Lecture 1

Bose-Einstein condensation.From discovery to modern results

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- Basic concepts. Role of the density of states
- BEC in trapped gases
- Dynamics of evolving condensates
- BEC in dipolar quantum gases
- Supersolid states of bosons

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Introduction

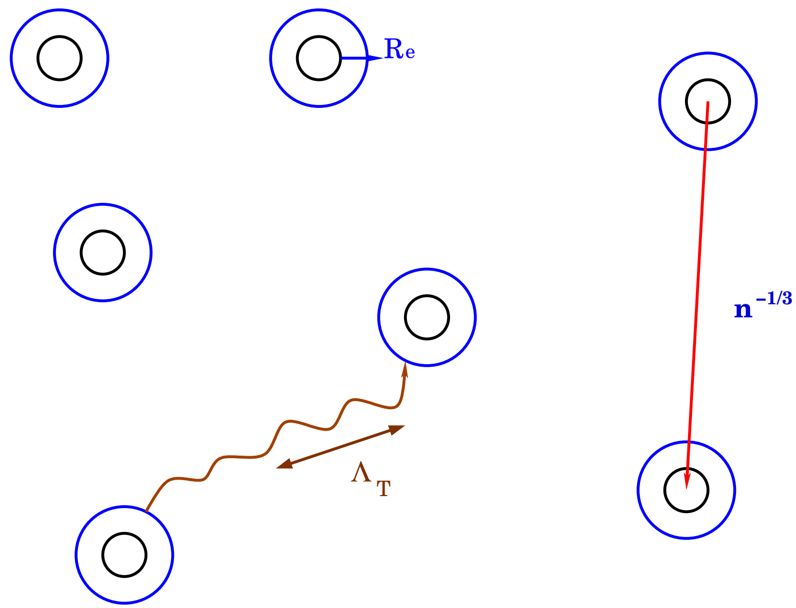
BEC $\rightarrow$ macroscopic number of identical bosons in a single quantum state

Prediction of BEC $\rightarrow$ Bose/Einstein, 1924

Discovery of superfluidity in liquid He $\rightarrow$ Kapitsa, 1937

Discovery of BEC in dilute atomic gases $\rightarrow$ Cornell/Wieman/Ketterle, 1995

Length scales



$$\Lambda_T = \left(\frac{2\pi\hbar^2}{mT}\right)^{1/2} \rightarrow$$ thermal de Broglie wavelength

$R_e \rightarrow$ characteristic radius of interparticle interaction

$n^{-1/3} \rightarrow$ mean interparticle distance

$n^{-1/3} \gg R_e \rightarrow \; nR_e^3 \ll 1$ dilute limit

$\Lambda_T \gg R_e \rightarrow \; k_T R_e \ll 1$ ultracold limit

First ultracold gas $\Rightarrow$ atomic H↑, Silvera/Walraven (1979)

