volovik

Landau Institute

RUSSIAN ACADEMY OF SCIENCES
L.D Landau INSTITUTE FOR THEORETICAL PHYSICS

Kourovka 26 February, 2014

1. Introduction

* Standard Model as a low-energy topological effective theory
* Gapless & gapped topological media

2. Fermi surface as topological object (vortex) in p-space

* Landau & non-Landau Fermi liquids, flat band

3. Fermi points - monopoles in p-space (Weyl, Majorana & Dirac points)

* superfluid 3He-A, topological semimetals, vacuum of Standard Model in massless phase
* chiral fermions, gauge fields, gravity as emergent phenomena; quantum vacuum as 4D graphene
* exotic fermions: quadratic, cubic & quartic dispersion; dispersionless fermions; Horava gravity
* chiral anomaly in terms of p-space topological invariants

4. Nodal lines & flat bands

* surface flat bands: 3He-A, semimetals , cuprate superconductors , graphene , graphite
* towards room-temperature superconductivity

5. Fully gapped topological media

* superfluid 3He-B, topological insulators , chiral superconductors, quantum spin Hall insulators, vacuum of Standard Model of particle physics in present massive phase, vacua of lattice QCD
* Majorana edge states & zero modes on vortices (planar phase , topological insulator & 3He-B)
* Higgs bosons & Nambu some rule in 3He-B & Standard Model

3+1 sources of effective Quantum Field Theories emerging in many-body system & in quantum vacuum

Lev Landau

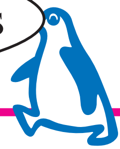
I think it is safe to say that no one understands **Quantum Mechanics**

Richard Feynman

Thermodynamics is the only physical theory of universal content

Albert Einstein

thermodynamics emerges in macroscopic systems



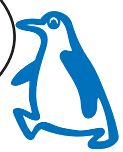
Symmetry: conservation laws,
invariance (gauge, translational, Lorentz, Galilean, ...)
spontaneously broken symmetry ...

effective theories
of quantum liquids:
two-fluid hydrodynamics
of superfluid ^{4}He
& Fermi liquid theory of
liquid ^{3}He

Standard Model & GUT are based on symmetry

Topology:
generic (robust to deformations), emergent symmetry,
emergent physical laws, anti-Grand-Unification

missing ingredient
in Landau theories
& Standard Model



Hierarchy problem

characteristic mass scale in our vacuum
is Planck energy

$$m_P c^2 \sim E_P \sim 10^{19}\ \text{GeV} \sim 10^{32}\text{K}$$

even at high temperature
of electroweak phase transition
everything should be completely frozen out

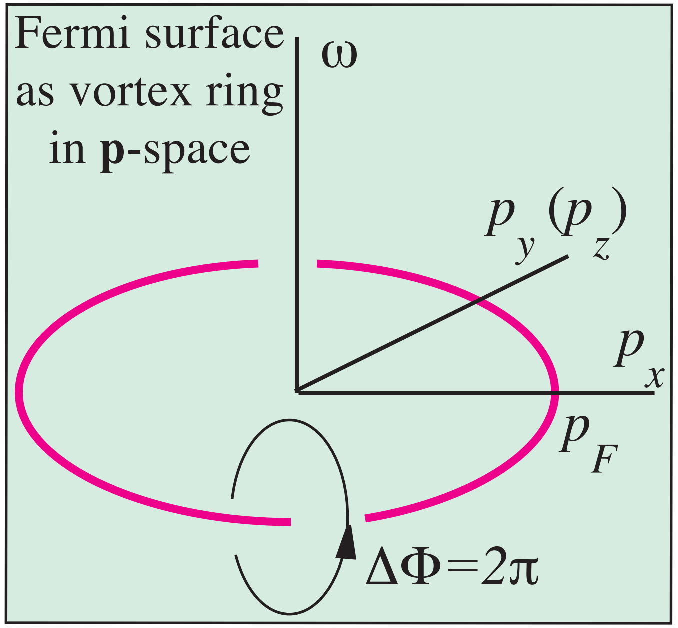
$$T_{ew} \sim 1\ \text{TeV} \sim 10^{16}\text{K}$$

$$e^{-mc^2/T_{ew}} = 10^{-10^{16}} = 0$$

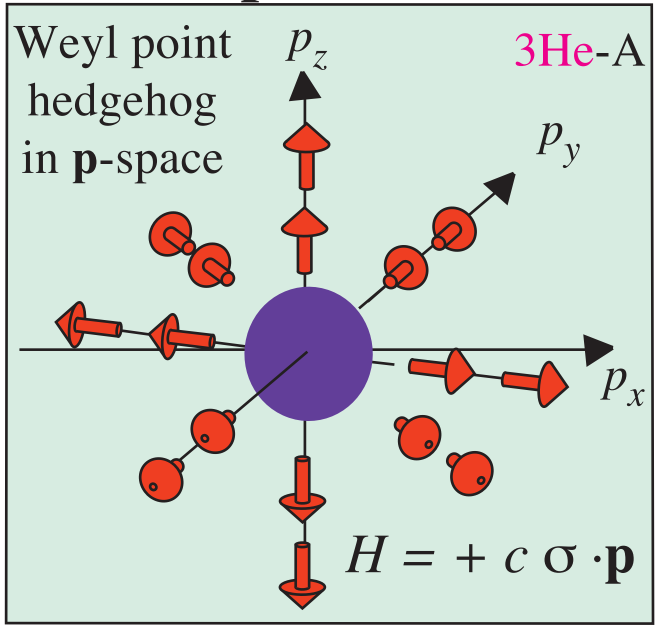
Why no freezing at low T?

gapless (massless) topological vacua as defects in momentum space

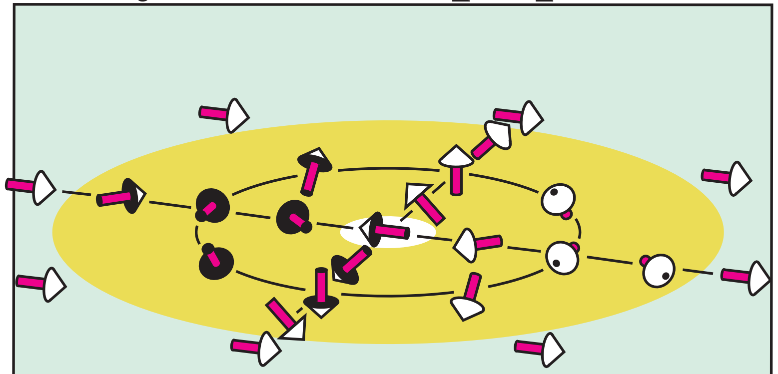
gapped topological matter as skyrmions in p-space



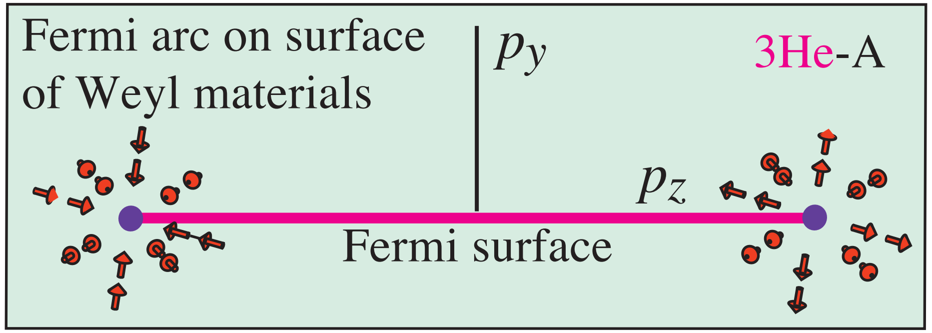
metals, normal 3He



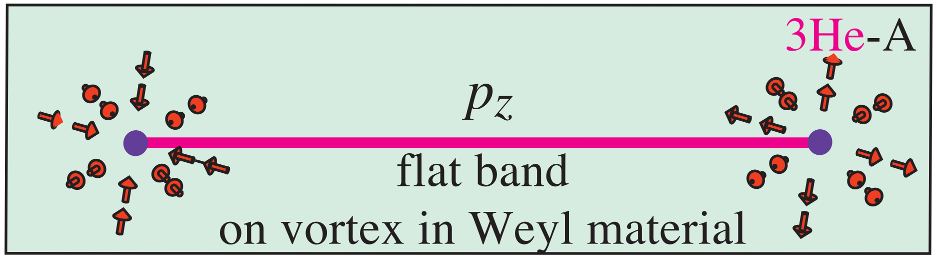
3He-A, vacuum of Standard Model, topological semimetals (Abrikosov)



3He-B, topological insulators, intrinsic QHE & spin QHE, 3He-A film, Standard Model vacuum in massive phase



Dirac strings in **p**-space terminating on monopole



origin of gapless (massless) fermions:
at topologically protected
singularities in Green's function
the energy of fermions is zero

bulk - edge correspondence:

topology in bulk protects
gapless fermions on surface of fully gapped systems;
higher order nodes in nodal topological materials

2D Quantum Hall insulator & 3He-A film	gapless chiral edge states (GV 1992)
3D topological insulator	gapless Dirac fermions (Volkov-Pankratov 1985)
superfluid 3He-B	gapless Majorana fermions (Salomaa-GV 1988)
3He-A, Weyl semimetal with point nodes	Fermi arc (nodal line) on surface (Tutsumi et al 2011)
graphene with point nodes	dispersionless 1D flat band (Ryu-Hatsugai 2002)
semimetal with Fermi lines	2D flat band on the surface (Heikkila-GV 2010)

bulk - defect correspondence:

topology in bulk protects gapless fermions inside topological defect

relativistic string	fermion zero modes in core (Jackiw-Rossi 1981)
3He-A with point nodes	1D flat band in the core (Kopnin-Salomaa 1991)
2D p+ip superconductor	Majorana fermions in the core (GV 1999)

80's

B.A. Volkov, A.A. Gorbatsevich, Yu.V. Kopaev and V.V. Tugushev,
Macroscopic current states in crystals,
JETP 54, 391 (1981)
B.A. Volkov and O.A. Pankratov,
Two-dimensional massless electrons in an inverted contact,
JETP Lett. 42, 178 (1985)
O.A. Pankratov, S.V. Pakhomov, B.A. Volkov,
Supersymmetry in heterojunctions:
band-inverting contact on the basis of $Pb_{1-x}Sn_xTe$ and $\mathbf{Hg_{1-x}Cd_xTe}$,
Solid State Communications 61, 93 (1986)

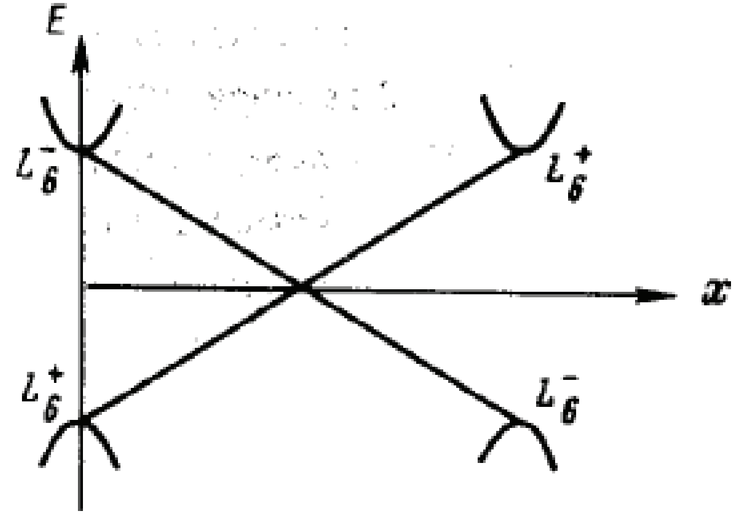


Рис. 1. Инверсия зон $L_6^{\pm}$ в $Pb_{1-x}Sn_xTe(Se)$ с изменением состава

H. So, Induced topological invariants by lattice fermions in odd dimensions, Prog. Theor. Phys. 74, 585 (1985)
G.E. Volovik, An analog of the quantum Hall effect in a superfluid 3He film, JETP 67, 1804 (1988)
F.D.M. Haldane, Model for a quantum Hall effect without Landau levels:
Condensed-matter realization of the "Parity Anomaly", PRL 61, 2015 (1988)
V.M. Yakovenko, Spin, statistics and charge of solitons in (2+1)-dimensional theories,
Fizika (Zagreb) 21, suppl. 3, 231 (1989)

****

00's

B.A. Bernevig and Shou-Cheng Zhang, Quantum spin Hall effect, PRL 96, 106802 (2006)
B.A. Bernevig, T.A. Hughes, and Shou-Cheng Zhang,
Quantum spin Hall effect and topological phase transition in **HgTe** quantum wells, Science 314, 1757 (2006)

two major universality classes of gapless fermionic vacua

Landau theory of Fermi liquid

vacuum with Fermi surface:
metals, normal ^{3}He

Standard Model + gravity

vacuum with Fermi (Weyl) point:
^{3}He-A, planar phase, Weyl semimetal, vacuum of SM

gravity emerges from
Fermi (Weyl) point
analog of
Fermi surface

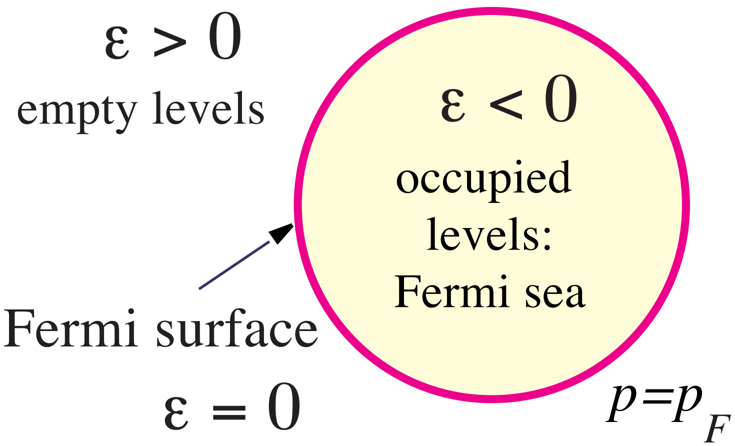
$$g^{\mu\nu}(p_\mu - eA_\mu - e\tau \cdot \mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau \cdot \mathbf{W}_\nu) = 0$$

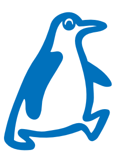
Theory of topological matter:
Nielsen, So, Ishikawa, Matsuyama, Haldane, Yakovenko, Horava, Kitaev, Ludwig, Schnyder, Ryu, Furusaki, S-C Zhang, Kane, Liang Fu, ...

Fermi surface as topological object

Energy spectrum of
non-interacting gas of fermionic atoms

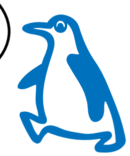
$$\varepsilon(p) = \frac{p^2}{2m} - \mu = \frac{p^2}{2m} - \frac{p_F^{\ 2}}{2m}$$





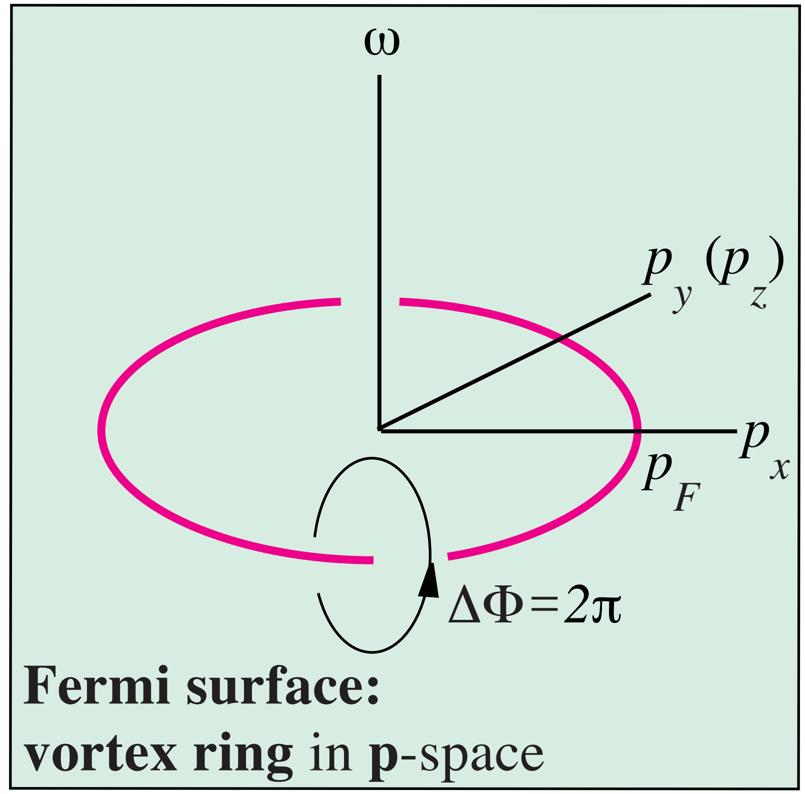
is Fermi surface a domain wall in momentum space?

no! it is a **vortex ring**



Green's function

$$G^{-1} = i\omega - \varepsilon(p)$$



phase of Green's function

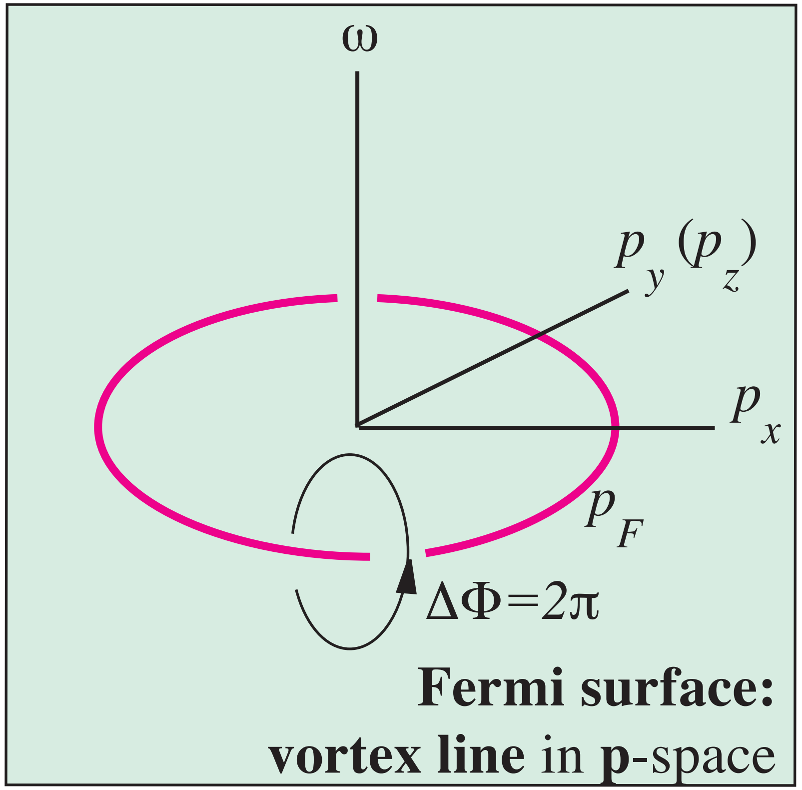
$$G(\omega,\mathbf{p}) = |G| e^{\,i\Phi}$$

has winding number $N = 1$

Fermi liquids & non-Fermi liquids

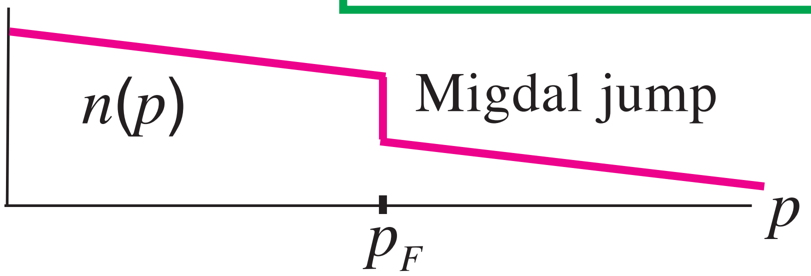
Singularity at Fermi surface is robust to perturbations:

winding number $N=1$ *cannot change continuously,*
interaction (perturbative) cannot destroy singularity
but behavior near FS can be different



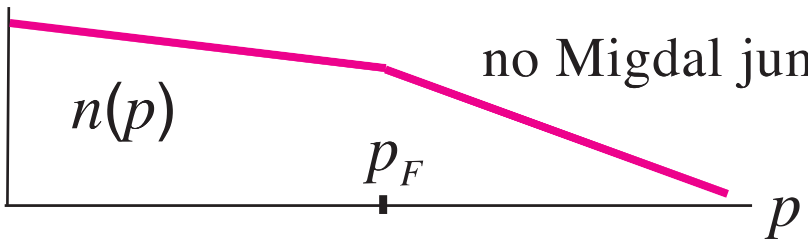
$$G(\omega,\mathbf{p})=|G|e^{\,i\Phi}$$

Landau Fermi-liquid



Fermi surface exists in superfluids/superconductors
examples: 3He-A in flow & Gubankova-Schmitt-Wilczek,
PRB74 (2006) 064505, but no Luttinger theorem

Luttinger Fermi liquid, marginal Fermi liquid



unparticles in particle physics

zeroes in $G(\omega,\mathbf{p})$ have the same $N=1$ as poles

$$G(\omega,\mathbf{p}) = \frac{Z(p,\omega)}{i\omega - \varepsilon(p)}$$

$$Z(p,\omega) = (\omega^2 + \varepsilon^2(p))^{\gamma}$$

zeroes in $G(\omega,\mathbf{p})$

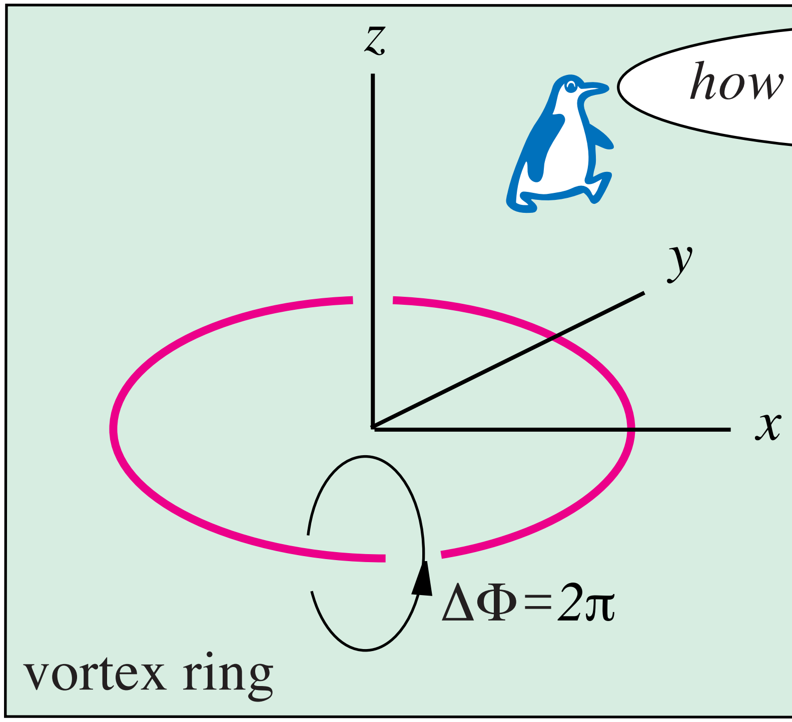
for $\gamma > 1/2$

quantized vortex in **r**-space ≡ **Fermi surface** in **p**-space

homotopy group π_1

how is it in **p**-*space ?*

Topology in **r**-space

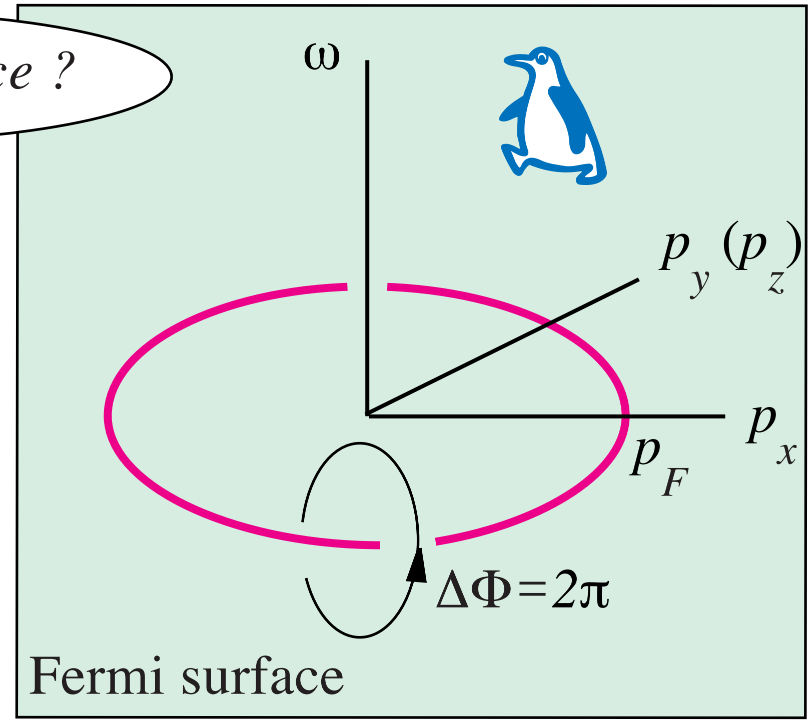


$$\Psi(\mathbf{r}) = |\Psi|\, e^{\,i\Phi}$$

scalar order parameter
of superfluid & superconductor

winding number $N_1 = 1$

Topology in **p**-space



$$G(\omega, \mathbf{p}) = |G|\, e^{\,i\Phi}$$

Green's function (propagator)

classes of mapping $S^1 \rightarrow U(1)$
manifold of
broken symmetry vacuum states

classes of mapping $S^1 \rightarrow GL(\text{n,C})$
space of
non-degenerate complex matrices

non-topological flat bands due to interaction

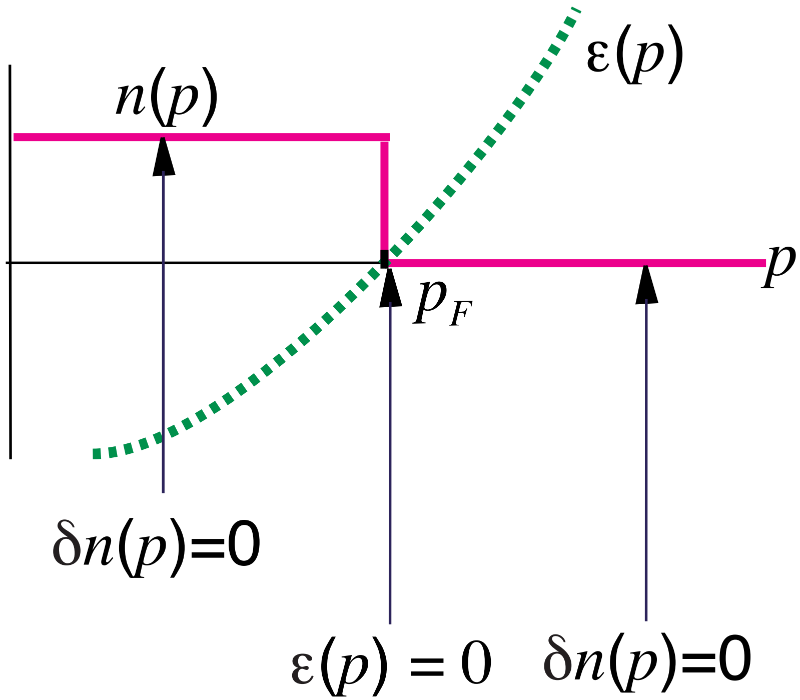
Khodel-Shaginyan fermion condensate

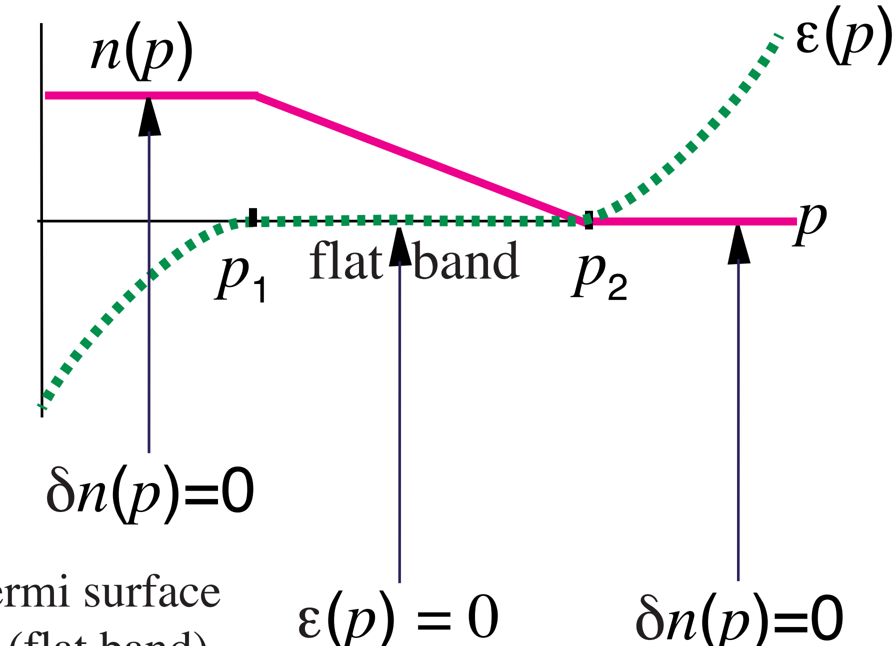
JETP Lett. **51**, 553 (1990)
GV, JETP Lett. **53**, 222 (1991)
Nozieres, J. Phys. (Fr.) **2**, 443 (1992)

$$E\{n(p)\}$$

$$\delta E\{n(p)\} = \int \varepsilon(p)\,\delta n(p)\,d^{\mathrm{d}}p = 0$$

solutions: $\varepsilon(p) = 0$ or $\delta n(p)=0$



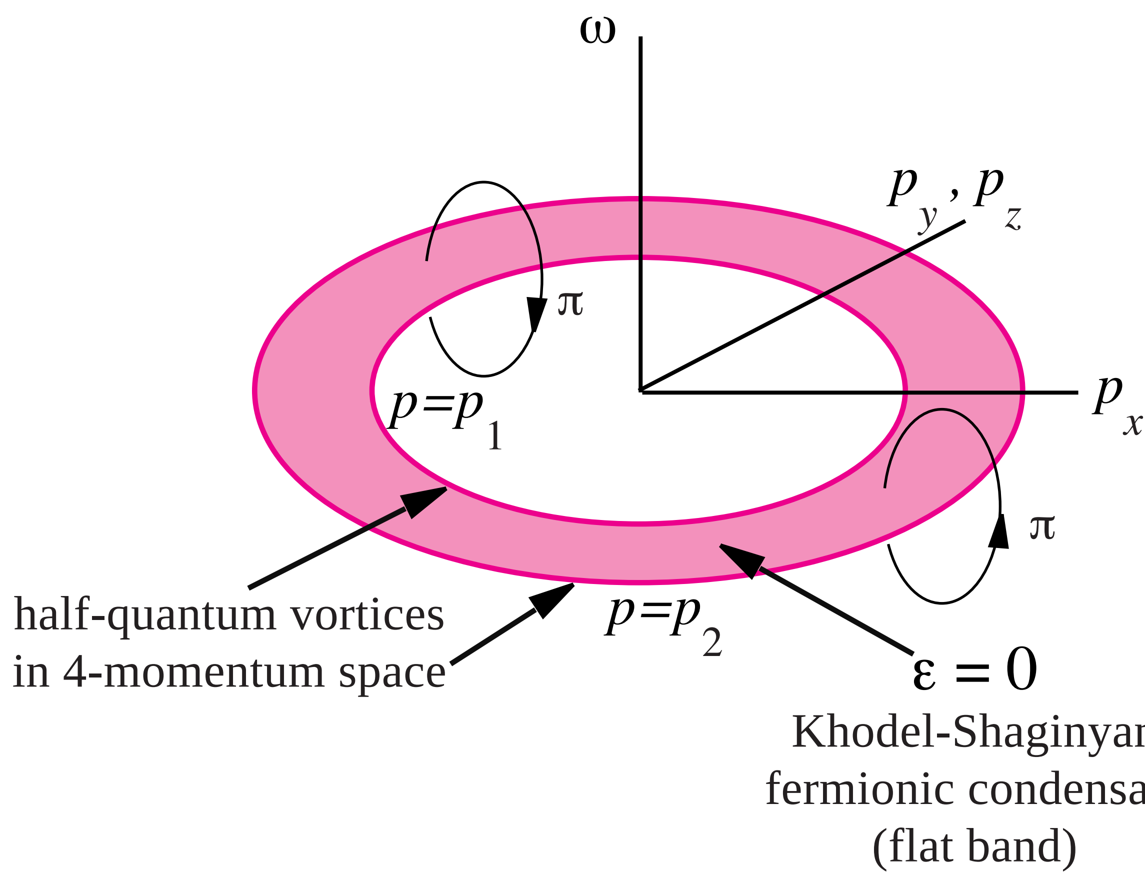


splitting of Fermi surface
to Fermi ball (flat band)

S.-S. Lee
Non-Fermi liquid from a charged black hole: A critical Fermi ball
PRD 79, 086006 (2009)

anti-de Sitter/conformal field theory correspondence

Flat band as momentum-space dark soliton terminated by half-quantum vortices



phase of Green's function changes by π across the "dark soliton"

Fermi Condensation Near van Hove Singularities Within the Hubbard Model on the Triangular Lattice

Dmitry Yudin,[1] Daniel Hirschmeier,[2] Hartmut Hafermann,[3] Olle Eriksson,[1]
Alexander I. Lichtenstein,[2] and Mikhail I. Katsnelson[4,5]

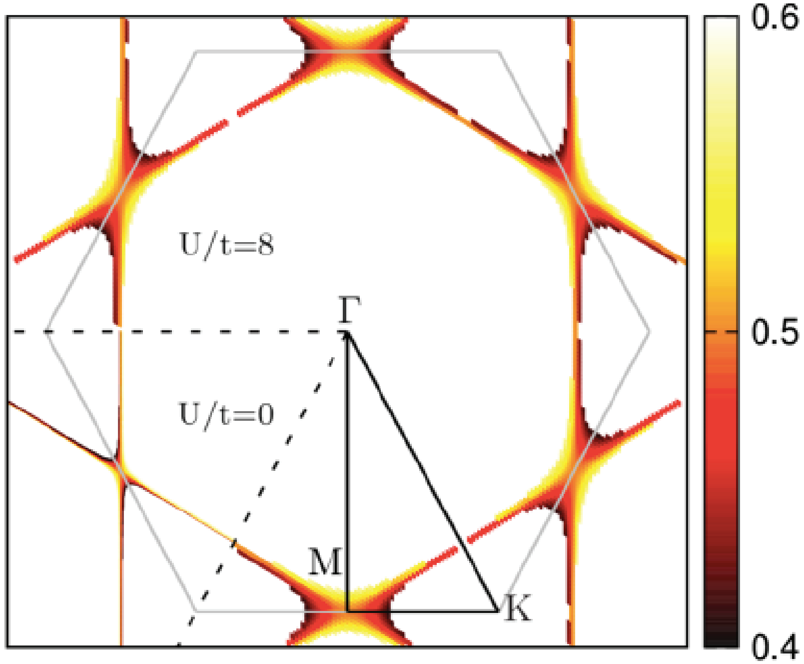
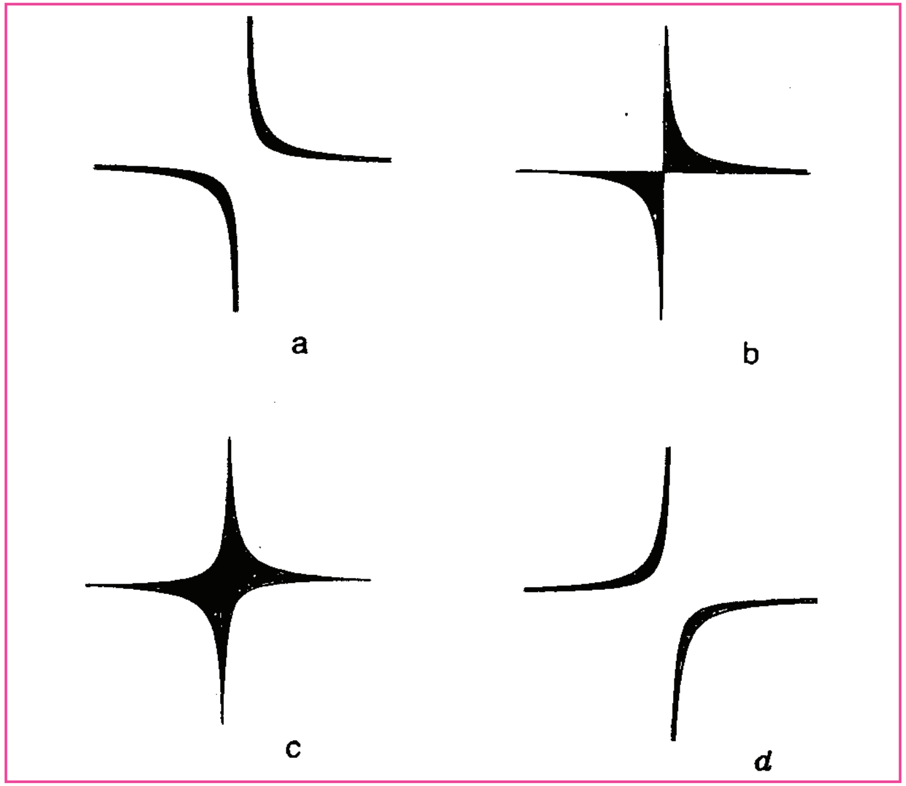


FIG. 4 (color online). Broadened Fermi surface within ±0.1 electrons for $U/t = 8$ and $T/t = 0.1$. The lower left sextant shows the noninteracting result.



Письма в ЖЭТФ, том 59, вып.11, стр.798 - 802, ©1994 г. 10 июня

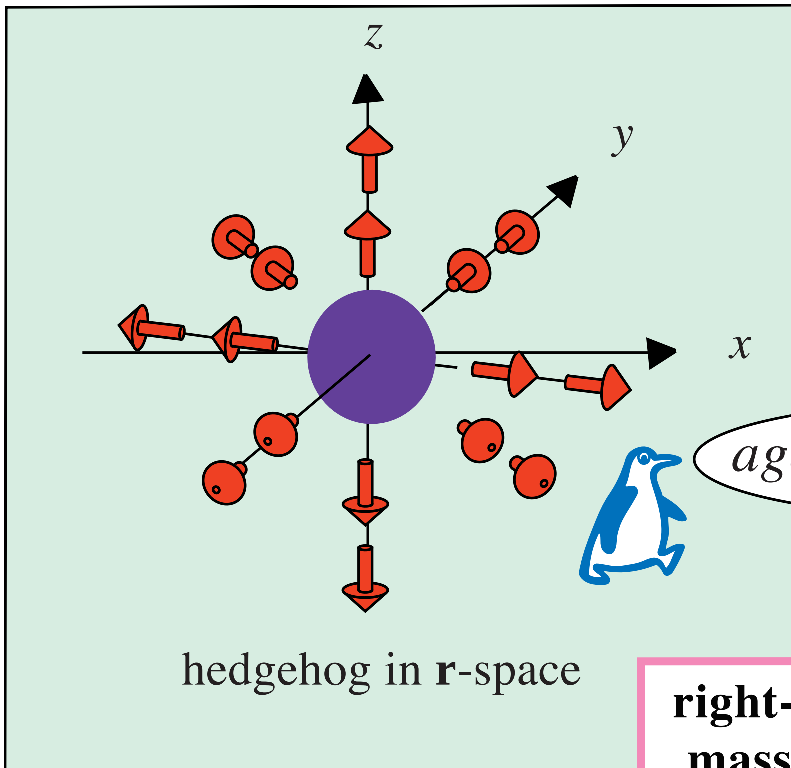
ON FERMI CONDENSATE: NEAR THE SADDLE POINT AND WITHIN THE VORTEX CORE

G.E. Volovik

3. Classes of Fermi points & nodal lines:

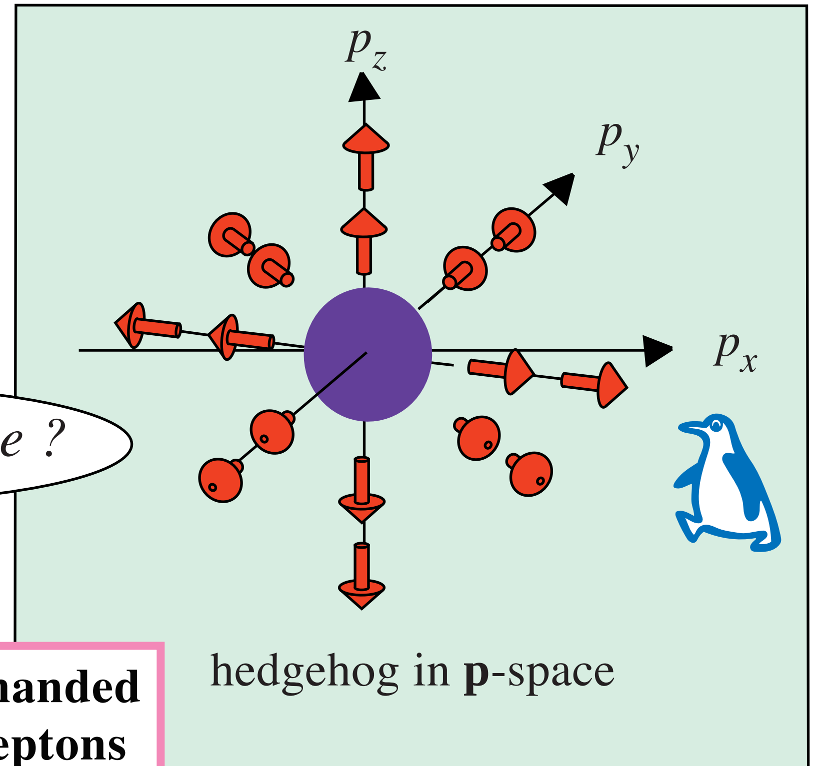
superfluid ^{3}He-A, Standard Model, semimetals, graphene, cuprate SC, ...
surface of ^{3}He-B & topological insulators

magnetic hedgehog



$\sigma(\mathbf{r})=\hat{\mathbf{r}}$

Weyl point
or *Berry phase Dirac monopole*



$\sigma(\mathbf{p}) = \hat{\mathbf{p}}$

again no difference ?

right-handed and left-handed massless quarks and leptons are elementary particles in Standard Model

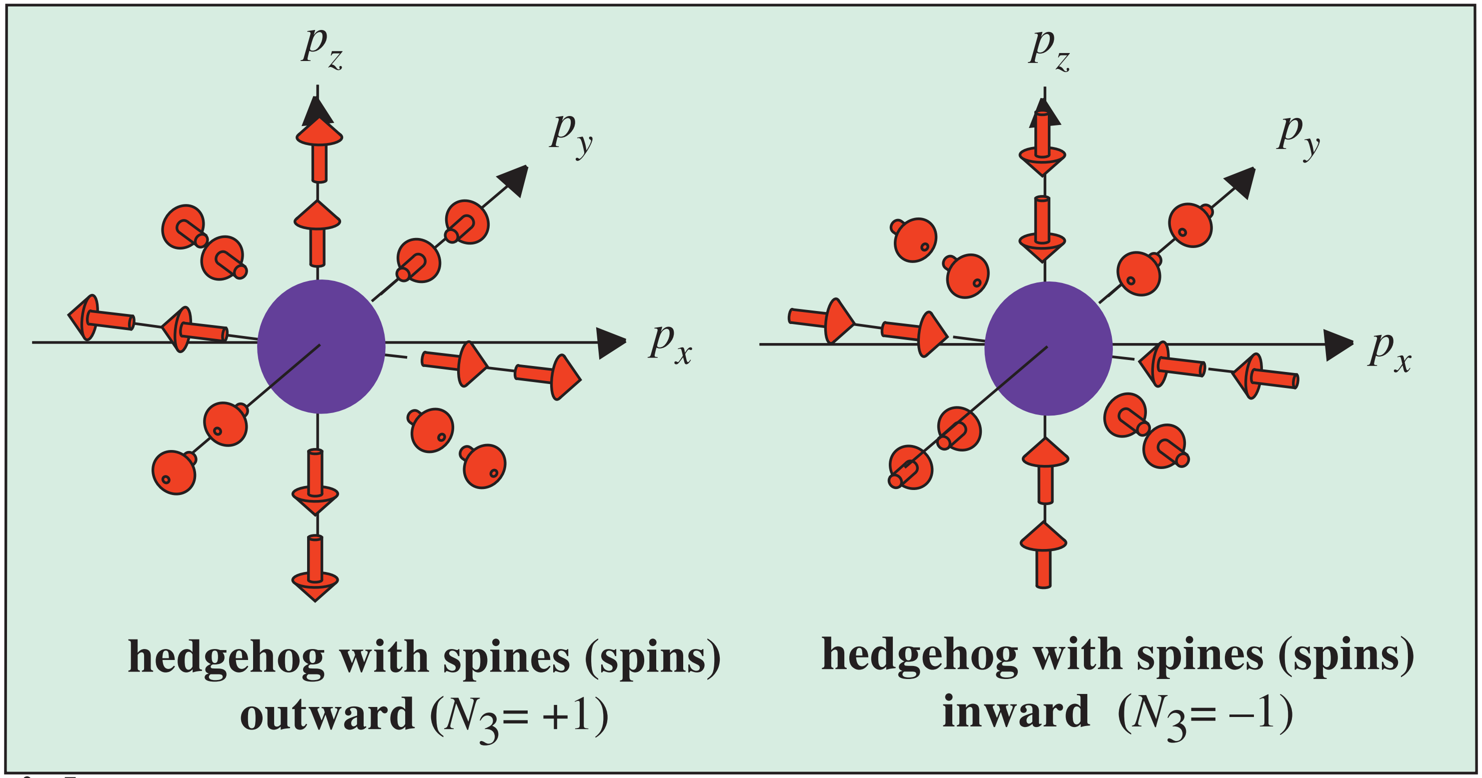
Landau CP symmetry is emergent

close to Fermi point

$$H = + c\, \sigma \cdot \mathbf{p}$$

right-handed Weyl electron =
hedgehog in **p**-space with spines = spins

Topological invariant for right and left elementary particles



right

$$H = +c\,\sigma \cdot \mathbf{p}$$
$$\mathbf{g}(\mathbf{p}) = +\,c\mathbf{p}$$

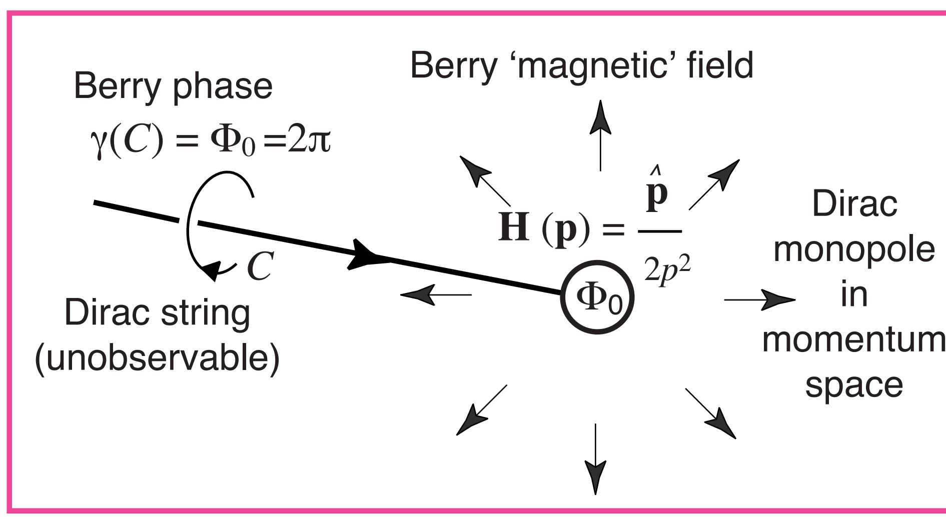
$$H = \sigma \cdot \mathbf{g}(\mathbf{p})$$

left

$$H = -\,c\,\sigma \cdot \mathbf{p}$$
$$\mathbf{g}(\mathbf{p}) = -\,c\mathbf{p}$$

$$N_3 = \frac{1}{8\pi}\, e_{ijk} \int dS^i\, \hat{\mathbf{g}} \cdot (\partial^j \hat{\mathbf{g}} \times \partial^k \hat{\mathbf{g}})$$

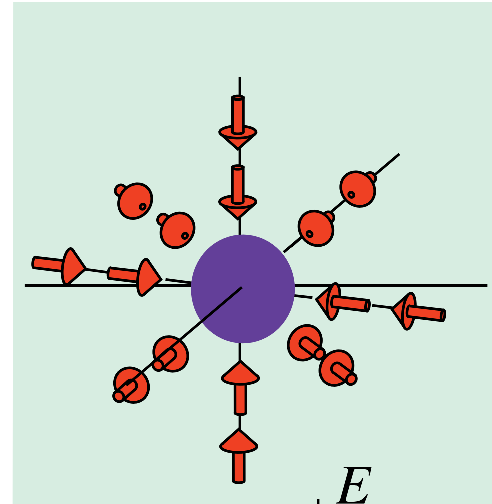
over 2D surface around Fermi point



Weyl fermions in 3+1 gapless topological cond-mat

topologically protected Weyl points in:

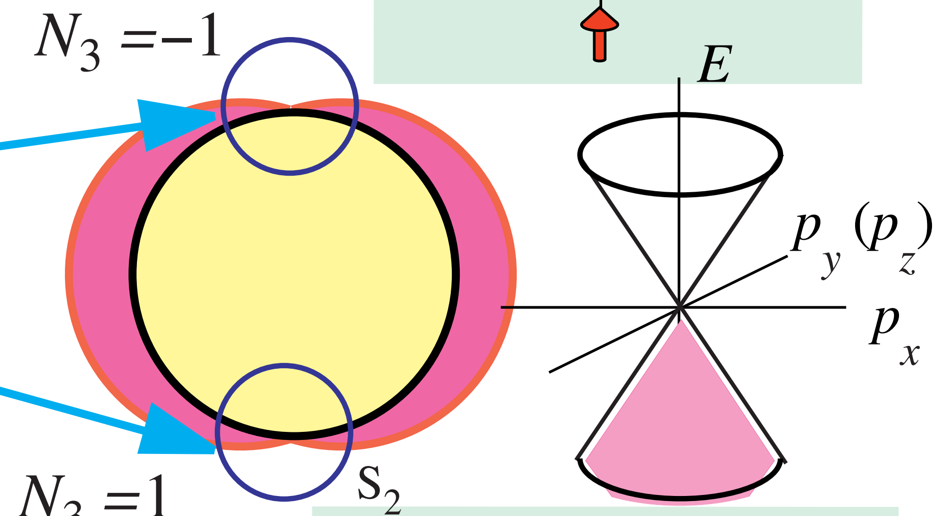
topological semi-metal or Weyl metals (Abrikosov-Beneslavskii 1971), ^{3}He-A (1982), triplet Fermi gases, $CoSb_3$ (arXiv:1204.5905)



Gap node - Weyl point (anti-hedgehog)

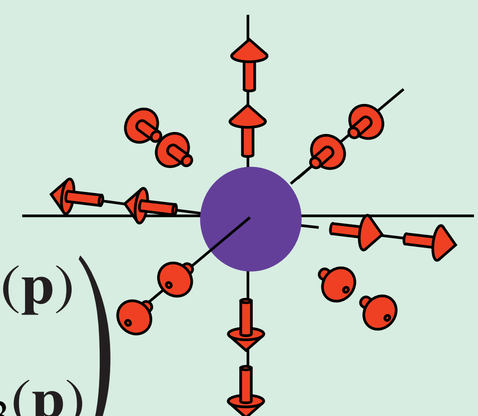
$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot (\nabla_{p_i} \hat{\mathbf{g}} \times \nabla_{p_j} \hat{\mathbf{g}})$$

over 2D surface S in 3D p-space



Gap node - Weyl point (hedgehog)

$$p^2 = p_x^2 + p_y^2 + p_z^2$$



$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix}$$

emergence of relativistic chiral Weyl fermions near Fermi points

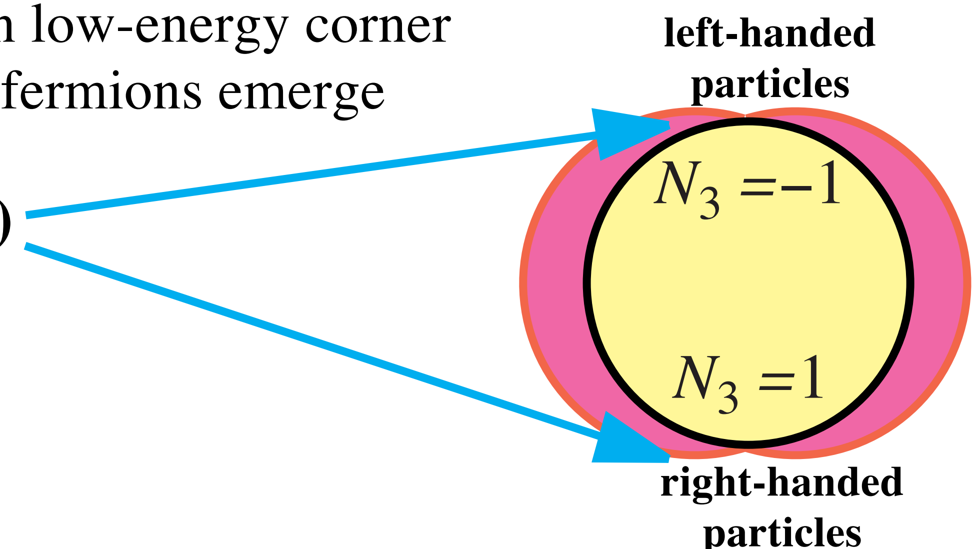
original non-relativistic Hamiltonian

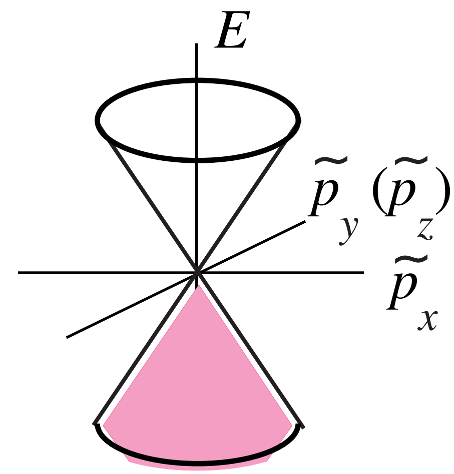
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

close to nodes, i.e. in low-energy corner
relativistic chiral fermions emerge

$$H = N_3\, c\, \boldsymbol{\tau} \cdot (\mathbf{p} - \mathbf{p}_0)$$

$$E = -\,c\tilde{p}$$





top. invariant determines chirality in low-energy corner

chirality is emergent ??