$n \sim 10^{16}$ cm^{-3}, $\quad T \sim 300$ mK

Energy scales

$T \rightarrow$ temperature

Interaction between particles

Interaction energy for 2 particles in the ultracold limit

$$\epsilon_{int} = \frac{g}{V}; \qquad g = \frac{4\pi\hbar^2}{m}a$$

$a \rightarrow$ scattering length

Total interaction energy for identical bosons

$$E_{int} = \frac{N^2}{2}\frac{g}{V}$$

Interaction energy per particle $\dfrac{\partial E}{\partial N} = ng$

Present experiments

$n \sim 10^{12} - 10^{14}$ cm^{-3}; $N \sim 10^5 - 10^8$

$ng \sim 1 - 100$ nK

$T \sim 10$ nK - 1μK

Statistics

$n^{-1/3} \gg \Lambda_T \rightarrow \; n\Lambda_T^3 \ll 1$ Boltzmann statistics

$n\Lambda_T^3 \gtrsim 1 \rightarrow$ Quantum statistics

$n\Lambda_T^3 \rightarrow$ degeneracy parameter

$n\Lambda_T^3 \simeq 1$ at degeneracy temperature

$$T_d = \frac{2\pi\hbar^2}{m} n^{2/3}$$

Weakly interacting regime

$$\frac{4\pi\bar{r}^3}{3} = \frac{1}{n}$$

$$\bar{r} = \left(\frac{3}{4\pi n}\right)^{1/3}$$

$T \to 0$ Kinetic energy in a box of 1 particle

$$\frac{\hbar^2}{m\bar{r}^2}$$

Particle wavefunction is not influenced by interaction with other particles

$$\frac{\hbar^2}{m\bar{r}^2} \gg ng \;\Rightarrow\; n|a|^3 \ll 1$$

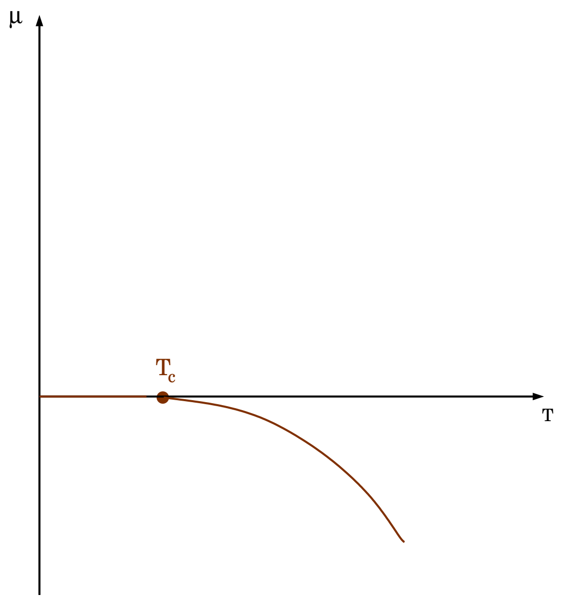
weakly interacting regime

BEC in an ideal gas

$$\hat{H} = \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}; \quad \mu \leq 0$$

$$N_k = \langle \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \rangle = \frac{1}{\exp\left(\frac{E_k - \mu}{T}\right) - 1}; \qquad E_k = \frac{\hbar^2 k^2}{2m}$$

$$N = \sum_{\mathbf{k}} N_k = \int \frac{V d^3 k}{(2\pi)^3} \frac{1}{\exp\left(\frac{E_k - \mu}{T}\right) - 1}$$



No solution at $T < T_c = \dfrac{3.31\hbar^2}{m} n^{2/3}$

BEC in an ideal gas

$n\Lambda^3_{T_c} = 2.62 \rightarrow$ quantum degenerate gas

The way to resolve $\rightarrow$

At $T < T_c$ a macroscopic number of particles goes to $\mathbf{k} = 0$

$$N = N(k=0) + N(k>0)$$

k-distribution $\rightarrow \mu = 0$

$$N(k>0) = N' = \int_0^\infty \frac{V\, 4\pi k^2 dk}{(2\pi)^3} \frac{1}{\exp\left(\frac{E_k}{T}\right) - 1} = N \left(\frac{T}{T_c}\right)^{3/2}$$

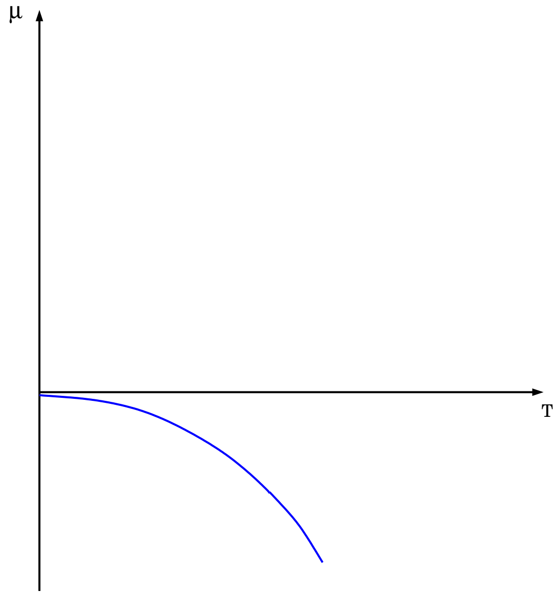
$$N_0 \equiv N(k=0) = N \left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]$$