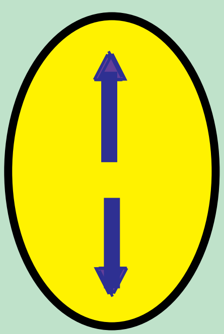
relativistic invariance as well

what else is emergent ?

bosonic collective modes in two generic fermionic vacua

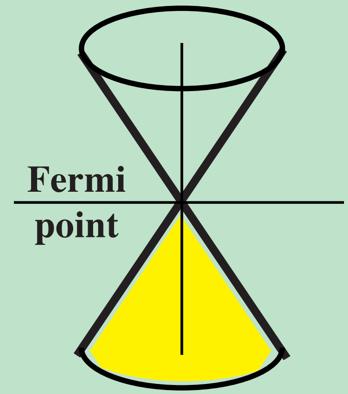
Landau theory of Fermi liquid

collective Bose modes of fermionic vacuum:
propagating
oscillation of **shape**
of Fermi surface



Landau, ZhETF **32**, 59 (1957)

Standard Model + gravity

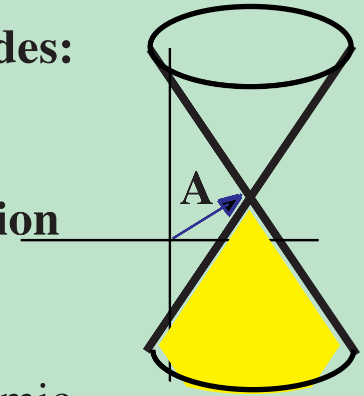


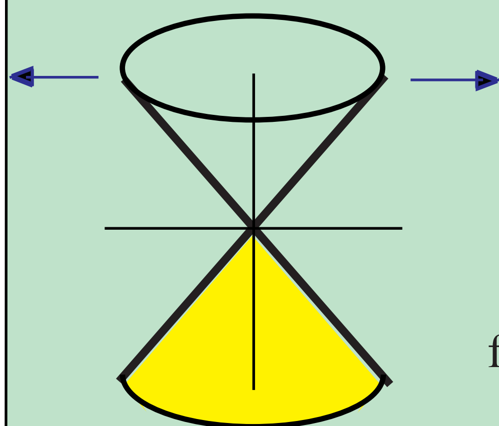
collective Bose modes:

propagating
oscillation of **position**
of Fermi point

$$\mathbf{p} \to \mathbf{p} - e\mathbf{A}$$

form effective dynamic
electromagnetic field

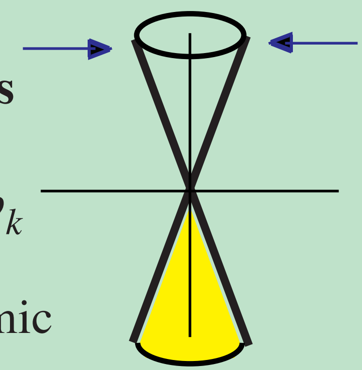




propagating
oscillation of **slopes**

$$E^2 = c^2p^2 \to g^{ik}p_i p_k$$

form effective dynamic
gravity field



two generic quantum field theories of interacting bosonic & fermionic fields

relativistic quantum fields & gravity emerging near Weyl point

Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac Γ-matrices

primary object:
tetrad
$e_a^{\ \mu}$

$$E = v_F\,(p - p_F)$$

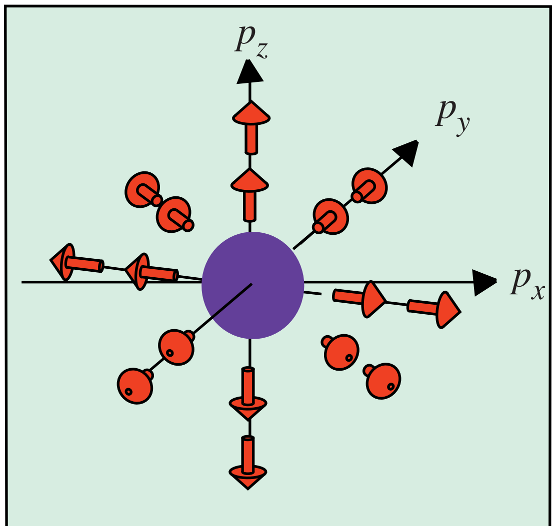
emergent relativity

$$H = e_a^{\ k}\,\Gamma^a \cdot (p_k - p_k^0)$$

linear expansion near Fermi surface

linear expansion near Weyl point

secondary object:
metric
$$g^{\mu\nu} = \eta^{ab}\, e_a^{\ \mu}\, e_b^{\ \nu}$$



hedgehog in **p**-space

$$g^{\mu\nu}(p_\mu - eA_\mu - e\tau \cdot \mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau \cdot \mathbf{W}_\nu) = 0$$

effective metric:
emergent gravity

effective
electromagnetic
field

effective
$SU(2)$ gauge
field

effective
electric charge
$e = +1$ or -1

effective
isotopic spin

all ingredients of Standard Model :
chiral fermions & gauge fields
emerge in low-energy corner
together with spin, Dirac Γ−matrices,
gravity & physical laws:
Lorentz & gauge invariance,
equivalence principle, etc

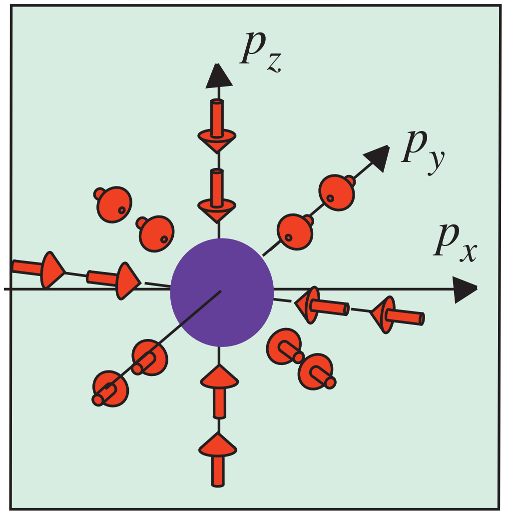
gravity & gauge fields
are collective modes
of vacua with Weyl point

Einstein-Cartan-Sciama-Kibble theory
with tetrads, spin connection & torsion

Chiral Weyl fermions in Standard Model

Family #1 of quarks and leptons

left particles



hedgehog with spines (spins) inward ($N_3 = -1$)

quarks — $SU(3)_C$ — $SU(2)_L$

+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6

+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3

leptons

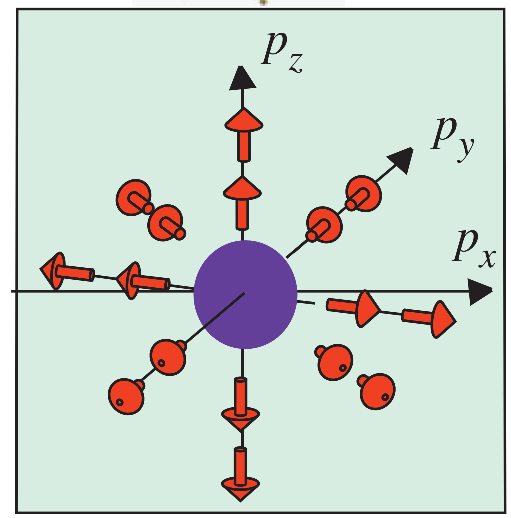
0 ν_L -1/2	-1 e_L -1/2

0 ν_R 0	-1 e_R -1

$$H = -c\,\sigma\cdot\mathbf{p}$$
$$N_3 = -1$$

$$H = +c\,\sigma\cdot\mathbf{p}$$
$$N_3 = +1$$

right particles



hedgehog with spines (spins) outward ($N_3 = +1$)

$$N_3 = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda\gamma}\, \mathbf{tr} \int dS^\gamma\, \mathbf{G}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}^\nu\, \mathbf{G}^{-1} \mathbf{G}^\lambda\, \mathbf{G}^{-1}$$

over 3D surface S in 4D momentum space

general topological invariant
in terms of Green's function
for interacting systems

Standard Model topological invariant

Topological invariant protected by symmetry

$$N_K = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int_{\text{over } S^3} dV \, \mathbf{K} \, \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1}$$

G is Green's function, K is symmetry operator

$$\mathbf{G}\mathrm{K} = +/- \, \mathrm{K}\mathbf{G}$$

for Standard Model vacuum K is Z_2 center group

$$\mathrm{K} = \exp 2\pi i \tau_3$$

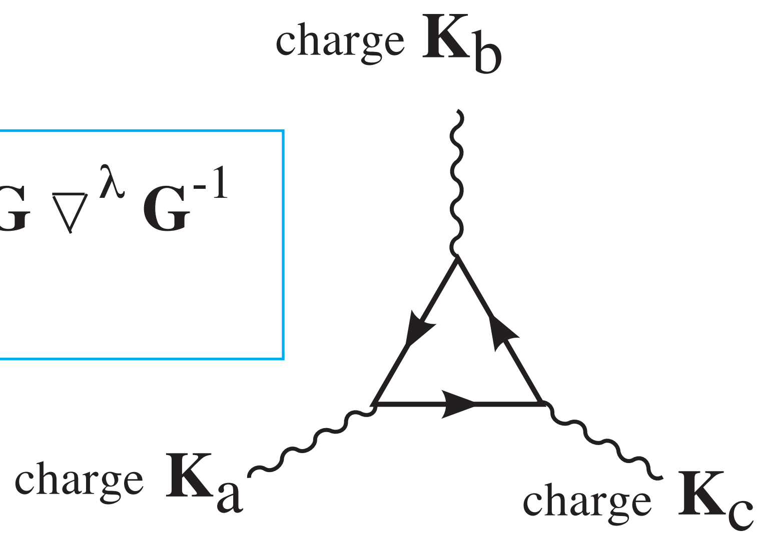
τ_3 weak isotopic spin of SU(2)

$$N_K = 16 \, n_g$$

16 massless Weyl particles in one generation are protected by combined symmetry and topology

Symmetry protected p-space topological invariants & chiral anomaly

$$K_{abc} = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr} \int_{\text{over } S^3} dS\, \mathbf{K}_a \mathbf{K}_b \mathbf{K}_c\, \mathbf{G}\, \nabla^\mu \mathbf{G}^{-1}\, \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G}\, \nabla^\lambda \mathbf{G}^{-1}$$



triangle
Feynman diagram

integral is around Weyl point

$\mathbf{G}(p_\mu)$ is Green's function matrix

$\mathbf{K}_c$ are charges of fields acting on fermions
(electric, weak, baryonic, hypercharge, etc.)

these invariants determine chiral anomaly effects due 3+1 Weyl points:
electroweak baryogenesis, chiral magnetic effect, Kopnin force, chiral vortical effect,
spin quantum Hall effect, ...

p-space invariants are prefactors of topological terms in ***r***-space action

topological origin
of quantization of physical parameters

chiral anomaly in topological Weyl vacua: Standard Model & 3He-A

electroweak baryogenesis in Standard Model of particle physics
baryon production from vacuum by hypermagnetic field in early Universe

chiral anomaly equation
(Adler, Bell, Jackiw)

$$\dot{B} = \frac{1}{4\pi^2} N_B \, \mathbf{B}_Y \cdot \mathbf{E}_Y$$

topological origin
of quantization of physical parameters

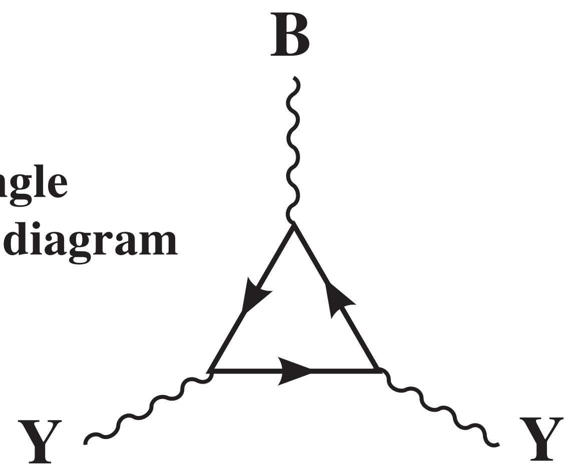
symmetry protected integer valued topological invariant

$$N_B = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int_{\text{over } S^3} dV \, \mathbf{B}\mathbf{Y}^2 \, \mathbf{G} \nabla^{\mu} \mathbf{G}^{-1} \, \mathbf{G} \nabla^{\nu} \mathbf{G}^{-1} \mathbf{G} \nabla^{\lambda} \mathbf{G}^{-1}$$

matrix of baryonic charge (B) — **matrix of hypecharge** (Y)

$$\dot{B} = \frac{1}{4\pi^2} \mathbf{B}_Y \cdot \mathbf{E}_Y \sum_a B_a C_a Y_a^2$$

triangle
Feynman diagram



B_a -- **baryonic charge**
Y_a -- **hypercharge**
C_a -- **chirality** $= +1$ **for right**
-1 **for left**

experimental verification of chiral anomaly equation
measurement of *Kopnin force*

$$\mathbf{A} = p_F \mathbf{l} \qquad \mathbf{B} = p_F \nabla \times \boldsymbol{l}$$
$$\mathbf{E} = p_F \dot{\boldsymbol{l}} \qquad B_a = \mathbf{P}_a$$

translation from SM to language of 3He-A

momentum from vacuum of fermion zero modes

baryogenesis in early Universe

$$\dot{\mathbf{P}} = \sum \mathbf{P}_a \dot{n}_a$$

$\mathbf{P}_a$ -- **momentum of Weyl point (fermionic charge)**
e_a -- **effective electric charge**

$$\dot{\mathbf{P}} = (1/4\pi^2)\, \mathbf{B} \cdot \mathbf{E} \sum_a \mathbf{P}_a C_a e_a^2$$

applied to ^{3}He-A

$C_a = +1$ **for right**
-1 **for left**

$$\dot{B} = \sum B_a \dot{n}_a$$

B_a -- **baryonic charge**
Y_a -- **hypercharge**

$$\dot{B} = (1/4\pi^2)\, \mathbf{B}_Y \cdot \mathbf{E}_Y \sum_a B_a C_a Y_a^2$$

applied to Standard Model

$C_a = +1$ **for right**
-1 **for left**

chiral anomaly equation (Adler, Bell, Jackiw)

quasiparticles move from vacuum to the positive energy world, where they are scattered by quasiparticles in bulk and transfer momentum from vortex to normal component

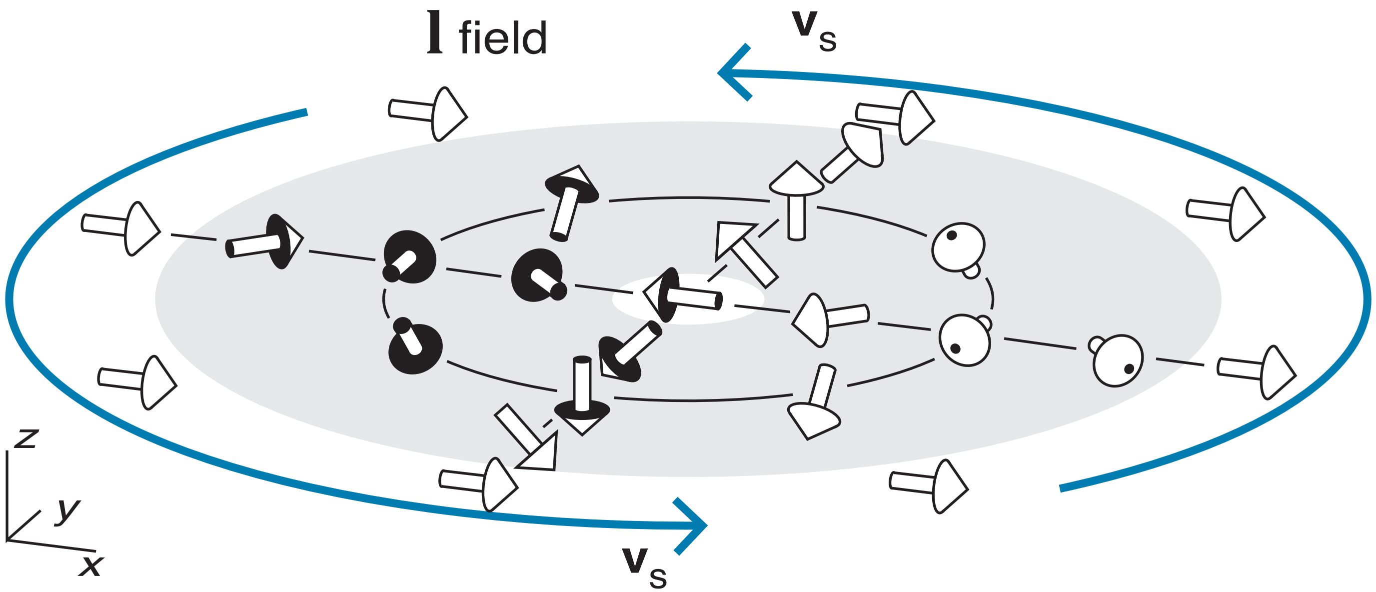
this is the source of Kopnin spectral flow force

Bevan, Manninen, Cook, Hook, Hall, Vachaspati & GV
Momentum creation by vortices in superfluid 3He as a model of primordial baryogenesis, Nature **386**, 689 (1997)

Kopnin force from chiral anomaly - momentogenesis by vortex-skyrmion

$$m = (1/4\pi) \iint dx\, dy\ \left(\mathbf{l} \cdot (\partial \mathbf{l} / \partial x \ \mathbf{x}\ \partial \mathbf{l} / \partial y) \right) = 1$$

vortex-skyrmion with *N*=2m=2 **circulation quanta**

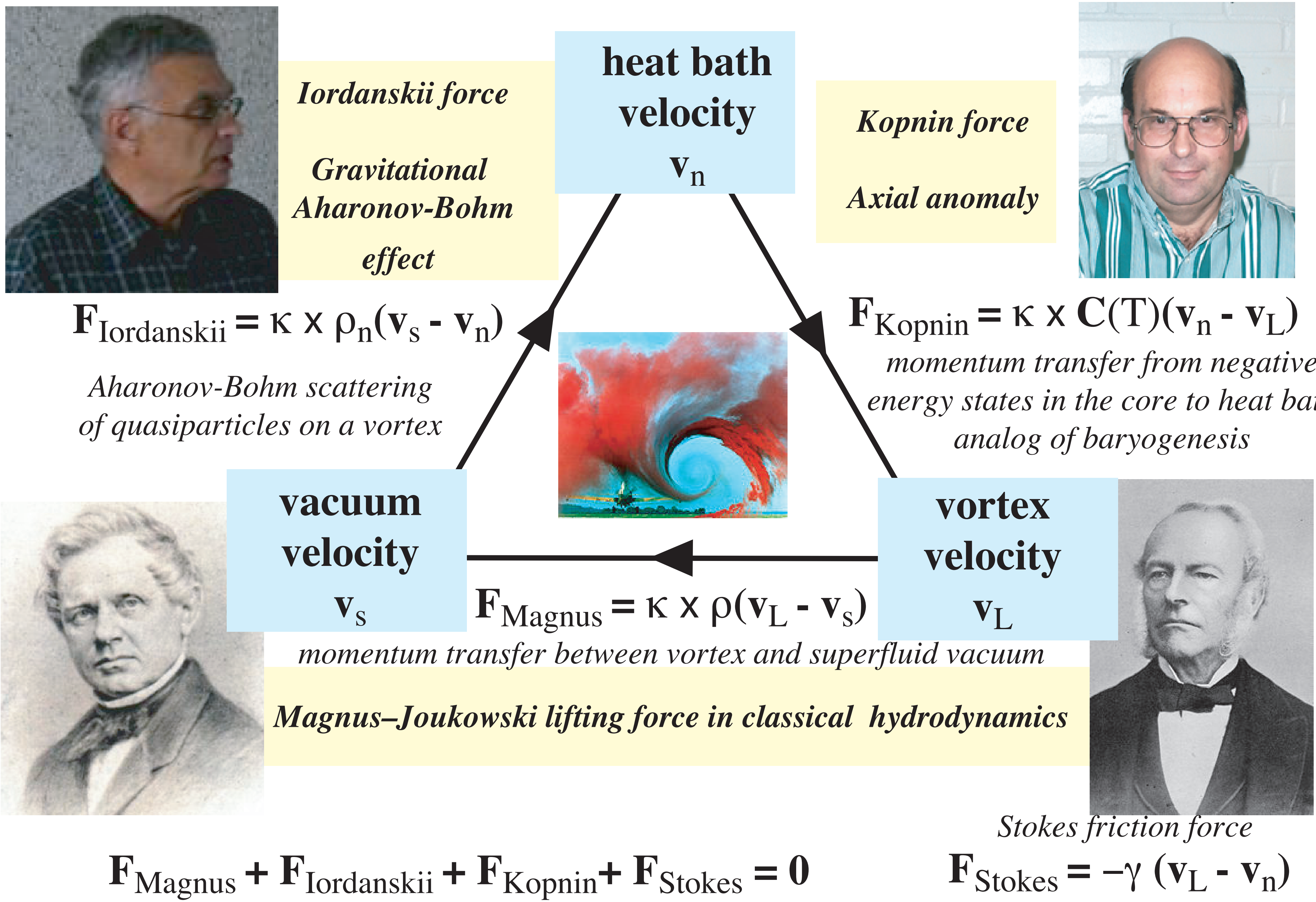


$\mathbf{l} = \mathbf{l}(\mathbf{r}\text{-}\mathbf{v}t)$

Momentum transfer from vacuum to the heat bath (matter) gives extra topological force on skyrmion (spectral-flow Kopnin force)

$$\mathbf{F} = \int d^3r\ \dot{\mathbf{P}} = (1/2\pi^2) \int d^3r\ \ (\mathbf{B} \cdot \mathbf{E})\, p_F\, \mathbf{l} = (1/2\pi^2)\, \hbar\, p_F^{\,3} \int d^3r\ \ (\nabla \times \mathbf{l} \cdot\ d\mathbf{l}/dt)\, \mathbf{l}$$

$$= 2\pi\hbar\, (1/3\pi^2)\, p_F^{\,3}\ \hat{\mathbf{z}} \times (\mathbf{v}_n - \mathbf{v}_L)$$


heat bath velocity $\mathbf{v}_n$
Iordanskii force
Gravitational Aharonov-Bohm effect
Kopnin force
Axial anomaly
$\mathbf{F}_{\mathrm{Iordanskii}} = \kappa \times \rho_n(\mathbf{v}_s - \mathbf{v}_n)$
Aharonov-Bohm scattering of quasiparticles on a vortex
$\mathbf{F}_{\mathrm{Kopnin}} = \kappa \times \mathbf{C}(T)(\mathbf{v}_n - \mathbf{v}_L)$
momentum transfer from negative energy states in the core to heat bath
analog of baryogenesis
vacuum velocity $\mathbf{v}_s$
vortex velocity $\mathbf{v}_L$
$\mathbf{F}_{\mathrm{Magnus}} = \kappa \times \rho(\mathbf{v}_L - \mathbf{v}_s)$
momentum transfer between vortex and superfluid vacuum
Magnus–Joukowski lifting force in classical hydrodynamics
Stokes friction force
$\mathbf{F}_{\mathrm{Stokes}} = -\gamma\,(\mathbf{v}_L - \mathbf{v}_n)$
$\mathbf{F}_{\mathrm{Magnus}} + \mathbf{F}_{\mathrm{Iordanskii}} + \mathbf{F}_{\mathrm{Kopnin}} + \mathbf{F}_{\mathrm{Stokes}} = 0$

Chiral effects in QCD quark matter & Weyl superfluid 3He-A

Important part of heavy-ion experimental programs
of RHIC at BNL & of LHC at CERN
is devoted to studies of effects of strong magnetic field.
The most interesting effect is CME

Chiral magnetic effect (CME) due to chiral imbalance

$$F_{CME} = (\mu_L - \mu_R) / 8\pi^2 \int dV\, \mathbf{A} \cdot \mathbf{B}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

current **J** is
along magnetic field **B**

$$\mathbf{J} = \delta F / \delta \mathbf{A} = \mathbf{B}\, (\mu_L - \mu_R) / 4\pi^2$$

current is quantized in terms of p-space topological invariant

$$F_{CME} = N_{CME} / 8\pi^2 \int dV\, \mathbf{A} \cdot \mathbf{B}$$

$$N_{CME} = \frac{1}{24\pi^2}\, e_{\alpha\beta\gamma}\, \mathbf{tr} \int dS\, \mathbf{Q}^2 \mu\, \mathbf{G}_{,\alpha}\, \mathbf{G}^{-1}\, \mathbf{G}_{,\beta}\, \mathbf{G}^{-1} \mathbf{G}_{,\gamma}\, \mathbf{G}^{-1}$$

μ matrix of chemical potentials

$\mathbf{Q}$ matrix of electric charges

Analog of chiral magnetic effect in 3He-A

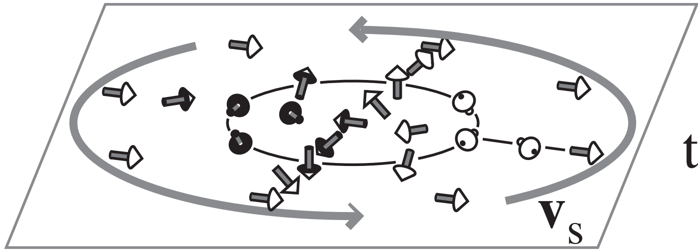
$$F_{CME} = (\mu_L - \mu_R) / 8\pi^2 \int dV\, \mathbf{A}\cdot\mathbf{B}$$

$$\mathbf{A} = p_F \mathbf{l} \qquad \mathbf{B} = p_F \nabla \times \mathbf{l}$$

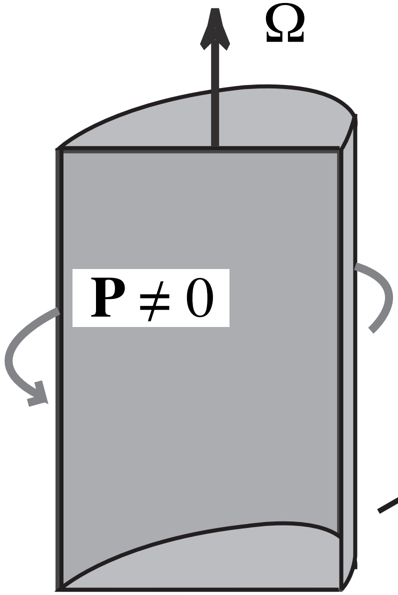
$$\mu_L - \mu_R = 2\, p_F\, \mathbf{l}\cdot(\mathbf{v}_n - \mathbf{v}_s)$$

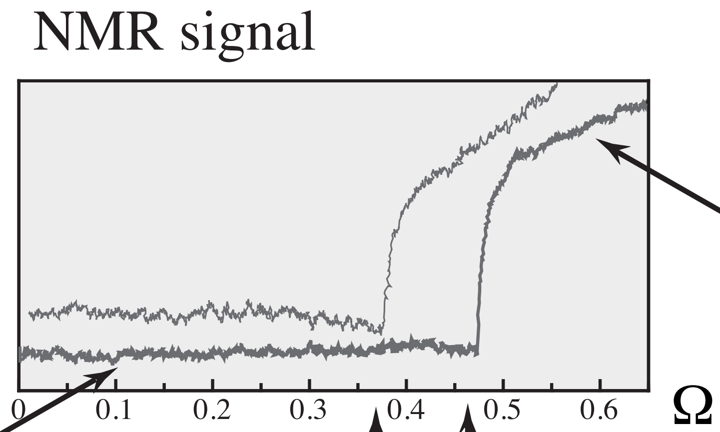
analog of CME leads to helical instability in 3He-A (magnetogenesis)

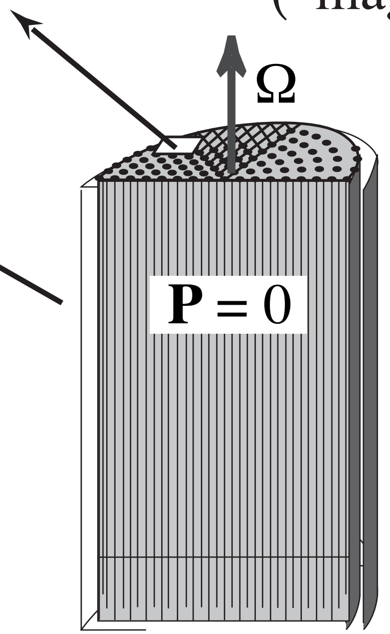
chiral imbalance is produced by counterflow:
$\mu_L - \mu_R = 2\, p_F\, \mathbf{l}\cdot(\mathbf{v}_n - \mathbf{v}_s)$
no skyrmions
(no "magnetic" field)



$\mu_L - \mu_R = 0$;
chiral instability
transforms counterflow to
lattice of skyrmions
("magnetic" field **B**)







helical instability in ^{3}He-A:
onset of magnetogenesis

Helsinki 1996

quantum turbulence in chiral 3He-A in future ROTA experiments
effect of Weyl fermions, chiral anomaly, CME, CVE, Majorana core fermions

Chiral vortical effect (CVE) in quark liquid due to chiral imbalance

$$F_{\mathrm{CVE}} = (\mu_L^2 - \mu_R^2) / 8\pi^2 \int dV\, \mathbf{A} \cdot \mathbf{B}_g$$

$\mathbf{B}_g$ is gravimagnetic field

current $\mathbf{J}$ is
along rotation axis $\boldsymbol{\Omega} = \mathbf{B}_g / c^2$

$$\mathbf{J} = \delta F / \delta \mathbf{A} = \boldsymbol{\Omega}\, (\mu_L^2 - \mu_R^2) / 4\pi^2 c^2$$

current is quantized
in terms of p-space topological invariant

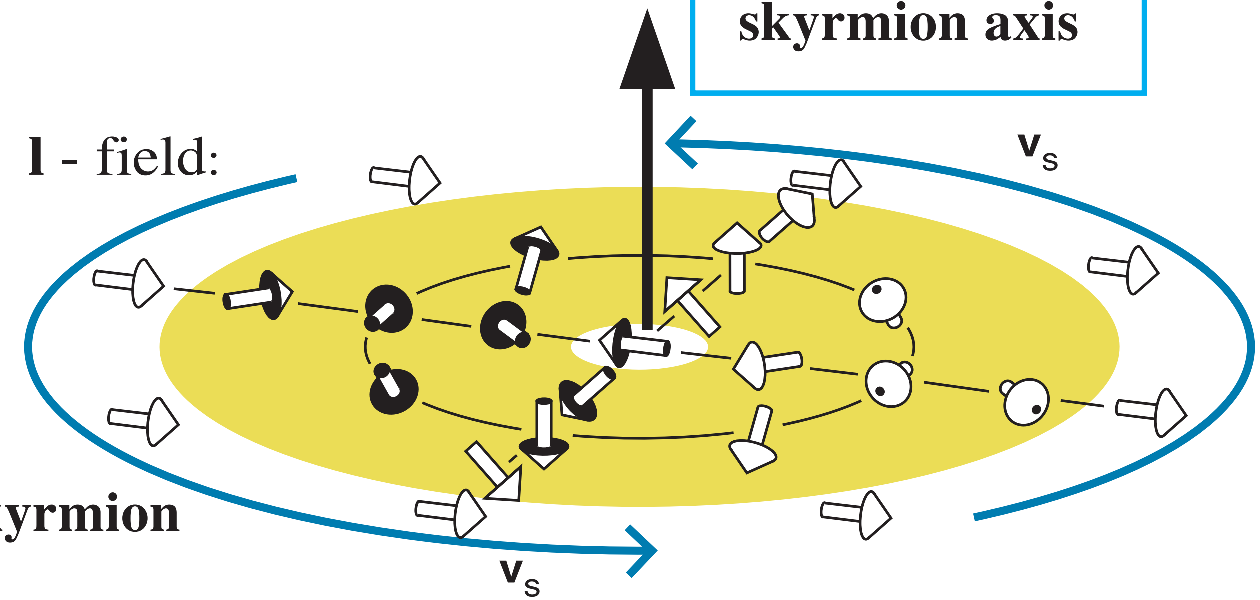
$$F_{\mathrm{CVE}} = N_{\mathrm{CVE}} / 8\pi^2 \int dV\, \mathbf{A} \cdot \mathbf{B}_g$$

$$N_{\mathrm{CVE}} = \frac{1}{24\pi^2}\, e_{\alpha\beta\gamma}\, \mathbf{tr} \int dS\, \mathbf{Q} \mu^2\, \mathbf{G}_{,\alpha}\, \mathbf{G}^{-1}\, \mathbf{G}_{,\beta}\, \mathbf{G}^{-1} \mathbf{G}_{,\gamma}\, \mathbf{G}^{-1}$$

μ matrix of chemical potentials
$\mathbf{Q}$ matrix of electric charges

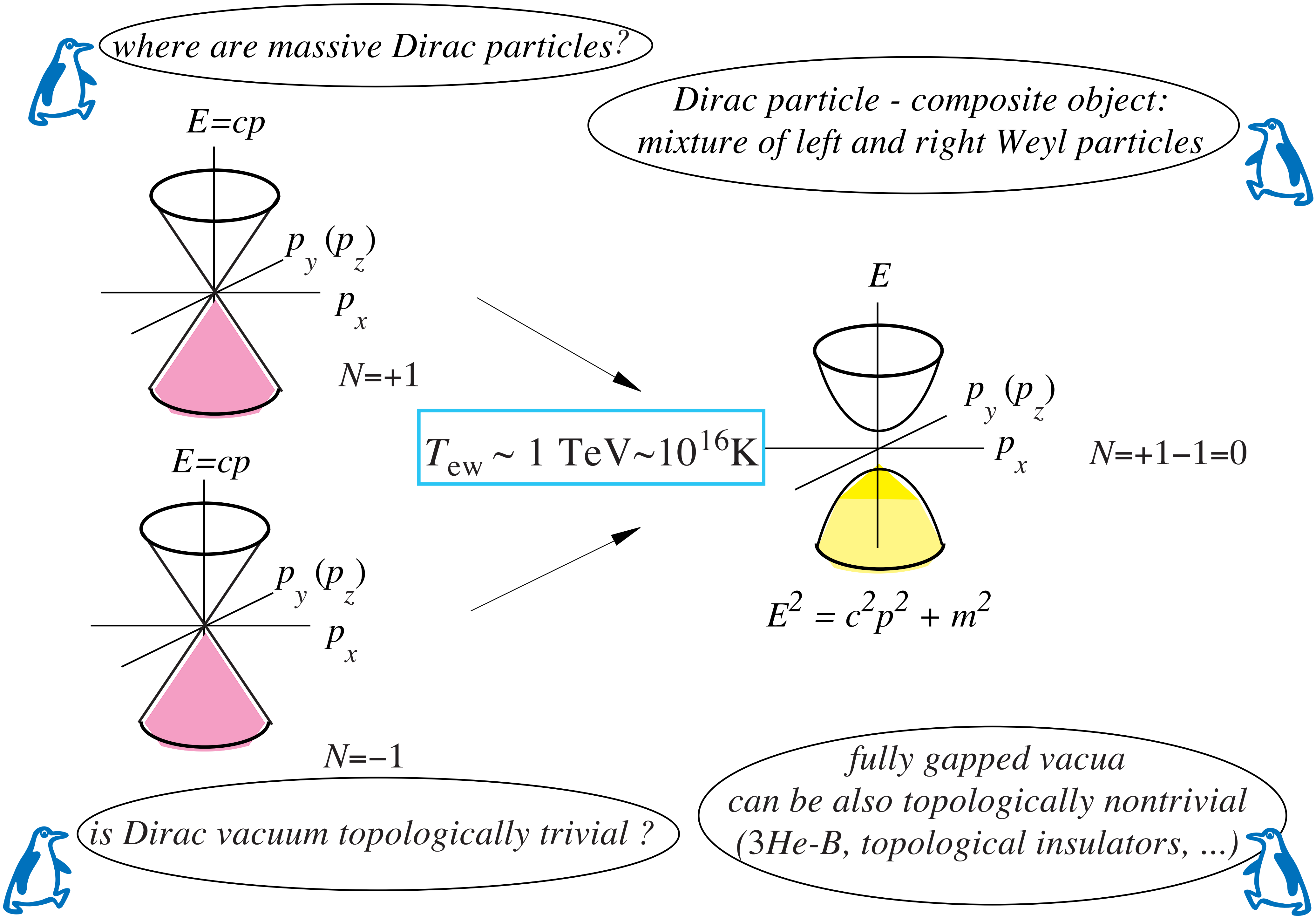
suggestion for ROTA group:
CVE in rotating 3He-A

current J along
skyrmion axis



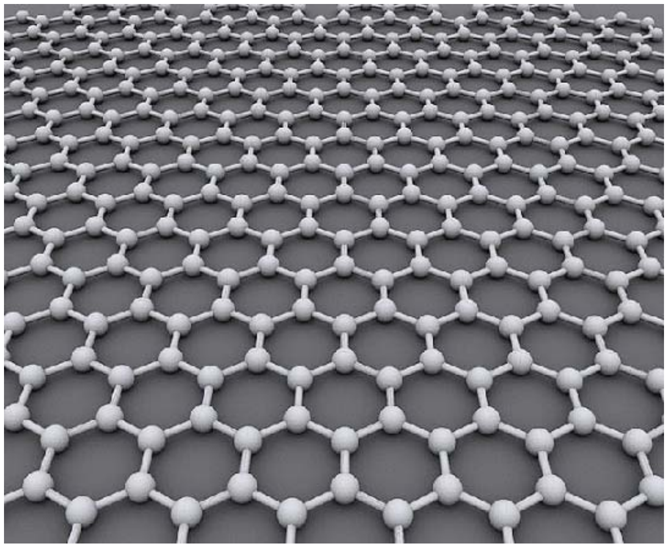
continuous vortex - skyrmion

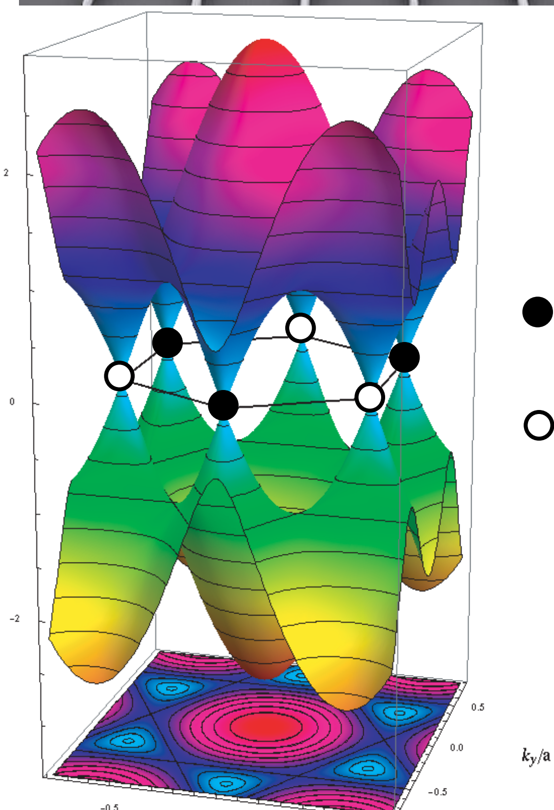
From massless Weyl particles to massive Dirac particles



p-space analogs of graphene

emergence of 2+1 gapless relativistic fermions in 2D graphene





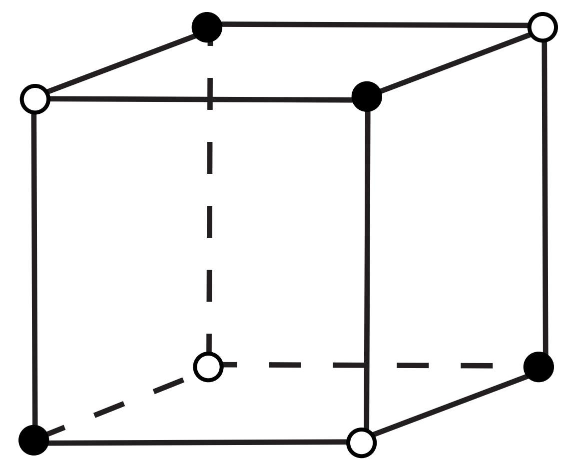
● Weyl/Dirac point $N = +1$

○ Weyl/Dirac point $N = -1$

p-space analog of
4D graphene in lattice QCD

Kaplan 1993, 2011, Creutz JHEP 04 (2008) 017

quantum vacuum as crystal



p-space analog of 3D graphene: superconductor α - phase

emergence of 2+1 relativistic fermions due to topology of graphene nodes

$$N = \frac{1}{4\pi i}\,\mathrm{tr}\left[\mathbf{K}\oint dl\,\mathbf{H}^{-1}\nabla_l\,\mathbf{H}\right]$$

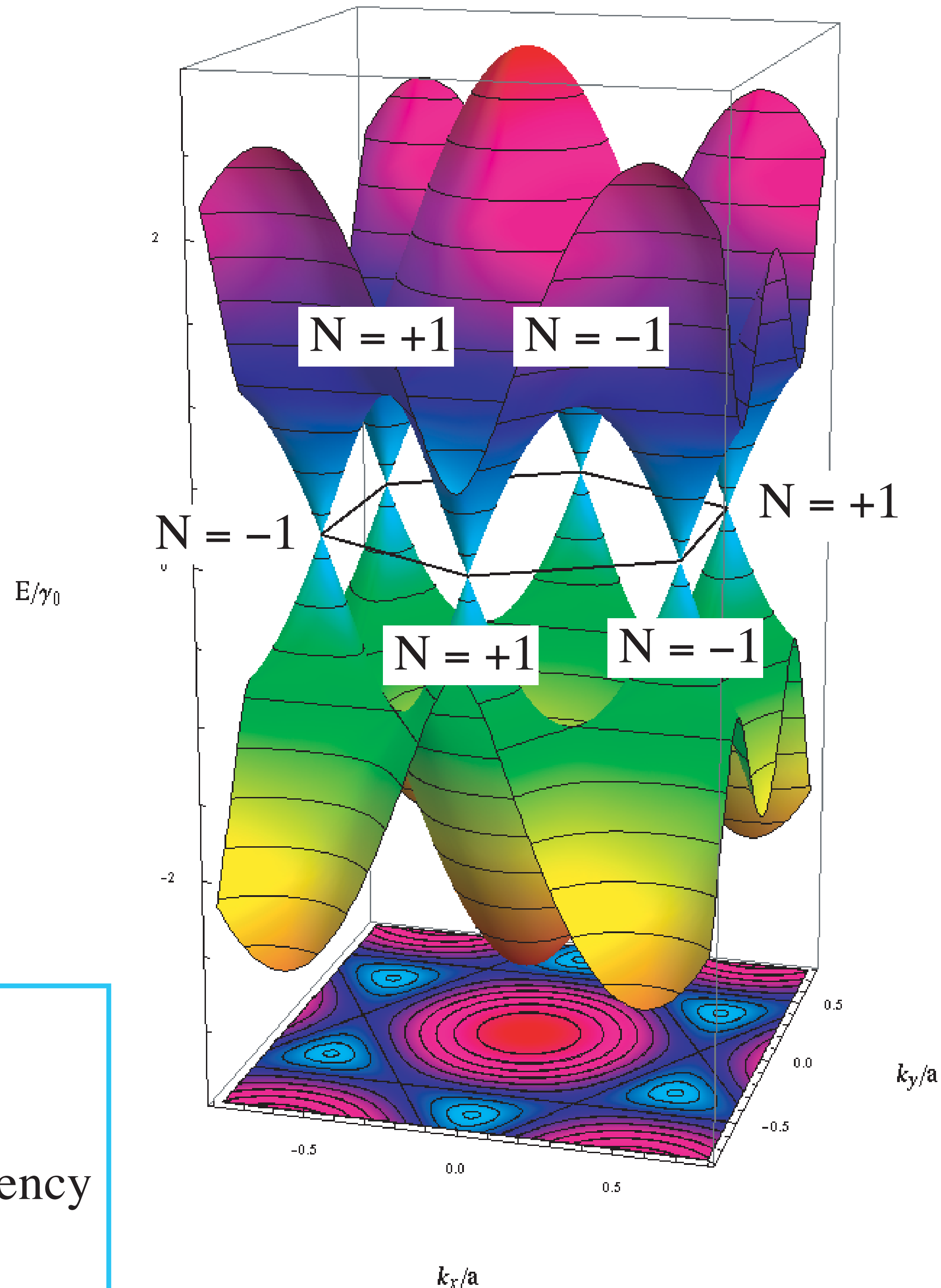
K - symmetry operator,
commuting or anti-commuting with **H**

close to nodes:

$$\mathbf{H}_{N=+1} = \tau_x p_x + \tau_y p_y$$

$$\mathbf{H}_{N=-1} = \tau_x p_x - \tau_y p_y$$

$$\mathbf{K} = \tau_z$$



for real interacting systems
the Hamiltonian $\mathbf{H}(\mathbf{p})$ is substituted
by inverse Green's function at zero frequency
$\mathbf{G}^{-1}(\omega=0,\mathbf{p})$

SU(2) gauge fields emerging near Weyl & Dirac points

Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac Γ-matrices

$$H = e_a^{\ k}\, \Gamma^a \cdot (p_k - eA_k - e\tau \cdot \mathbf{W}_k)$$

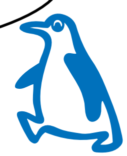
effective tetrad: emergent gravity

effective electromagnetic field

effective isotopic spin

effective $SU(2)$ gauge field

$SU(2)$ gauge field is collective mode of vacuum with Weyl point



isotopic spin comes from spin or from band indices

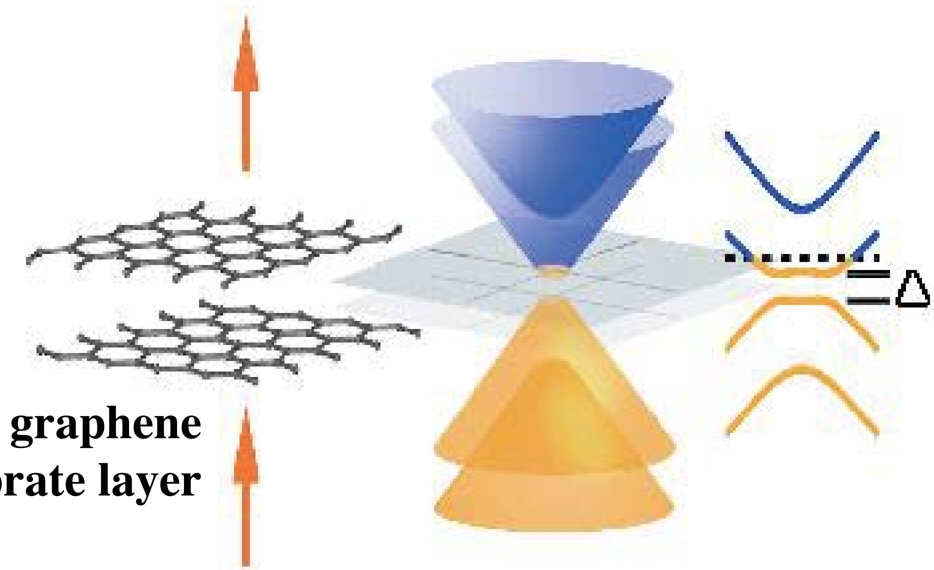
SU(2) field near Dirac points in graphene

$$\mathbf{H}_{N=+1} = \tau_x(p_x - A_x - \sigma \cdot \mathbf{W}_x) + \tau_y(p_y - A_y - \sigma \cdot \mathbf{W}_y)$$