"Condensation in the momentum space"$\Rightarrow$ BEC

2D ideal gas

Free space $N = \int_0^\infty \frac{A\,2\pi k dk}{(2\pi)^2} \frac{1}{\exp\left(\frac{E_k-\mu}{T}\right) - 1}$

$$\mu = -T \ln\left[\frac{1}{1-\exp(-n\Lambda_T^2)}\right]$$



$$n\Lambda_T^2 \ll 1 \rightarrow \quad \mu = -T\ln\left[\frac{1}{n\Lambda_T^2}\right]$$

$$n\Lambda_T^2 \gg 1 \rightarrow \quad \mu = -T\exp(-n\Lambda_T^2)$$

No BEC at finite T. Only at $T = 0$

Density of states

$$\nu(E) = \int \delta(E - E(k)) \frac{V d^d k}{(2\pi)^d}$$

$$N = \int_0^\infty \frac{\nu(E) dE}{\exp\left(\frac{E-\mu}{T}\right) - 1}$$

$$\mu \to 0$$

3D$\Rightarrow$ $\nu(E) \propto \sqrt{E}$

2D$\Rightarrow$ $\nu = \mathrm{const}$

1 D$\Rightarrow$ $\nu(E) \propto 1/\sqrt{E}$

The emergence of BEC $\to$

for ν decreasing with E one can not distribute a given number of particles according to the Bose distribution below a certain critical temperature

External potential $V(r)$. Quasiclassical approach

$$\nu(E) = \int \delta(E - E(k) - V(r)) \frac{d^d k d^d r}{(2\pi)^d}$$

BEC in powerlaw potentials Bagnato/Kleppner, 1991

Role of interactions

Weakly interacting regime. At $T \ll T_c$ almost all particles in BEC

$$E_{int} = \frac{N^2}{2V} g$$

$g < 0 \Rightarrow$ E_{int} decreases with $V \rightarrow$ collapse

Now $g > 0$

$$\hat{H} = \int d^3 r \left[-\frac{\hbar^2}{2m} \hat{\Psi}^\dagger(\mathbf{r}) \Delta_{\mathbf{r}} \hat{\Psi}(\mathbf{r}) + \hat{\Psi}^\dagger(\mathbf{r}) V(\mathbf{r}) \hat{\Psi}(\mathbf{r}) + \frac{g}{2} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right]$$

Heisenberg equation of motion

$$i\hbar \frac{\partial \hat{\Psi}}{\partial t} = (\hat{\psi}\hat{H} - \hat{H}\hat{\psi}) = \left(-\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} + V(\mathbf{r}) + g \hat{\Psi}^\dagger \hat{\Psi} \right) \hat{\Psi}$$

$\hat{\Psi} = \Psi_0 + \hat{\Psi}' \rightarrow$ omit small $\hat{\Psi}'$

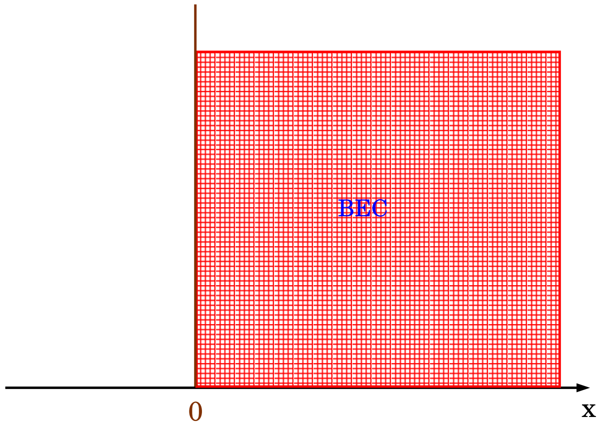
Gross-Pitaevskii equation

$$i\hbar \frac{\partial \Psi_0}{\partial t} = \left(-\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} + V(\mathbf{r}) + g|\Psi_0|^2 \right) \Psi_0$$