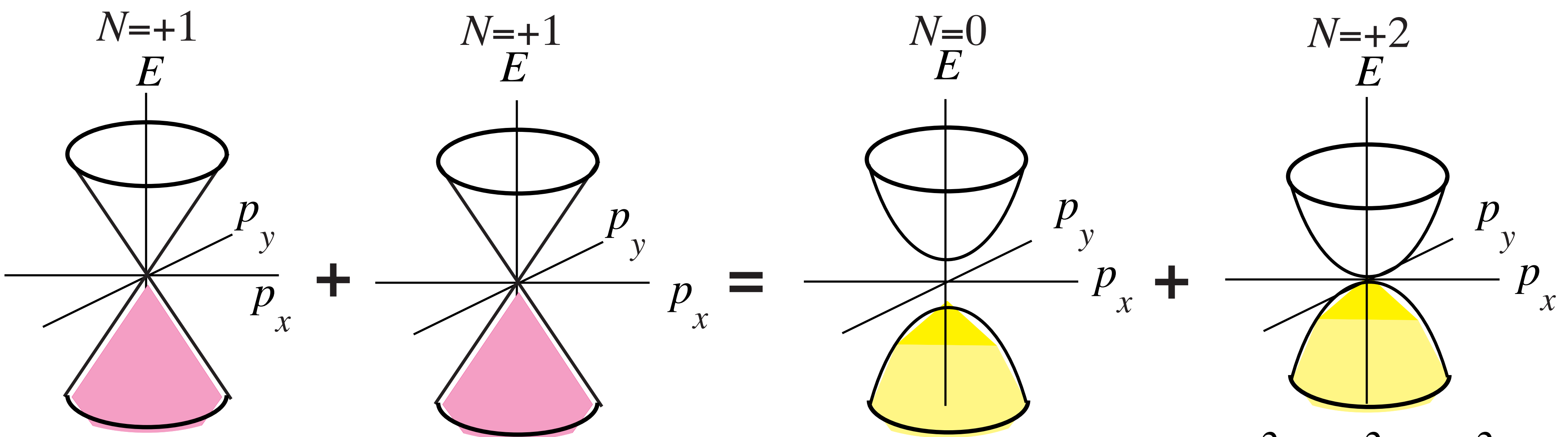
Summation of topological charges in action

exotic fermions:
massless fermions with quardatic,
cubic & quartic dispersion
semi-Dirac fermions

$$N = \frac{1}{4\pi i}\,\mathrm{tr}\left[\mathbf{K}\oint dl\, \mathbf{H}^{-1}\nabla_l\, \mathbf{H}\right]$$



bilayer graphene
double cuprate layer



$E=cp$ $\qquad$ $E=cp$ $\qquad$ $E^2 = 2c^2p^2 + 4m^2$ $\qquad$ $E^2 = (p^2/\,2m)^2$

relativistic massless fermions

Dirac fermion

nonrelativistic Dirac fermion with quadratic dispersion

re-entrant violation of Lorentz invariance & neutrino physics Klinkhamer & GV arXiv:1109.6624

nonlinear Dirac fermions

N=1: Dirac fermions with linear dispersion

$$H = \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix} = \sigma_x p_x + \sigma_y p_y$$

N=2: Dirac fermions with quadratic dispersion

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^2 \\ (p_x - ip_y)^2 & 0 \end{pmatrix}$$

N=3: Dirac fermions with cubic dispersion

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^3 \\ (p_x - ip_y)^3 & 0 \end{pmatrix}$$

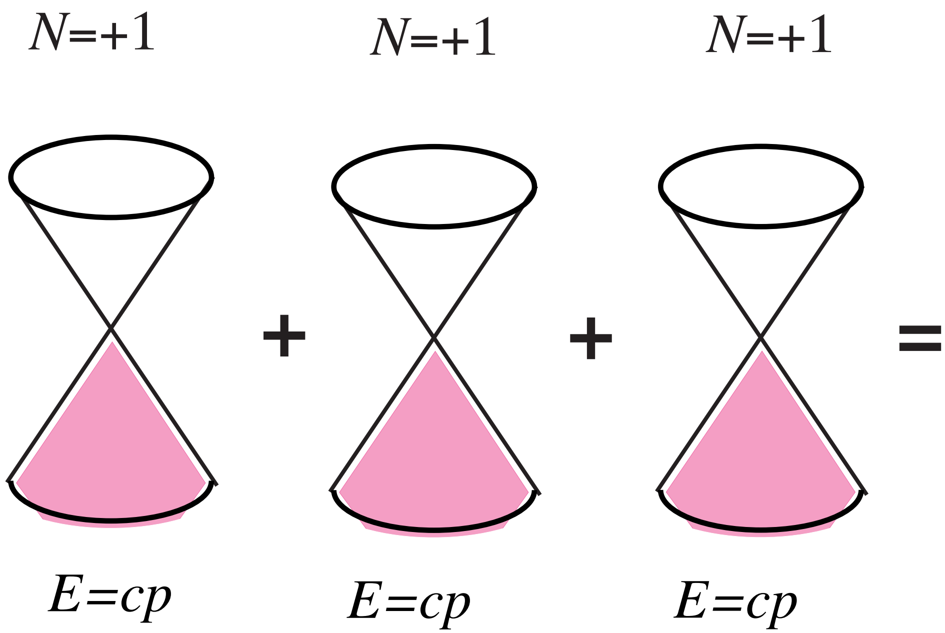
Dirac fermions with nonlinear dispersion

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^N \\ (p_x - ip_y)^N & 0 \end{pmatrix}$$

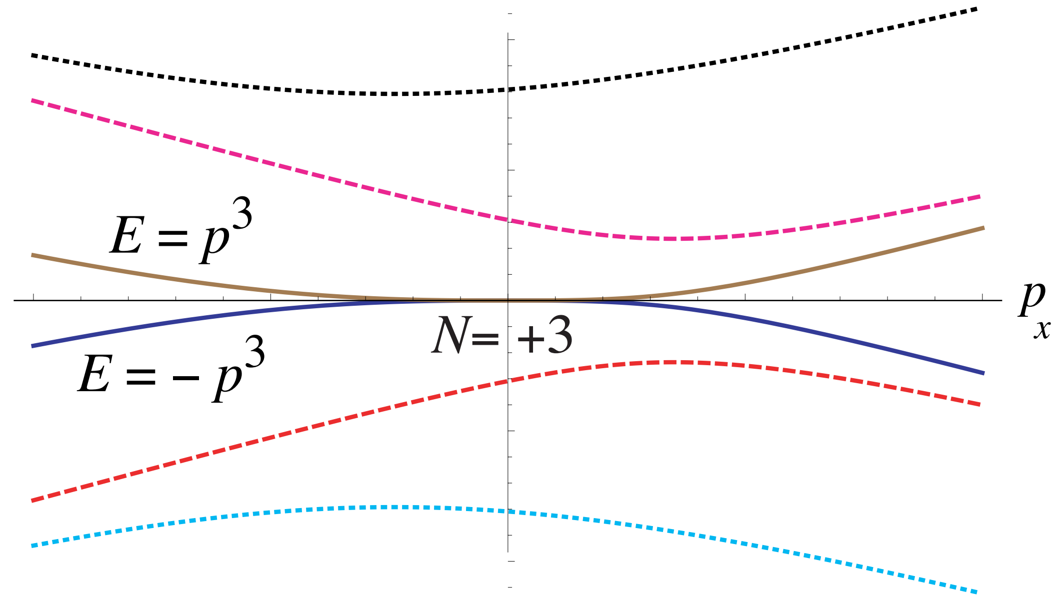
multiple Fermi point

T. Heikkila & GV arXiv:1010.0393

cubic spectrum in trilayer graphene



$$N = 1 + 1 + 1 = 3$$



multilayered graphene

$$N = 1 + 1 + 1 + \ldots$$

spectrum in the outer layers

$$E = p^N$$

$$E = - p^N$$

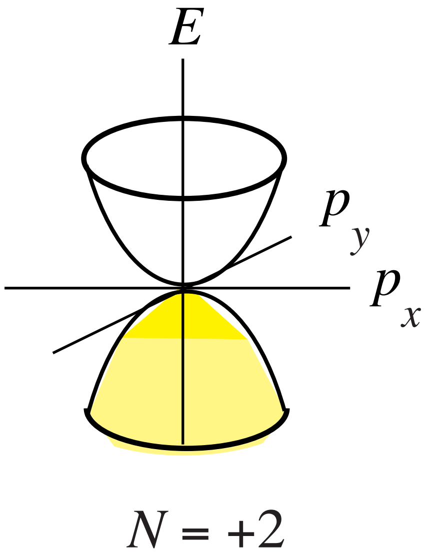
what kind of induced gravity emerges near degenerate Fermi point?

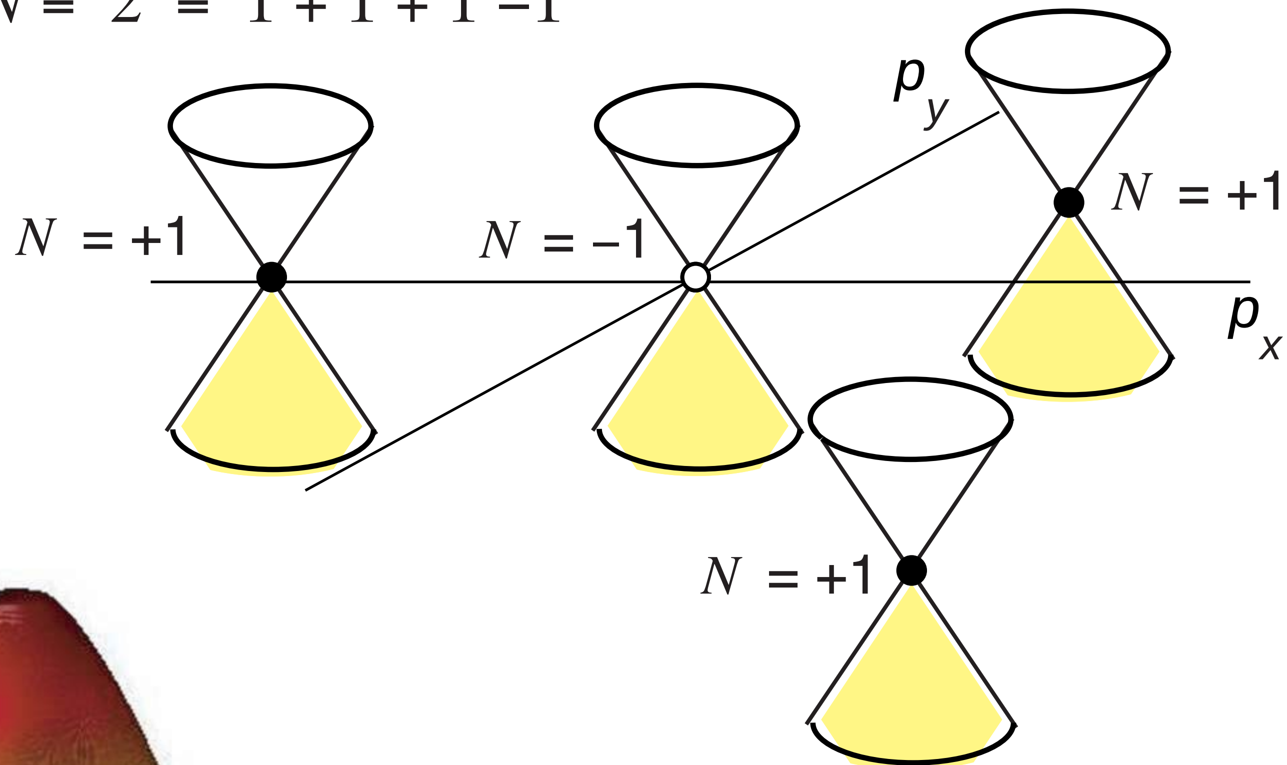
route to topological flat band on the surface of 3D material

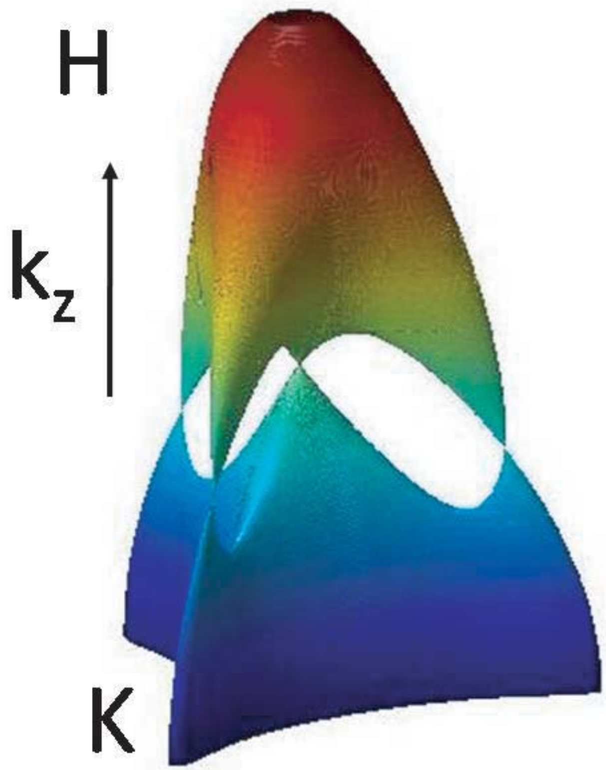
Splitting of Dirac and Weyl points

Splitting of quadratic point or trigonal warping

$$N = 2 = 1 + 1 + 1 - 1$$







Horava anisotropic scaling gravity

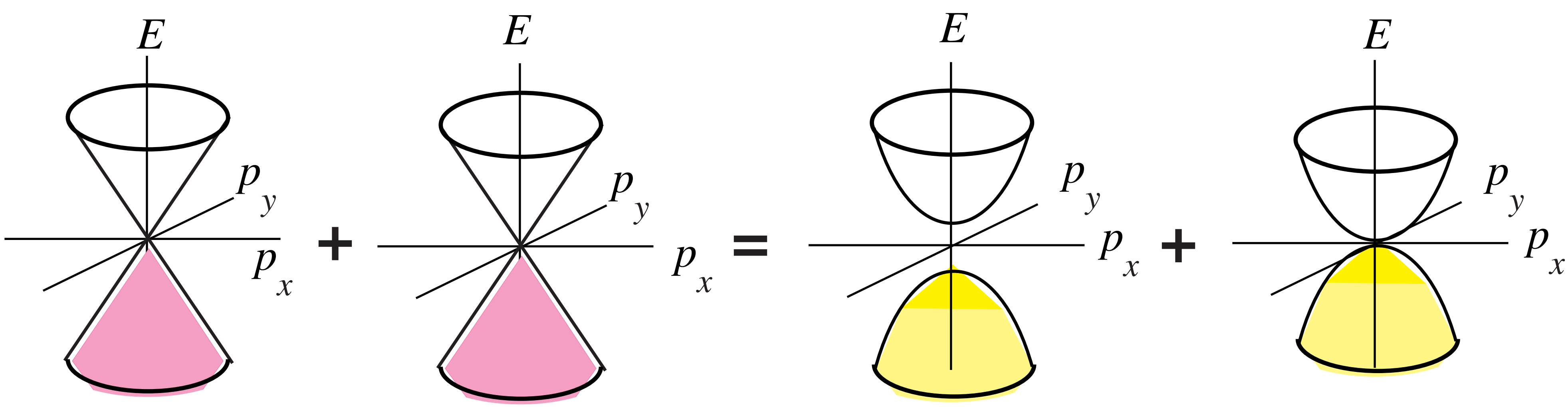
anisotropic z=3 scaling: x = b x' , t = b^3t'

$$S_{grav} = \int \underset{b^3}{d^3x}\ \underset{b^3}{dt}\ \underset{b^{-6}}{R^3}$$

Horava anisotropic z=2 scaling in bilayered graphene

$$N = \frac{1}{4\pi i}\,\mathbf{tr}\,[\mathbf{K}\oint dl\,\mathbf{H}^{-1}\nabla_l\,\mathbf{H}]$$

N=+1 | N=+1 | N=0 | N=+2



$E=cp$ | $E=cp$ | $E^2 = 2c^2p^2 + 4m^2$ | $E^2 = (p^2/\,2m)^2$

2+1 massless Dirac fermions

massive fermions

massless Dirac fermions with quadratic dispersion

Fermions in 2+1 bylayer graphene

single layer

$$H = \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix} = \sigma_x p_x + \sigma_y p_y = \begin{pmatrix} 0 & (\mathbf{e}_1 + i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A}) \\ (\mathbf{e}_1 - i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A}) & 0 \end{pmatrix}$$

zweibein

double layer

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^2 \\ (p_x - ip_y)^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & [(\mathbf{e}_1 + i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A})]^2 \\ [(\mathbf{e}_1 - i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A})]^2 & 0 \end{pmatrix}$$

zweibein

anisotropic scaling: $x = b\,x'$, $t = b^2\, t'$

Horava-Lifshitz gravity:
Horava, Quantum gravity at a Lifshitz point
PRD **79**, 084008 (2009)

Heisenberg-Euler action in massive QED

non-linear corrections to Maxwell equations due to vacuum polarization

$$S_{\rm HE} = \int d^3x\, dt\, [(B^2 - E^2)^2 /M^4 + (\mathbf{B}\cdot\mathbf{E})^2 /M^4]$$

M is rest energy of electrons $\quad B,E << M^2$

What is the Heisenberg-Euler action for relativistic massless QED
emerging in condensed matter ?

What is the Heisenberg-Euler action for massless QED
with exotic Dirac fermions with quadratic and cubic spectrum ?

relation to Horava quantum gravity with anisotropic scaling

isotropic QED emerging in 3He-A and single layer graphene

isotropic scaling: $x = b\,x'$, $t = b\,t'$, $B = b^{-2}\,B'$, $E = b^{-2}\,E'$, $S = S'$

3+1 isotropic QED & emerging in Weyl superfluids & semimetals

$$S_{\rm QED} = \int \underset{b^3}{d^3x}\ \underset{b}{dt}\ \underset{b^{-4}}{(B^2 - E^2)}\ \underset{b^{-4}}{\ln}\ 1/(B^2 - E^2)$$

imaginary action (Schwinger pair production) at $B^2 < E^2$

2+1 isotropic QED emerging in single layer graphene

$$S_{\rm QED} = \int \underset{b^2}{d^2x}\ \underset{b}{dt}\ \underset{b^{-3}}{(B^2 - E^2)^{3/4}}$$

imaginary action (Schwinger pair production) at $B^2 < E^2$

2+1 anisotropic QED emerging in bylayer graphene

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^2/2m \\ (p_x - ip_y)^2/2m & 0 \end{pmatrix} = \begin{pmatrix} 0 & [(\mathbf{e}_1 + i\,\mathbf{e}_2)(\mathbf{p} - \mathbf{A})]^2/2m \\ [(\mathbf{e}_1 - i\,\mathbf{e}_2)(\mathbf{p} - \mathbf{A})]^2/2m & 0 \end{pmatrix}$$

Heisenberg-Euler action

anisotropic scaling: $x = b\,x'$, $t = b^2\,t'$, $B = b^{-2}\,B'$, $E = b^{-3}\,E'$, $S = S'$

$$S = 1/m \int \underbrace{d^2x}_{b^2}\, \underbrace{dt}_{b^2}\, \underbrace{B^2}_{b^{-4}}\, g(\mu)$$

$g(\mu)$ – dimensionless fuction of dimensionless parameter $\mu = m^2 \underbrace{E^2}_{b^{-6}} / \underbrace{B^3}_{b^6}$

magnetic field asymptote

$$S_B = 1/m \int d^2x\, dt\, B^2\, ln\; 1/B^2$$

electric field asymptote

$$S_E = 1/m \int \underbrace{d^2x}_{b^2}\, \underbrace{dt}_{b^2}\, (-\, m^2 \underbrace{E^2}_{b^{-4}})^{2/3}$$

Schwinger pair production $\sim E^{4/3}$

pair production mainly occurs at $\mu > 1$ **i.e at** $E^2 > B^3/m^2$

Schwinger pair production

$$\mathrm{Im}\, S = \pi^{-2} B^2 \mu \int_0^1 dx \exp(-\mu f(x))$$

$\mu = m^2 E^2 / B^3$ $\qquad f(x) = x - (1+x)/2 \ln(1+x) - (1-x)/2 \ln(1-x)$

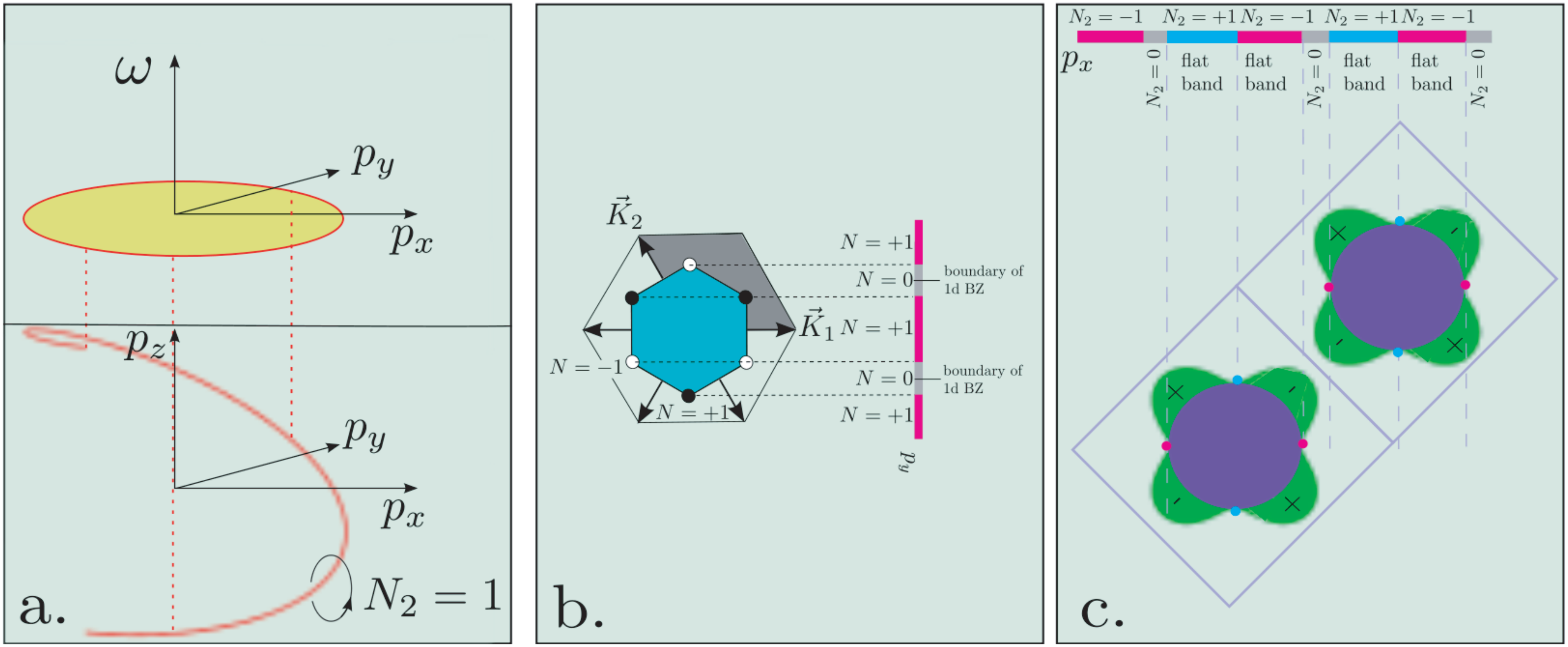
pair production mainly occurs at $\mu > 1$ **i.e at** $E^2 > B^3/m^2$

at $\mu >> 1$ $\quad f(x) = x^3/6$

Schwinger pair production $\qquad \mathrm{Im}\, S \sim E^{4/3} m^{1/3}$

M.I. Katsnelson & G.E. Volovik,
Quantum electrodynamics with anisotropic scaling:
Heisenberg-Euler action and Schwinger pair production in the bilayer graphene,
Pis'ma ZhETF 95, 457 (2012); arXiv:1203.1578

4. Flat bands & Fermi arcs in topological matter



nodal spiral in multilayered graphene generates flat band with zero energy in the top and bottom layers

Hekilla, Kopnin, GV

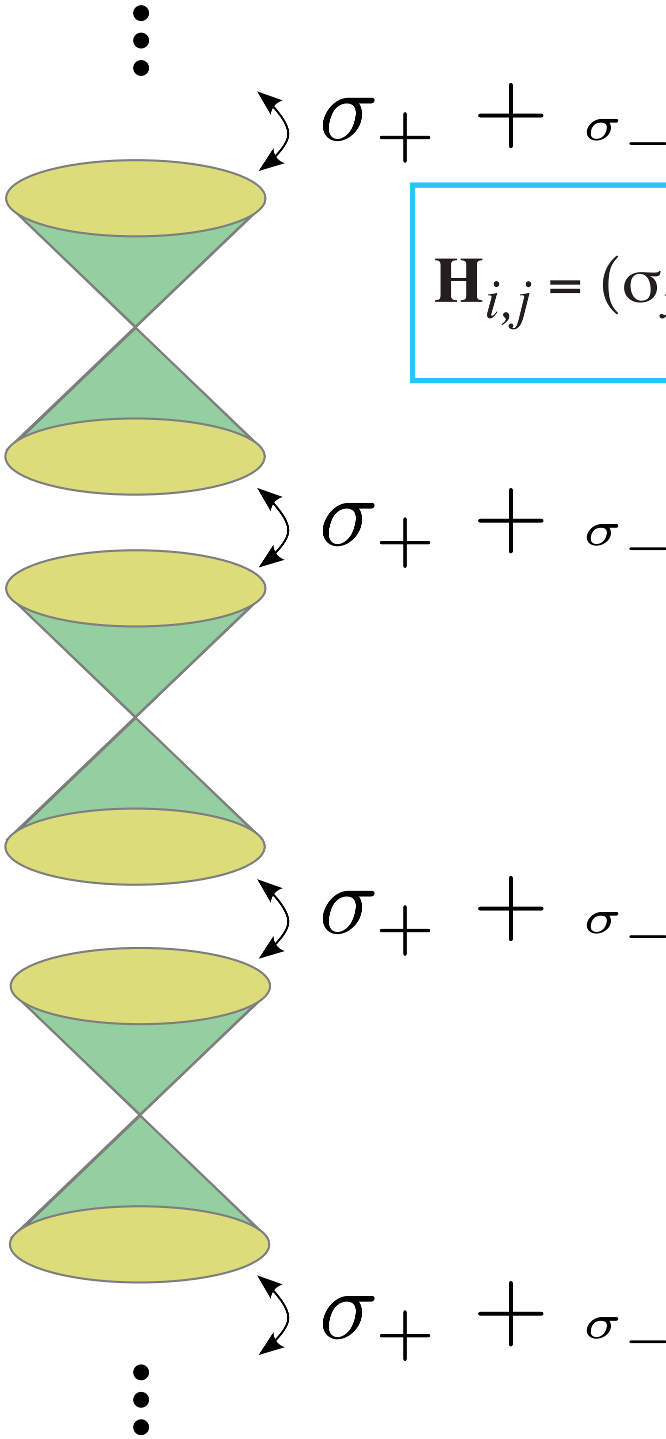
nodes in graphene generate flat band on zigzag edge

nodal lines in cuprate superconductors generate flat band on side surface

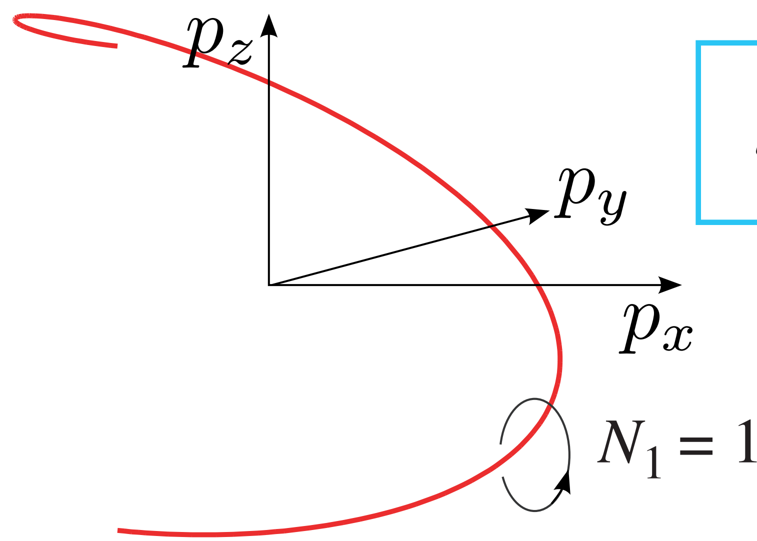
Shinsei Ryu

approximate flat band on side surface of graphite

formation of nodal spiral in bulk (together with flat band on the surface) by stacking of graphene layers



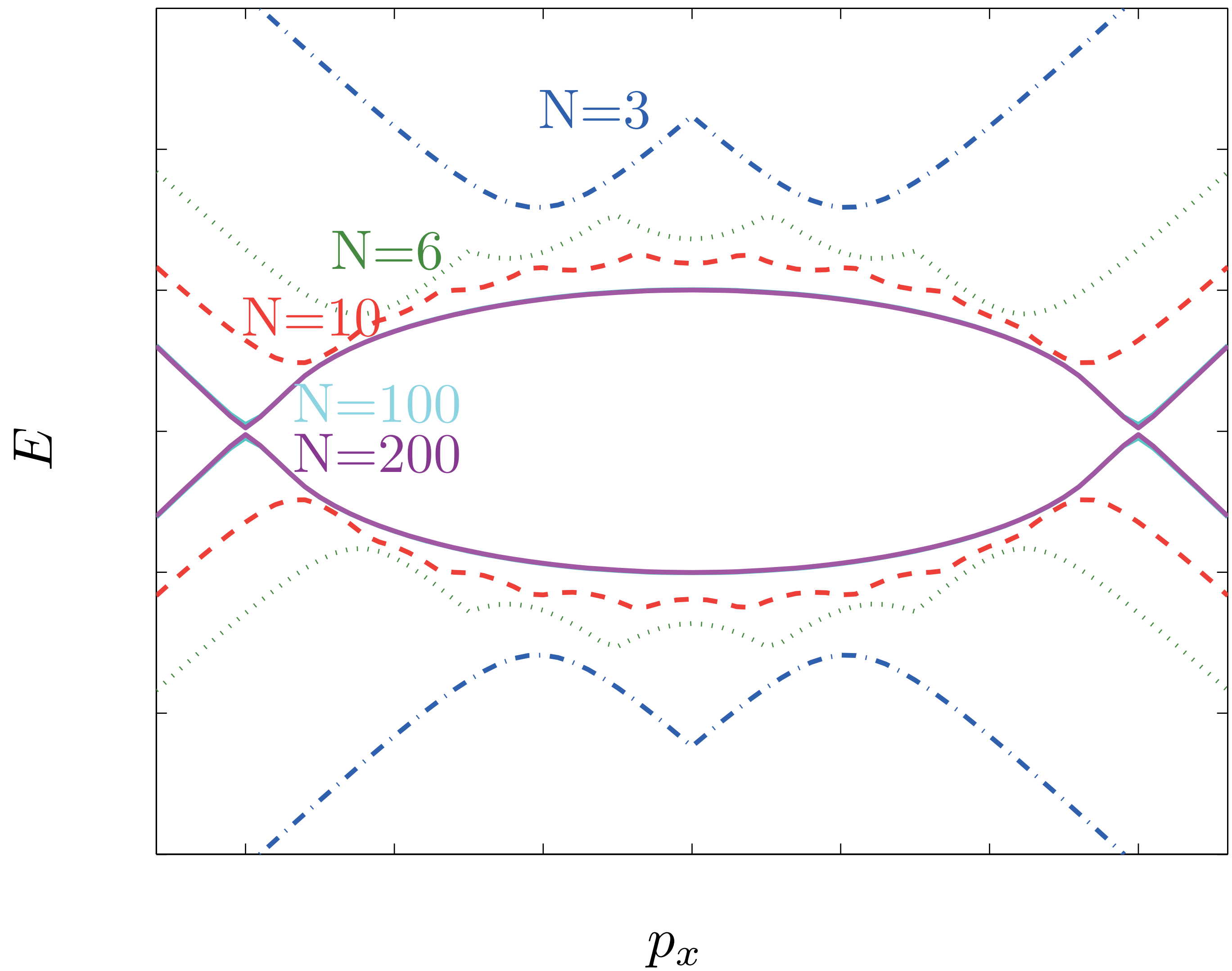
$$\mathbf{H}_{i,j} = (\sigma_x p_x + \sigma_y p_y)\delta_{i,j} + (\sigma_+ t_+ + \sigma_- t_-)\delta_{i,j+1}$$



$$t_+ > t_-$$

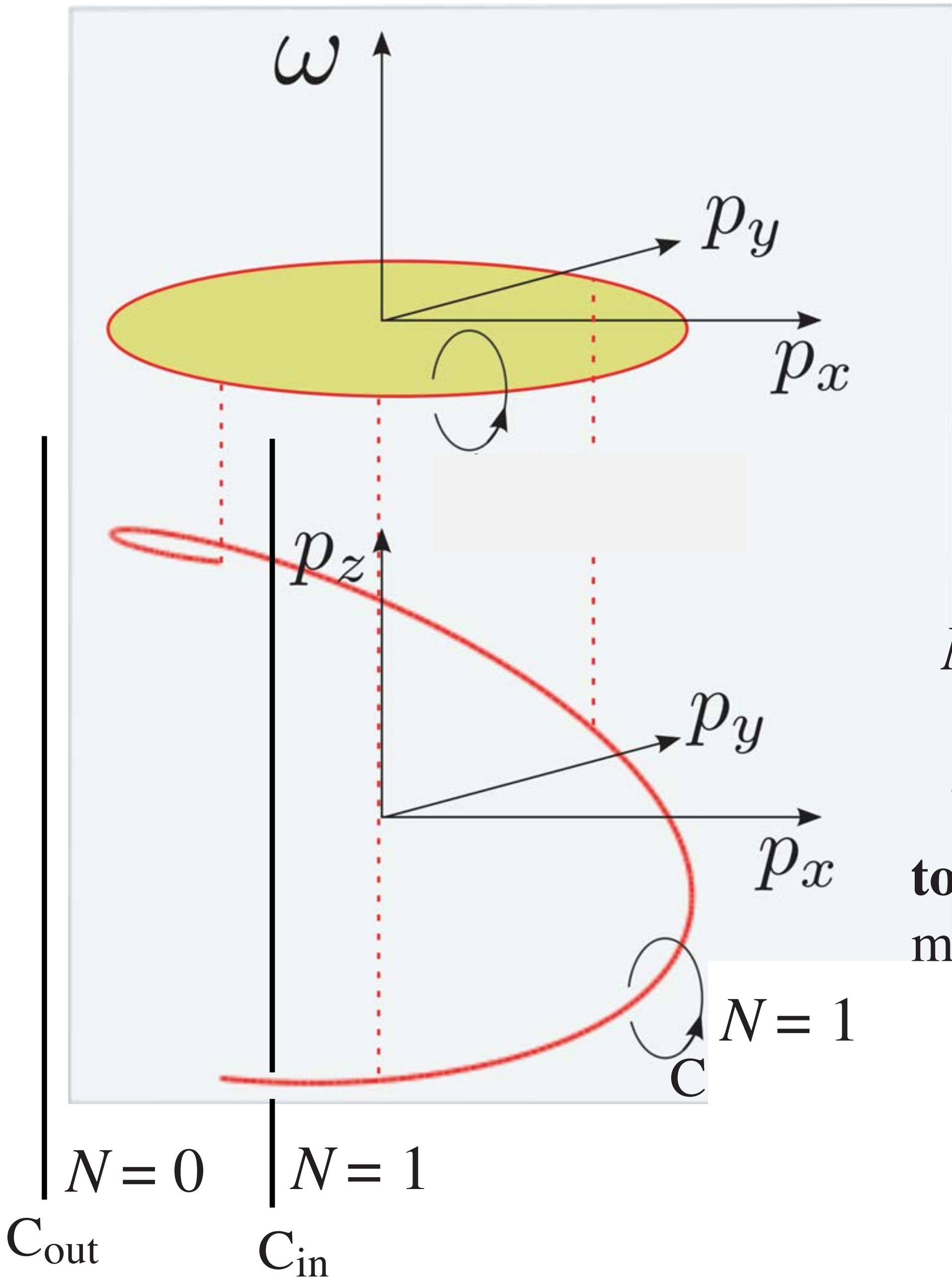
$$N_1 = \frac{1}{4\pi i}\,\mathbf{tr}\left[\mathbf{K}\oint dl\,\mathbf{H}^{-1}\nabla_l\,\mathbf{H}\right]$$

Emergence of nodal line from gapped branches by stacking graphene layers



example of topological bulk-surface correspondense:
Nodal spiral generates flat band on the surface

projection of spiral on the surface determines boundary of flat band



$$N = \frac{1}{4\pi i} \, \mathbf{tr}\, [\mathbf{K} \oint_C dl\, \mathbf{H}^{-1} \nabla_l \mathbf{H}]$$

at each (p_x,p_y) except the boundary of circle one has 1D fully gapped state (insulator)

$N_{out}(p_x,p_y) = 0$ trivial 1D insulator

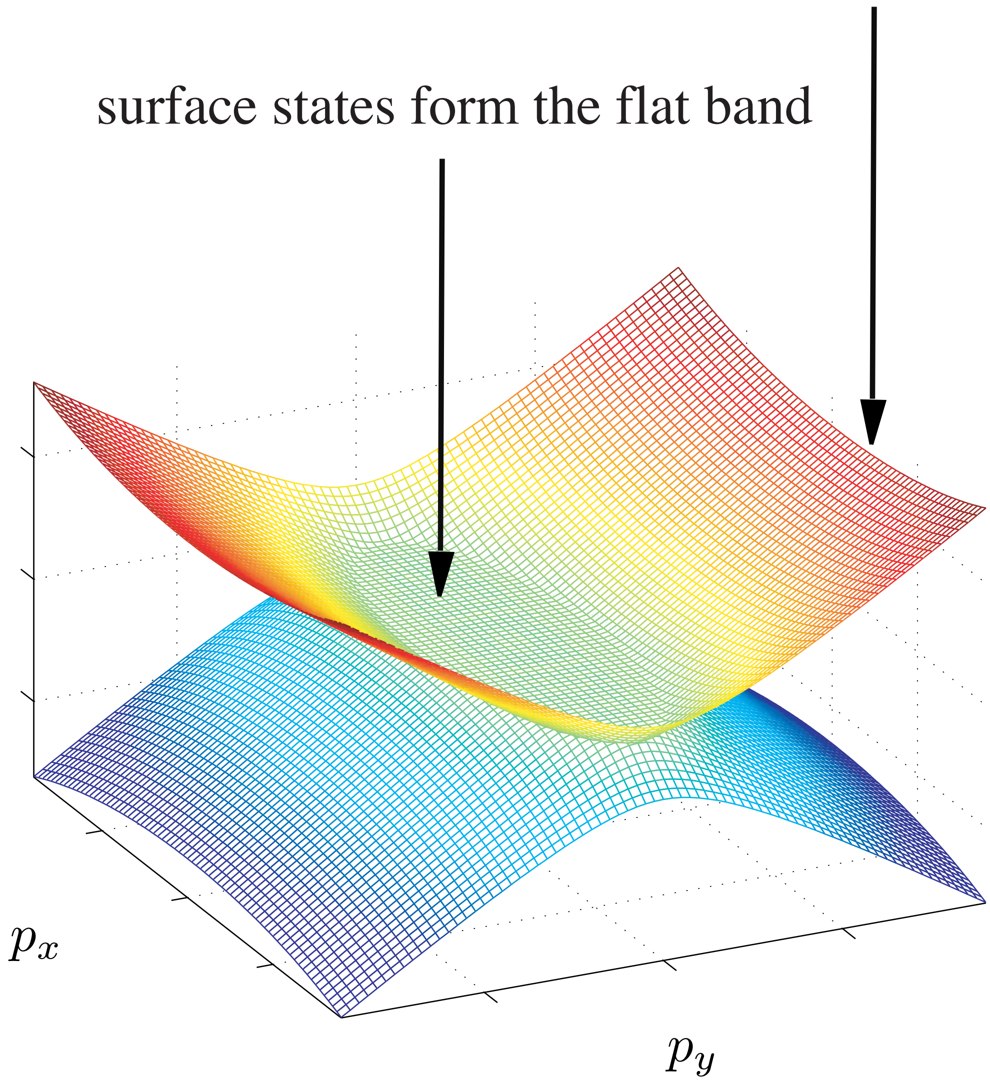
$N_{in}(p_x,p_y) = 1$ topological 1D insulator

topological insulator has 1D gapless **edge state**
manifold (p_x,p_y) of edge states forms **flat band**

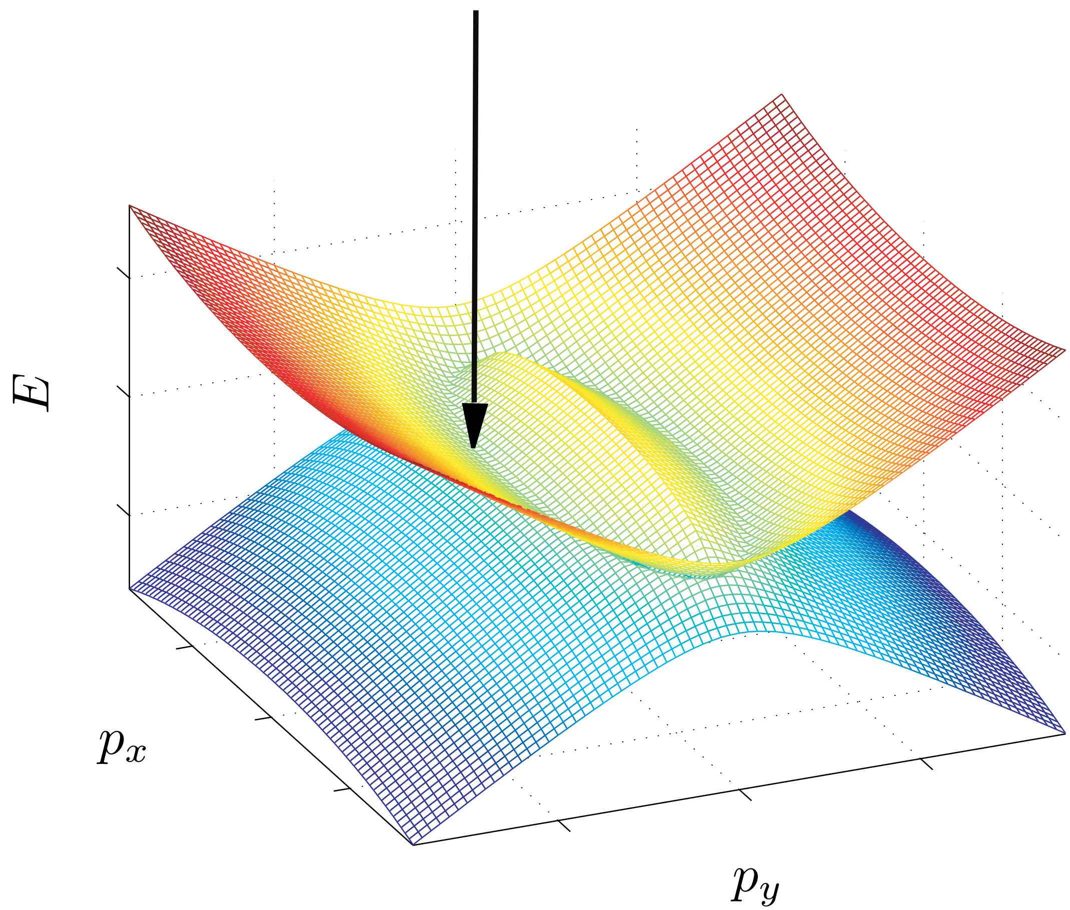
Nodal spiral generates flat band on the surface

projection of nodal spiral on the surface determines boundary of flat band

lowest energy states:



energy spectrum in bulk
(projection to p_x , p_y plane)



Modified nodal spiral in rhombohedral graphite: spiral of Fermi surfaces (McClure 1969)

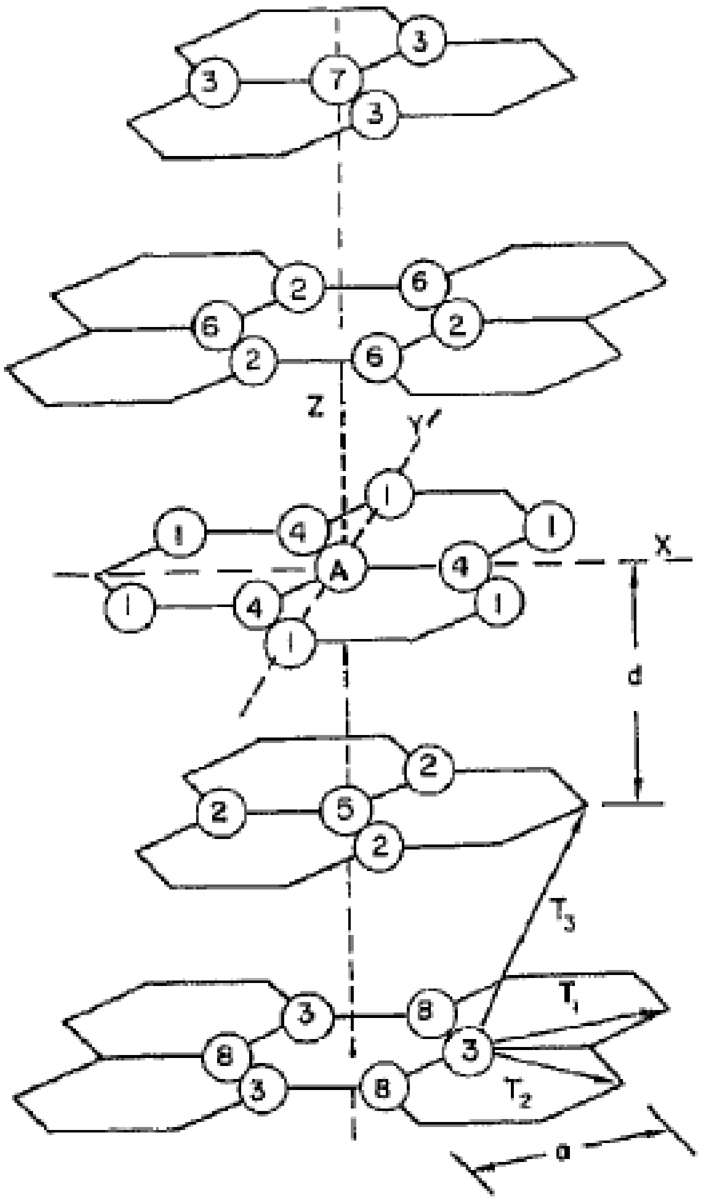


Fig. 1. The crystal lattice of rhombohedral graphite. The numbering of the groups of neighbors of the central *A* atom is explained in the text.

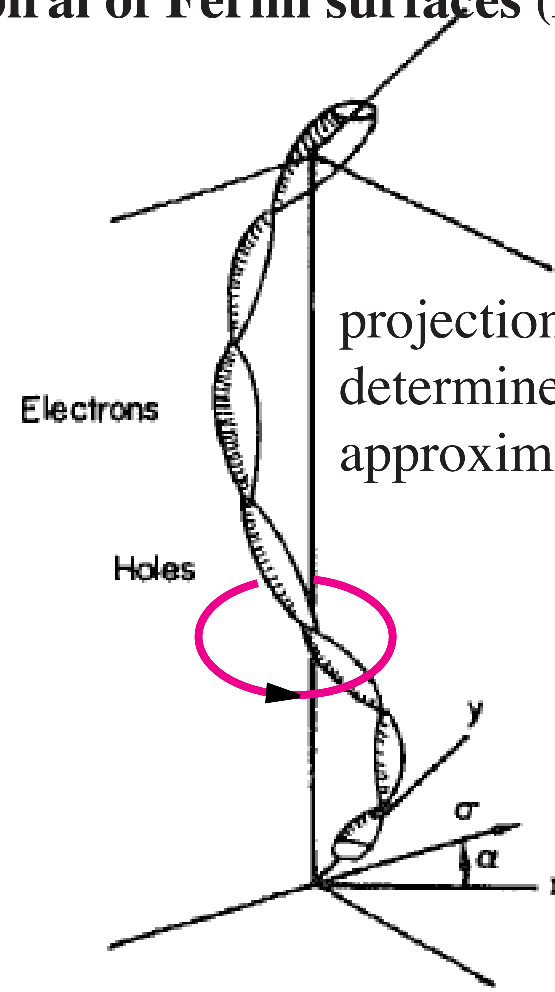
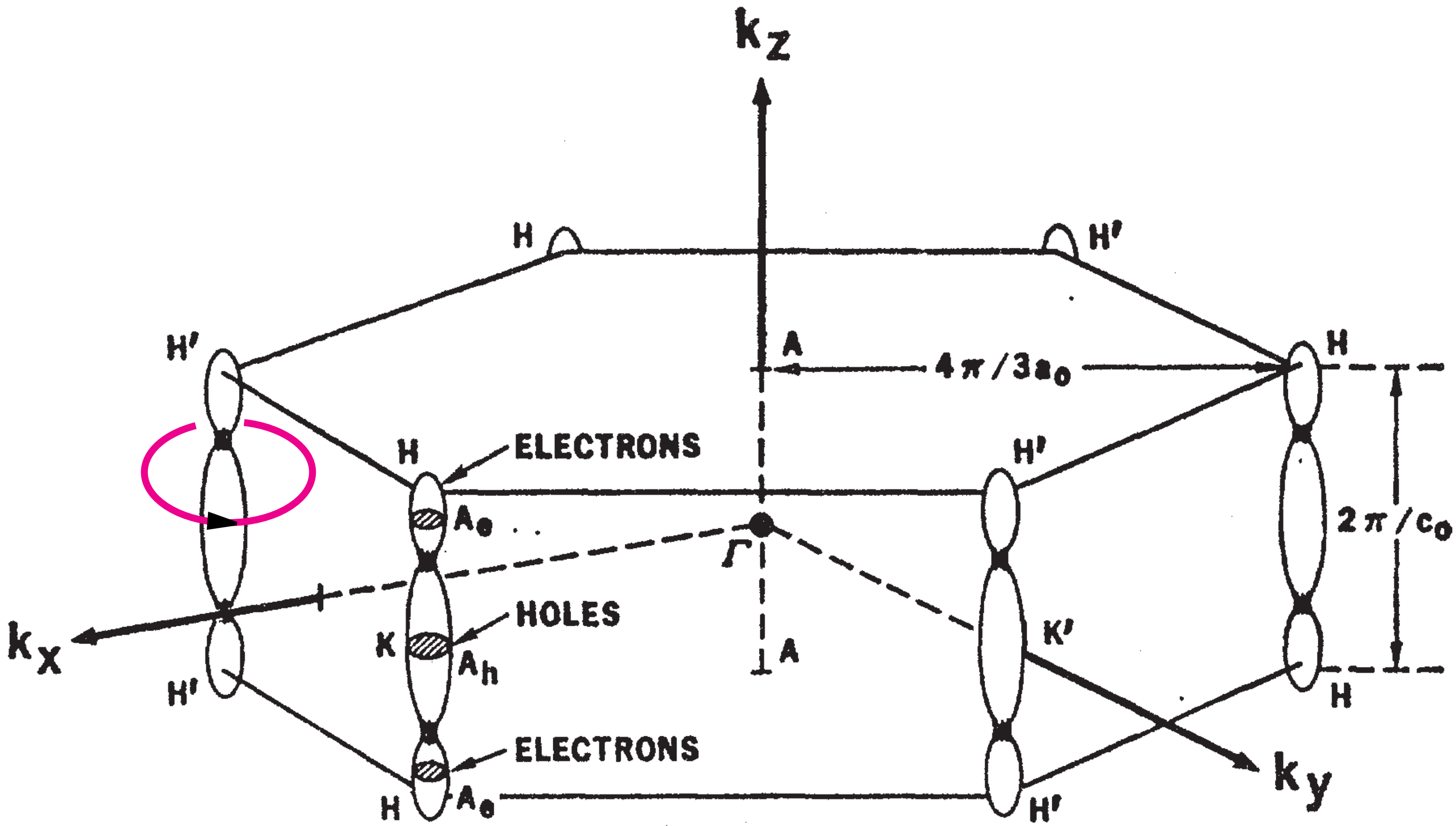


Fig. 2. The Fermi surface of rhombohedral graphite. The surface is centered on one of the six vertical zone edges. The widths of the surfaces have been magnified by more than an order of magnitude.

projections on surfaces determine approximate flat band

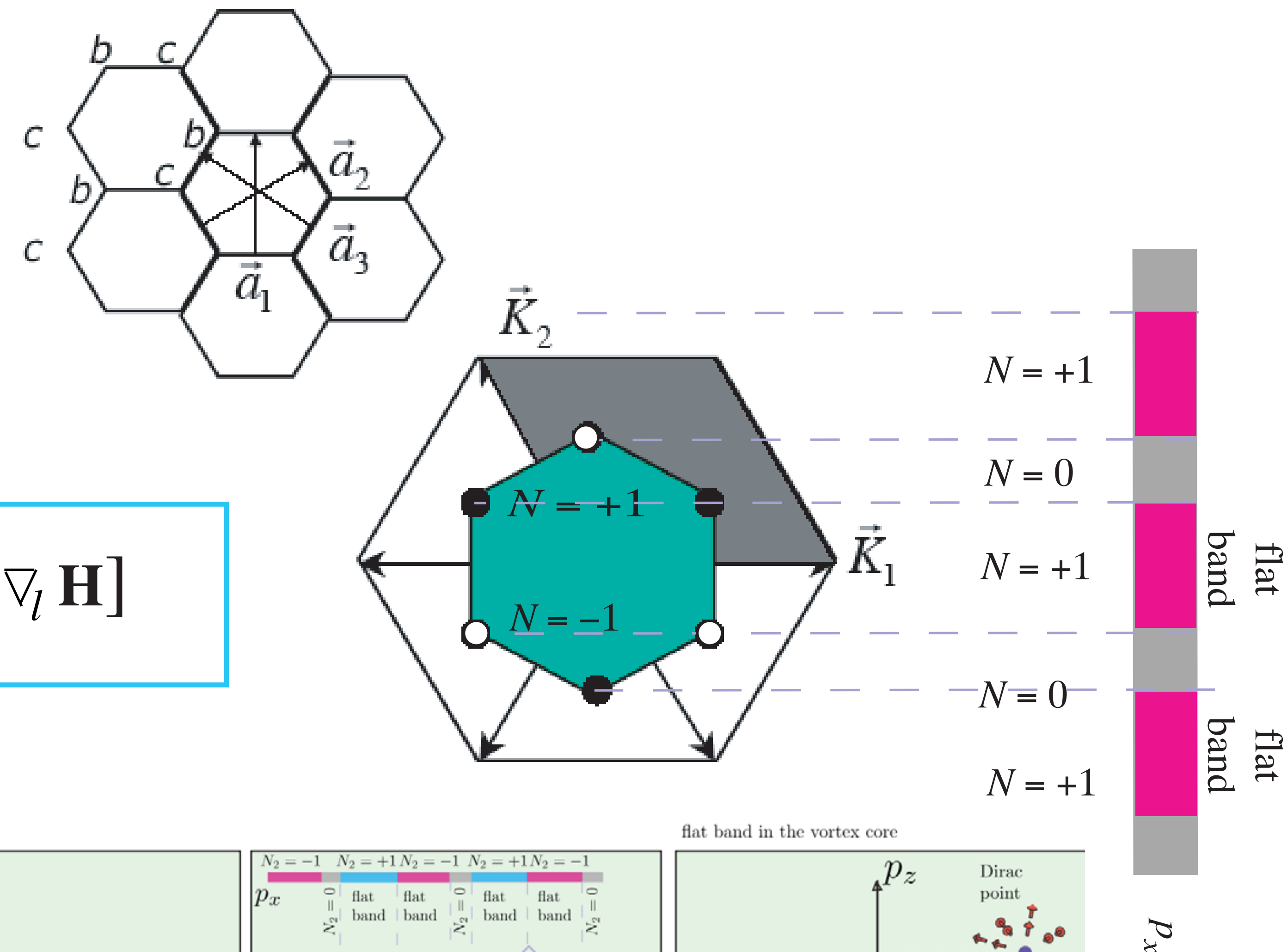
for conventional graphite: approximate flat band on the lateral surface

Nodal lines in graphite tranformed to chain of electron and hole FS

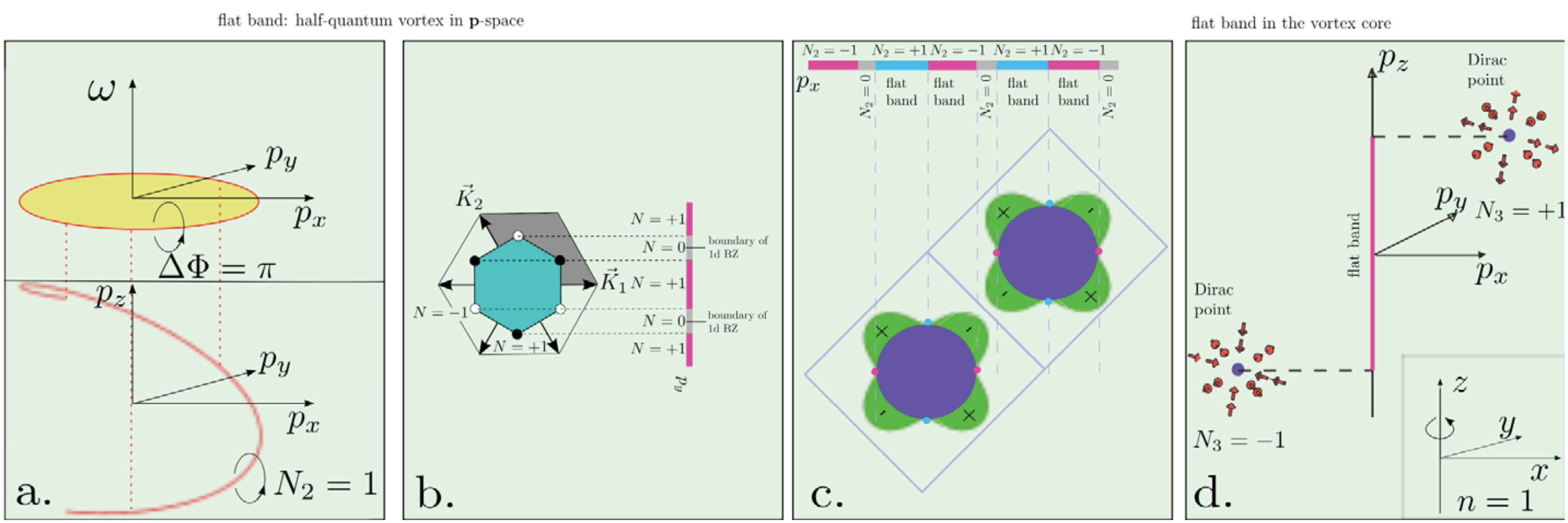


for conventional graphite:
approximate flat band
on the lateral surface

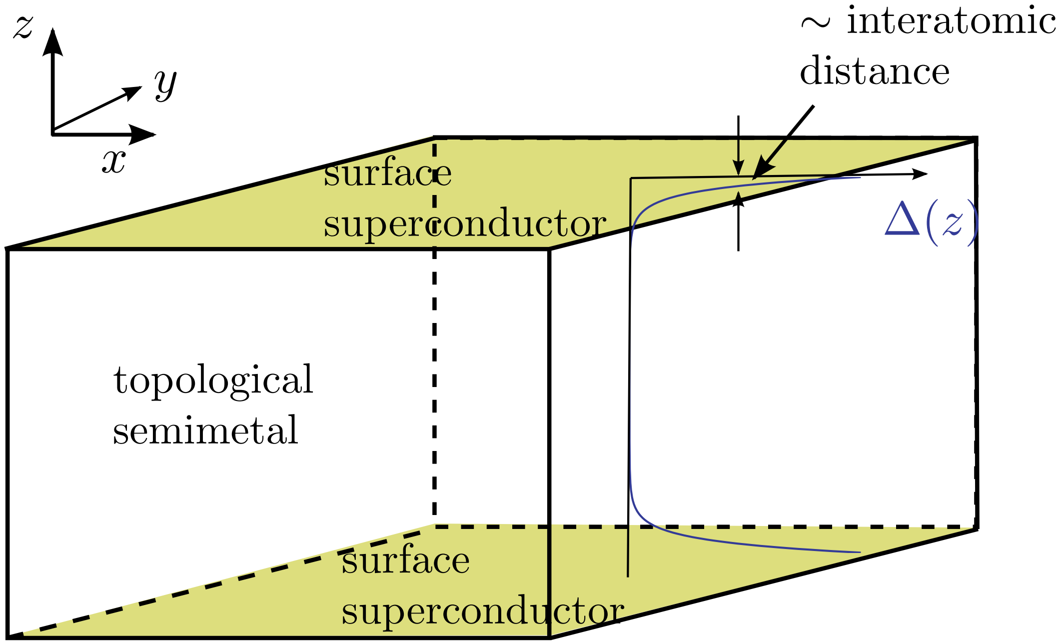
Flat band on the graphene edge

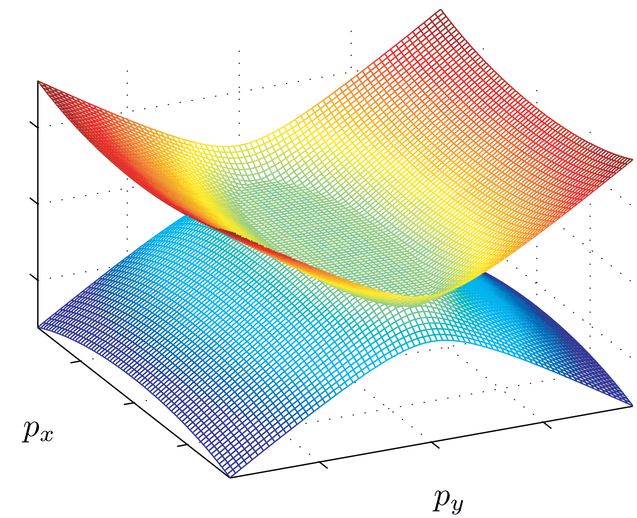


$$N = \frac{1}{4\pi i}\, \mathbf{tr}\, [\mathbf{K} \oint dl\, \mathbf{H}^{-1} \nabla_l\, \mathbf{H}]$$



Surface superconductivity in topological semimetals: route to room temperature superconductivity





Extremely high DoS of flat band gives high transition temperature:

$$1 = g \int \frac{d^2 p}{2h^2} \frac{1}{E(p)}$$

normal superconductors: exponentially suppressed transition temperature

$$T_c = T_F \exp(-1/g\nu)$$

interaction ***DOS***

$$\text{DoS} = \nu(\varepsilon) \sim \varepsilon^{2/N - 1}$$

N is number of layers

flat band superconductivity: linear dependence of T_c on coupling g

$$T_c \sim g S_{FB}$$

coupling ***flat band area***

$N = 4$: $\nu(\varepsilon) \sim \varepsilon^{-1/2}$ Kopaev (1970); Kopaev-Rusinov (1987)

evidence of room-temperature superconductivity?

2000: Kopelevich Y, Esquinazi P, Torres J H S and Moehlecke S
J. of Low Temp. Phys. **11**, 691–702

2002: Kempa H, Esquinazi P and Kopelevich Y
Phys. Rev. B **65** 241101

2007: Kopelevich Y and Esquinazi P
J. of Low Temp. Phys. **146** 629

2012: Scheike T, Böhlmann W, Esquinazi P, Barzola--Quiquia J, Ballestar A and Setzer A.
Advanced Materials **24** 5826

2013: Scheike T, Esquinazi P, Setzer A and Böhlmann W , arXiv:1301.4395

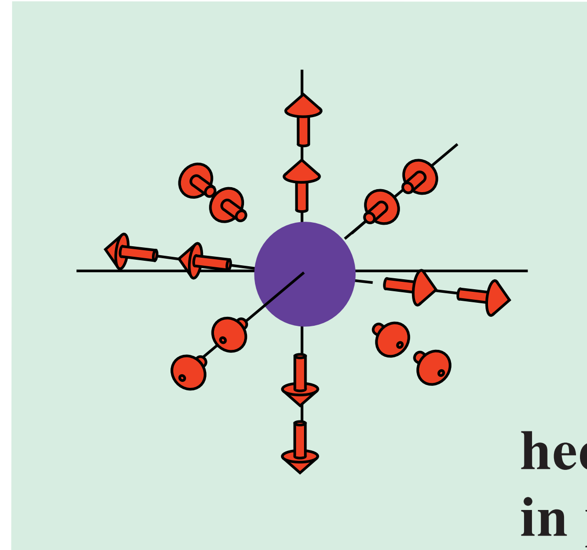
Ballestar A, Barzola-Quiquia J, Scheike T and Esquinazi P
New J. Phys. **15** 023024

G. Larkins, Y. Vlasov, K. Holland, arXiv:1307.0581

From Weyl point to quantum Hall topological insulators

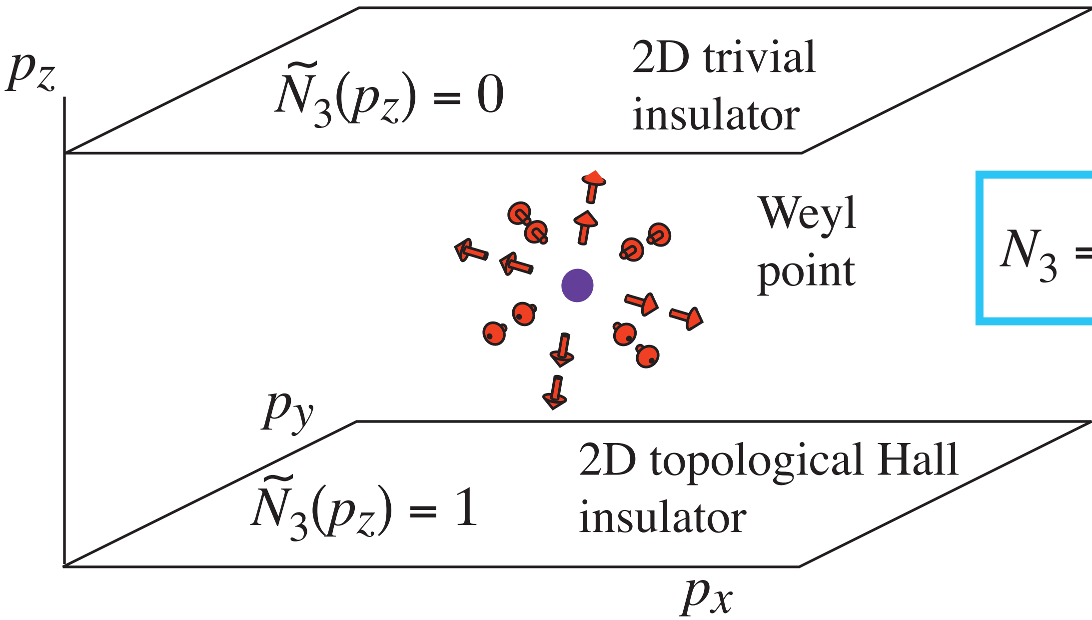
$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot (\nabla_{p_i} \hat{\mathbf{g}} \times \nabla_{p_j} \hat{\mathbf{g}})$$

over 2D surface S
in 3D momentum space



hedgehog in p-space

top. invariant for Weyl point in 3+1 system



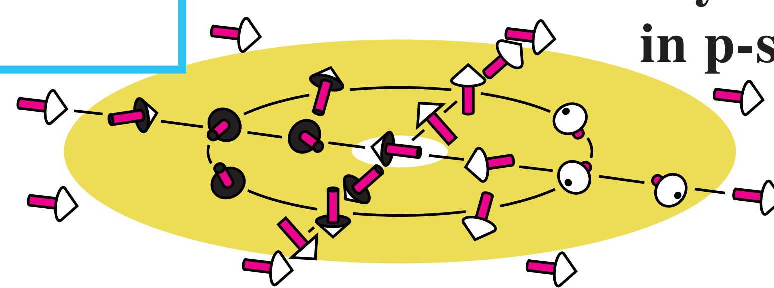
$$N_3 = \tilde{N}_3(p_z < p_0) - \tilde{N}_3(p_z > p_0)$$

at each p_z one has 2D insulator or fully gapped 2D superfluid

$$\tilde{N}_3(p_z) = \frac{1}{8\pi} e_{ijk} \int dp_x dp_y \; \hat{\mathbf{g}} \cdot (\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}})$$

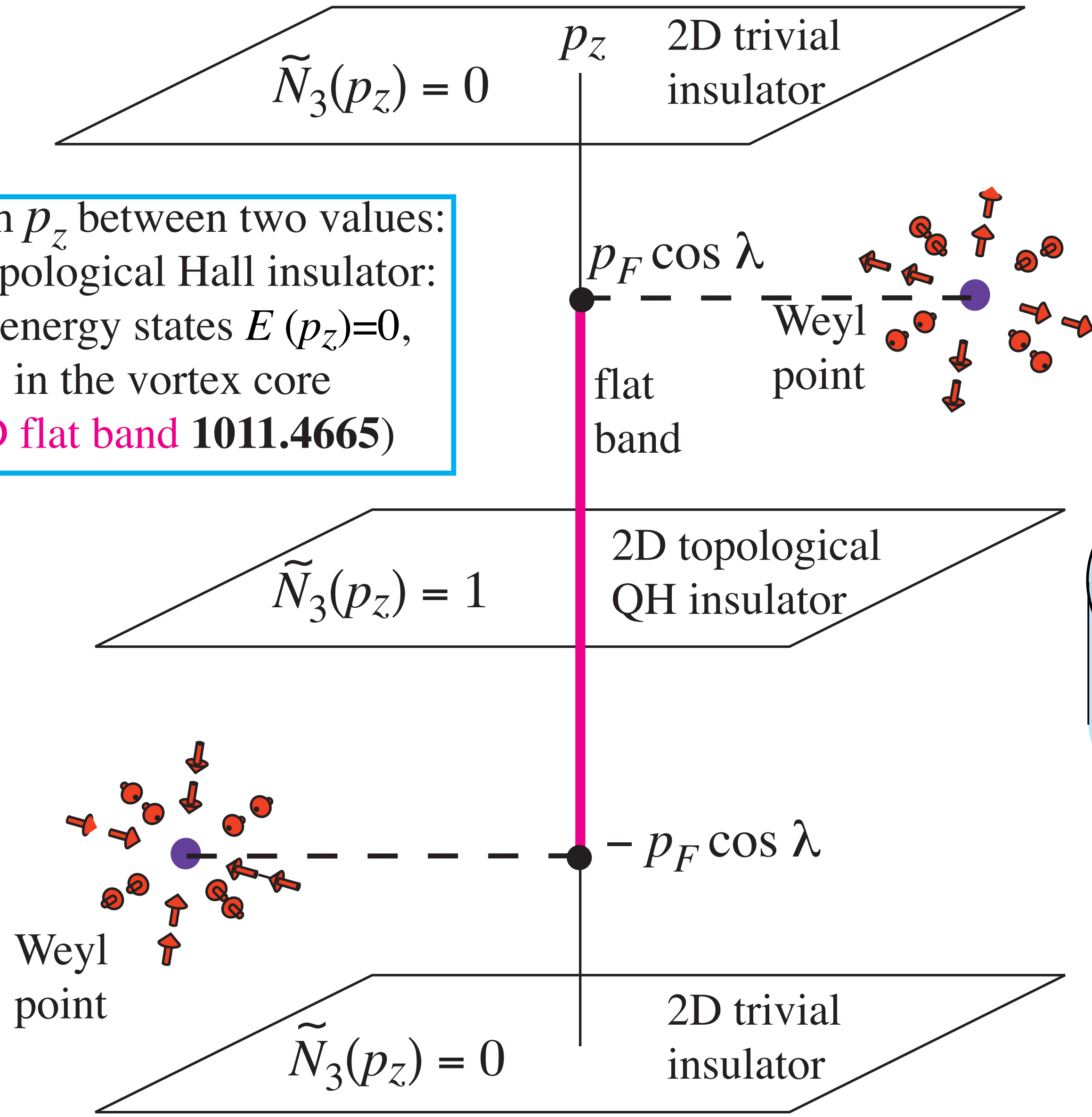
over the whole 2D momentum space
or over 2D Brillouin zone

top. invariant for fully gapped 2+1 system



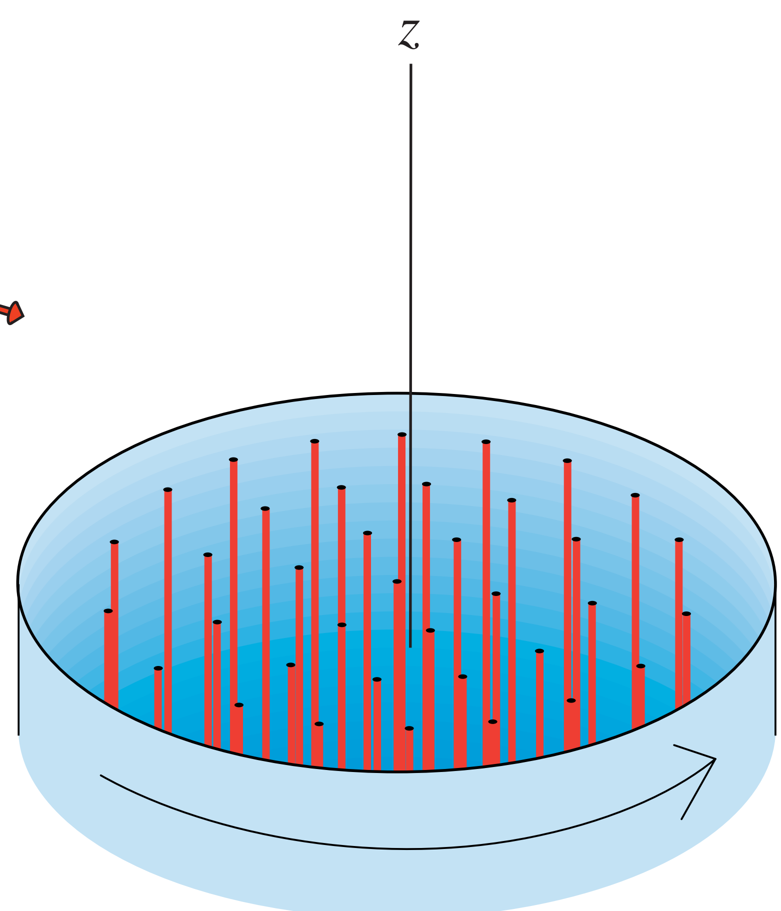
skyrmion in p-space

3D matter with Weyl points:
Topologically protected flat band in vortex core



at each p_z between two values:
2D topological Hall insulator:
zero energy states $E\,(p_z)$=0,
in the vortex core
(1D flat band **1011.4665**)

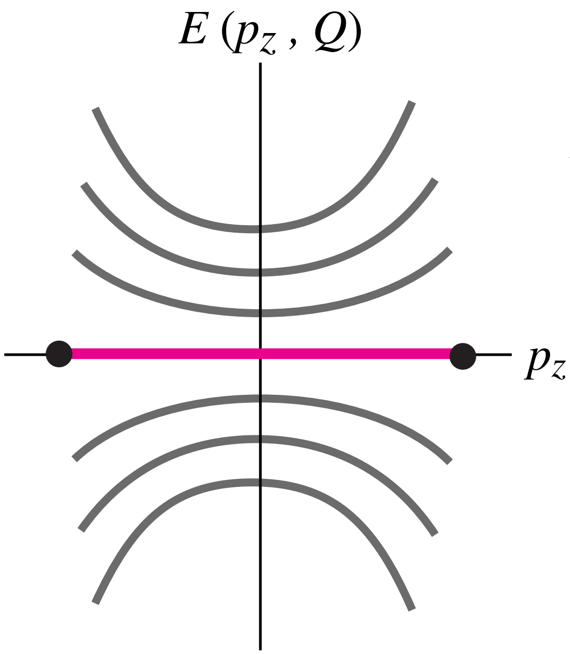
vortices in **r**-space



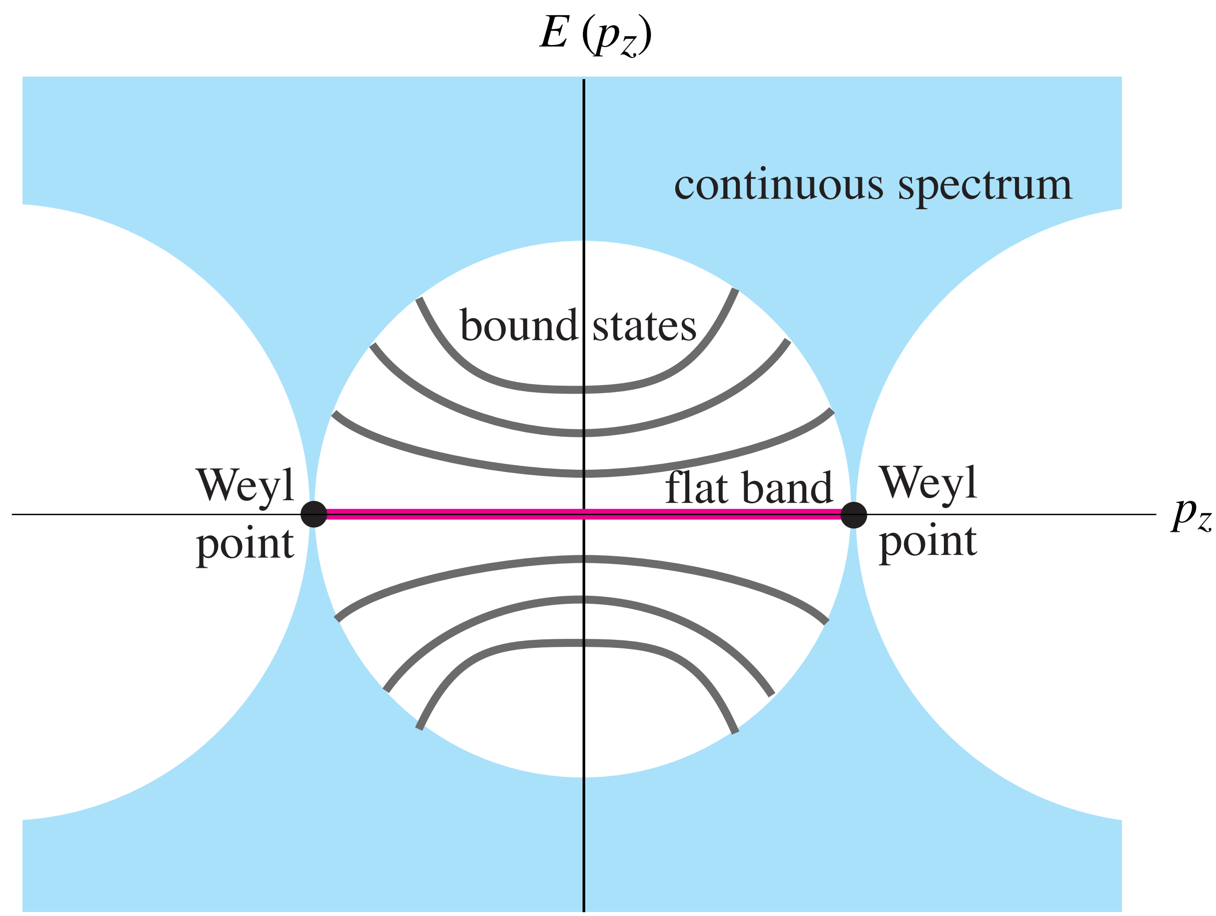
$$\tilde{N}_3(p_z) = \frac{1}{4\pi^2}\,\mathbf{tr}\int dp_x\, dp_y d\omega\; \mathbf{G}\,\nabla_\omega\, \mathbf{G}^{-1}\, \mathbf{G}\,\nabla_{p_x}\, \mathbf{G}^{-1}\mathbf{G}\,\nabla_{p_y}\, \mathbf{G}^{-1}$$

GV & Yakovenko (1989)

Topologically protected flat band in vortex core of superfluids with Weyl points



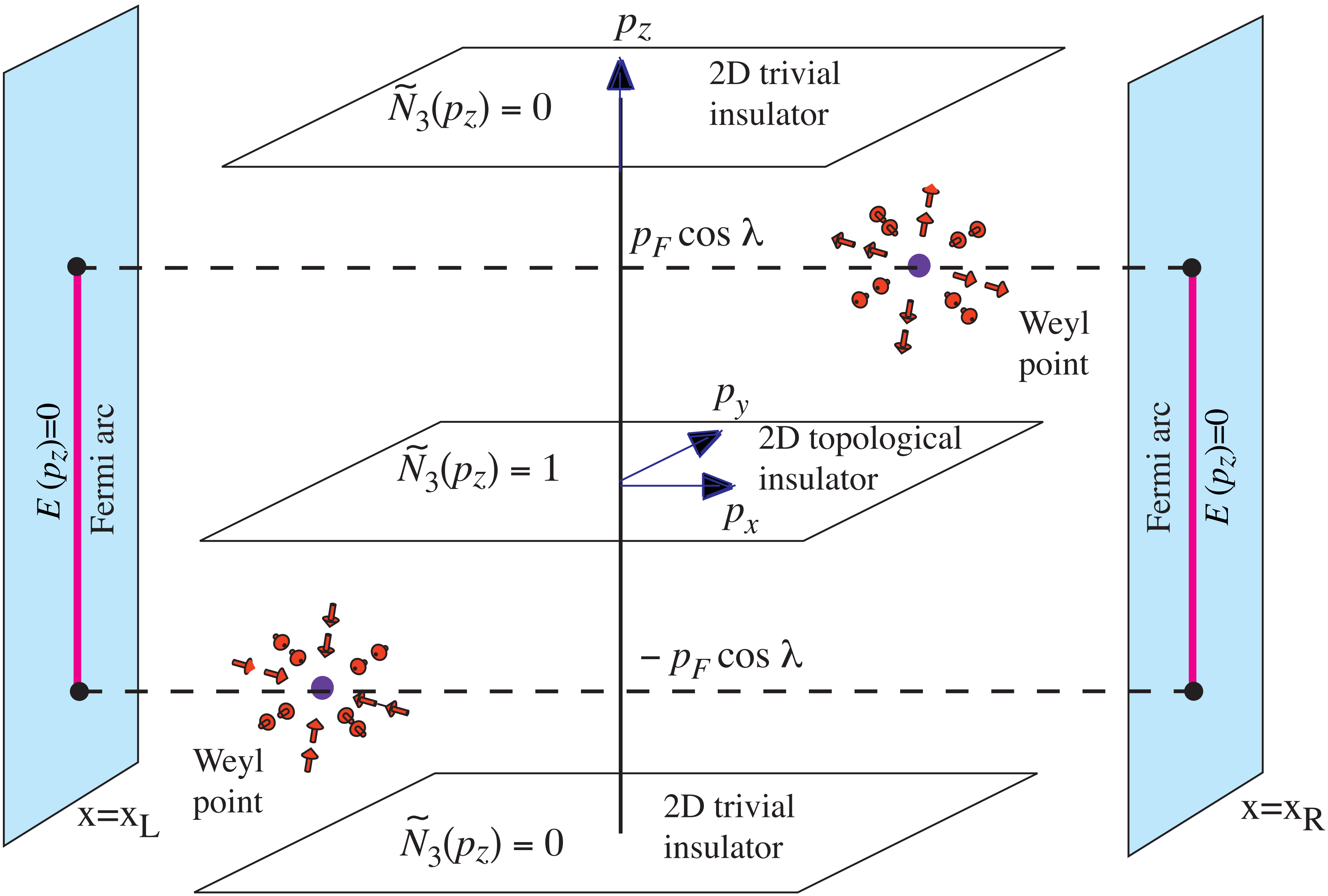
flat band
in spectrum of fermions
bound to core of 3He-A vortex
(Kopnin-Salomaa 1991)



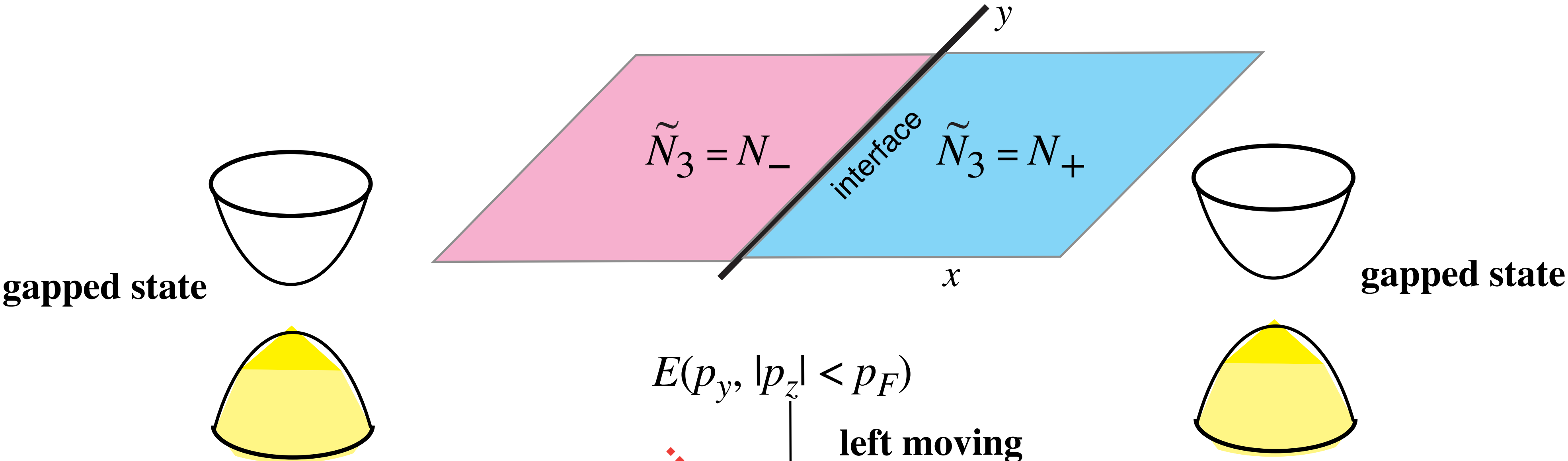
flat band of bound states
terminates on zeroes
of continuous spectrum
(i.e on Weyl points)

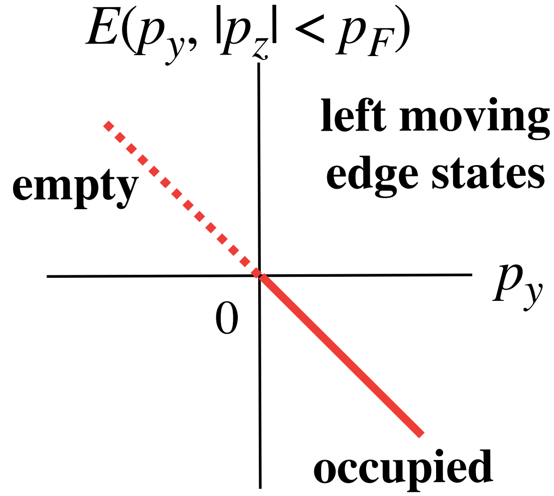
3He-A with Weyl points: Topologically protected Dirac valley (Fermi arc) on surface

for each $|p_z| < p_F \cos\lambda$
one has 2D topological Hall insulator with
zero energy edge states $E(p_z)=0$
(Dirac valley **PRB 094510** or Fermi arc **PRB 205101**)



Edge states at interface between effective two 2+1 topological insulators & Fermi arc





Index theorem:
number of fermion zero modes at interface:

$$\nu = N_+ - N_-$$

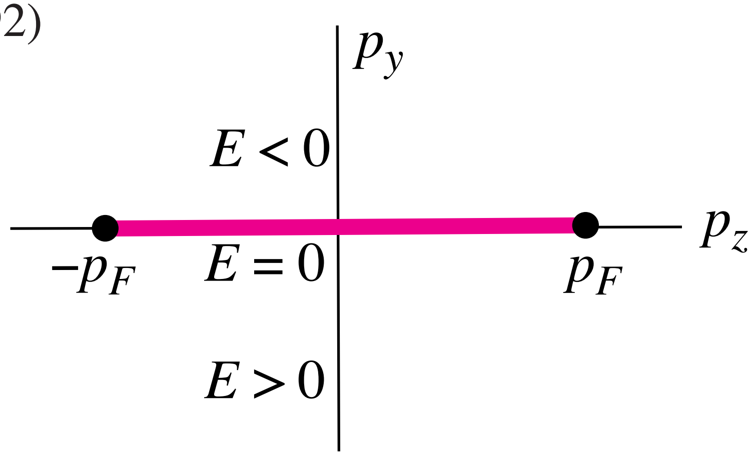
GV JETP Lett. **55**, 368 (1992)

on the edge of insulator with

$$\widetilde{N}_3 = 1$$

one fermion zero mode

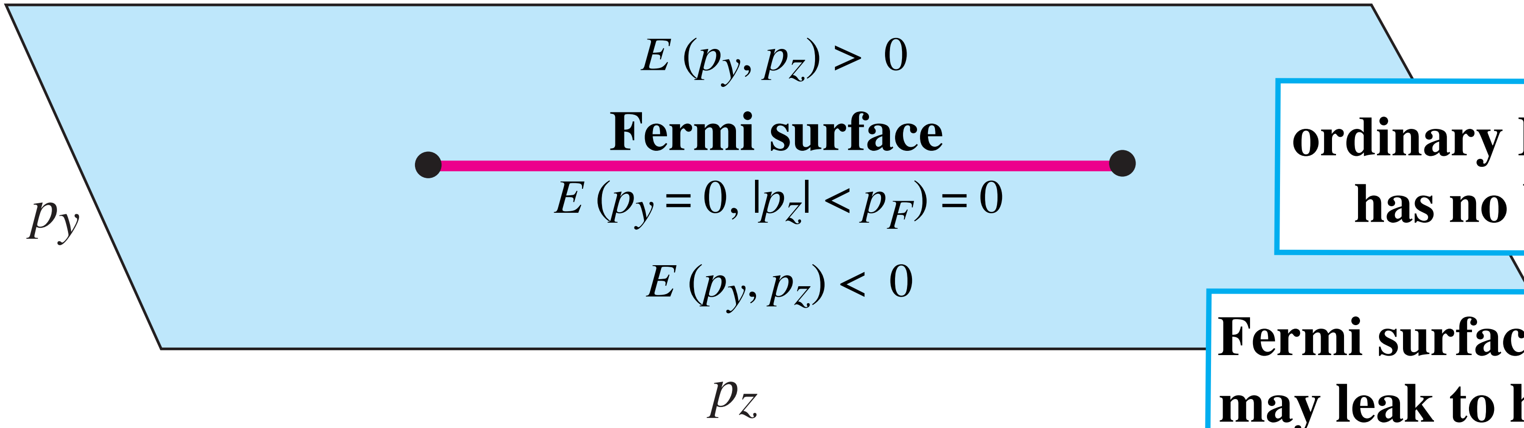
$$\nu = 1$$



Fermi arc in 2D:
Fermi surface which terminates on two points:
projections of Weyl points

Fermi arc:

Fermi surface separates positive and negative energies, but has boundaries

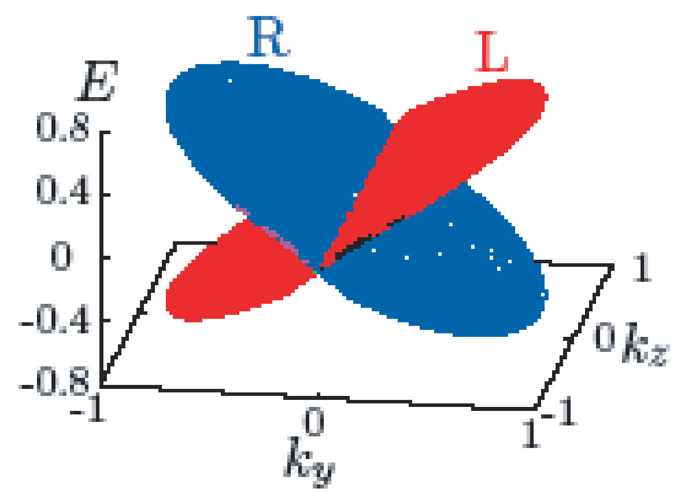


ordinary Fermi surface has no boundaries

Fermi surface of bound states may leak to higher dimension

Fermi surface of localized states is terminated by projections of Weyl points when localized states merge with continuous spectrum

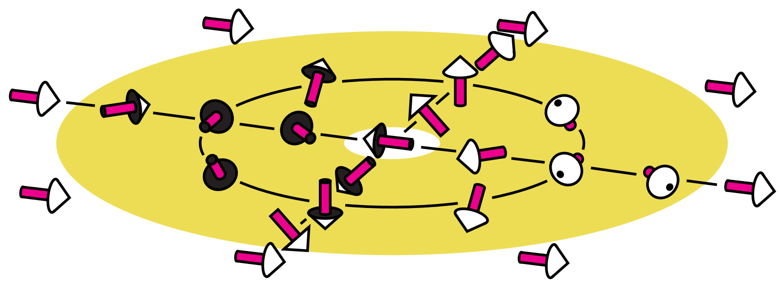
L spectrum of edge states on left wall



Tsutsumi, et al **PRB 094510**

R spectrum of edge states on right wall

5. Fully gapped topological matter



skyrmions in p-space

emergent relativistic fermions as edge states

3+1vacuum with massless fermions

↓ **dimensional reduction** ↓

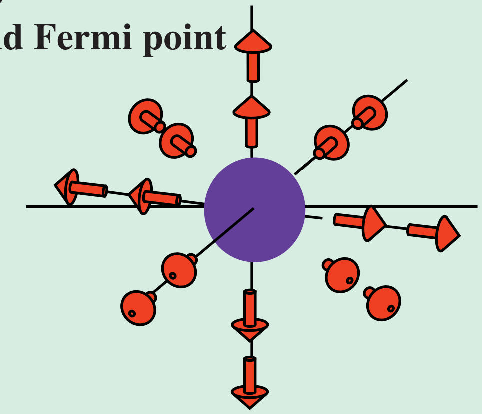
Fully gapped 2+1 vacuum

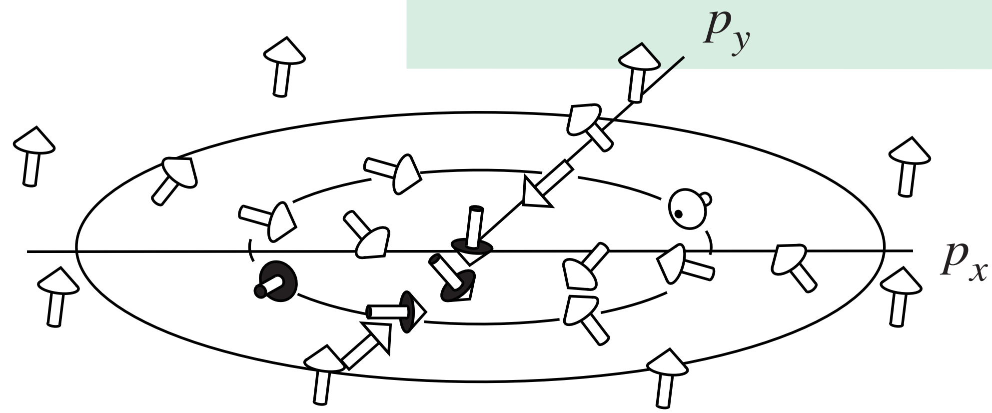
↓ **dimensional reduction** ↓

gapless 1+1 edge states

$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \ \hat{\mathbf{g}} \cdot (\nabla_{p_i} \hat{\mathbf{g}} \times \nabla_{p_j} \hat{\mathbf{g}})$$

around Fermi point





$$\tilde{N}_3 = \frac{1}{8\pi} e_{ijk} \int dp_x dp_y \ \hat{\mathbf{g}} \cdot (\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}})$$

over the whole 2D momentum space
or over 2D Brillouin zone

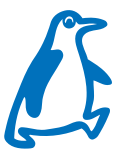
Fully gapped 4+1 vacuum gives 3+1 relativistic fermions (Kaplan, arXiv:1112.0302)

topological insulators & gapped superconductors in 2+1

topological insulator =
bulk insulator
with topologically protected
gapless states on the boundary

topological gapped superconductor =
superconductor with gap in bulk
but with topologically protected
gapless states on the boundary

p-wave 2D superconductor (Sr_2RuO_4 ?), ^{3}He-A thin film,
CdTe/HgTe/Cd insulator quantum well, planar phase film



who protects gapless states?

generic example:

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \qquad p^2 = p_x^{\,2} + p_y^{\,2}$$

How to extract useful information on energy states from this Hamiltonian
without solving equation

$$H\psi = E\psi$$

Topological invariant in momentum space

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$H = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

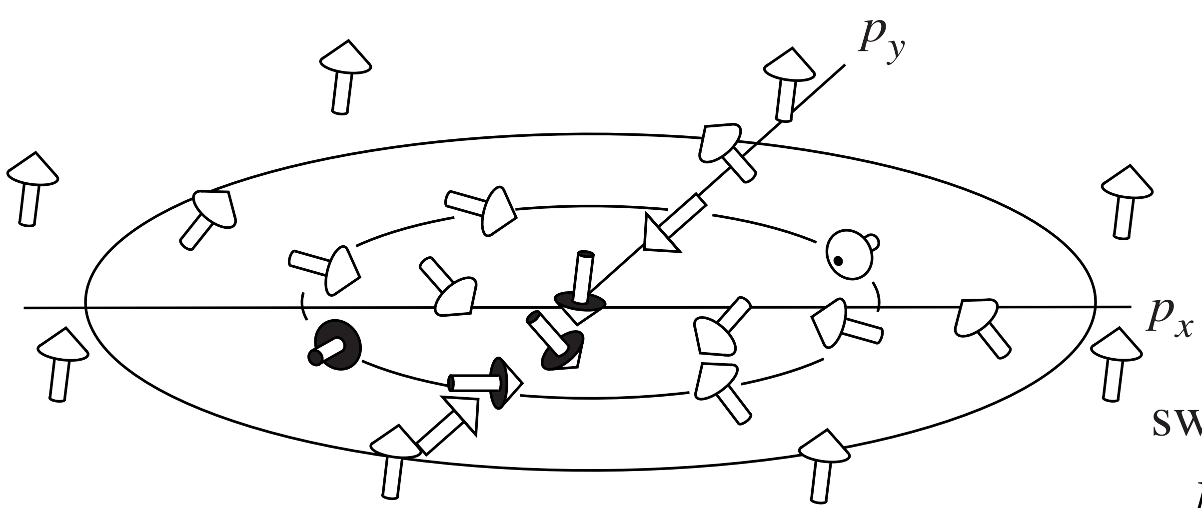
$$p^2 = p_x^{\,2} + p_y^{\,2}$$

fully gapped 2D state at $\mu \neq 0$

$$\tilde{N}_3 = \frac{1}{8\pi} e_{ijk} \int dp_x dp_y \;\; \hat{\mathbf{g}} \cdot (\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}})$$

GV, JETP **67**, 1804 (1988)

Skyrmion (coreless vortex) in momentum space at $\mu > 0$

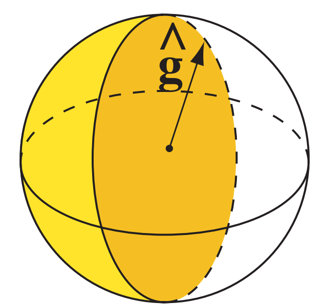


unit vector

$\hat{\mathbf{g}}(p_x, p_y)$

sweeps unit sphere

$\tilde{N}_3\ (\mu > 0) = 1$

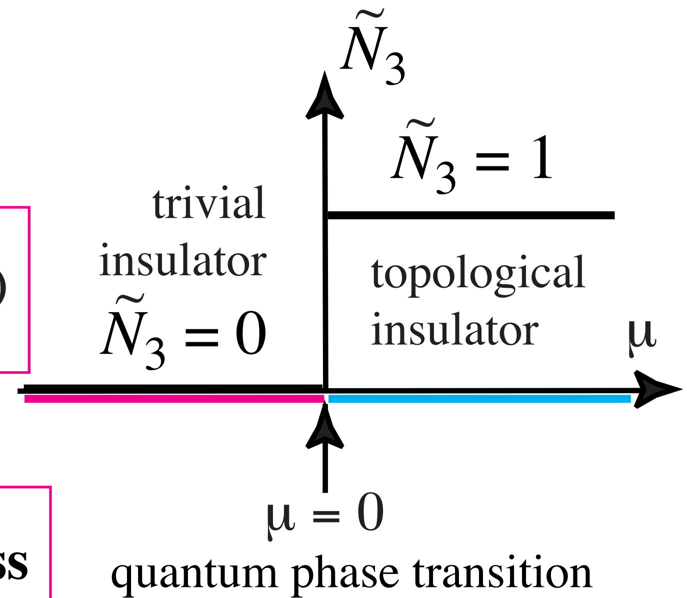


quantum phase transition:
from topological to non-topologicval insulator/superconductor

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

Topological invariant in momentum space

$$N_3 = \frac{1}{8\pi} e_{ijk} \int dp_x dp_y \;\; \hat{\mathbf{g}} \cdot (\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}})$$