Equilibrium $\Psi_0(\mathbf{r}, t) = \psi_0(\mathbf{r}) \exp(-i\mu t/\hbar)$

$$\left(-\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} + V(\mathbf{r}) + g|\psi_0|^2 - \mu \right) \psi_0 = 0$$

Healing length

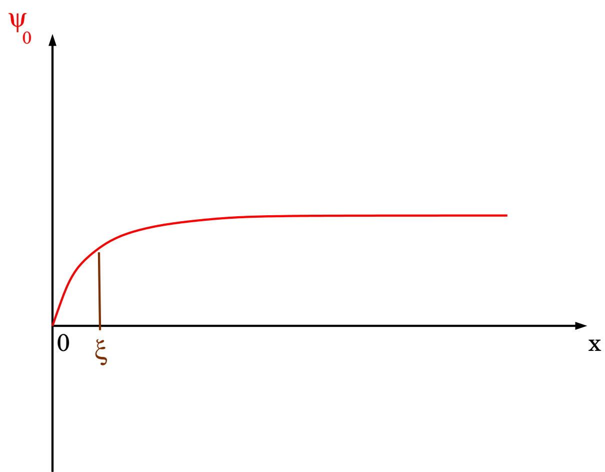


$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + g|\psi_0|^2\psi_0 - \mu\psi_0 = 0$$

$$\psi_0(0) = 0; \;\; \psi_0(\infty) = \sqrt{n_0}$$

Set ψ_0 real $\rightarrow \;\; \psi_0(x) = \pm\sqrt{n_0}\tanh\left(\frac{x}{\xi}\right)$

$$\xi = \frac{\hbar}{\sqrt{mn_0 g}} \Rightarrow$$ healing length



Harmonically trapped BEC

Spherical trap $V(r) = m\omega^2 r^2/2$

Stationary condensate

$$-\frac{\hbar^2}{2m}\Delta_{\mathbf{r}}\psi_0 + \frac{m\omega^2 r^2}{2}\psi_0 + g|\psi_0|^2\psi_0 - \mu\psi_0 = 0$$

$$g = 0 \rightarrow \quad \psi_0 = \frac{\sqrt{N_0}}{\pi^{3/4} l_h^{3/2}} \exp\left(-\frac{r^2}{2l_h^2}\right); \;\; \mu = \frac{3}{2}\hbar\omega$$

Valid for $n_{max}|g| \ll \hbar\omega$

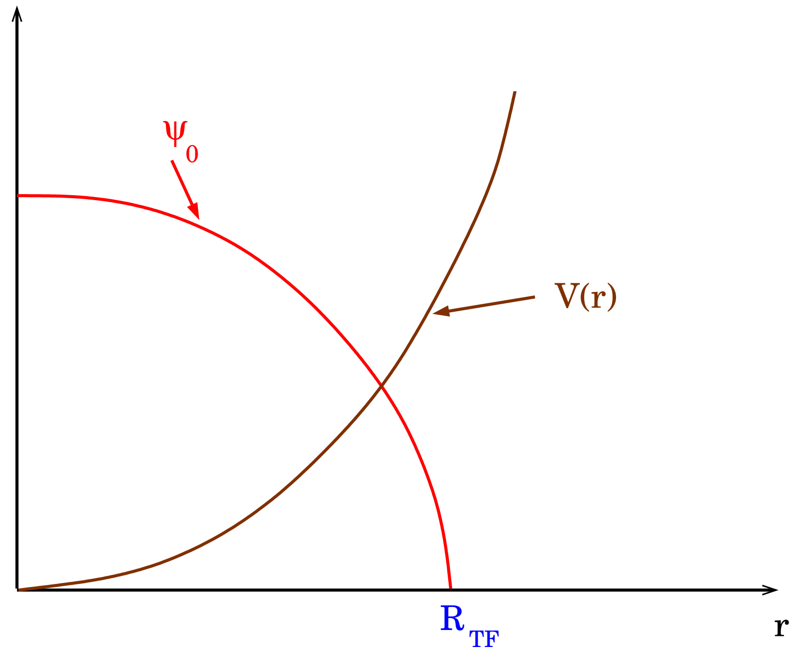
$g > 0$ and $n_{max} g \gg \hbar\omega \rightarrow$ Drop the kinetic energy term