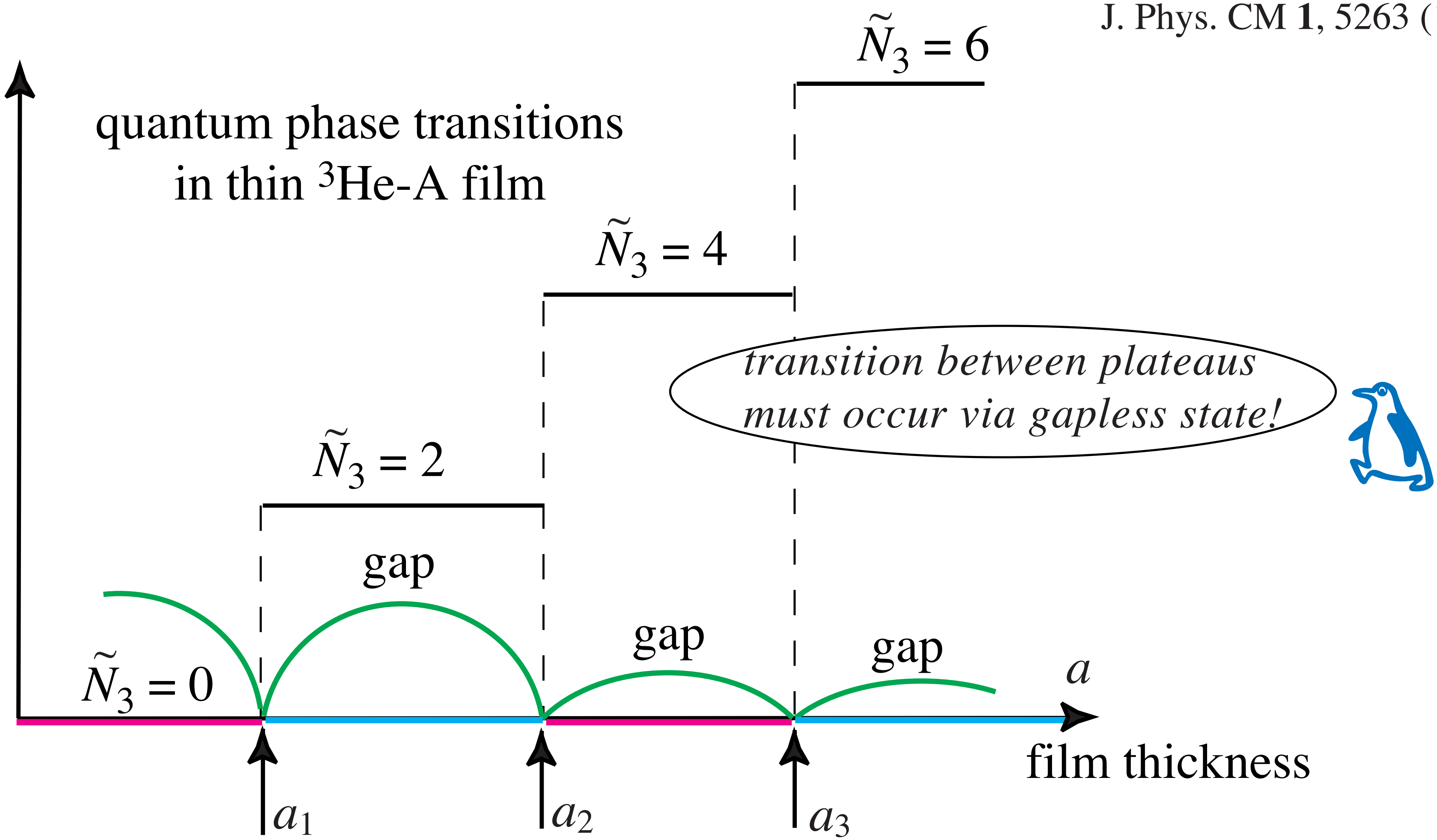
intermediate state at $\mu = 0$ must be gapless

$\Delta\widetilde{N}_3 \neq 0$ is origin of fermion zero modes
at the interface between states with different $\widetilde{N}_3$

p-space invariant in terms of Green's function & topological QPT

$$\tilde{N}_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int d^2p \, d\omega \, \mathbf{G}\nabla^\mu \mathbf{G}^{-1} \mathbf{G}\nabla^\nu \mathbf{G}^{-1} \mathbf{G}\nabla^\lambda \mathbf{G}^{-1}$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)



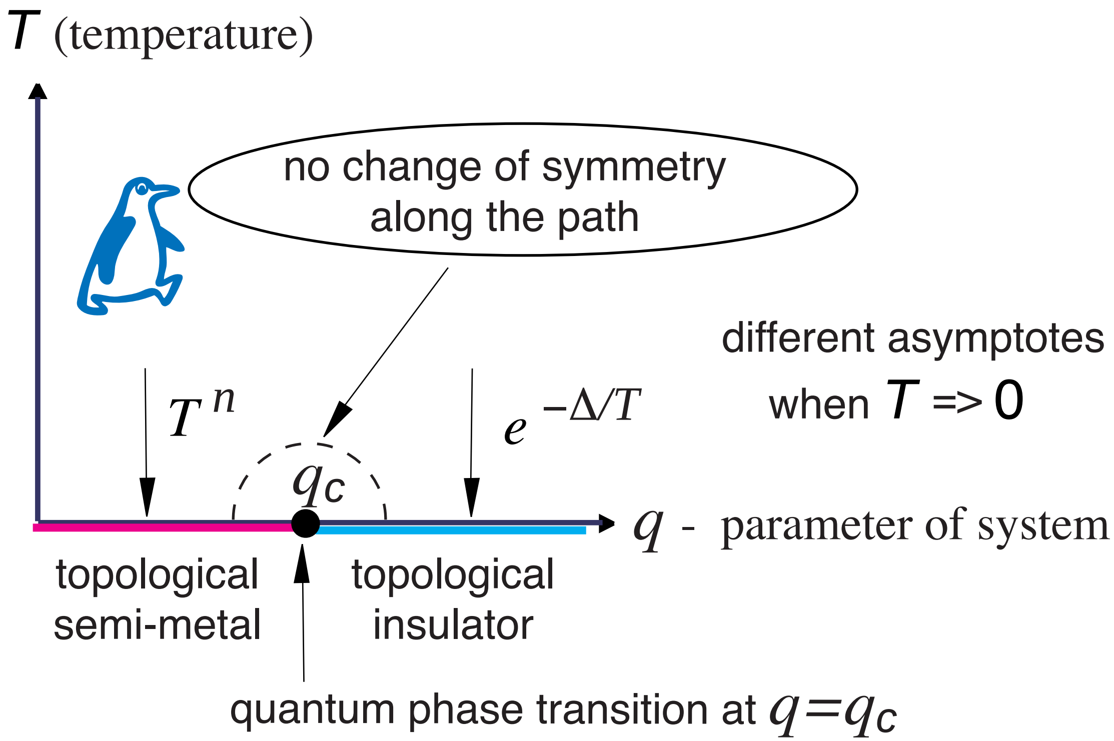
QPT from trivial
to topological state

plateau-plateau QPT
between topological states

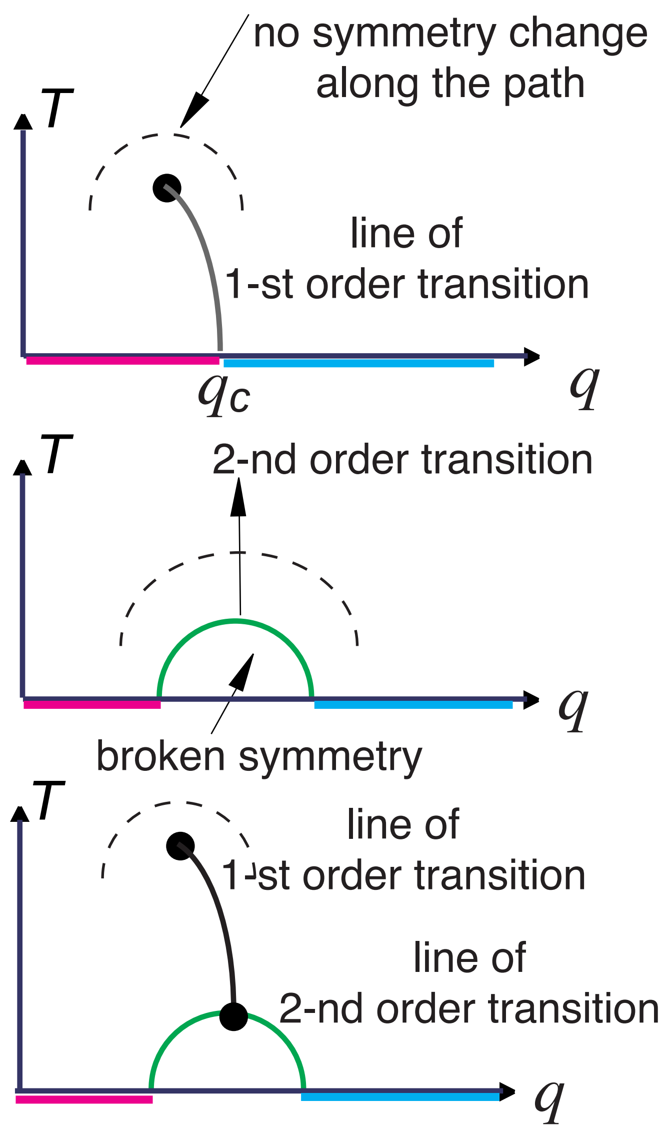
topological quantum phase transitions

transitions between **ground states (vacua)** of the **same symmetry**,
but **different topology** in **momentum space**

example: QPT between gapless & gapped matter

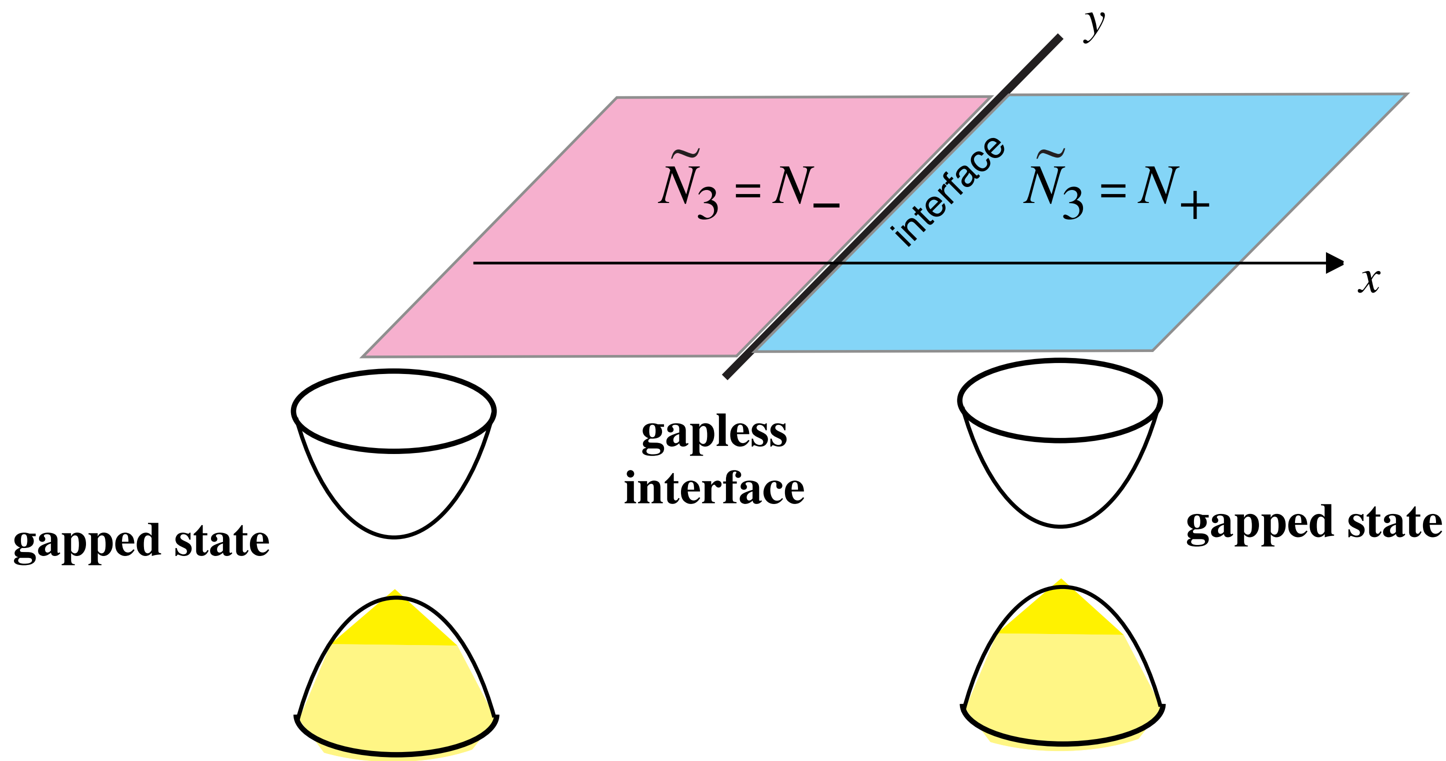


QPT interrupted by thermodynamic transitions



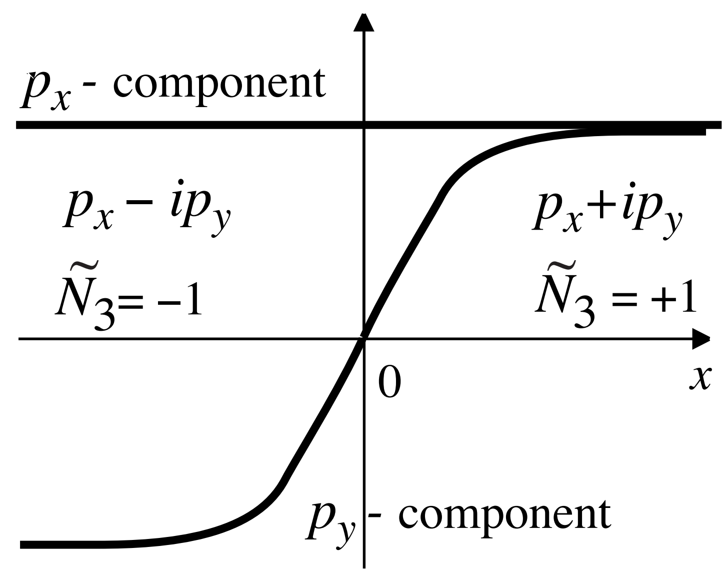
other topological QPT:
Lifshitz transition,
transtion between topological and nontopological superfluids,
plateau transitions,
confinement-deconfinement transition, ...

interface between two 2+1 topological insulators or gapped superfluids

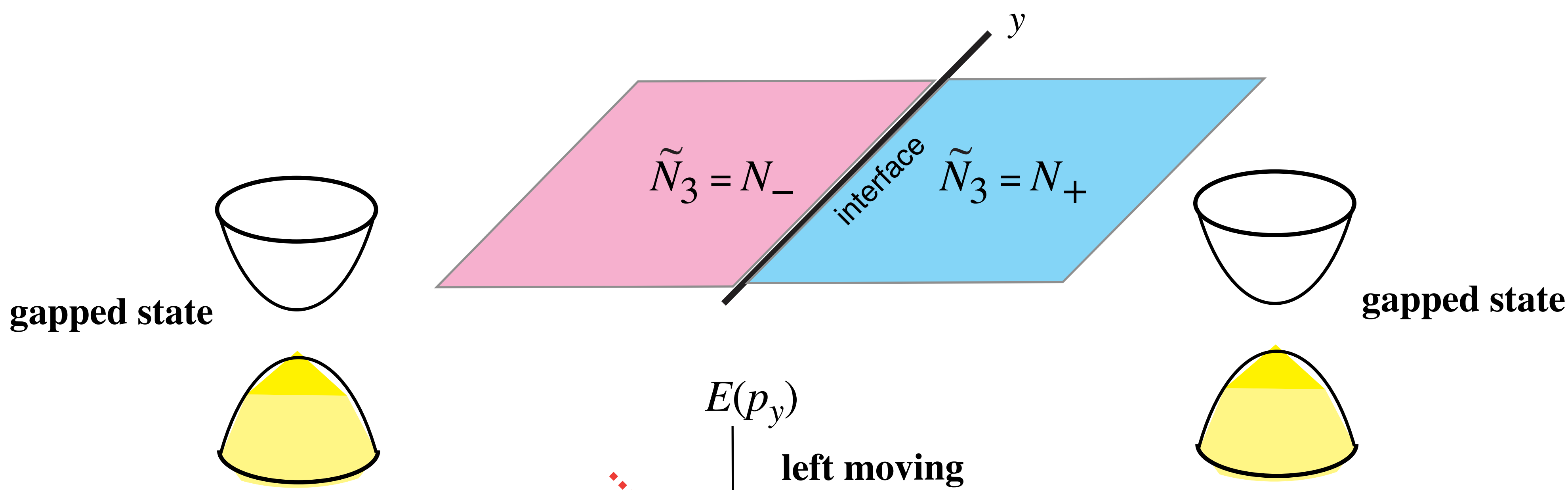


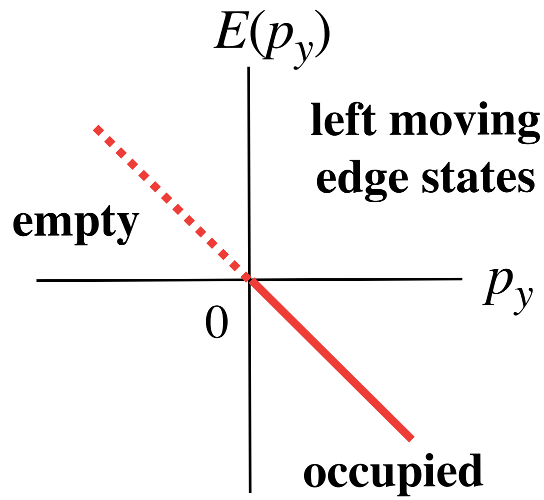
* domain wall in 2D chiral superconductors:

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\, p_y \tanh x) \\ c(p_x - i\, p_y \tanh x) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$



Edge states at interface between two 2+1 topological insulators or gapped superfluids



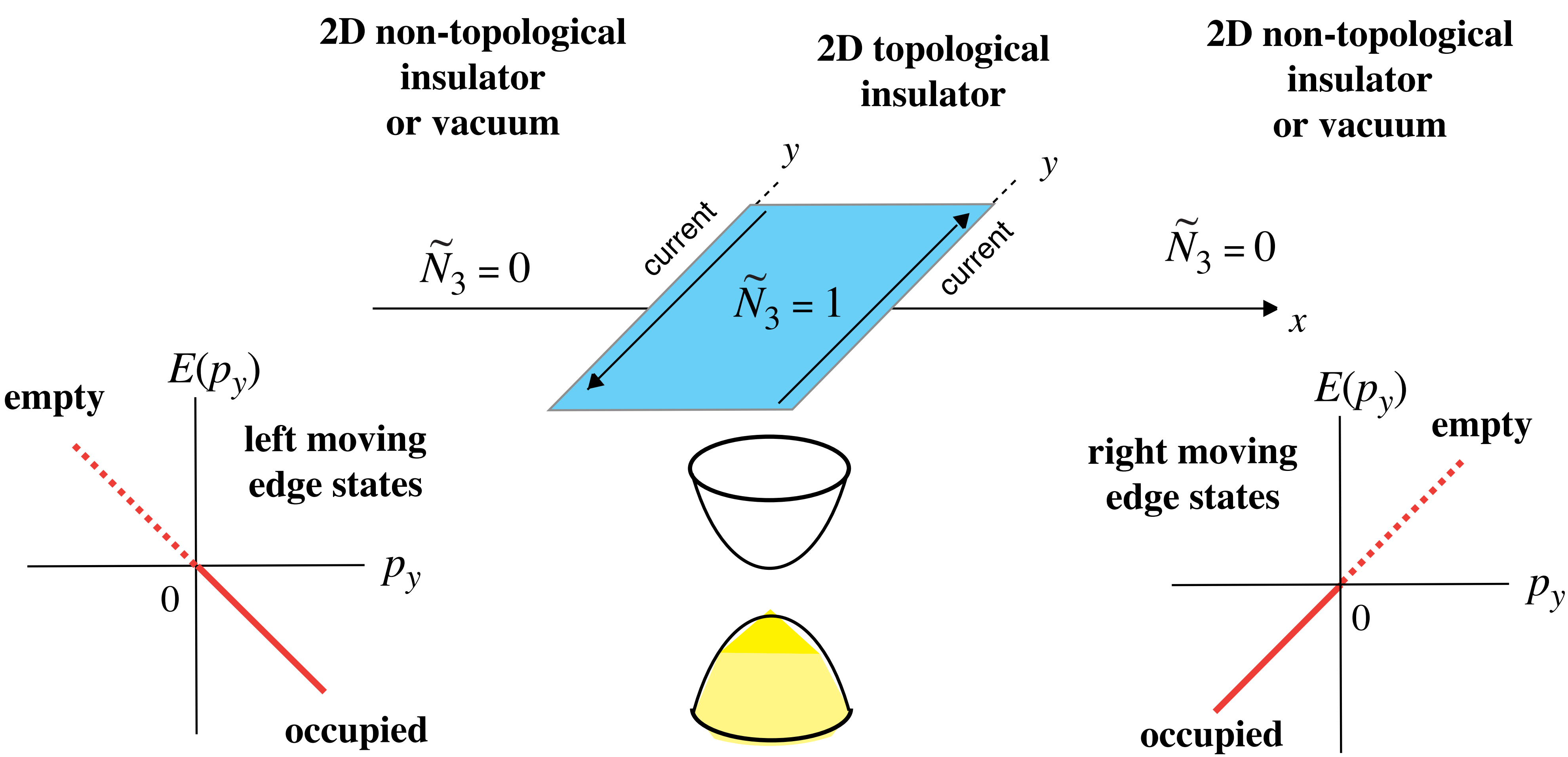


GV JETP Lett. **55**, 368 (1992)

Index theorem:
number of fermion zero modes
at interface:

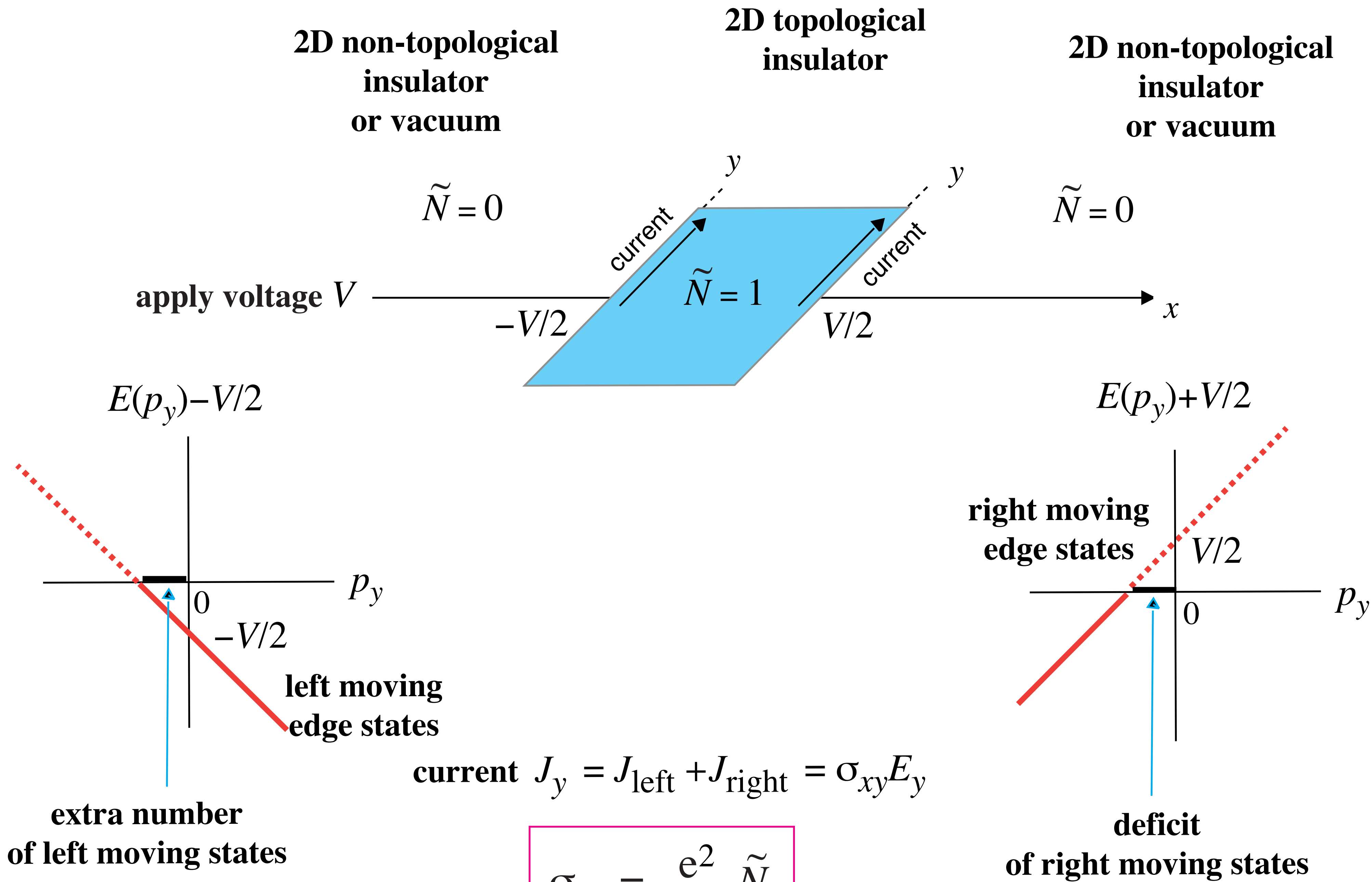
$$\nu = N_+ - N_-$$

Edge states and currents



current $J_y = J_{\text{left}} + J_{\text{right}} = 0$

Edge states & intrinsic QHE: topological invariant determines Hall quantization



current $J_y = J_{\text{left}} + J_{\text{right}} = \sigma_{xy} E_y$

$$\sigma_{xy} = \frac{e^2}{4\pi} \tilde{N}$$

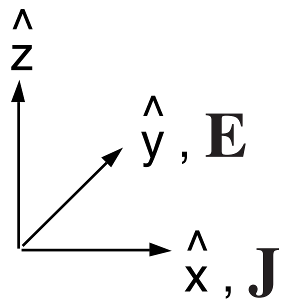
Intrinsic quantum Hall effect & momentum-space invariant

$$S_{CS} = \frac{e^2}{16\pi} \widetilde{N}_3 e^{\mu\nu\lambda} \int d^2x \, dt \, A_\mu F_{\nu\lambda}$$

p-space invariant ($\widetilde{N}_3$) — **r**-space invariant ($\int d^2x \, dt \, A_\mu F_{\nu\lambda}$)

A_μ - electromagnetic field

electric current $J_x = \delta S_{CS} / \delta A_x = \dfrac{e^2}{4\pi} \widetilde{N}_3 E_y$

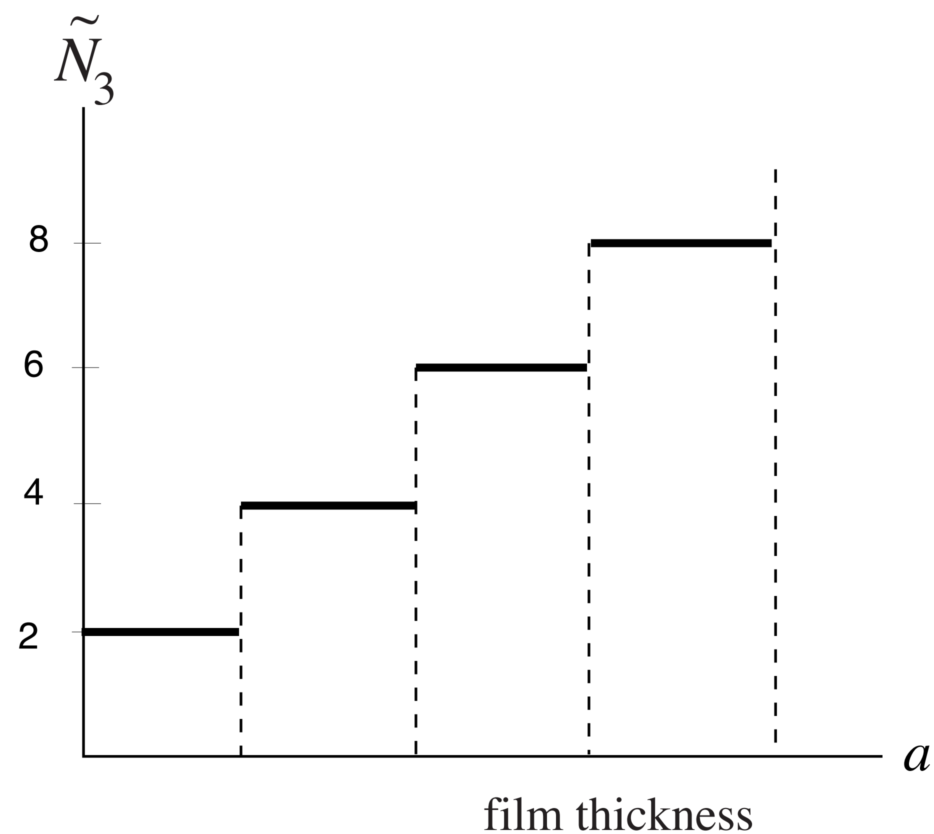


quantized intrinsic Hall conductivity
(without external magnetic field)

$$\sigma_{xy} = \frac{e^2}{4\pi} \widetilde{N}_3$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

film of topological quantum liquid



general Chern-Simons terms & momentum-space invariant

(interplay of ***r***-space and ***p***-space topologies)

$$S_{CS} = \frac{1}{16\pi} \tilde{N}_{3IJ}\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu^{\ I} F_{\nu\lambda}^{\ J}$$

***r*-space invariant**

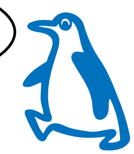
***p*-space invariant protected by symmetry**

dimensional reduction
of chiral anomaly in 3+1

$$\tilde{N}_{3IJ} = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr} \left[\int d^2p\, d\omega\, K_I K_J\, \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1} \right]$$

K_I - charge interacting with gauge field $A_\mu^{\ I}$

gauge fields can be
real, artificial or auxiliary



$K=e$ for electromagnetic field A_μ

$K=\hat{\sigma}_z$ for effective spin-rotation field $A_\mu^{\ z}$ $\quad (A_0^{\ z} = \gamma H^{\ z})$

$$id/dt - \gamma\hat{\sigma}\cdot\mathbf{H} = id/dt - \hat{\sigma}\cdot\mathbf{A}_0$$

applied Pauli magnetic field plays the role of components of effective SU(2) gauge field $A_\mu^{\ i}$

Intrinsic spin-current quantum Hall effect & momentum-space invariant

spin current $J_x^z = \frac{1}{4\pi}(\gamma N_{ss}\, dH^z/dy + N_{se} E_y)$

spin-spin QHE (points to $\gamma N_{ss}\, dH^z/dy$)

spin-charge QHE (points to $N_{se} E_y$)

2D singlet superconductor:

$$\sigma_{xy}^{spin/spin} = \frac{N_{ss}}{4\pi}$$

s-wave: $N_{ss} = 0$

$p_x + ip_y$: $N_{ss} = 2$

$d_{xx-yy} + id_{xy}$: $N_{ss} = 4$

film of planar phase of superfluid ^{3}He

$$\sigma_{xy}^{spin/charge} = \frac{N_{se}}{4\pi}$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

spin quantum Hall effect: planar phase film of 3He & 2D topological insulator

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\, p_y \sigma_z) \\ c(p_x - i\, p_y \sigma_z) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$\tilde{N}_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr} \left[\int d^2p\, d\omega\, \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1} \right] = 0$$

$$\tilde{N}_{se} = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr} \left[\int d^2p\, d\omega\, \sigma_z \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1} \right]$$

$$\tilde{N}_3^+ = +1 \qquad \tilde{N}_3^- = -1$$

$$\tilde{N}_3 = \tilde{N}_3^+ + \tilde{N}_3^- = 0 \qquad \tilde{N}_{se} = \tilde{N}_3^+ - \tilde{N}_3^- = 2$$

spin quantum Hall effect

spin current $J_x^z = \frac{1}{4\pi} N_{se} E_y$

spin-charge QHE

$$\sigma_{xy}^{spin/charge} = \frac{N_{se}}{4\pi}$$

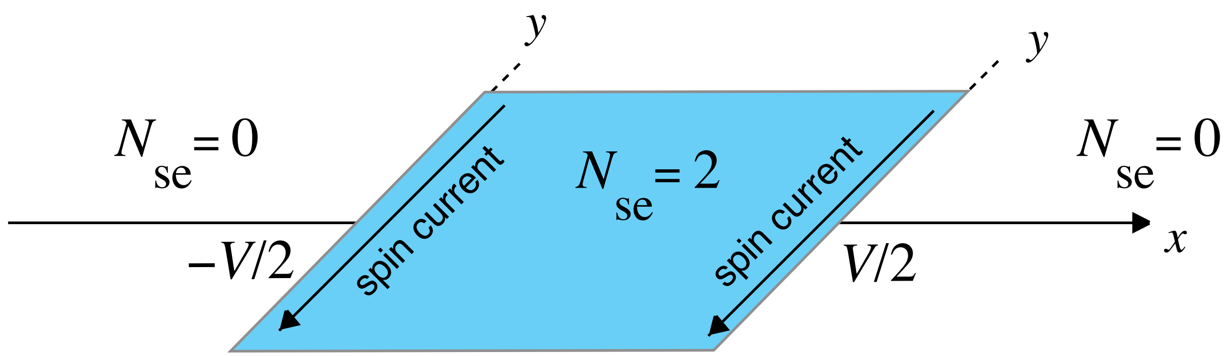
$N_{se} = 2$

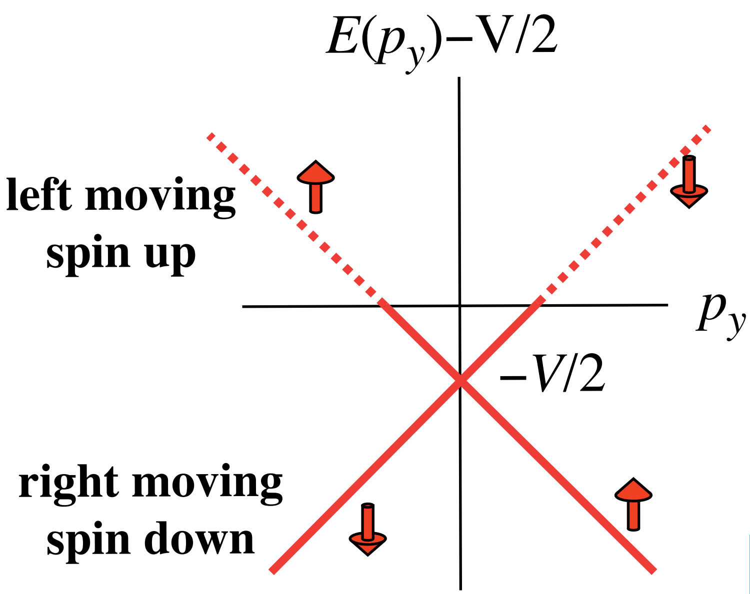
GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

Intrinsic spin-current quantum Hall effect & edge state

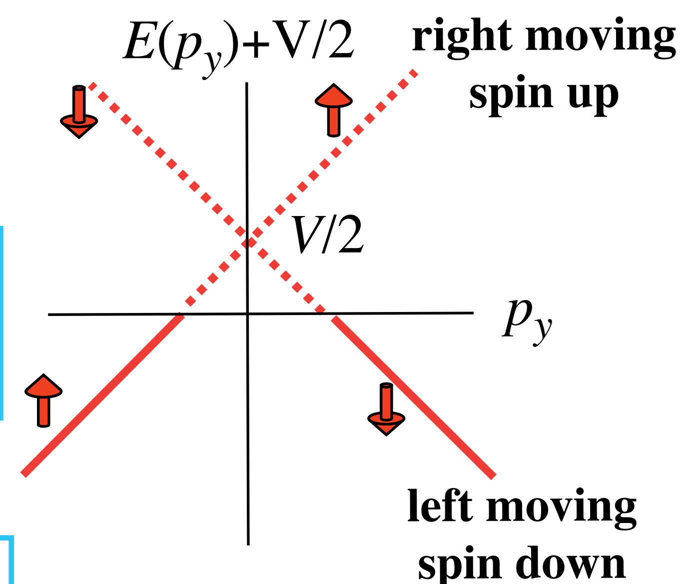
spin current $J_x^{\,z} = \dfrac{1}{4\pi}\left(\gamma N_{ss}\, dH^z/dy + N_{se}\, E_y\right)$

spin-charge QHE





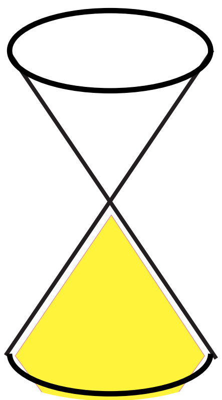
$$\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}$$



electric current is zero
spin current is nonzero

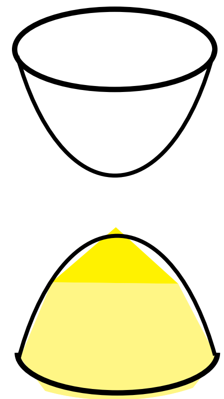
3D topological superfluids / insulators / semiconductors / vacua

gapless topologically
nontrivial vacua

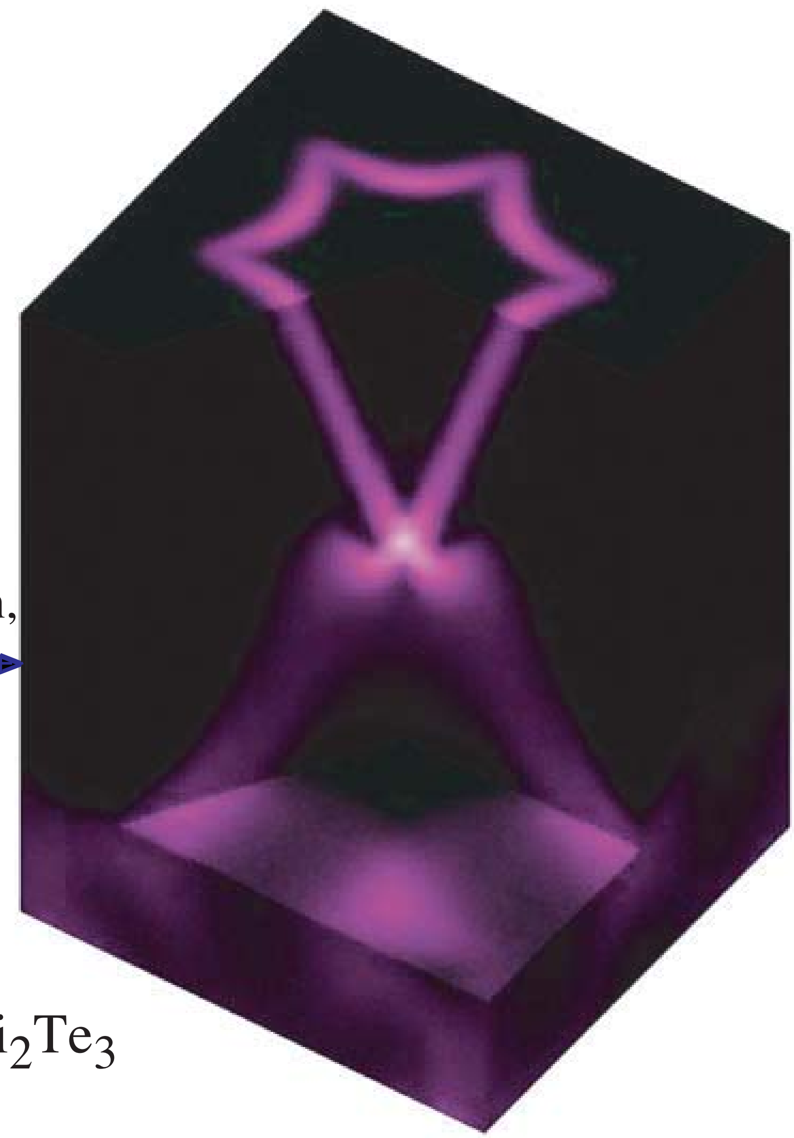


3He-A,
Standard Model
above electroweak transition,
semimetals,
4D graphene
(cryocrystalline vacuum)

fully gapped topologically
nontrivial vacua

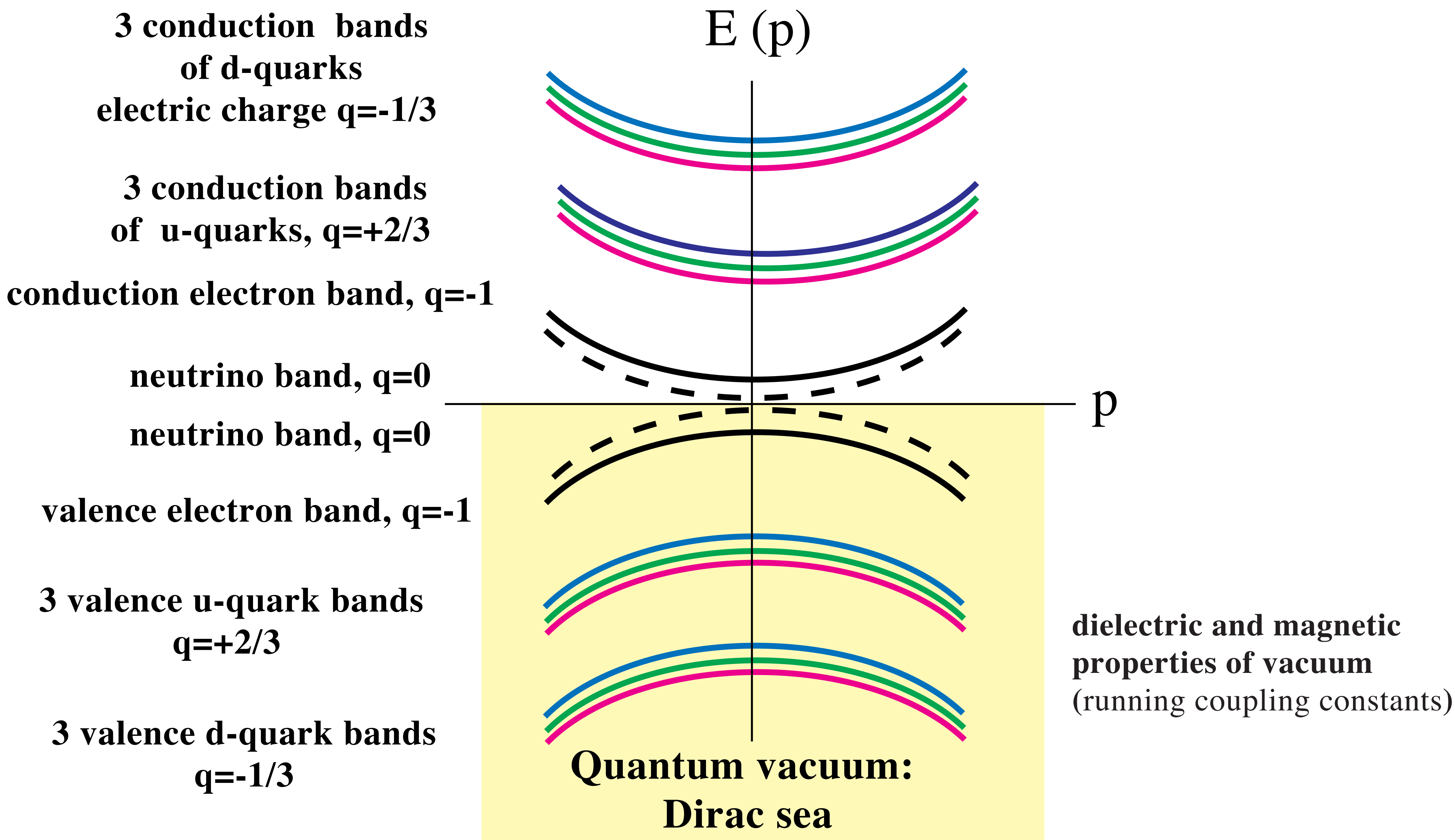


3He-B,
Standard Model
below electroweak transition,
topological insulators,
triplet & singlet
chiral superconductor, ...



Bi_2Te_3

Present vacuum as semiconductor or insulator



electric charge of quantum vacuum

$$Q = \sum_a q_a = N\,[-1 + 3\times(-1/3) + 3\times(+2/3)\,] = 0$$

fully gapped 3+1 topological matter

superfluid ^{3}He-B, topological insulator Bi_2Te_3 , present vacuum of Standard Model

* Standard Model vacuum as topological insulator

Topological invariant protected by symmetry

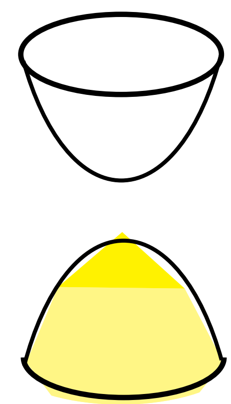
$$N_K = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int dV \, K \, \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \, \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1}$$

over 3D momentum space

G is Green's function at ω=0, K is symmetry operator **GK** =+/− **KG**

Standard Model vacuum: $K=\gamma_5$ $\mathbf{G}\gamma_5 = -\gamma_5\mathbf{G}$

$$N_K = 8n_g$$



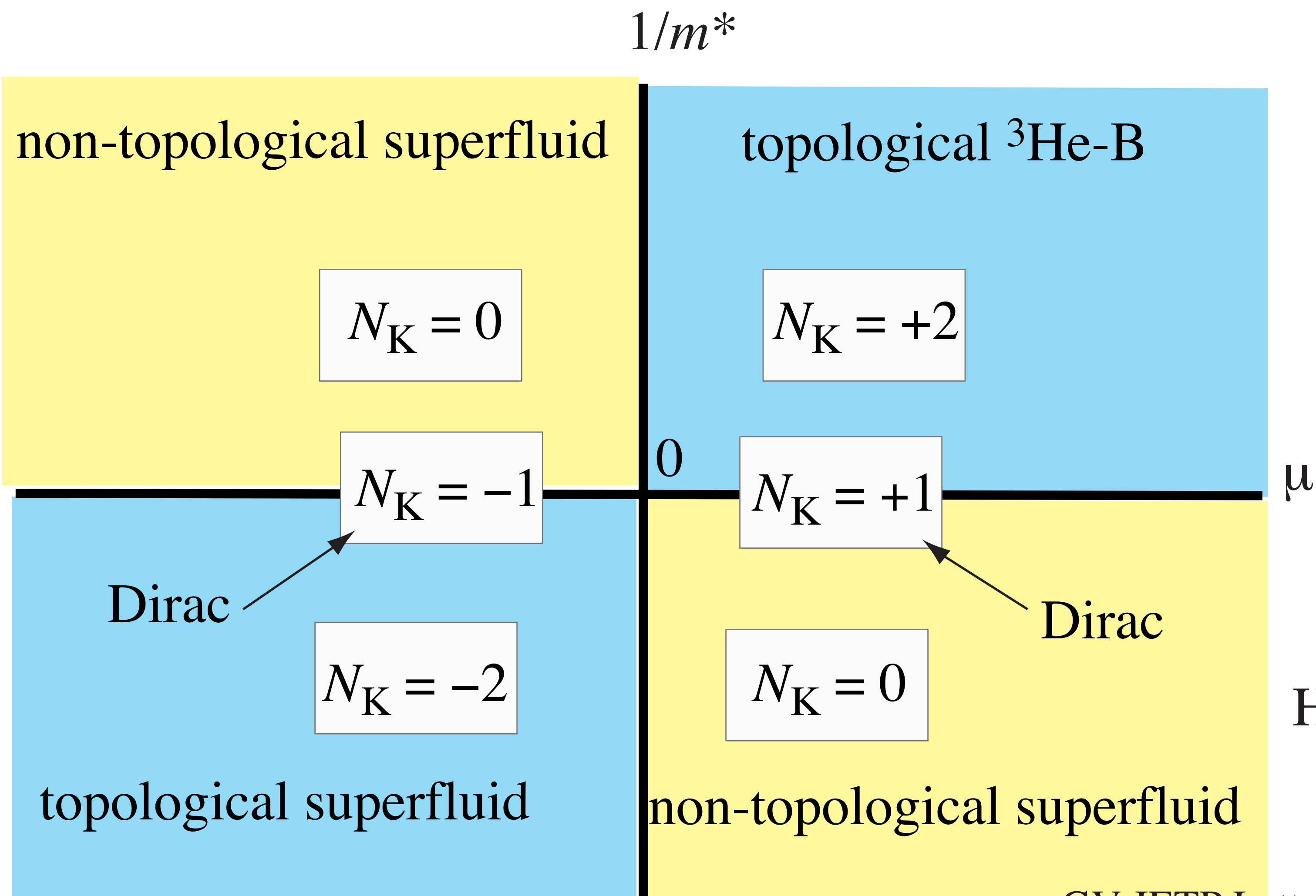
8 massive Dirac particles in one generation

topological superfluid ^{3}He-B

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix} = \left(\frac{p^2}{2m^*} - \mu\right)\tau_3 + c_B \sigma\cdot\mathbf{p}\,\tau_1$$

$$\mathbf{H}\tau_2 = -\tau_2\mathbf{H}$$

$$K = \tau_2$$



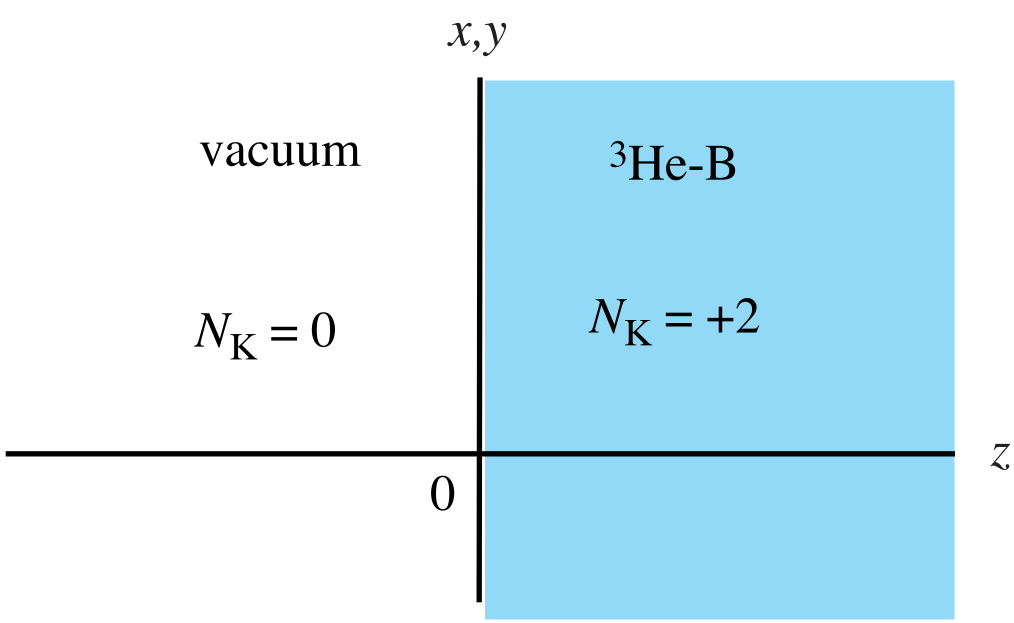
Dirac vacuum

$$1/m^* = 0$$

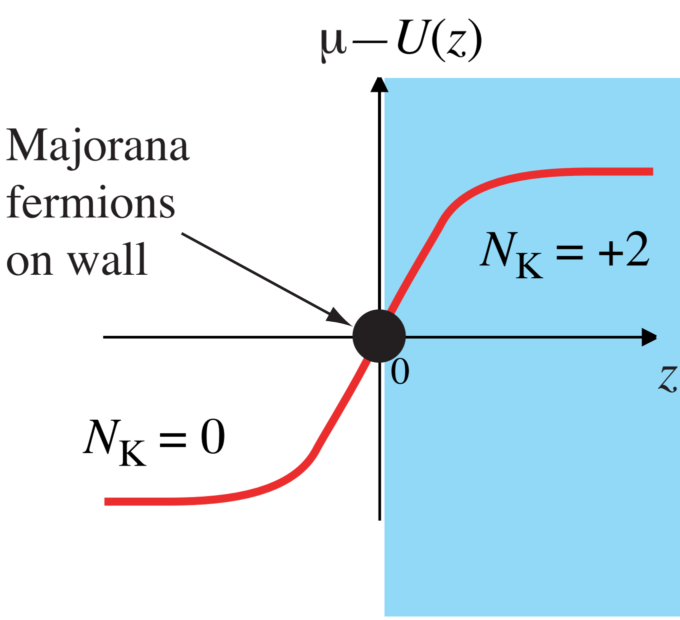
$$H = \begin{pmatrix} -M & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & +M \end{pmatrix}$$

GV JETP Lett. **90**, 587 (2009)

Boundary of 3D gapped topological superfluid



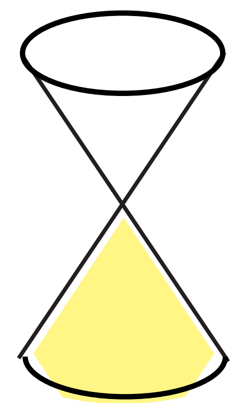
$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$



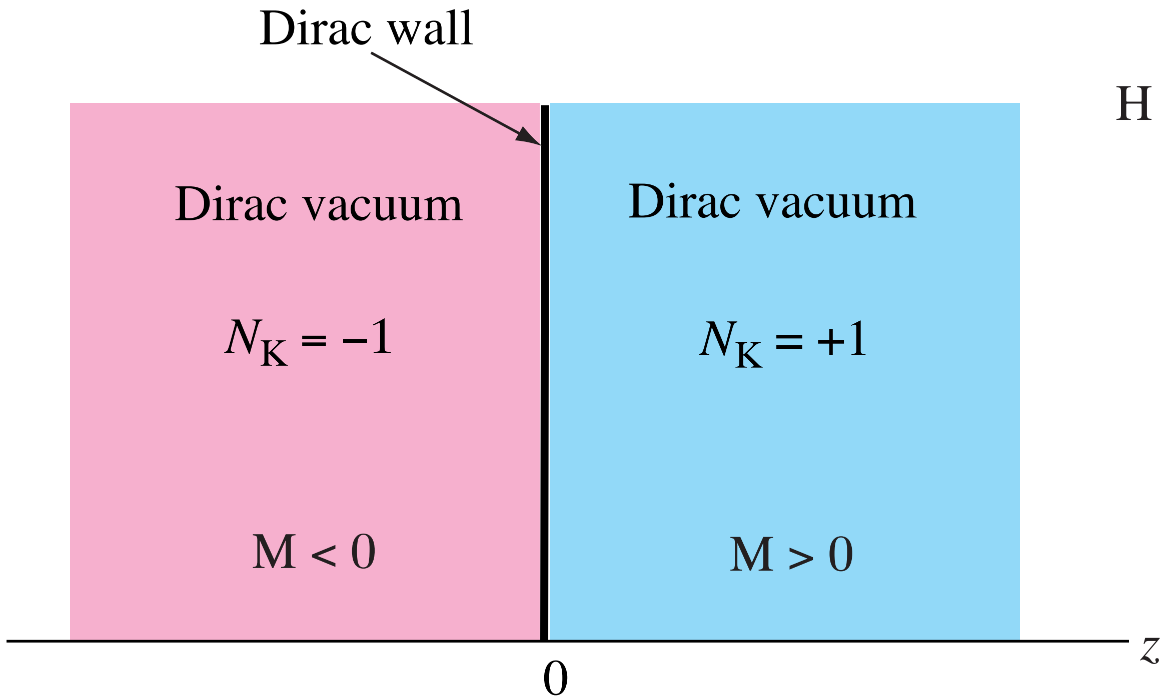
spectrum of Majorana fermion zero modes

$$H_{zm} = c_B\, \hat{\mathbf{z}} \cdot \sigma \times \mathbf{p} = c_B\, (\sigma_x p_y - \sigma_y p_x)$$

helical fermions

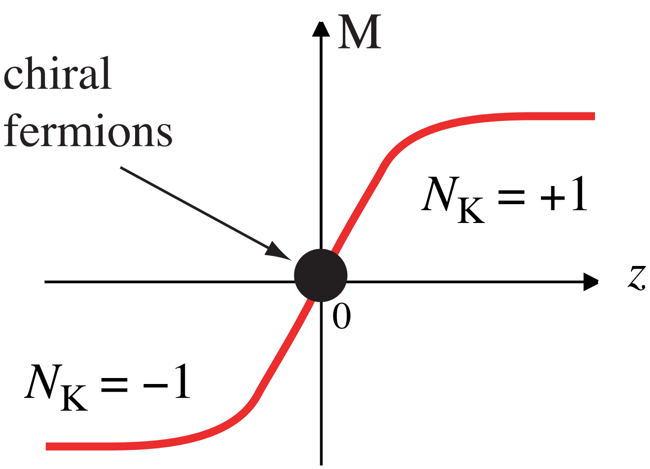


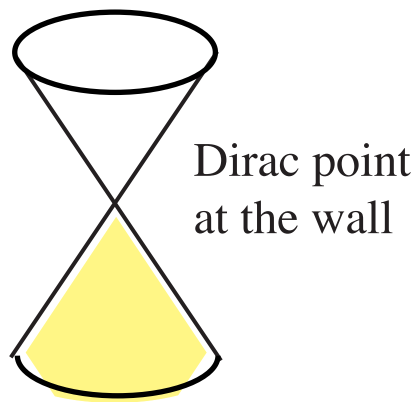
fermion zero modes on Dirac wall

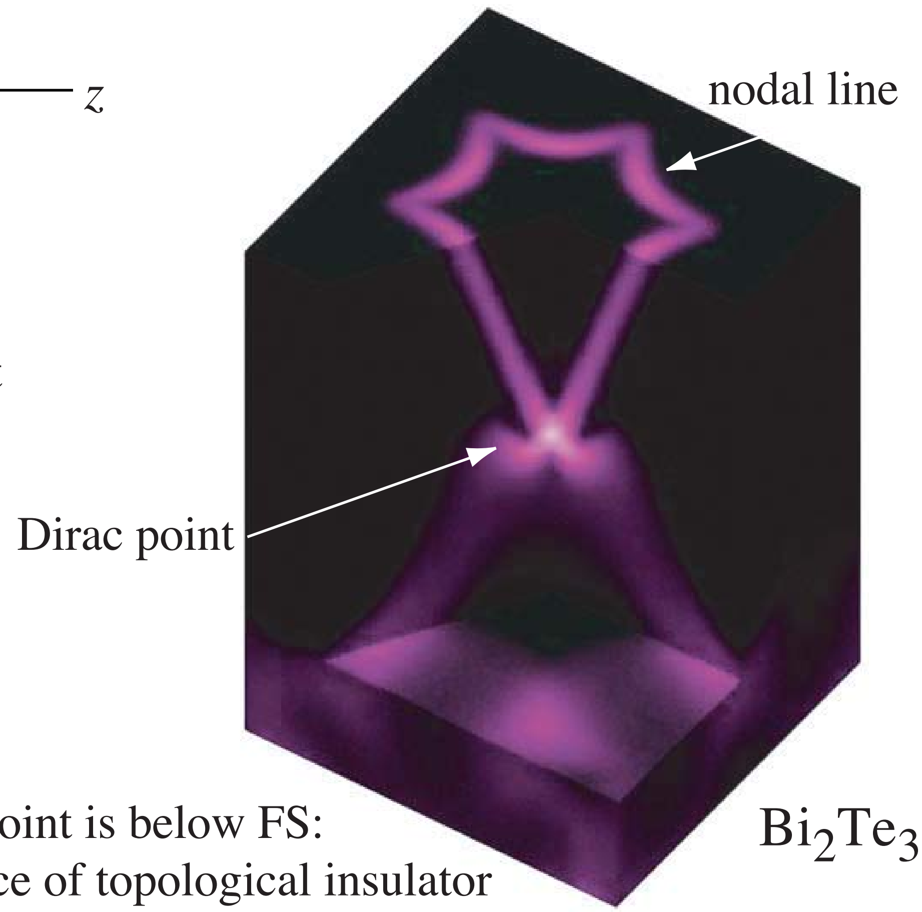


$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)





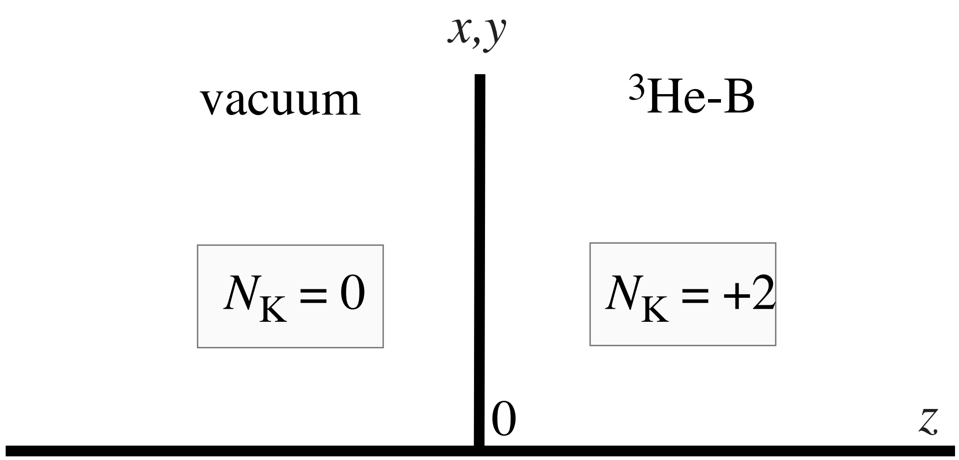


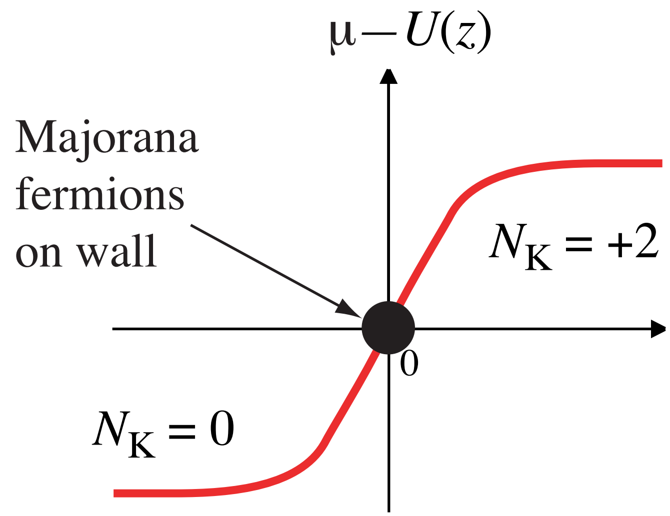
in Bi_2Te_3 Dirac point is below FS:
nodal line on surface of topological insulator

Majorana fermions: edge states
on the boundary of 3D gapped topological matter

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$

* boundary of topological superfluid ^{3}He-B



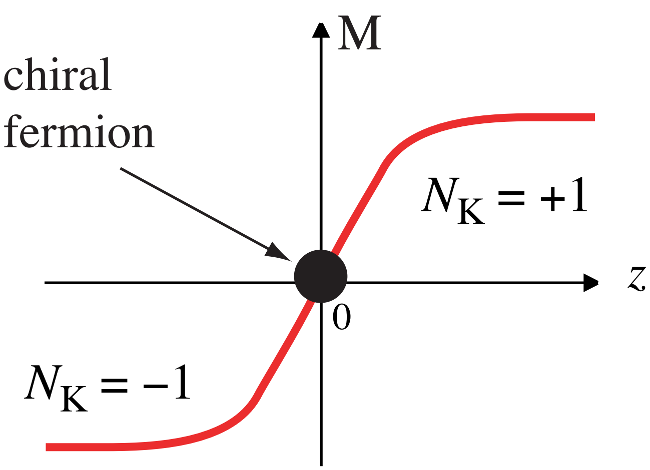


spectrum of fermion zero modes

$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

helical fermions

* Dirac domain wall

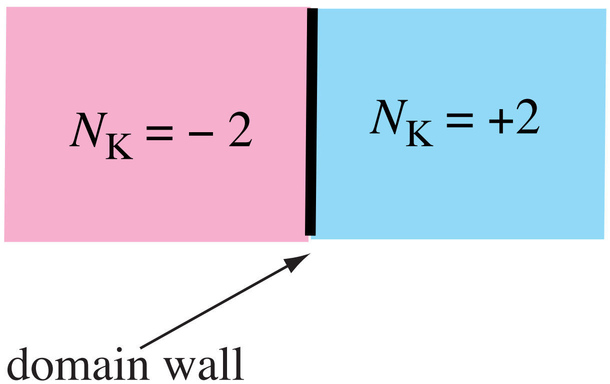


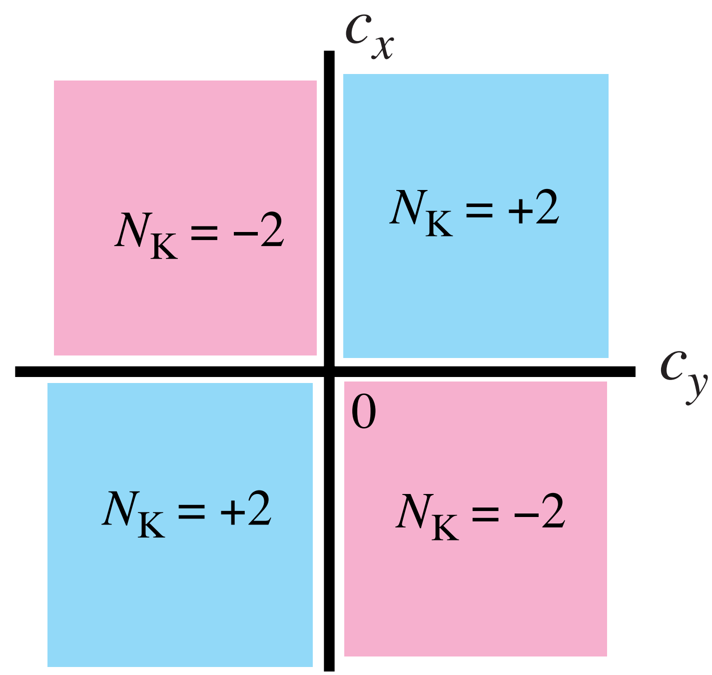
$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)

Majorana fermions on interface in topological superfluid ^{3}He-B

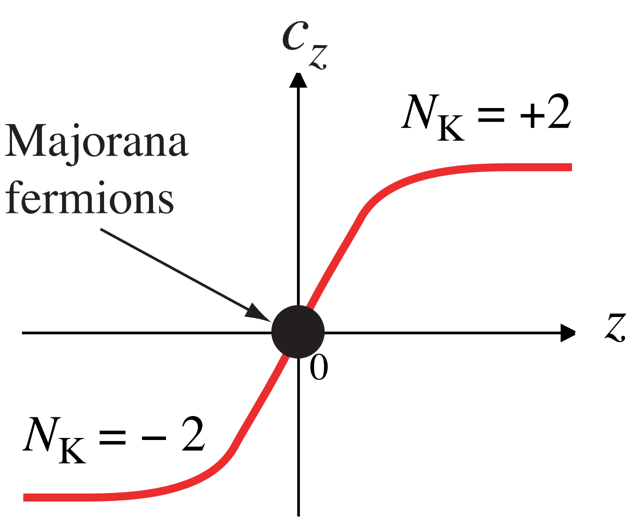
$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z \\ \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z & -\frac{p^2}{2m^*} + \mu \end{pmatrix}$$

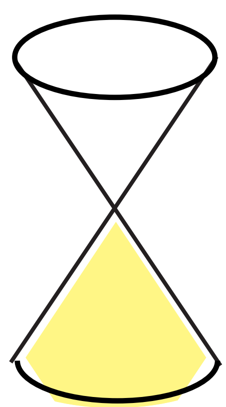




phase diagram

one of 3 "speeds of light" changes sign across wall





spectrum of fermion zero modes

$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

Zero energy states in the core of vortices in topological superfluids

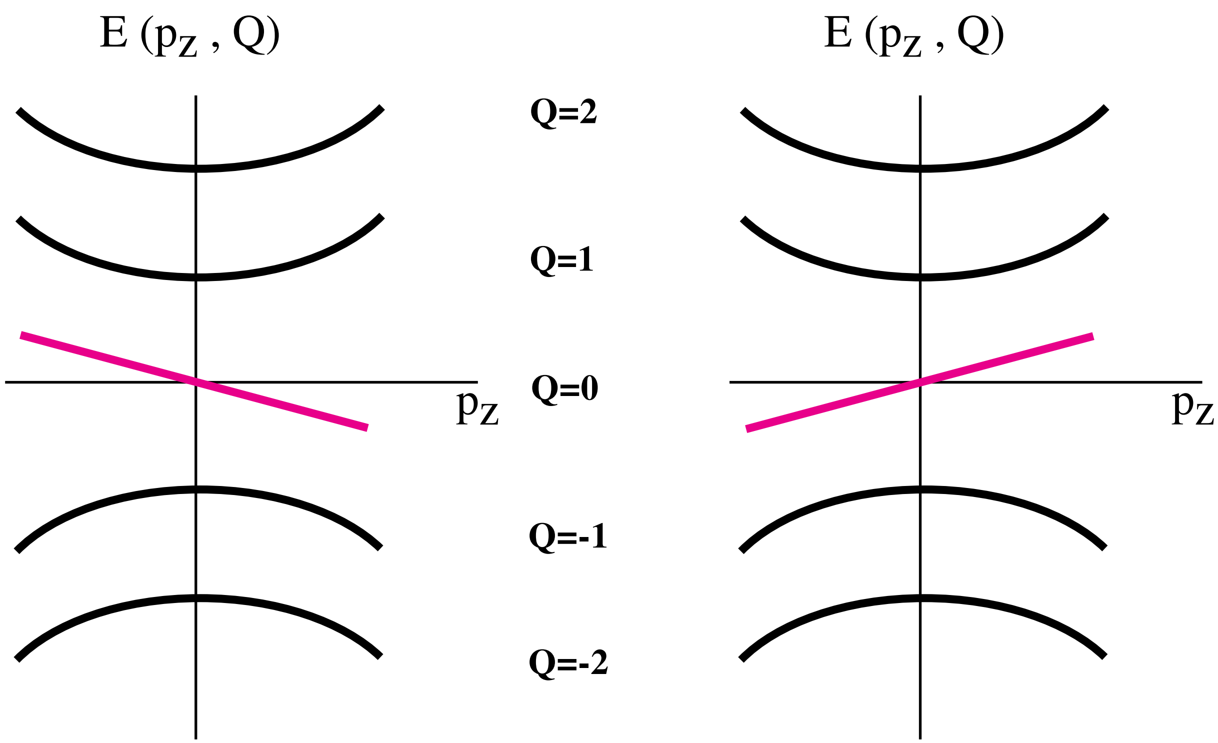
vortices in fully gapped 3+1 system

fermion zero modes in vortex core

Bound states of fermions on cosmic strings and vortices

Spectrum of quarks in core of electroweak cosmic string

quantum numbers: Q - angular momentum & p_z - linear momentum



$E(p_z) = - cp_z$ for d quarks $E(p_z) = cp_z$ for u quark

asymmetric branches cross zero energy

Index theorem:

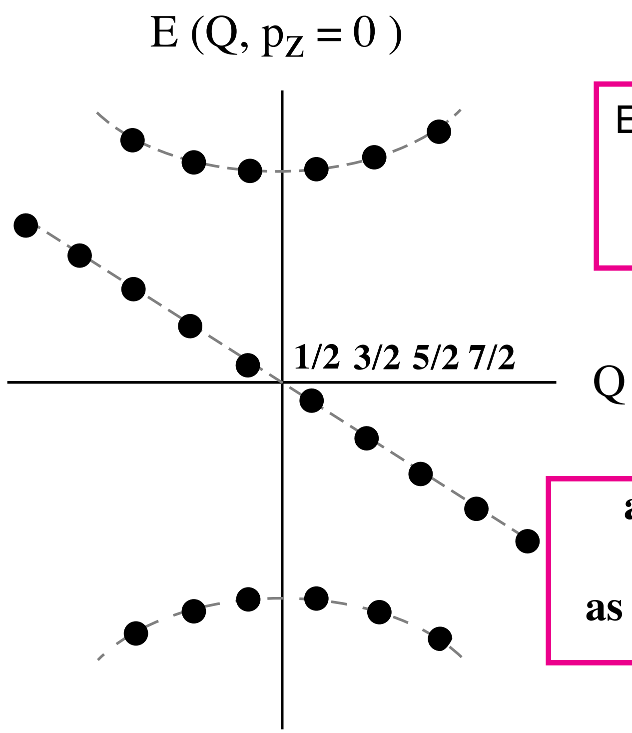
Number of asymmetric branches = N
N is vortex winding number

Jackiw & Rossi
Nucl. Phys. B**190**, 681 (1981)

Bound states of fermions on vortex in s-wave superconductor

Caroli, de Gennes & J. Matricon, Phys. Lett. **9** (1964) 307

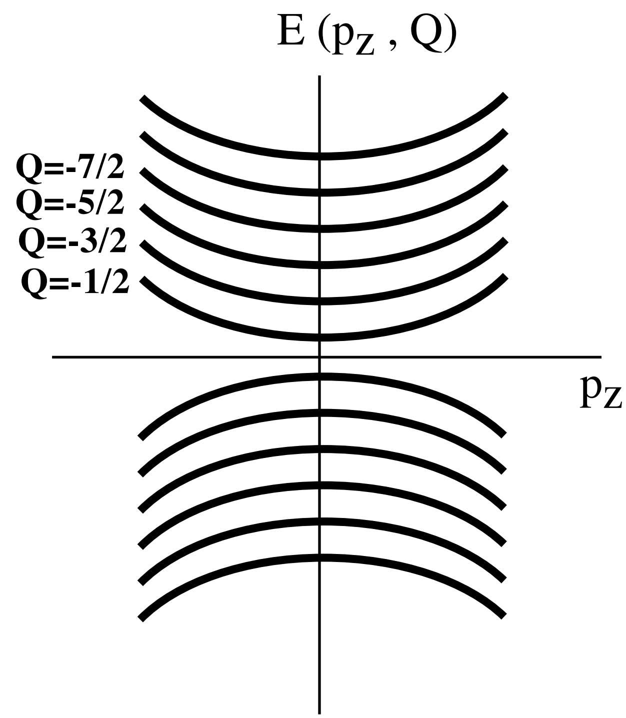
$N_K = 0$



$E(Q,p_z) = -Q\,\omega_0(p_z)$

$\omega_0 = \Delta^2 / E_F \ll \Delta$

asymmetric branch as function of Q



Angular momentum Q is half-odd integer
in s-wave superconductor

no true fermion zero modes:
no asymmetric branch as function of p_z

Index theorem for approximate fermion zero modes:

Number of asymmetric Q-branches = 2N
N is vortex winding number

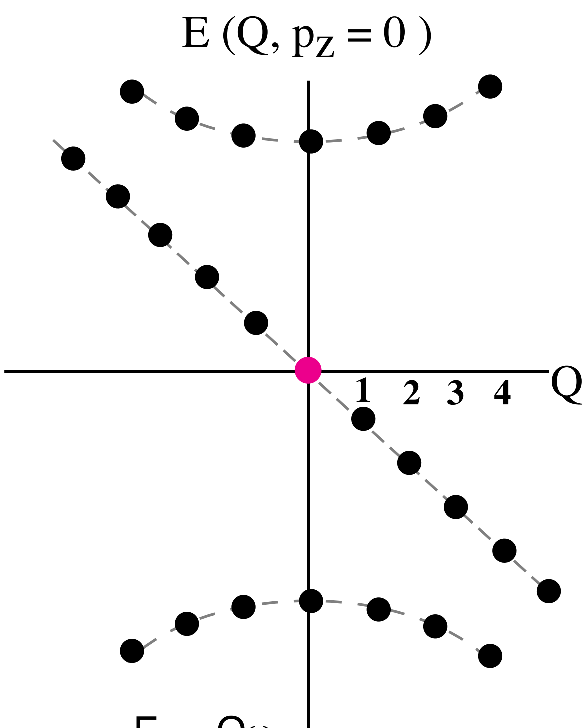
GV JETP Lett. **57**, 244 (1993)

Index theorem for true fermion zero modes?

is the existence of fermion zero modes
related to topology in bulk?

fermions zero modes on symmetric vortex in 3He-B

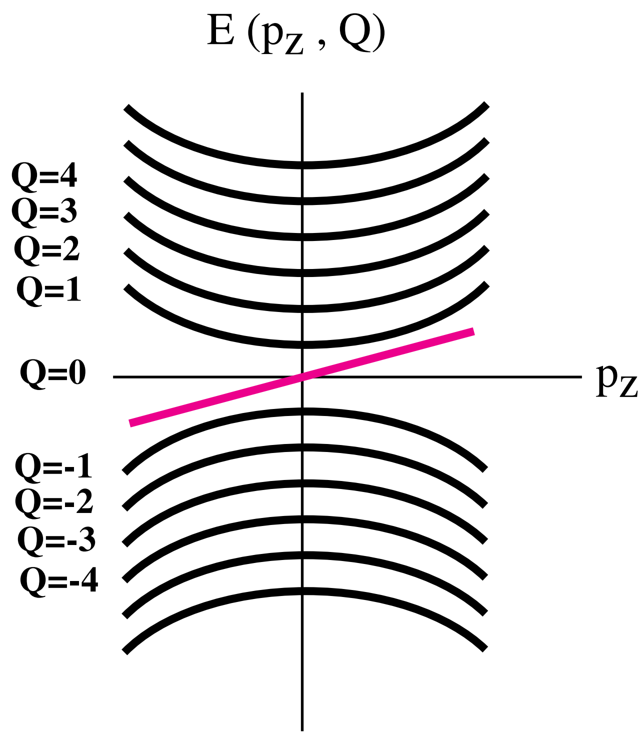
topological ^{3}He-B at $\mu > 0$: $N_K = 2$



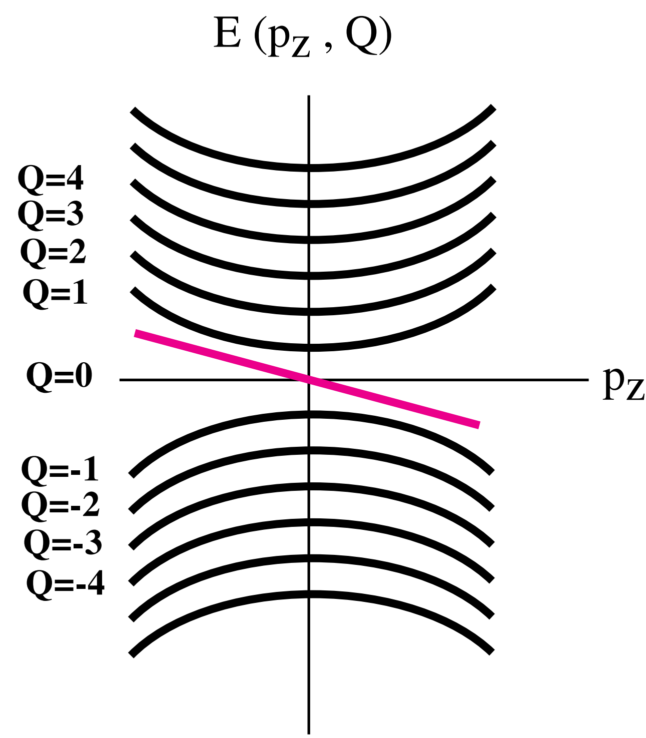
$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F << \Delta$

Q is integer
for p-wave superfluid ^{3}He-B



helicity +



helicity –

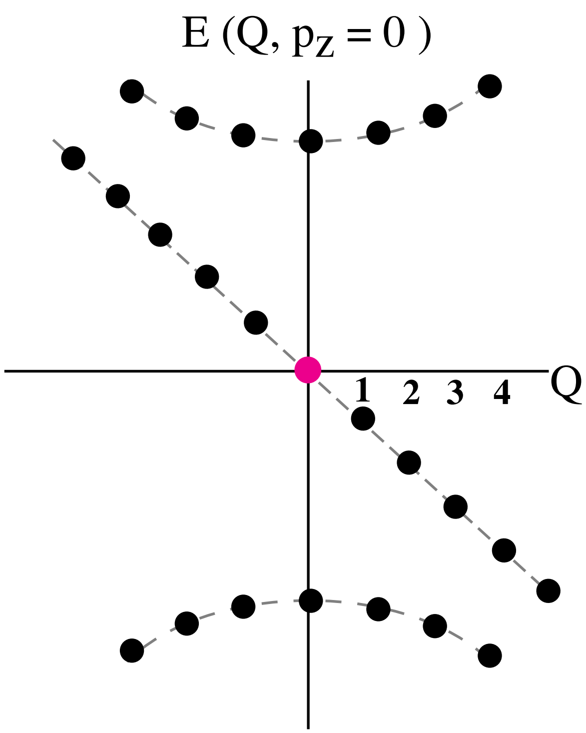
gapless fermions on Q=0 branch form

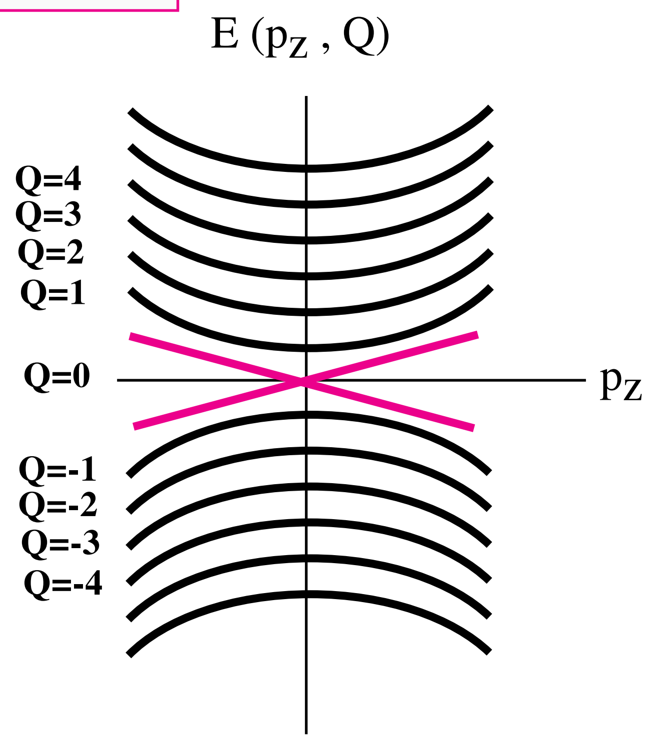
1D Fermi-liquid

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
Physica B **210**, 338 (1995)

fermions zero modes on symmetric vortex in 3He-B

topological ^{3}He-B at $\mu > 0$: $N_K = 2$





$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F << \Delta$

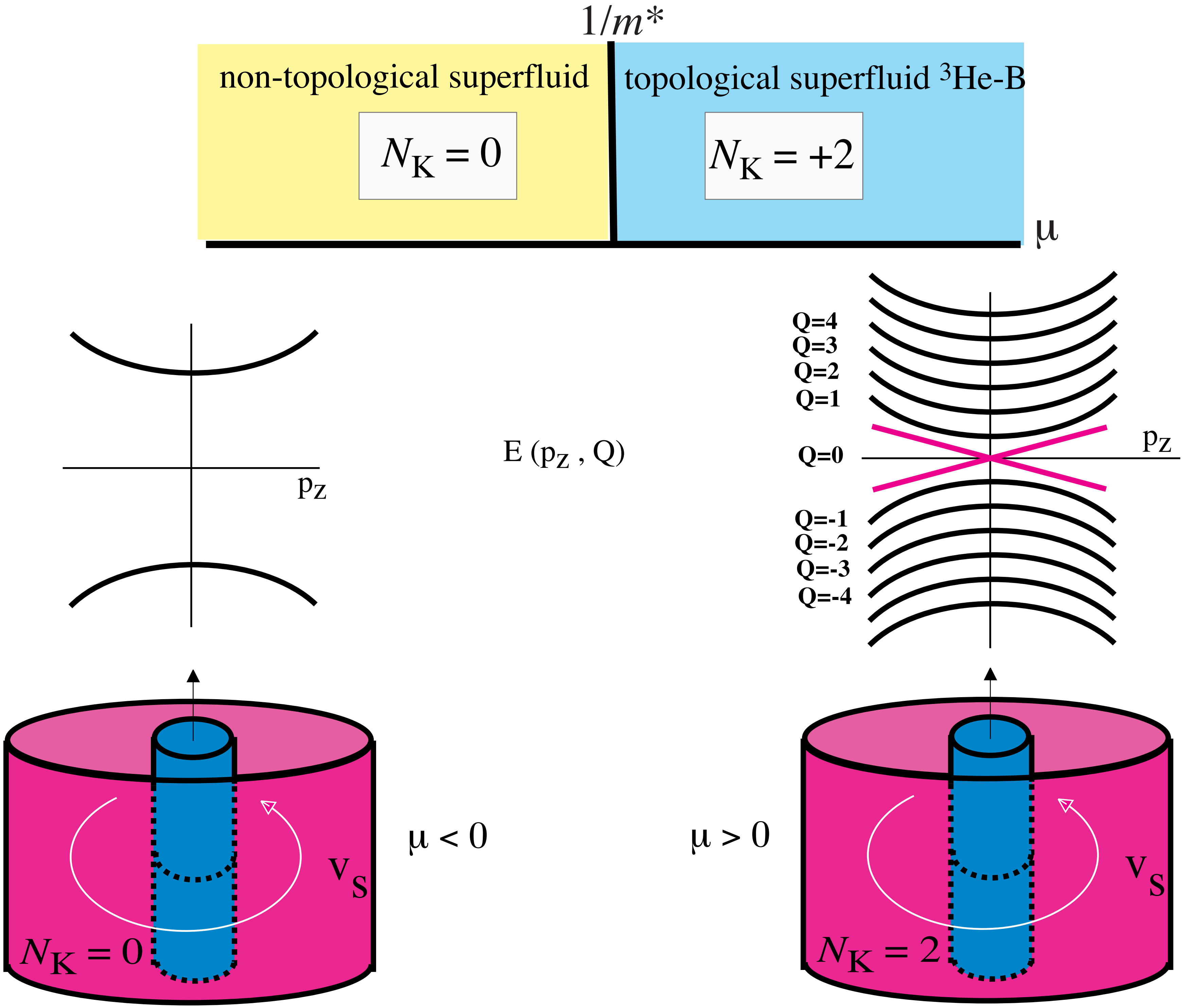
Q is integer
for p-wave superfluid ^{3}He-B

gapless fermions on Q=0 branch form

1D Fermi-liquid

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
Physica B **210**, 338 (1995)

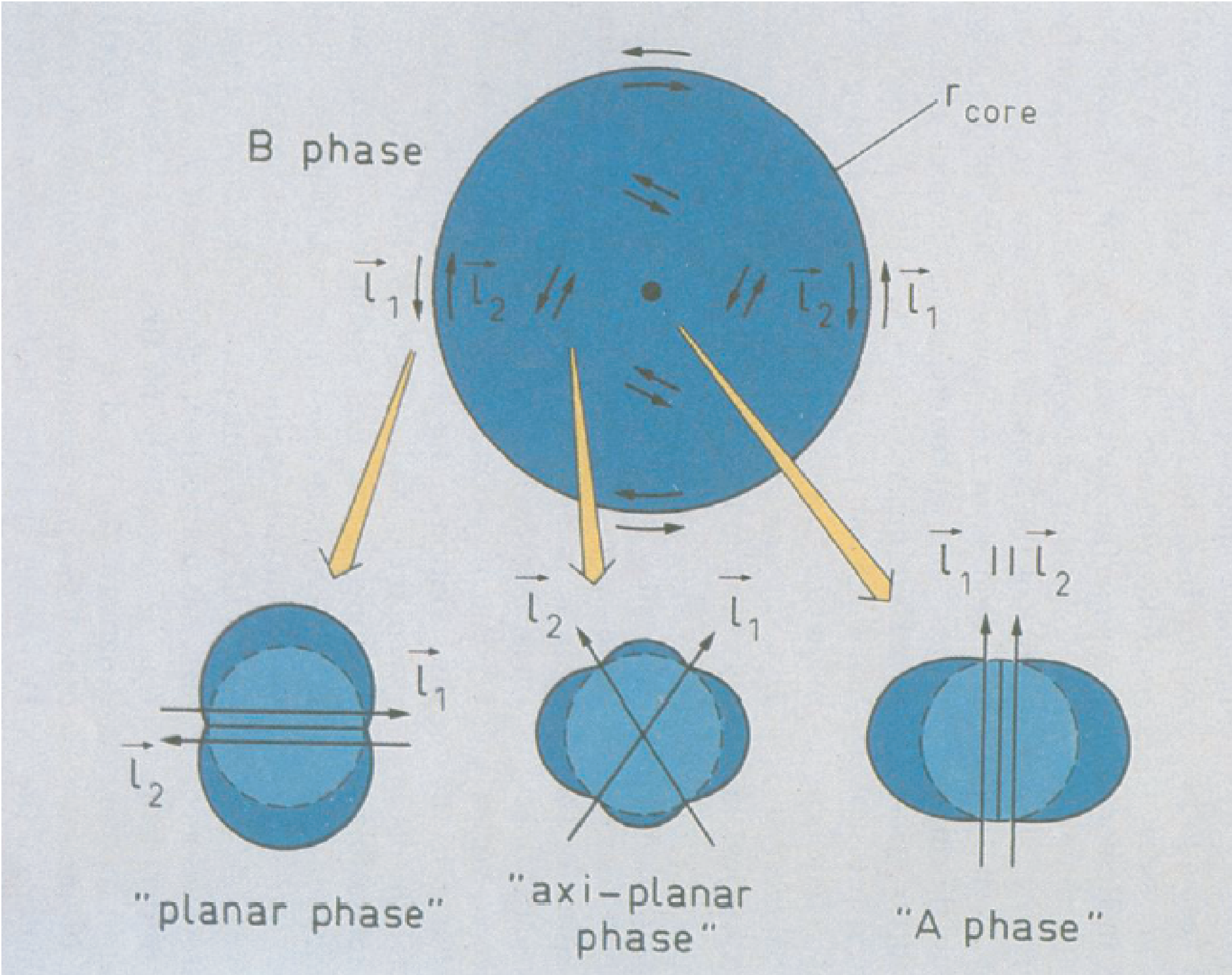
topological quantum phase transition in bulk & in vortex core



Kopnin force on singular vortex & chiral anomaly

extended core of B-phase singular vortex, $R_{core} >> \xi$

four Weyl points in axi-planar phase



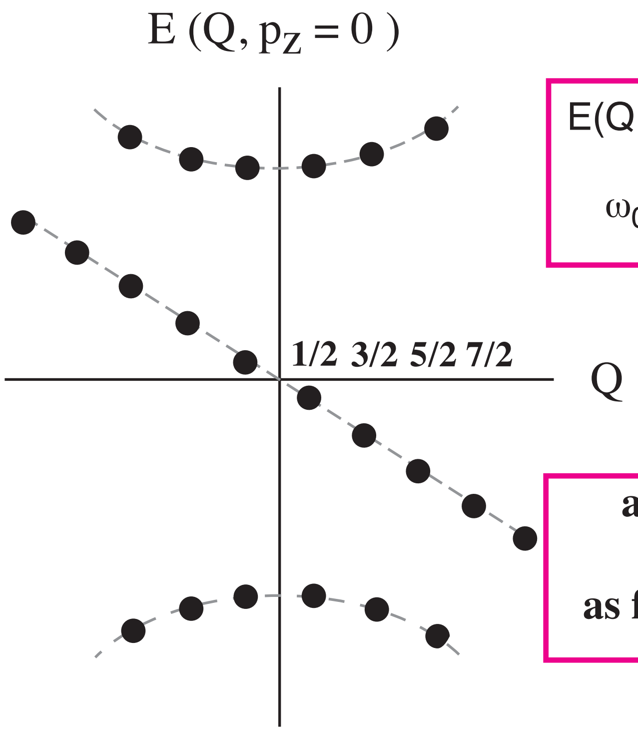
chiral anomaly equation for 4 species of Weyl fermions gives again Kopnin force with $C(T) = \rho$

fermion bound states on vortex in topologically trivial superconductor

Caroli, de Gennes & J. Matricon, Phys. Lett. **9** (1964) 307

Angular momentum Q is half-odd integer
in s-wave superconductor

level width $1/\tau$ due to collisions

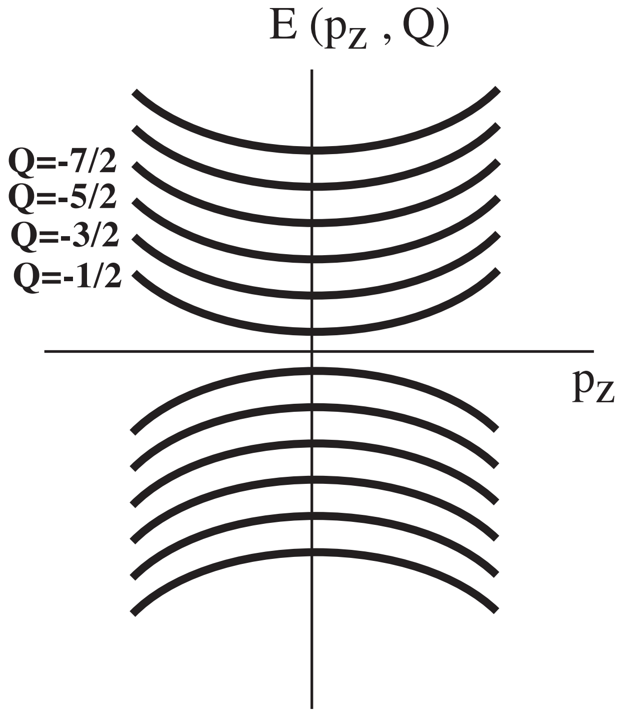


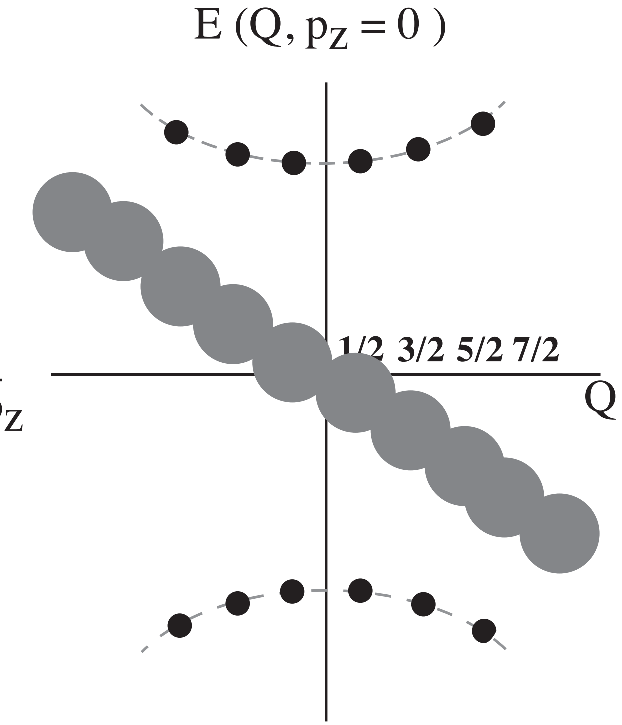
$E(Q,p_z) = -Q\,\omega_0\,(p_z)$

$\omega_0 = \Delta^2/E_F << \Delta$

minigap

asymmetric branch as function of Q





$\omega_0\tau >> 1$

no spectral flow between discrete levels

no true fermion zero modes:
no asymmetric branch as function of p_Z

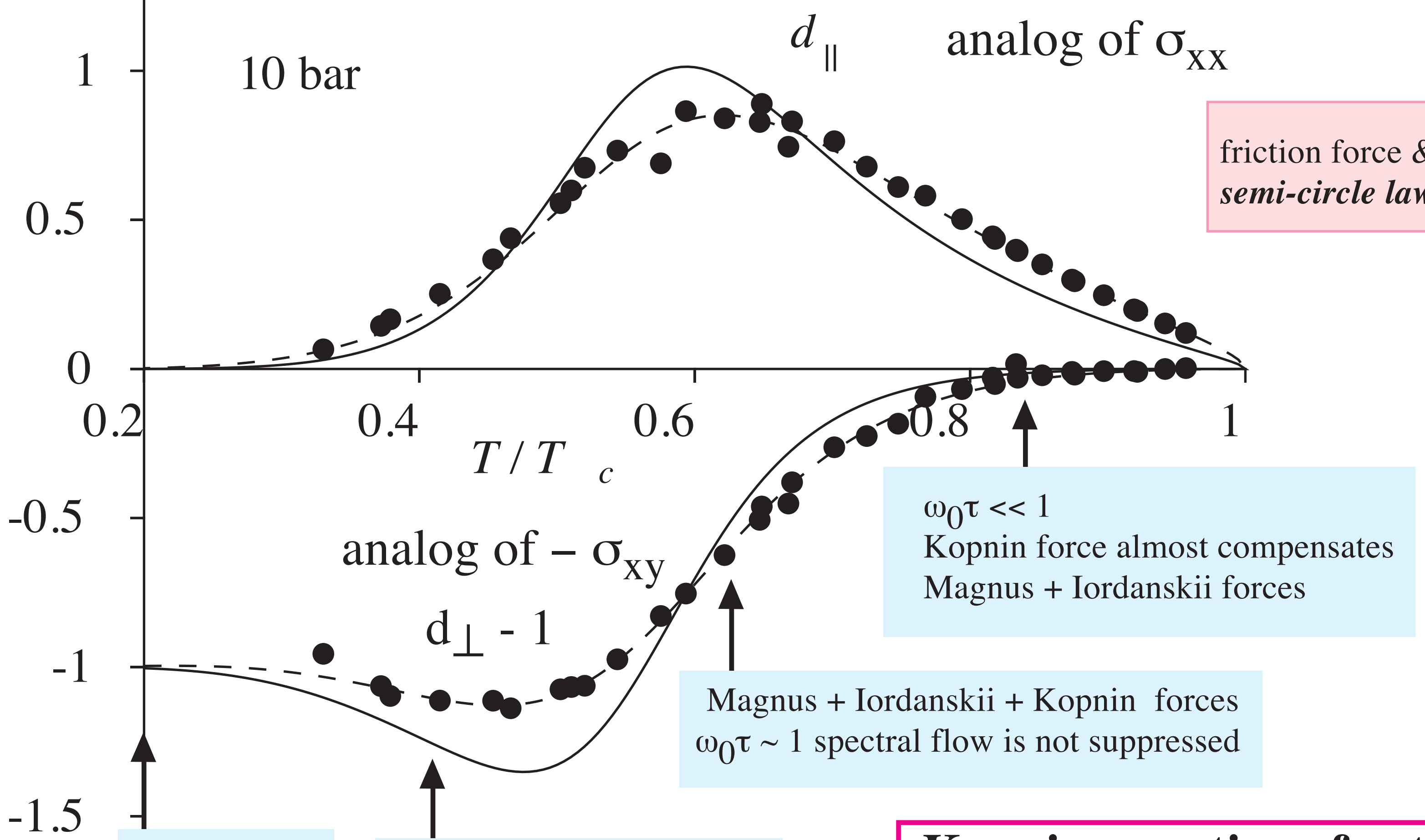
$\omega_0\tau << 1$

maximal spectral flow

spectral flow along asymmetric branch
is the source of Kopnin force on singular vortex
efficiency of spectral flow is determined by Kopnin number $\omega_0\tau$
spectral flow is maximal when minigap is smaller than level width, $\omega_0\tau << 1$
this limit corresponds to axial anomaly equation for extended core

Observation of Kopnin force in Manchester experiments on ^{3}He-B vortices

$$\hat{z} \times (\mathbf{v}_L - \mathbf{v}_s) + d_\perp \hat{z} \times (\mathbf{v}_n - \mathbf{v}_L) + d_\parallel (\mathbf{v}_n - \mathbf{v}_L) = 0$$



friction force & nondissipative forces obey ***semi-circle law*** as σ_{xx} and σ_{xy} in QHE

T=0
pure vacuum:
Magnus force

Magnus + Iordanskii forces
$\omega_0\tau >> 1$
spectral flow is suppressed

$$\alpha + i(1 - \alpha') = 1/(d_\parallel - i(1 - d_\perp))$$

Kopnin equations for transport parameters
reproduced via chiral anomaly by Stone (1996)

$$1 - d_\perp = (\rho / \rho_s) \tan(\Delta/2T) (\omega_0\tau)^2 / [1+(\omega_0\tau)^2]$$

$$d_\parallel = (\rho / \rho_s) \tan(\Delta/2T) \omega_0\tau / [1+(\omega_0\tau)^2]$$

Kopnin number for superfluid hydrodynamics

equations of superfluid hydrodynamics with vorticity

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu - \mathbf{v}_s \times (\nabla \times \mathbf{v}_s) = \mathbf{F}$$

$$\mathbf{F} = -\alpha'(\mathbf{v}_s - \mathbf{v}_n) \times (\nabla \times \mathbf{v}_s) - \alpha\, \hat{\mathbf{n}} \times ((\mathbf{v}_s - \mathbf{v}_n) \times (\nabla \times \mathbf{v}_s))$$

two dimensionless parameters in superfluid hydrodynamics:
non-dissipative $1-\alpha'$ & frictional α
their ratio is analog of Reynolds number for superfluid

$$\mathrm{Ko} = (1-\alpha')/\alpha$$

$$\mathrm{Ko} \sim \omega_0 \tau$$

Kopnin number for superfluid turbulence

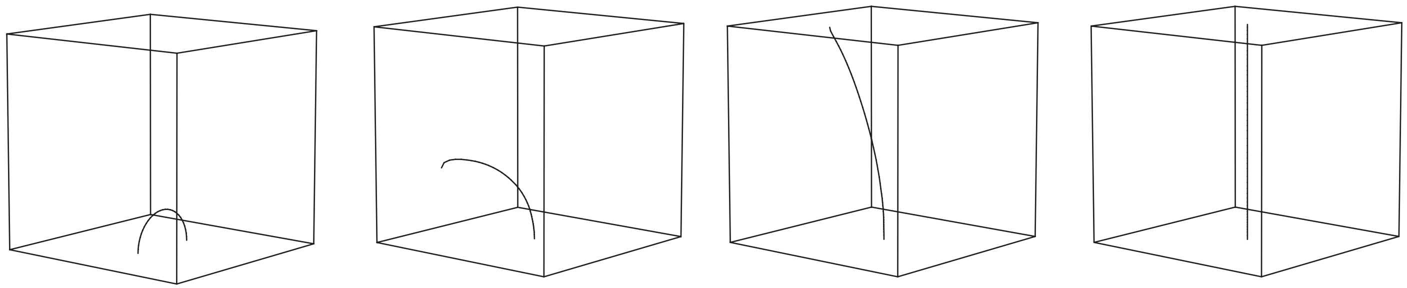
two dimensionless parameters in superfluid hydrodynamics:

non-dissipative $1-\alpha'$ & frictional α

their ratio is analog of Reynolds number for superfluid

$$Ko = (1-\alpha')/\alpha$$

$$Ko \sim \omega_0\tau$$



0 4 s 8 s 24 s

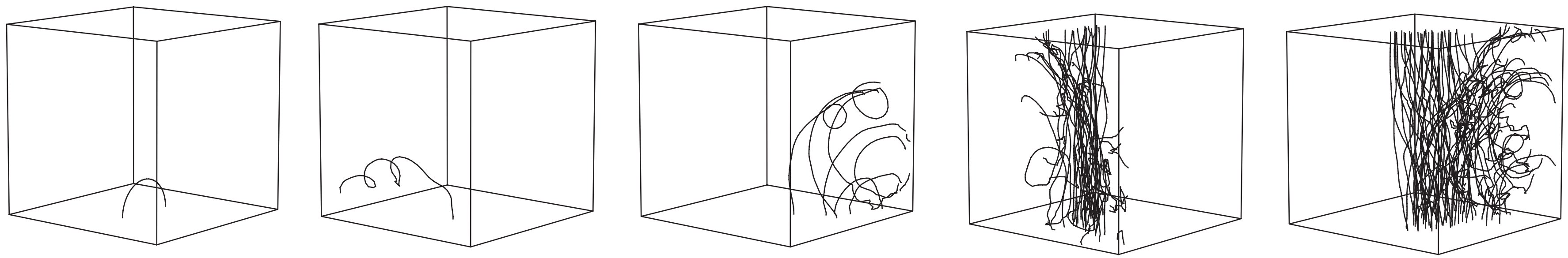
Ω

T=0.8 T_c

$Ko < 1$ $\omega_0\tau < 1$

spectral flow supresses turbulence

transition to superfluid turbulence occurs at $Ko \sim 1$



0 44 s 68 s 128 s 160 s

T=0.4 T_c

$Ko > 1$ $\omega_0\tau > 1$

phase diagram of turbulent flow in Fermi superfluids (^{3}He-B)

3 Reynolds numbers in 2-fluid hydrodynamics

$$Ko = (1 - \alpha')/\alpha$$

Kopnin number $Ko \sim \omega_0 \tau$

$$Re_\nu = UR / \nu_n$$ $<< 1$

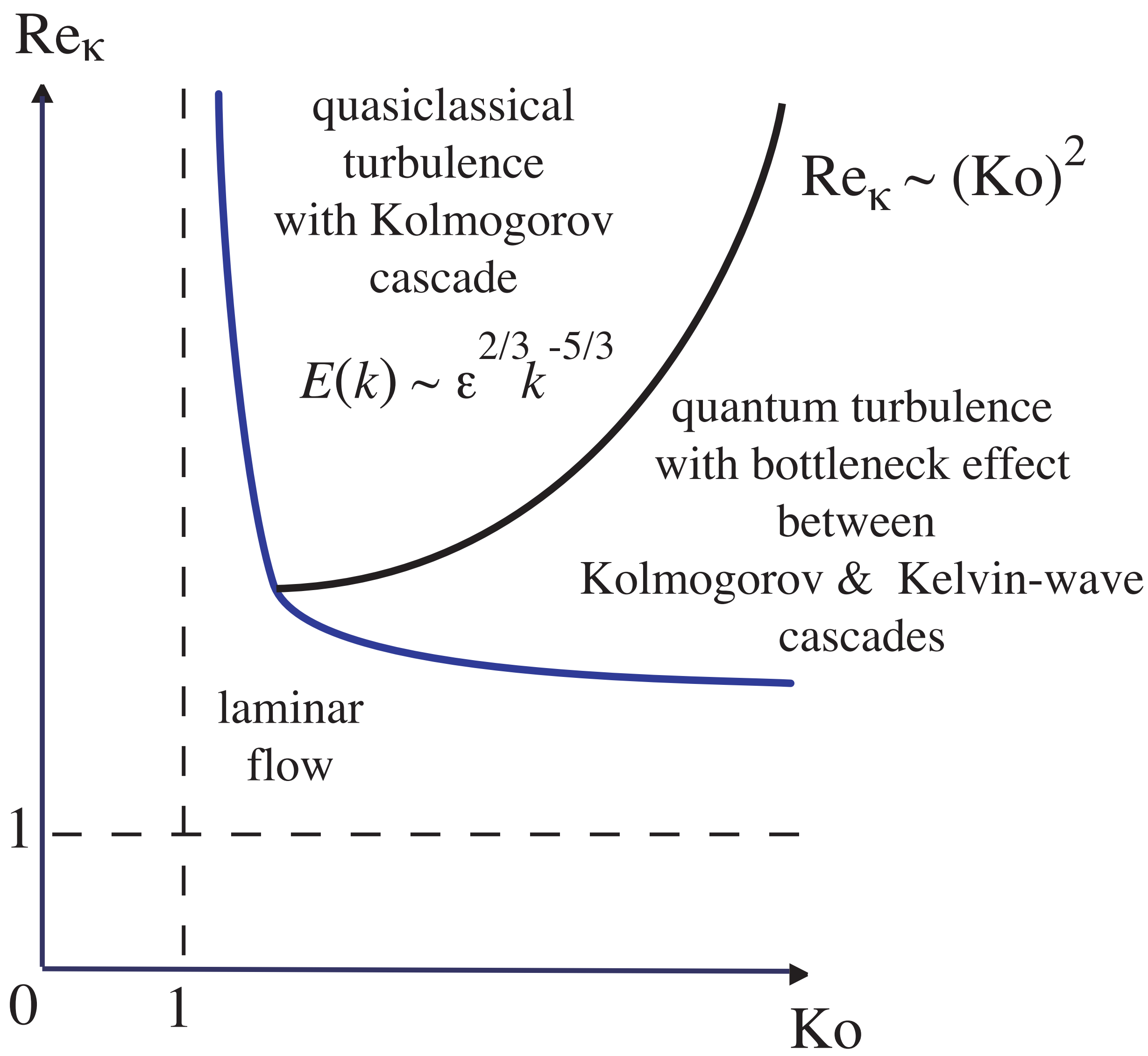
conventional Reynolds number

ν_n – viscosity of normal component

$$Re_\kappa = UR / \kappa$$

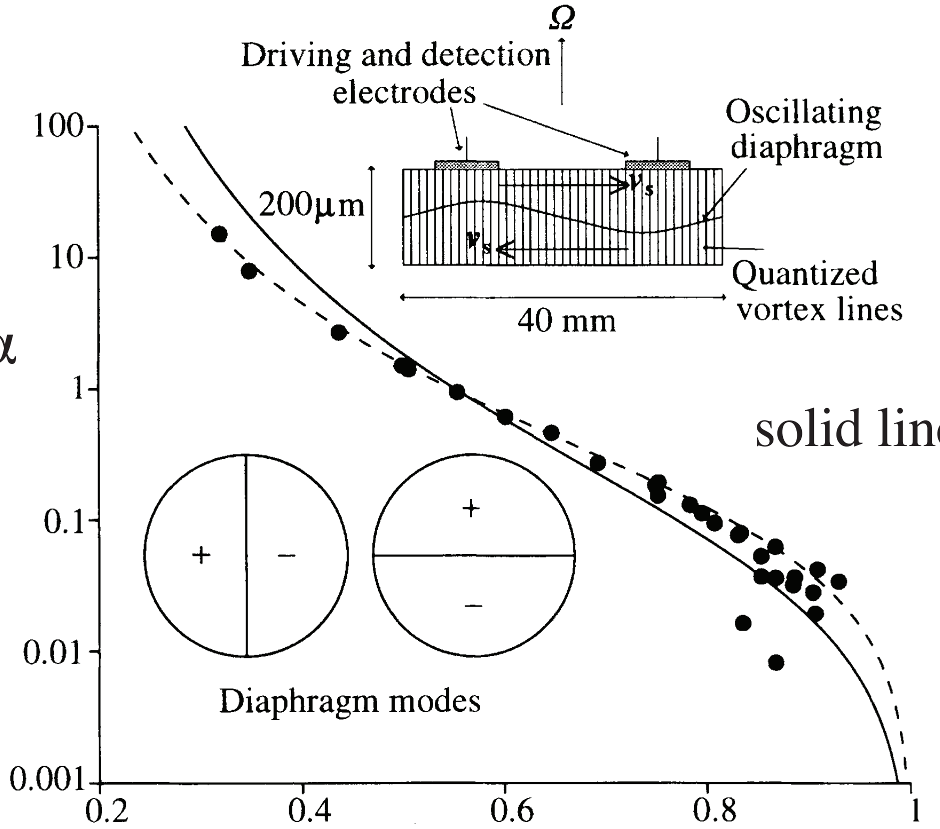
vorticity Reynolds number

κ – circulation quantum



measured temperature dependence of Kopnin number in 3He-B (Manchester)

$Ko = (1- \alpha')/ \alpha$



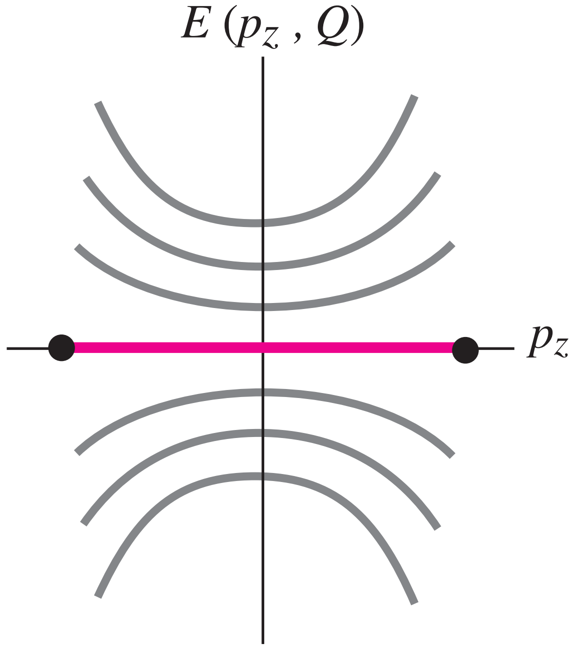
T/T_c

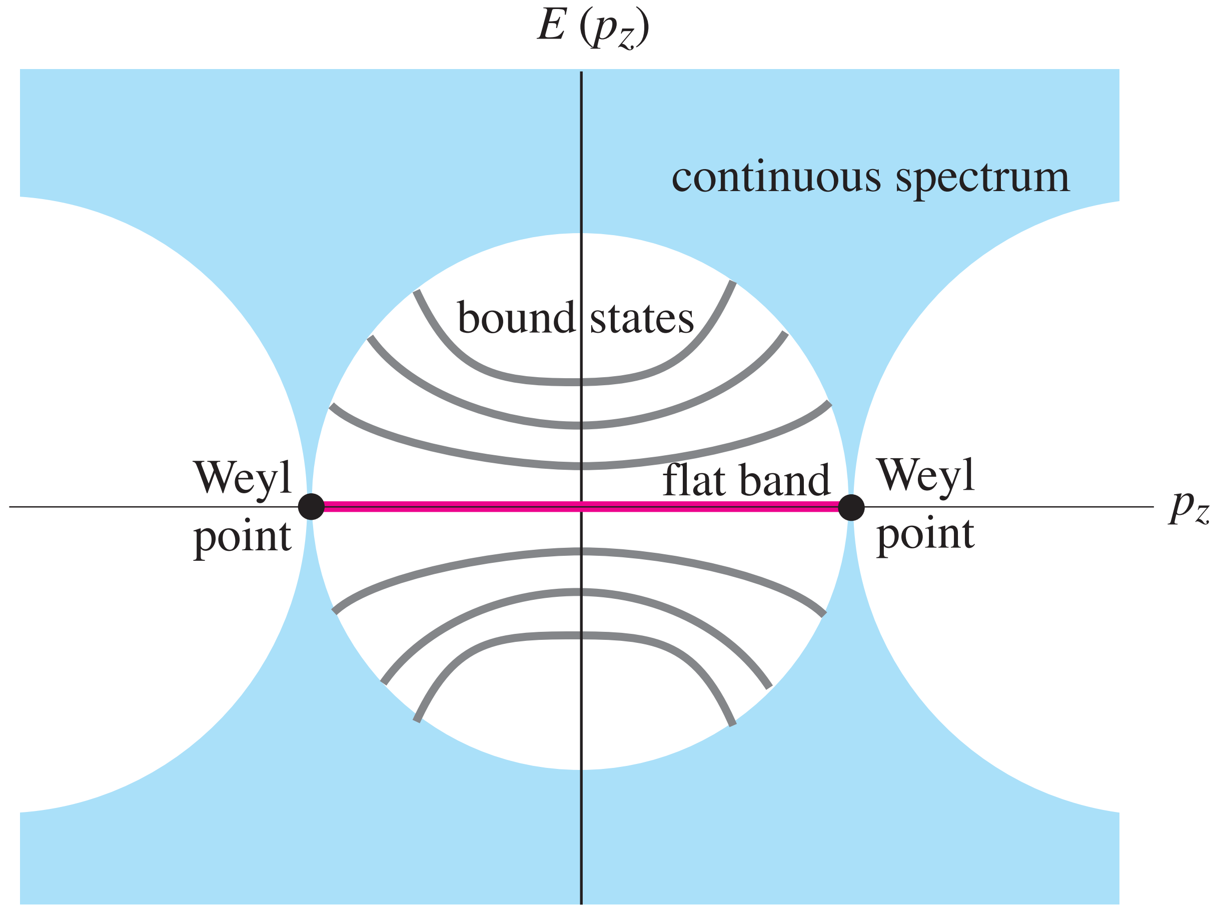
solid line - from Kopnin theory

Start of Majorana cond-mat physics + start of topologically protected flat bands

topologically protected Majorana flat band in vortex core of superfluids with Weyl points

(Kopnin-Salomaa 1991)



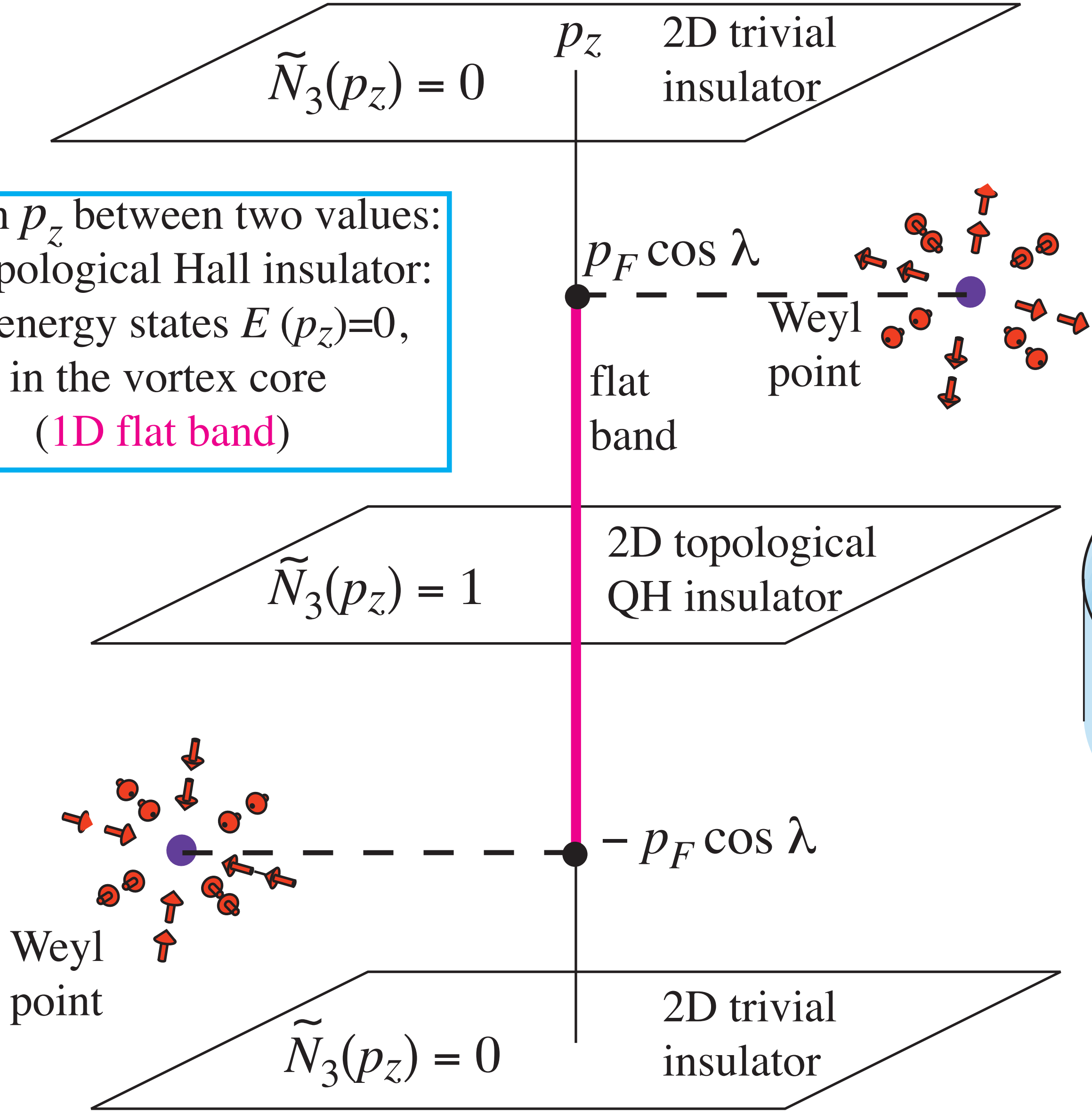


flat band of bound states terminates on zeroes of continuous spectrum (i.e on Weyl points)

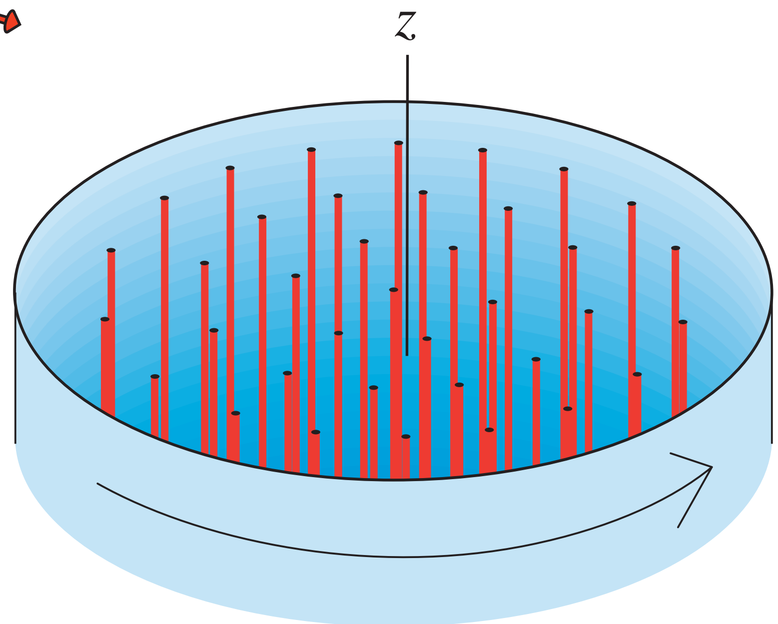
Topologically protected flat band of Majorana modes in vortex core of Weyl superfluid

Kopnin & Salomaa
Mutual friction in superfluid 3He:
effects of bound states in the vortex core
PRB **44**, 9667 (1991)

at each p_z between two values:
2D topological Hall insulator:
zero energy states $E\,(p_z)$=0,
in the vortex core
(1D flat band)



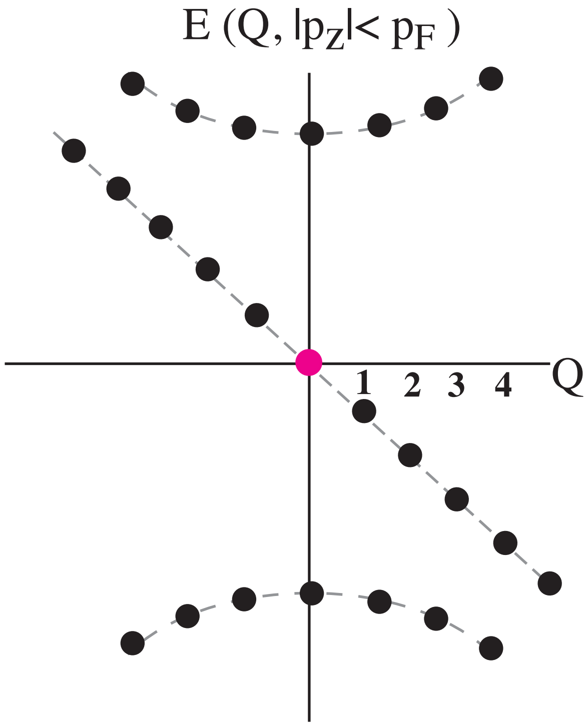
vortices in **r**-space



Chern number
for interacting systems
(So, Ishikawa, ...)
GV & Yakovenko
(1989)

$$\widetilde{N}_3(p_z) = \frac{1}{4\pi^2}\,\mathbf{tr}\int dp_x\, dp_y d\omega\; \mathbf{G}\, \nabla_\omega\, \mathbf{G}^{-1}\, \mathbf{G}\, \nabla_{p_x}\, \mathbf{G}^{-1} \mathbf{G}\, \nabla_{p_y}\, \mathbf{G}^{-1}$$

Majorana mode on a vortex in 2D topological superfluid



$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F << \Delta$

Q is integer
for p-wave superfluids

In 3He-A vortex there is a chain of Majorana modes:
they form flat band at $|p_z| < p_F$

Kopnin mass

Vortex mass in Bose superfluid (4He)

$$M_{vortex} = E_{vortex} / c^2$$

c → speed of sound

$$M_{vortex} \sim \rho a^2 \ln (R/\xi)$$

$a \sim h/mc$ ~ interatomic space

R → intervortex space

ξ → coherence length

Kopnin vortex mass in Fermi superfluid (3He-B & superconductor)

$$M_{vortex} \sim \rho \xi^2 >> \rho a^2$$

ξ → coherence length

what is the origin of Kopnin mass ?

Kopnin mass of skyrmion

Kopnin vortex mass in Fermi superfluid
(skyrmion in 3He-A)

$$M_{\text{skyrmion}} \sim \rho\, \xi\, R_{\text{core}} \gg \rho \xi^2$$

core radius (R_{core})

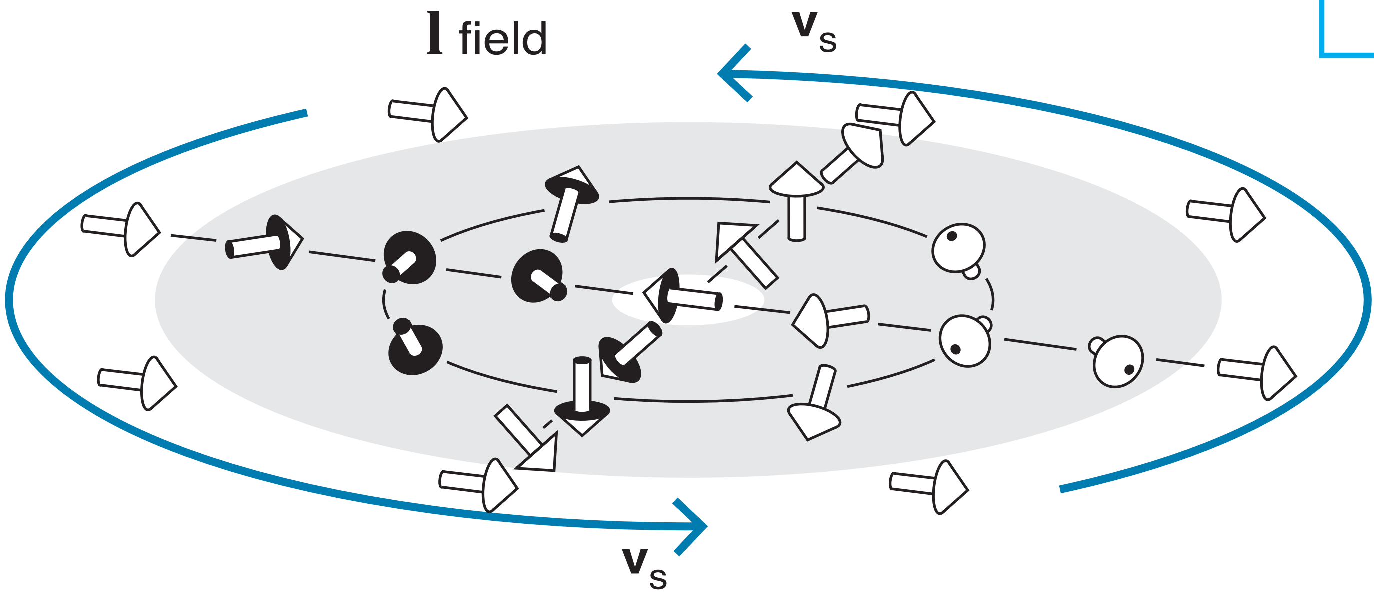
coherence length (ξ)

what is the origin of Kopnin mass ?

Kopnin mass of skyrmion from effective magnetic field

Kopnin vortex mass in Fermi superfluid
(skyrmion in 3He-A & extended core in 3He-B)

$\mathbf{A} = p_F \boldsymbol{l}$ $\mathbf{B} = p_F \nabla \times \boldsymbol{l}$



DoS in effective magnetic field
leads to normal component density at T=0

GV & Mineev, JETP **54**, 524 (1981)

$$\rho_n \sim |\mathbf{B}| \sim \rho\, \xi / R_{core}$$

$$M_{vortex} \sim \rho_n R_{core}^2 \sim \rho\, \xi\, R_{core}$$

Kopnin mass comes from excitations
localized in the vortex core

superfluid ^{3}He-B as non-relativistic limit of relativistic triplet superconductor

$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

relativistic triplet superconductor

$$\downarrow \quad cp \ll M, \quad \mu \ll M$$

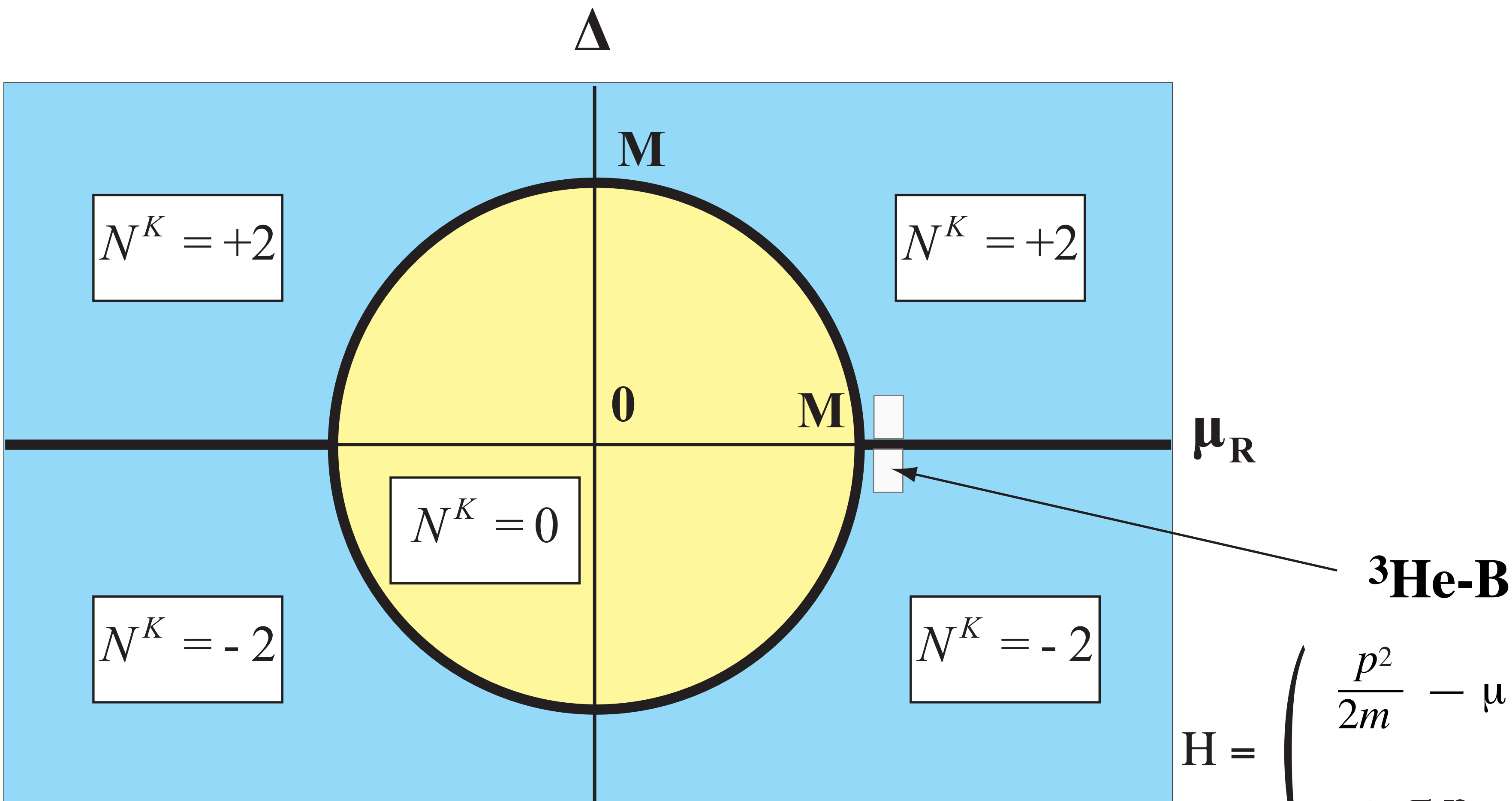
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B\sigma\cdot\mathbf{p} \\ c_B\sigma\cdot\mathbf{p} & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

superfluid ^{3}He-B

$$c_B = c\,\Delta/M \qquad m = M/c^2$$

$$(\mu + M)^2 = \mu_R^2 + \Delta^2$$

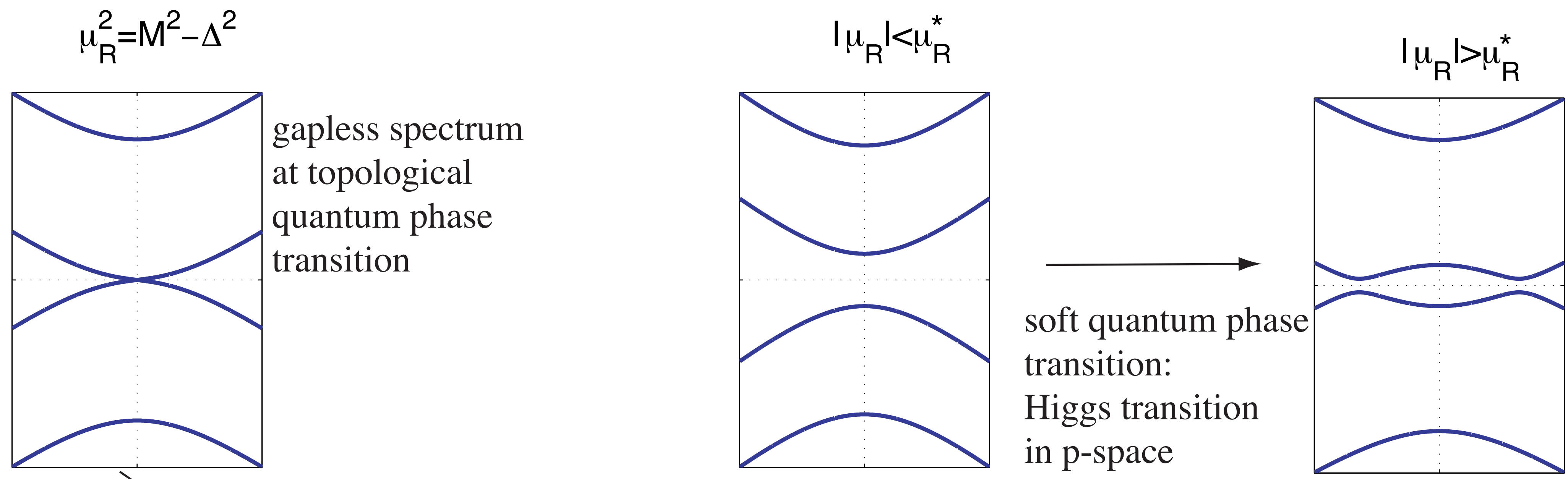
phase diagram of topological states of relativistic triplet superconductor

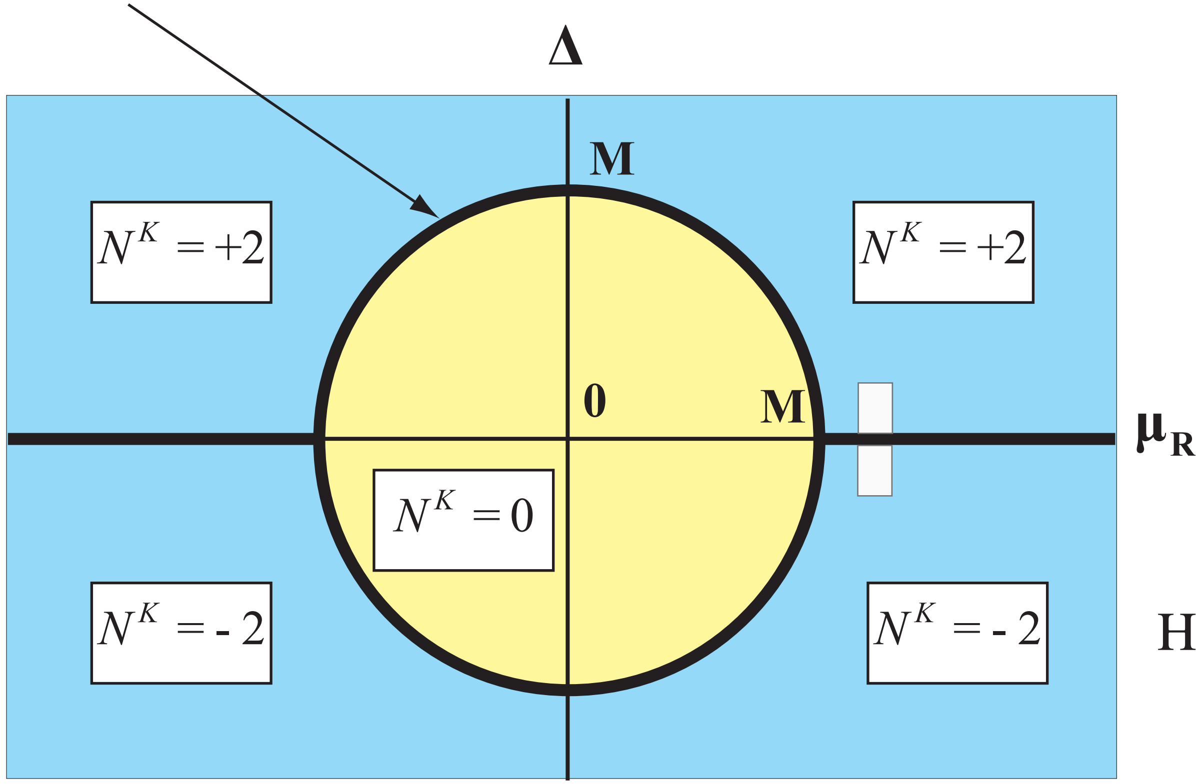


$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B \sigma \cdot \mathbf{p} \\ c_B \sigma \cdot \mathbf{p} & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$H = \begin{pmatrix} c\alpha \cdot \mathbf{p} + \beta M - \mu_R & \gamma_5 \Delta \\ \gamma_5 \Delta & - c\alpha \cdot \mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

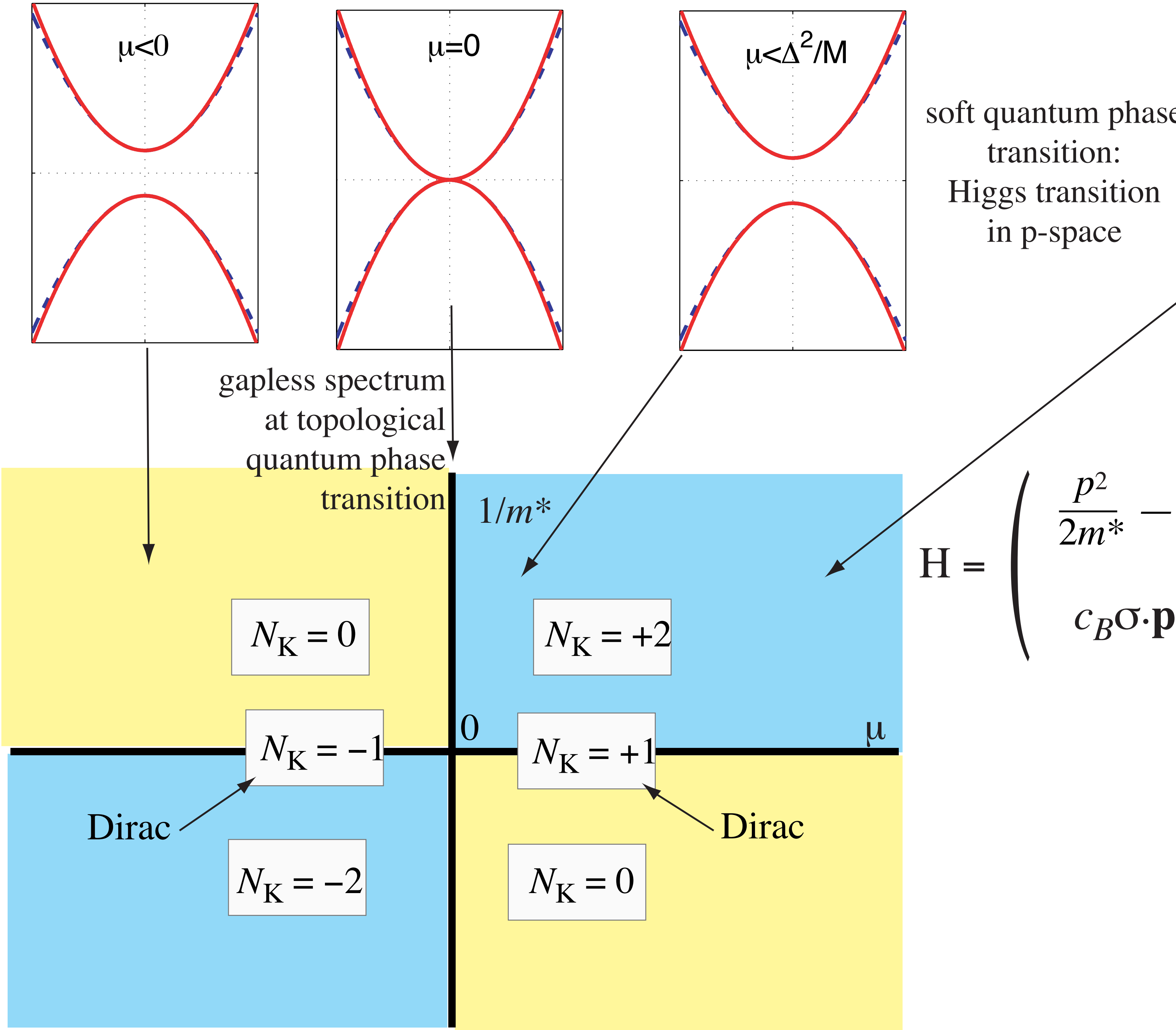
energy spectrum in relativistic triplet superconductor





$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

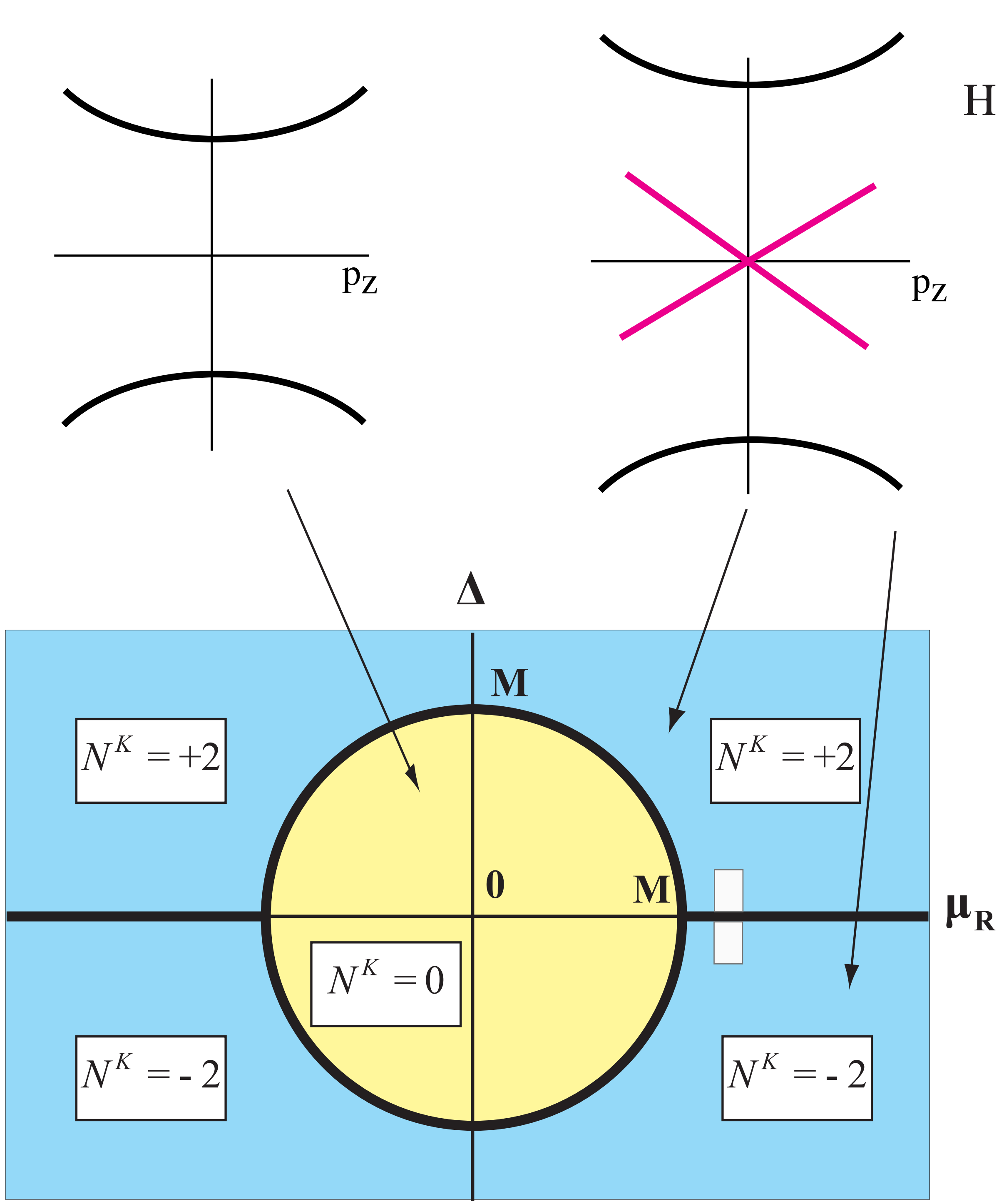
spectrum of non-relativistic ^{3}He-B



$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma \cdot \mathbf{p} \\ c_B \sigma \cdot \mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix}$$

fermion zero modes in relativistic triplet superconductor

$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$



vortices in topological superconductors have fermion zero modes

generalized index theorem ?

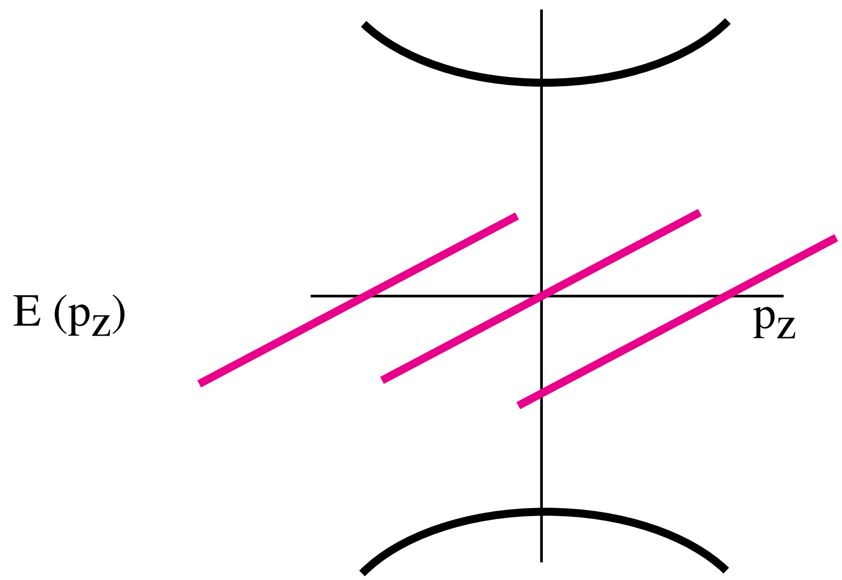
index theorem for fermion zero modes on vortices

(interplay of ***r***-space and ***p***-space topologies)

$$N_5 = \frac{1}{4\pi^3 i} \mathbf{tr} \left[\int d^3p \, d\omega \, d\phi \; \mathbf{G} \nabla_\omega \mathbf{G}^{-1} \mathbf{G} \nabla_\phi \mathbf{G}^{-1} \mathbf{G} \nabla_{p_x} \mathbf{G}^{-1} \mathbf{G} \nabla_{p_y} \mathbf{G}^{-1} \mathbf{G} \nabla_{p_z} \mathbf{G}^{-1} \right]$$

for vortices in Dirac vacuum

$N_5 = N$ **winding number**



N_5 invariant was introduced by Golterman, Jansen & Kaplan for lattice fermions Phys. Lett. B **301** (1993) 219

see also M.A. Zubkov & GV
Momentum space topological invariants for the 4D relativistic vacua with mass gap
Nucl. Phys. B **860** (2012) 295

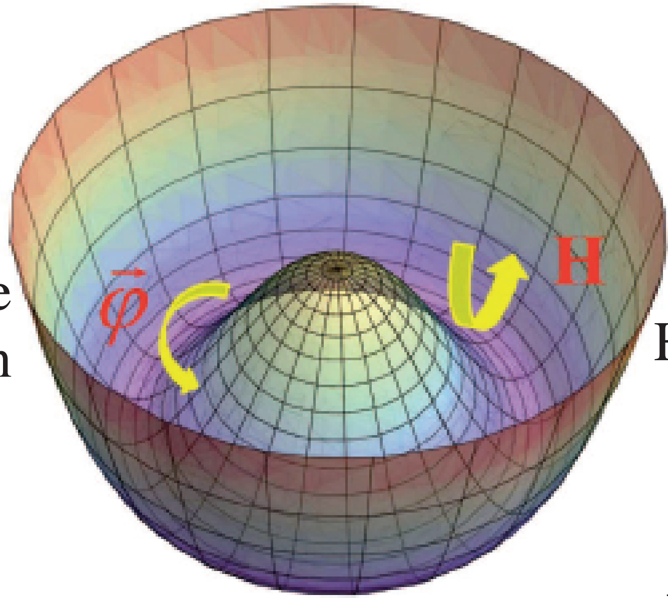
Higgs modes in gapped systems

3He-B vs Standard Model

Nambu conjecture

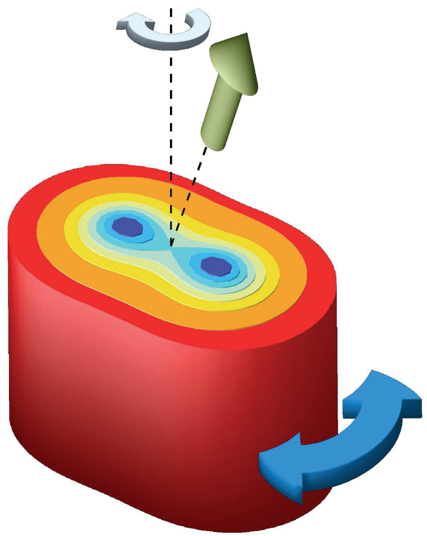
Symmetry breaking, Higgs mechanism & massive particles

Nambu-Goldstone (NG) boson



Higgs boson

twistino on double-core vortex



Standard Model

cond-mat

NG bosons

no NG bosons !

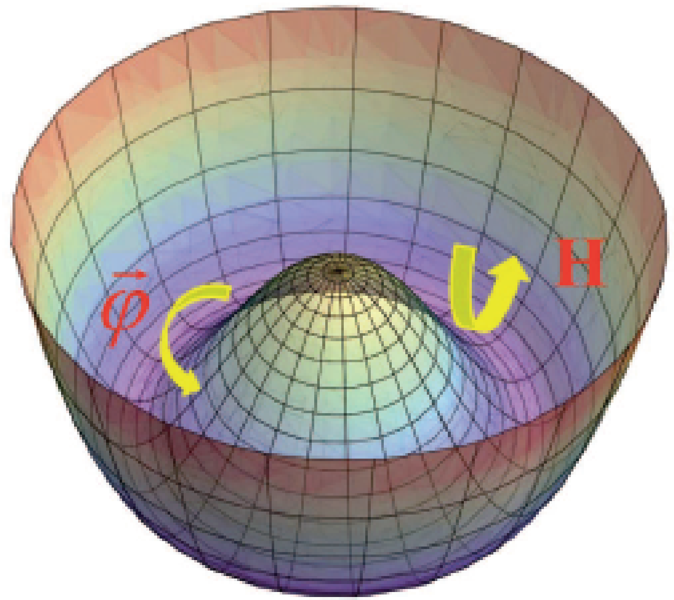
phonon, spin wave, orbital wave, Kelvon (Kelvin wave), twistino, Tkachenko wave, ripplon, Parshin (crystallization) wave, ...

massive particles

Higgs boson(s)	order parameter modes
W & Z gauge bosons	Meissner effect (expulsion of magnetic field)
quarks & leptons	gapped quasiparticles

Nambu-Goldstone (NG) bosons

Goldstone's theorem:



spontaneous breakdown of a global continuous symmetry
leads to massless particle – Nambu-Goldstone boson

Y. Nambu, Phys. Rev. Lett. **4** (1960) 380
J. Goldstone, Nuovo Cim. **19** (1961) 154
J. Goldstone, A. Salam, S. Weinberg, Phys. Rev. **127** (1962) 965

cond-mat order parameter = Higgs field

$$\Psi = |\Psi|\, e^{i\varphi}$$

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

cond-mat NG bosons = gapless modes of order parameter:
phonons, spin waves, orbital waves

no NG bosons had been observed in particle physics

Why ?

Higgs mechanism

spontaneous breaking of gauge symmetry does not require Goldstone bosons,
their degrees of freedom deliver the longitudinal polarization modes of gauge bosons,
gauge bosons become massive (Z-boson, W-boson)

P. Anderson, Plasmons, gauge invariance and mass, Phys. Rev. **130** (1963) 439
F. Englert, R. Brout, Phys. Rev. Lett. **13** (1964) 321
P. W. Higgs, Phys. Lett. **12** (1964) 132 and Phys. Rev. Lett. **13** (1964) 508
G. S. Guralnik, C. R. Hagen, T. W. B. Kibble, Phys. Rev. Lett. **13** (1964) 585

cond-mat analog: Cooper pairing and Meissner effect in superconductors

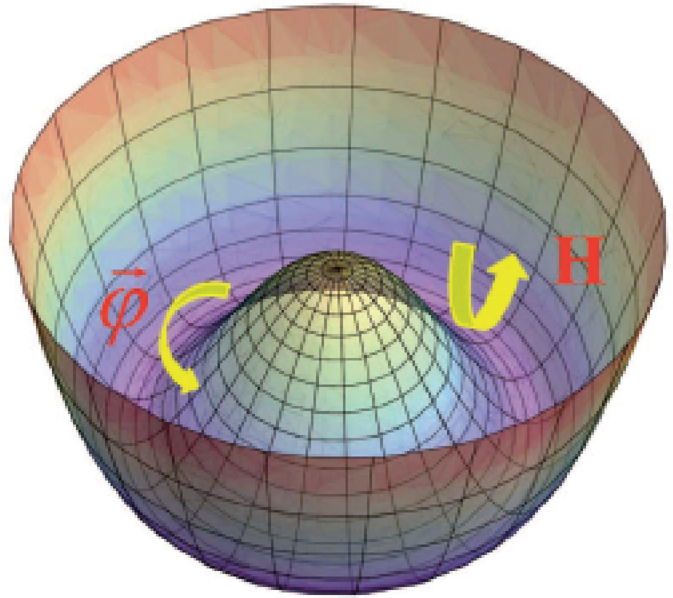
Higgs, PRL **13** (1964) 508:

"It is to be expected that this feature will appear also in theories
in which the symmetry-breaking scalar fields are not elementary dynamic variables
but bilinear combinations of Fermi fields."

"In the theory of superconductivity the scalar fields are associated with fermion pairs;
the doubly charged excitation responsible for the quantization of magnetic flux
is then the surviving member of a U(1) doublet."

How many Higgs bosons ?

amplitude H-modes of Higgs field



P. W. Higgs, PRL **13** (1964) 508:

“The model of the most immediate interest is that in which the scalar fields form an **octet** under **SU(3)**... There are **2** massive scalar bosons ... (**2 *Higgs bosons***) ... ; the remaining **6** components of the scalar octet combine with the corresponding components of the gauge-field octet to describe massive vector bosons (**6 *massive gauge bosons***).”

Vdovin (1963)

Bosons in superfluid ^{3}He-B: collective modes of order parameter

9-plet $A_{\alpha i}$ under **SO(3)** x **SO(3)** x **U(1)**

14 *Higgs bosons* + 4 *NG modes* (**4 *massive gauge bosons*** in gauged ^{3}He-B)

Standard Model

Order parameter Higgs field: complex spinor $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$

$$L = L_{GL}\{\Psi\} + L_{fermions} + L_{interaction}$$

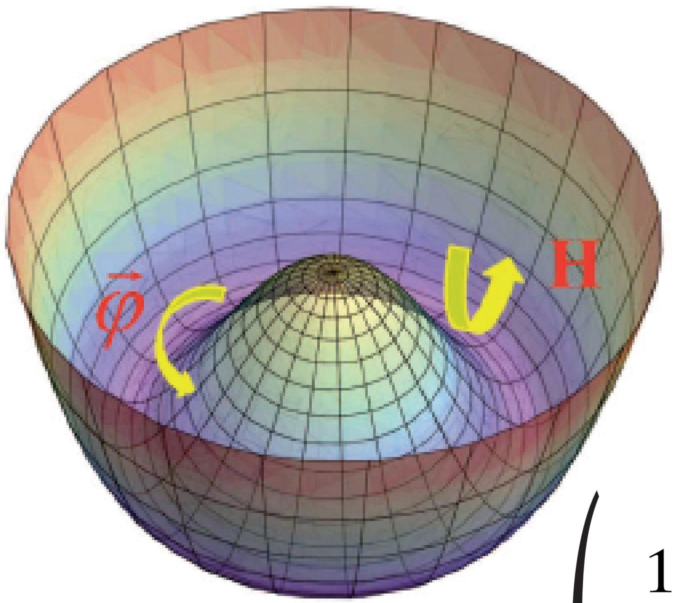
$$L_{GL} = |\nabla_\alpha \Psi - iqA_\alpha \Psi|^2 - \mu^2|\Psi|^2 + \lambda|\Psi|^4$$ Ginzburg-Landau functional

$$L_{fermions} = \bar{\chi}\, D\, \chi$$ Lagrangian for massless fermions

$$L_{interaction} = \Psi\, \bar{\chi}_L\, \chi_R$$ interaction of fermions with Higgs field

nonzero vacuum value of Ψ gives mass to fermions: $M = |\Psi|$

Ginzburg-Landau vs Standard Model



Order parameter:
complex scalar field

$$\Psi = |\Psi|\, e^{i\varphi}$$

2 = 1+1 collective modes:

$$\Psi = 1 + (H + i\varphi)$$

1 NG mode (phase φ mode)

1 amplitude H mode (Higgs boson)

Meissner effect:
mass of gauge vector boson = inverse penetration length

Higgs field: complex spinor

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

4 = 3+1 collective modes

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + H \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underset{\text{3 rotations}}{\delta\Psi}$$

3 NG modes

1 amplitude H mode (Higgs boson H)

Standard Model requires only one Higgs boson

3 NG modes are absorbed into longitudinal modes of 3 massive vector bosons (charged W^+, W^- and neutral Z)
this is called **Higgs mechanism**

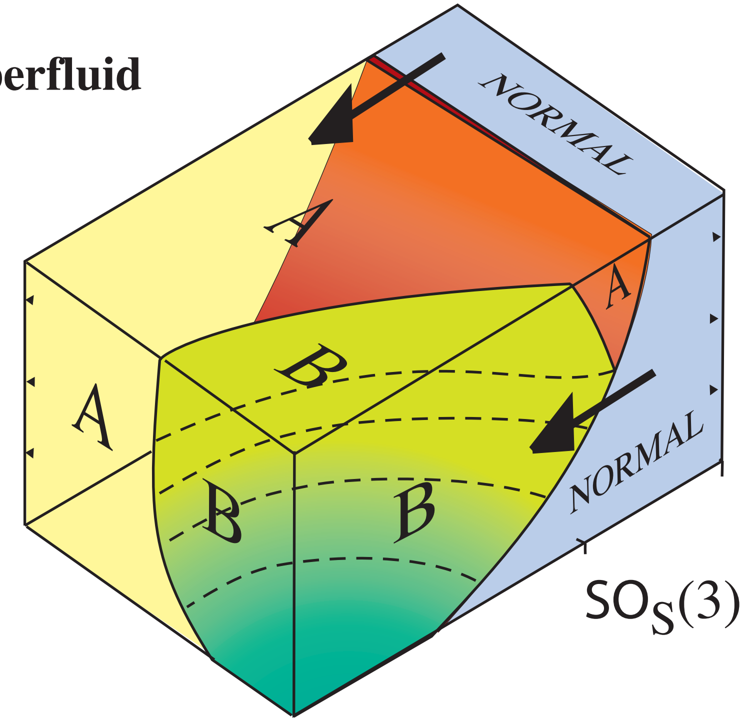
***GUT in* Standard Model**
symmetry breaking phase transitions

$SO(10) \rightarrow SU(3) \times SU_L(2) \times U(1) \rightarrow SU(3) \times U_Q(1)$

symmetry breaking phase transitions
***in* superfluid ^{3}He**

$SO_S(3) \times SO_L(3) \times U(1) \rightarrow SO_S(2) \times U_Q(1)$ Q is analog of electric charge

^{3}He-A
topological chiral Weyl superfluid
two Dirac cones



^{3}He-B
time-reversal symmetric
topological superfluid

$SO_S(3) \times SO_L(3) \times U(1) \rightarrow SO_J(3)$

from Standard Model Ginzburg-Landau to Standard Model BEC

nobody now believes that SM is complete theory,
all believe SM is effective low energy theory

composite Higgs is more natural than fundamental

Higgs, PRL **13** (1964) 508:

"... the symmetry-breaking scalar fields
are not elementary dynamic variables
but ***bilinear combinations of Fermi fields***."

Higgs field can be composite object as Cooper pair in superconductors

$$\Psi = \langle \bar{\chi}_L \chi_R \rangle$$

Standard Model as BCS theory in semimetal

BCS Cooper pairing

Order parameter as composite object

$$\Psi = \langle \chi \chi \rangle$$

phase & amplitude modes

$$\Psi = \langle \chi \chi \rangle_{vac} + \delta\Psi$$

Ψ mixes electron and hole

Hamiltonian for Bogoliubov-Nambu quasiparticles

$$H = \begin{pmatrix} \varepsilon(\mathbf{p}) & \Psi \\ \Psi^* & -\varepsilon(\mathbf{p}) \end{pmatrix}$$

order parameter induces gap

$$E^2(\mathbf{p}) = \varepsilon^2(\mathbf{p}) + |\Psi|^2$$

dynamical mixing of left & right particles

Higgs field as composite object

$$\Psi = \langle \bar{\chi}_L \chi_R \rangle$$

Higgs bosons

$$\Psi = \langle \bar{\chi}_L \chi_R \rangle_{vac} + \delta\Psi$$

Ψ mixes left and right Weyl particles

Hamiltonian for Dirac particles

$$H = \begin{pmatrix} c\sigma\cdot\mathbf{p} & \Psi \\ \Psi^+ & -c\sigma\cdot\mathbf{p} \end{pmatrix}$$

Higgs field induces Dirac mass $M = |\Psi|$

$$E^2(\mathbf{p}) = c^2 p^2 + |\Psi|^2$$

BCS

Standard Model (Weyl semimetal)

Order parameter $\Psi = < \chi \hat{O} \chi >$

Higgs field $\Psi = < \chi_L \hat{O} \chi_R^+ >$

Meissner effect

Gauge boson mass

Abrikosov vortex

Nielsen-Olesen vortex

pair breaking & squashing modes

Higgs bosons

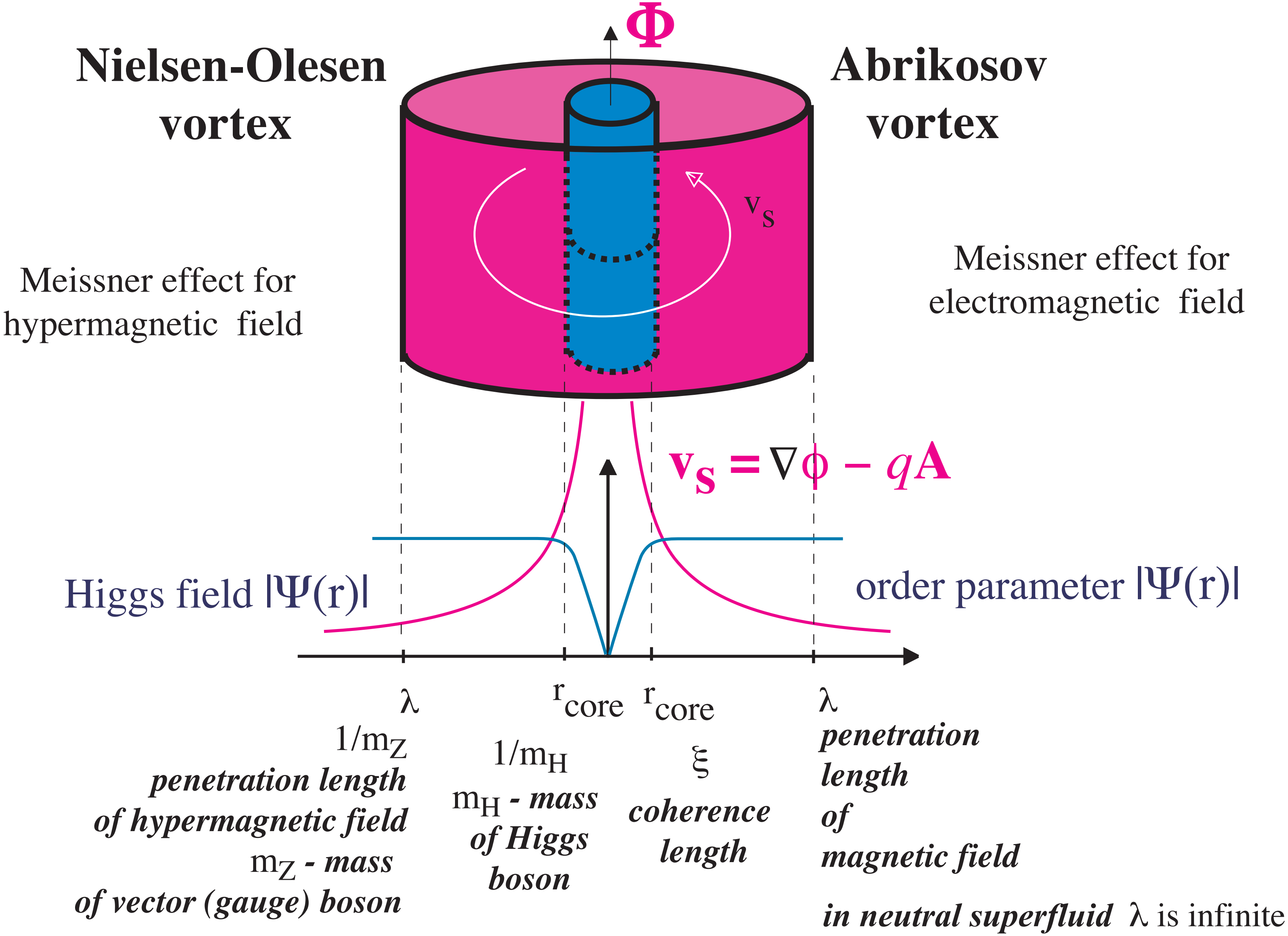
$$\Psi = < \chi \hat{O} \chi >_{vac} + \delta\Psi$$

$$\Psi = < \chi_L \hat{O} \chi_R^+ >_{vac} + \delta\Psi$$

$$H = \begin{pmatrix} \varepsilon(\mathbf{p}) & \Psi \\ \Psi^+ & -\varepsilon(\mathbf{p}) \end{pmatrix}$$

$$H = \begin{pmatrix} c\sigma\cdot\mathbf{p} & \Psi \\ \Psi^+ & -c\sigma\cdot\mathbf{p} \end{pmatrix}$$

Higgs mechanism in Standard Model & superconductor



$$E = |\nabla\Psi - iq\mathbf{A}\Psi|^2 = |\Psi|^2(\nabla\phi - q\mathbf{A})^2 = \lambda^{-2}(\nabla\phi - q\mathbf{A})^2 = \rho_s \mathbf{v_s}^2$$

4 Goldstone & 14 Higgs modes in ^{3}He-B

$SO_S(3) \times SO_L(3) \times U(1) \longrightarrow SO_J(3)$

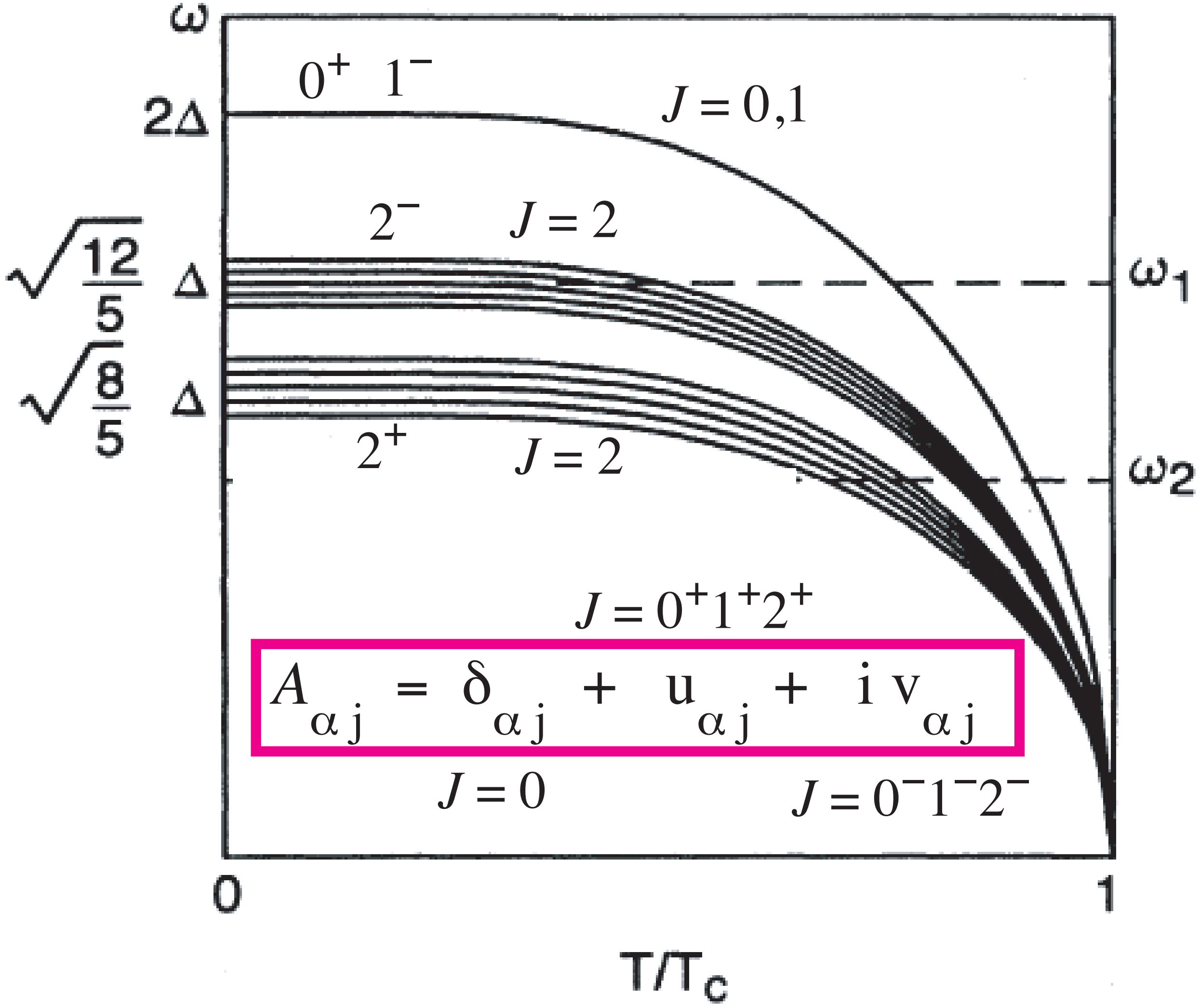


FIG. 3. A schematic plot in ω vs T space for pair breaking, the squashing mode, and the real squashing modes. The Zeeman splitting (not to scale) of the collective modes in an applied magnetic field is shown in the plot. The dashed lines labeled ω_1 and ω_2 correspond to two sound frequencies.

4 Goldstone modes

$$E_{0-} = E_{1+} = 0$$

14 Higgs bosons

$$E_{1-} = E_{0+} = 2\Delta$$

$$E_{2-} = (12/5)^{1/2}\Delta$$

$$E_{2+} = (8/5)^{1/2}\Delta$$

9 pairs of Nambu partners

$$E_{2-}^2 + E_{2+}^2 = 4\,\Delta^2$$

$$E_{0-} + E_{0+} = 4\,\Delta^2$$

$$E_{1-} + E_{1+} = 4\,\Delta^2$$

Nambu sum rule for ^{3}He-B

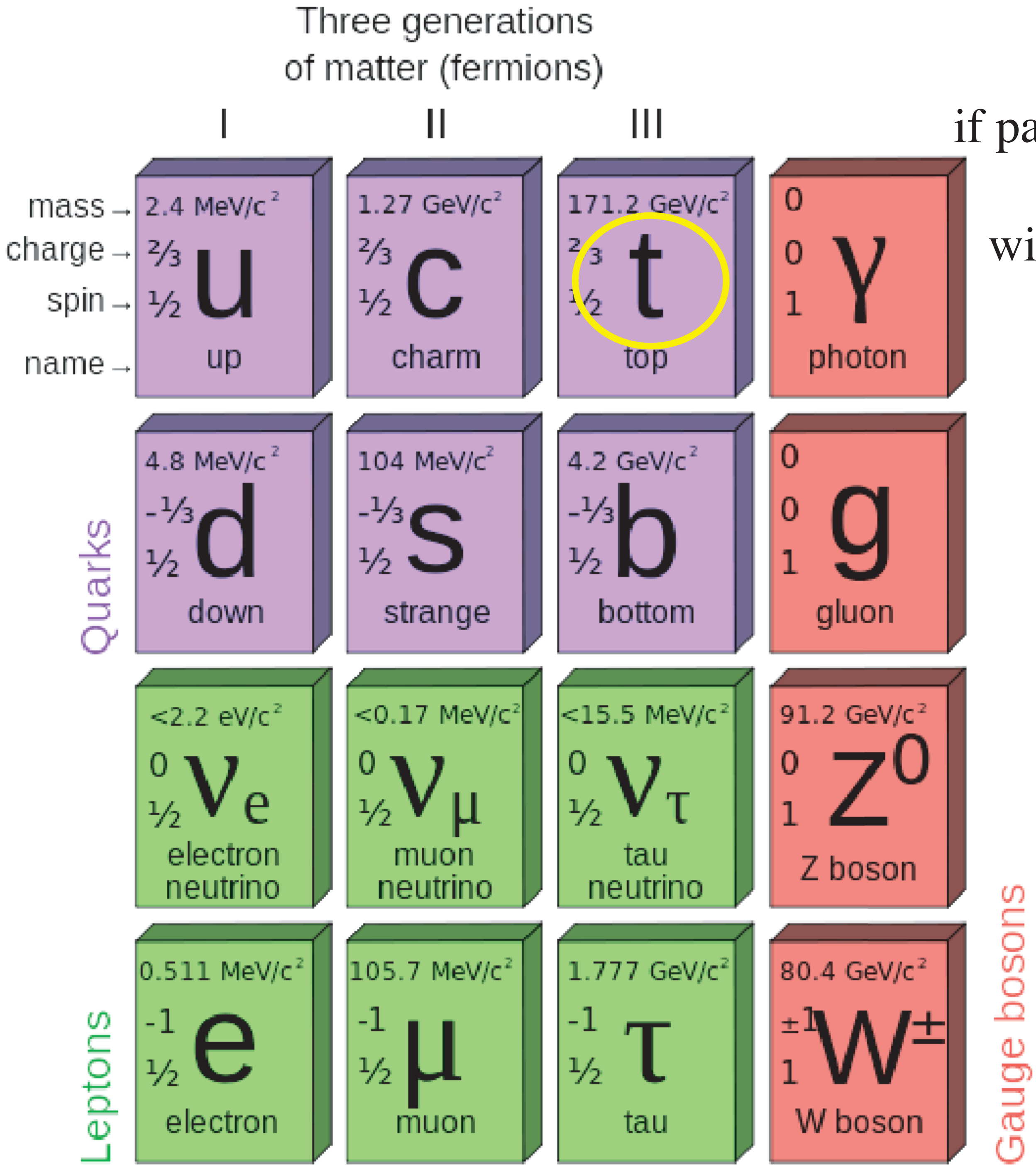
relation between energies E_B & E_F of bosonic & fermionic excitations in BCS type theories

$$E_{B1}^2 + E_{B2}^2 = 4\,E_F^2$$

3He-B	$E_{B1} = (8/5)^{1/2}\Delta$	$E_{B2} = (12/5)^{1/2}\Delta$	$E_F = \Delta$
	real squashing mode	squashing mode	gap in quasiparticle spectrum

which fermion is responsible?

Standard Model



if pairing occurs in one channel
fermion in this channel
will have largest gap (mass)

Nambu sum rule: from 3He-B to Standard Model

relation between energies E_B & E_F of bosonic & fermionic excitations in BCS type theories

$$\boxed{E_{B1}^2} + \boxed{E_{B2}^2} = \boxed{4\,E_F^2}$$

3He-B	$E_{B1} = (8/5)^{1/2}\Delta$	$E_{B2} = (12/5)^{1/2}\Delta$	$E_F = \Delta$
	real squashing mode	squashing mode	gap in quasiparticle spectrum

Application of Nambu rule to masses of Higgs fields and top quarks

$$\boxed{m_{H1}^2} + \boxed{m_{H2}^2} = \boxed{4\,m_t^2}$$

125 GeV	325 GeV	174 GeV
discovered Higgs	Nambu partner Higgs	top quark

hints of 325 GeV Higgs boson in earlier experiments

* CDF/PUB/EXOTICS/PUBLIC/10603 July 17, 2011
Search for High-Mass Resonances Decaying
into ZZ in $p\bar{p}$ Collisions at $s^{1/2} = 1.96$TeV

"The invariant masses of four events are clustered around 325 GeV"

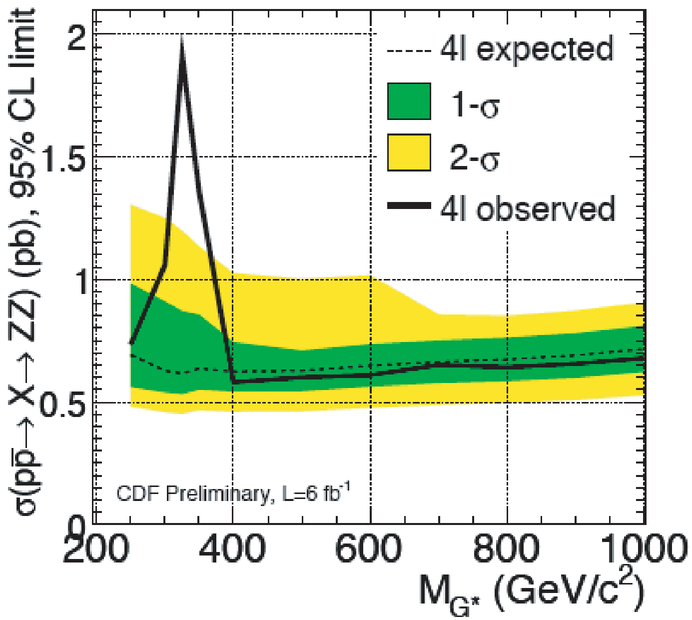
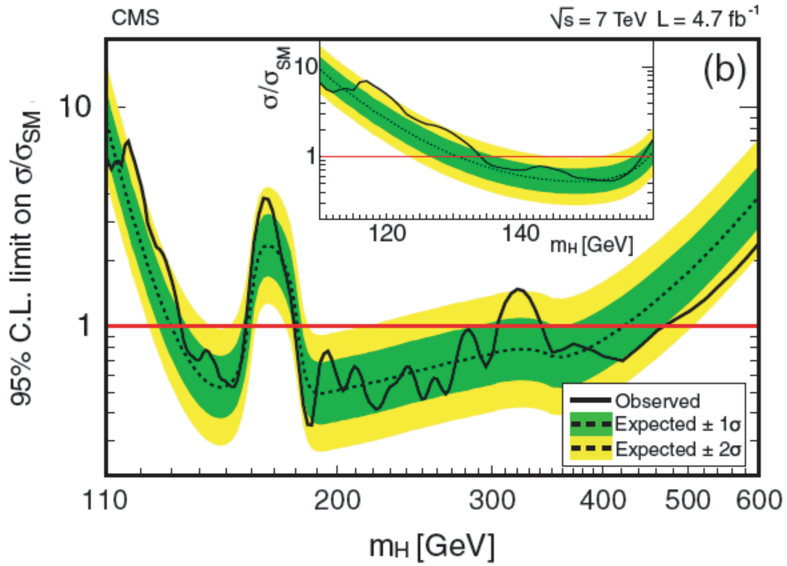


FIG. 13: Expected and observed 95% CL limits on $\sigma(p\bar{p} \to X \to ZZ)$ from the $ZZ \to \ell^+\ell^-\ell^+\ell^-$ channel; the four events with M_{ZZ}=327 GeV/c^2 result in a deviation from the expected limit.

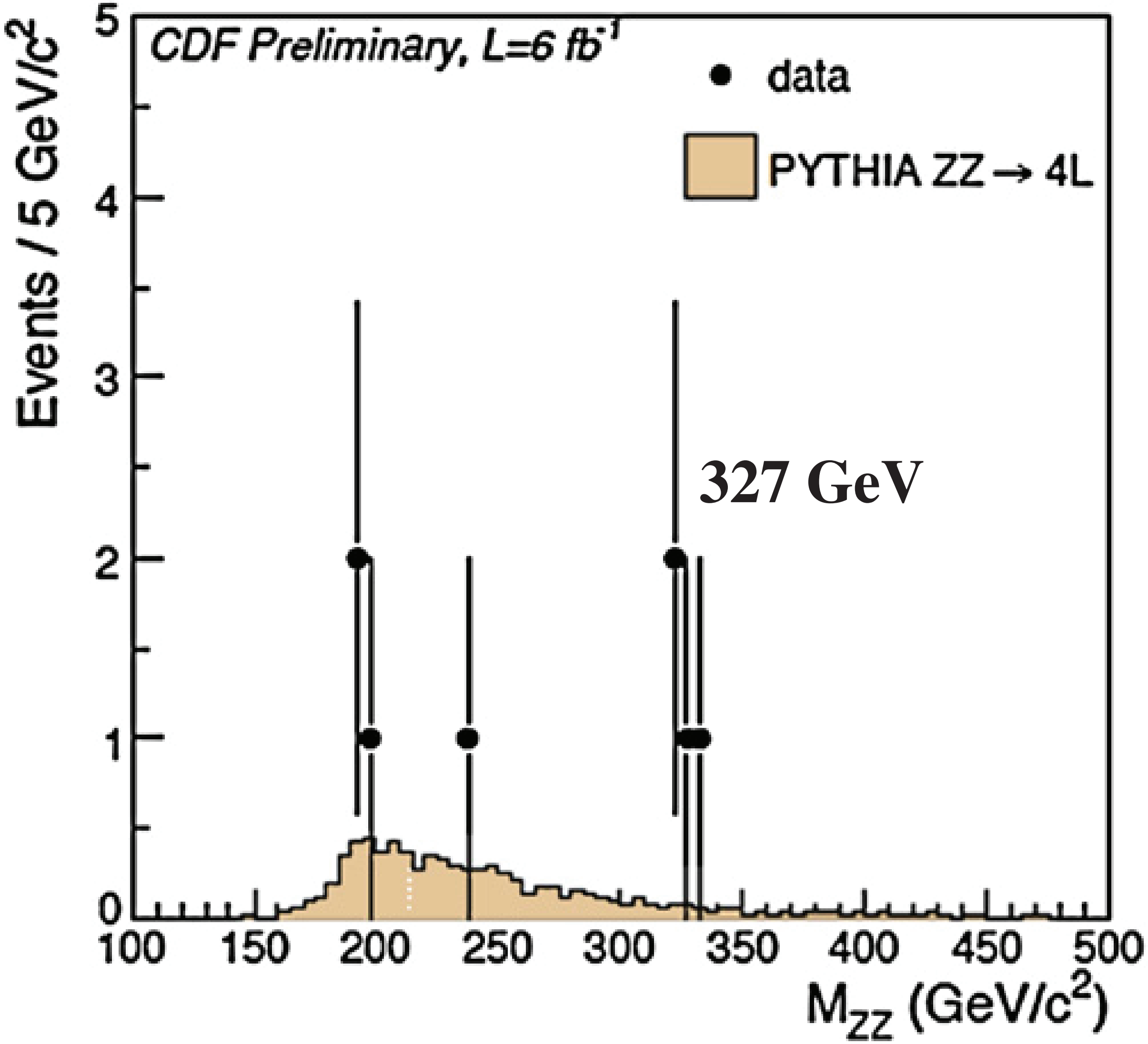
* Search for the Standard Model Higgs Boson
in CMS Collaboration @ LHS
(Compact Muon Solenoid))

PRL 108, 111804 (2012)

"Small excesses of events are observed around masses of 119, 126 & 320 GeV "



A narrow scalar resonance at 325 GeV?
Krzysztof A. Meissner & Hermann Nicolai
Phys. Lett. B **718** (2013) 943



"We propose to identify the excess of events with four charged leptons at E ~ 325 GeV seen by the CDF Collaboration (2012) [1] & CMS Collaboration (2012) [2] with a new 'sterile' scalar particle characterized by a very narrow resonance of the same height and branching ratios as the Standard Model Higgs boson, as predicted in the framework of the so-called Conformal Standard Model (K.A. Meissner & H. Nicolai 2007) [3]."

Fig. 1. The four lepton events reported by the CDF Collaboration [1].

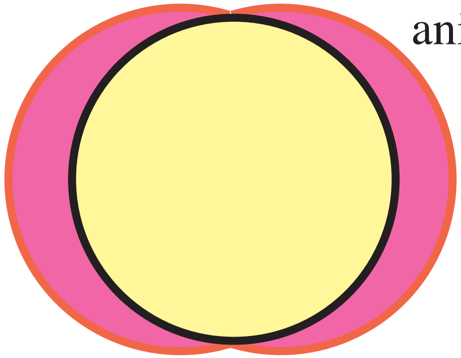
9 Goldstone + 9 Higgs modes in bulk ^{3}He-A

$U_Q(1)$ - analog of electromagnetic group

quantum number Q - analog of electric charge

$SO_S(3) \times SO_L(3) \times U(1) \longrightarrow SO_S(2) \times U_Q(1)$

Modes	Variables	In the absence of dipole interaction and magnetic field: Q	P^2	S_z	P^1
Sound $E = 0$	$u_{23}-v_{13}$	0	−1	0	+1
$E = 0$ Spin waves	$u_{11}+v_{21}$, $u_{12}+v_{22}$	0	+1	±1	–
Orbital modes	u_{33}, v_{33}	±1	–	0	−1
$E = 0$ Spin-orbit modes	u_{31}, v_{31} u_{32}, v_{32}	±1	–	±1	–
	$u_{13}+v_{23}$	0	+1	0	+1
$E^2 = 8/3\ \Delta_0^2$	$u_{22}-v_{12}$, $v_{11}-u_{21}$	0	−1	±1	–
$E^2 = 4/3\ \Delta_0^2$ "Clapping" modes	$u_{23}+v_{13}$, $u_{13}-v_{23}$	±2	–	0	+1
	$u_{11}-v_{21}$, $u_{21}+v_{11}$, $u_{12}-v_{22}$, $u_{22}+v_{12}$	±2	–	±1	–



anisotropic gap with nodes

$\Delta(\theta) = \Delta_0 \sin\theta$

$< \Delta^2 > = 2/3\ \Delta_0^2$

3 Goldstones + 9 Higgs form 6 Nambu pairs

$E_{0-}^2 + E_{0+}^2 = 4 < \Delta^2 >$

$E_{0-}^2 + E_{0+}^2 = 4 < \Delta^2 >$

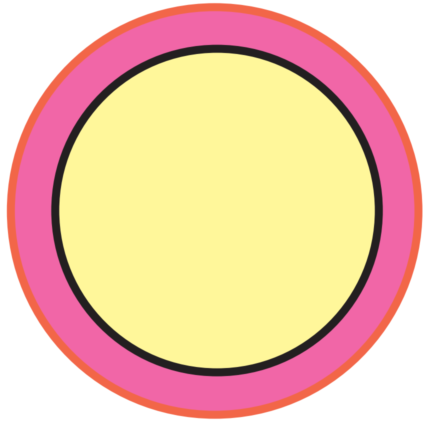
$E_{-2}^2 + E_{+2}^2 = 4 < \Delta^2 >$

6 Goldstones (2 "photons" + 4 from hidden symmetry) have no Nambu partners

3 Goldstone + 9 Higgs modes in ^{3}He-A film

$SO_S(3) \times SO_L(2) \times U(1) \longrightarrow SO_S(2) \times U_Q(1)$

$$m_{Q+}^2 + m_{Q-}^2 = 4\, m_f^2$$



isotropic gap Δ in 2D

3He-A film: quantum number Q is analog of electric charge, $m_f = \Delta$ gap

Q=0 $E_{0-} = 0$ $E_{0+} = 2\,\Delta$ $E_{0+}^2 + E_{0-}^2 = 4\,\Delta^2$

Goldstone (sound mode)

pair breaking mode

|Q|=1 $E_{-1} = 2^{1/2}\Delta$ $E_{+1} = 2^{1/2}\Delta$ $E_{-2}^2 + E_{+2}^2 = 4\,\Delta^2$

clapping modes
analogs of charged Higgs bosons

Nambu sum rule: from ^{3}He-A to Standard Model

relation between energies E_B & E_F of bosonic & fermionic excitations in BCS type theories

$$\boxed{E_{B1}^2} + \boxed{E_{B2}^2} = \boxed{4\,E_F^2}$$

2D 3He-A $E_{B+} = 2^{1/2}\Delta$ $E_{B-} = 2^{1/2}\Delta$ $E_F = \Delta$

clapping modes

gap in quasiparticle spectrum

Application of Nambu rule to masses of charged Higgs fields and top quarks

$$\boxed{m_{H+}^2} + \boxed{m_{H-}^2} = \boxed{4\,m_t^2}$$

245 GeV	245 GeV	174 GeV
charged Higgs	charged Higgs	top quark

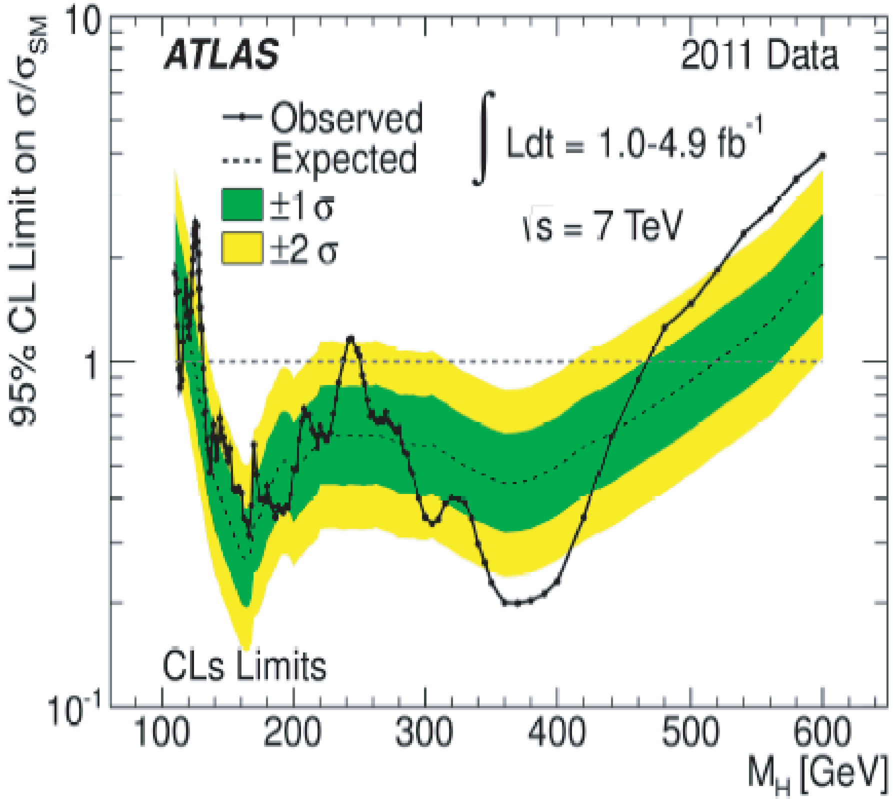
experimental evidence ?

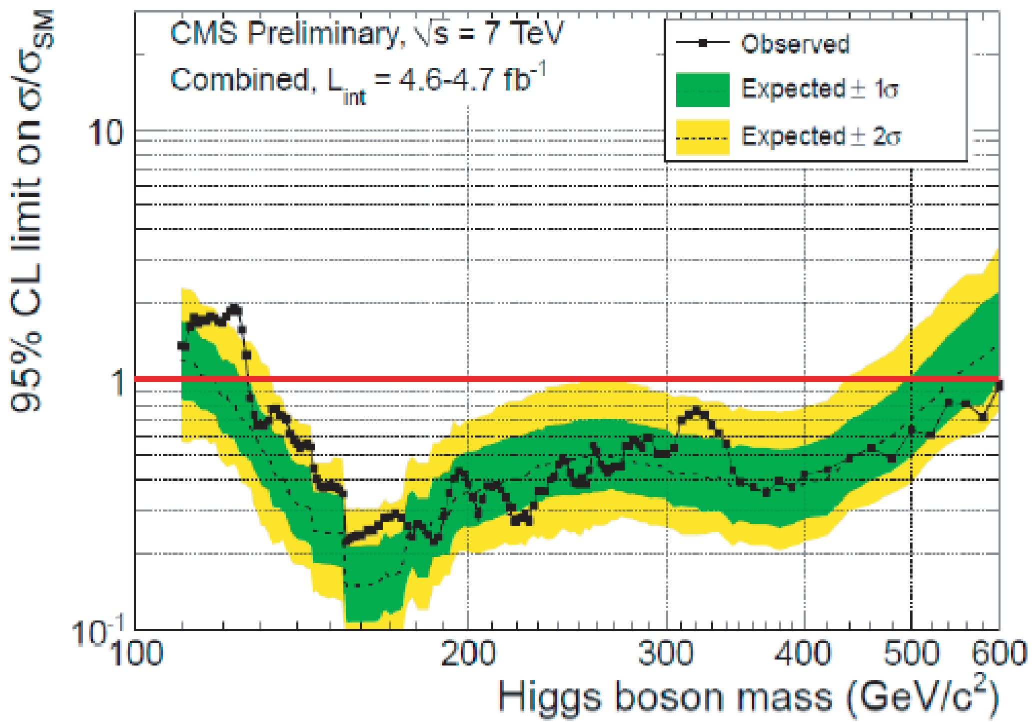
ATLAS @ LHC
PLB **710** 49 (2012)

hints of 245 GeV Higgs bosons in earlier experiments

* ATLAS-CONF-2011-135 August 21, 2011
Update of the Combination of Higgs Boson Searches

"The second was peaking at a Higgs boson mass hypothesis of 245 GeV"





Excess of events in $H \to \gamma\gamma, H \to ZZ$

ATLAS near M_H ~ 126 GeV
and near M_H ~ 245 GeV

CMS near M_H ~ 124 GeV
and near M_H ~ 119.5 GeV

excitation of Higgs boson by quench

dynamics of vacuum energy Λ in cosmology

$$\Lambda(t) \sim \omega^2 \frac{\sin^2 \omega t}{t^2}$$

$\omega \sim M_{\text{inflaton}}$

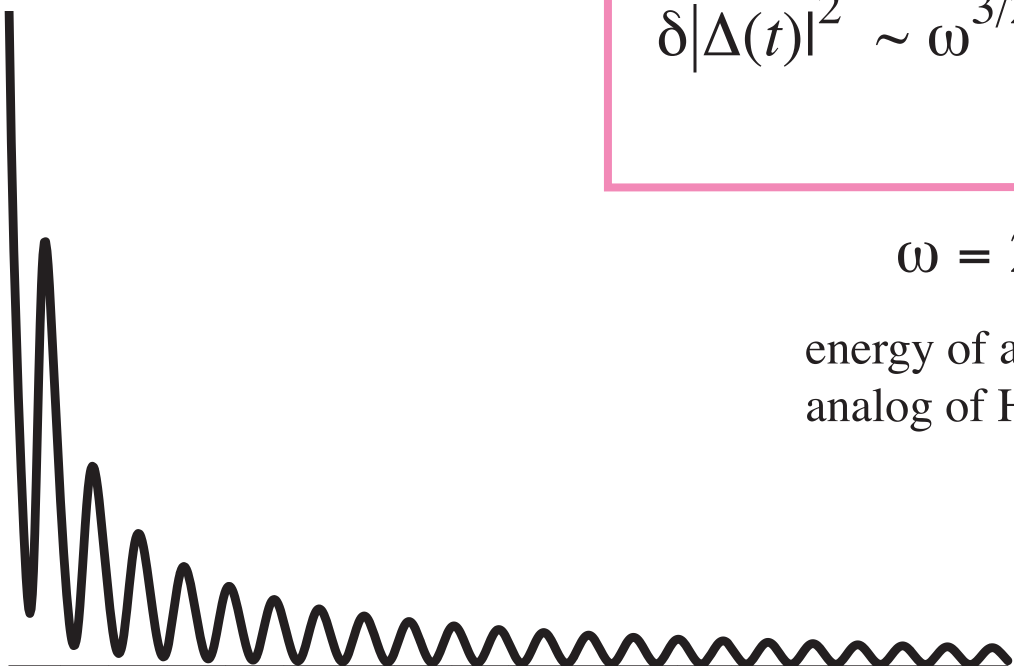
mass of Higgs inflation

dynamics of gap Δ in superconductor

$$\delta|\Delta(t)|^2 \sim \omega^{3/2} \frac{\sin \omega t}{t^{1/2}}$$

$\omega = 2\Delta$

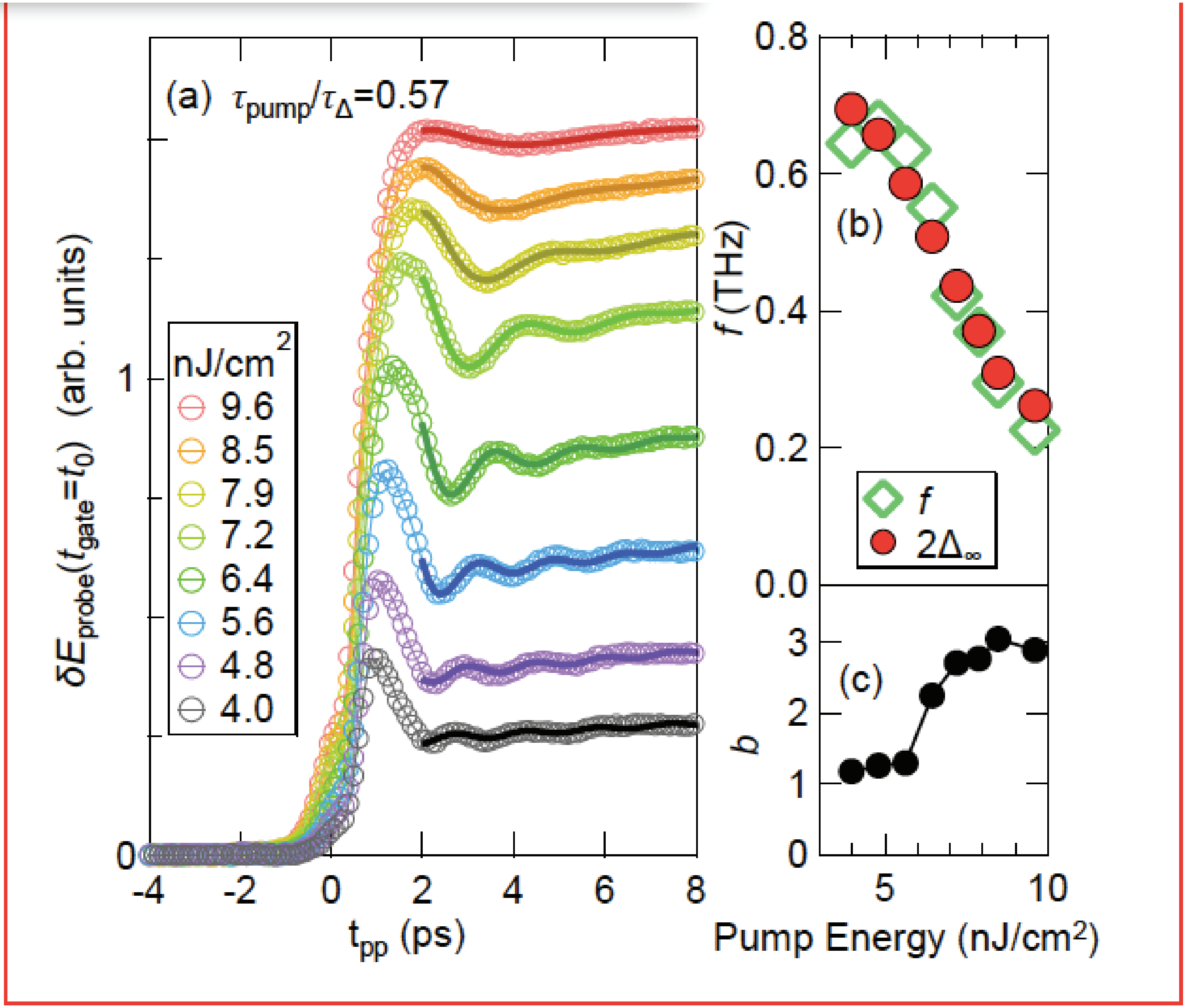
energy of amplitude mode:
analog of Higgs mass



Starobinsky Higgs inflation

V. Gurarie, 0905.4498, 1307.1485, 1307.2256
A.F. Volkov & S.M. Kogan, JETP **38**, 1018 (1974)
Barankov & Levitov, ...

Higgs amplitude mode after quench



Higgs Amplitude Mode in BCS Superconductors $Nb_{1-x}Ti_x$ induced by Terahertz Pulse Excitation
R. Matsunaga, et al. arXiv:1305.0381

Conclusion

Momentum-space topology determines:

universality classes of quantum vacua

effective field theories in these quantum vacua

topological quantum phase transitions (Lifshitz, plateau, etc.)

quantization of Hall and spin-Hall conductivity

topological Chern-Simons & Wess-Zumino terms

quantum statistics of topological objects

edge states (bulk-surface correspondence)

fermion zero modes on quantum vortices (bulk-vortex correspondence)

chiral anomaly, chiral magnetic effect, spectral flow force in vortex dynamics ...

exotic fermions: Dirac fermions with quadratic, cubic, quartic ... spectrum,
flat band, Fermi arc, Majorana fermions, etc.

effective gravity, where tetrad e_a^{μ} is more fundamental than metric $g^{\mu\nu}$

new type of gravity (Horava gravity with anisotropic scaling)