$$g|\psi_0|^2 = \mu - \frac{m\omega^2 r^2}{2}$$

Thomas-Fermi (TF) regime; A. Leggett, 1981

$$\psi_0 = \sqrt{\frac{\mu}{g}\left(1 - \frac{r^2}{R_{TF}^2}\right)}\,\theta(R_{TF} - r); \;\; R_{TF} = \left(\frac{2\mu}{m\omega^2}\right)^{1/2}$$

Harmonically trapped BEC



$$\int_0^{R_{TF}} |\psi_0(r)|^2 d^3r = N_0 \Rightarrow \quad N_0 = \frac{8\pi}{15}\frac{\mu R_{TF}^3}{g}$$

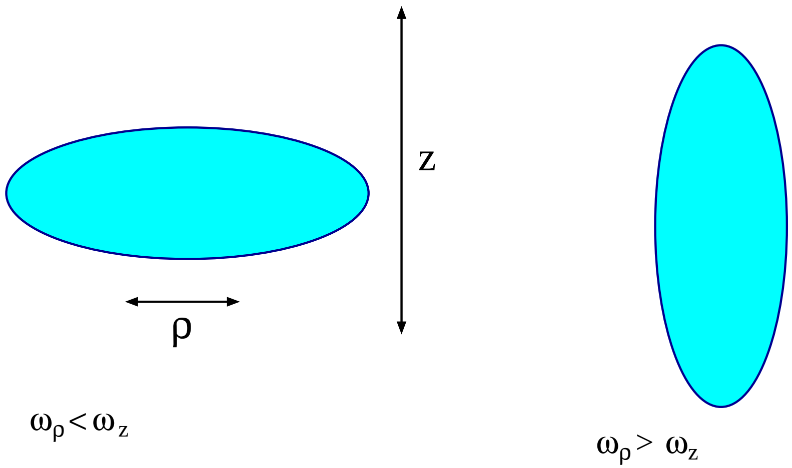
$$\mu = \left(N_0 g \frac{15}{16\sqrt{2}\pi}\right)^{2/5} (m\omega^2)^{3/5}$$

$$n_{0max} = \psi_0^2 = \frac{\mu}{g}$$

$$\xi = \frac{\hbar}{\sqrt{m n_{0max}}} \ll l_h = (\left(\frac{\hbar}{m\omega}\right)^{1/2}$$

Harmonically trapped BEC

Cylindrical trap $V = \frac{m\omega_z^2 z^2}{2} + \frac{m\omega_\rho^2}{2}$



$$\psi_0 = \sqrt{\frac{\mu}{g}\left(1 - \frac{z^2}{R_z^2} - \frac{\rho^2}{R_\rho^2}\right)}; \quad |z| \le R_z, \ \rho \le R_\rho$$

$$R_z = \left(\frac{2\mu}{m\omega_z^2}\right)^{1/2}; \quad R_\rho = \left(\frac{2\mu}{m\omega_\rho^2}\right)^{1/2}$$

Aspect ratio $\frac{R_\rho}{R_z} = \frac{\omega_z}{\omega_\rho}$

Dynamics of evolving condensates

3D spherical trap. Stationary BEC at $t = 0$

TF regime $\dfrac{m\omega_0^2 r^2}{2} + g|\psi_0|^2 - \mu = 0$

Make the frequency time-dependent: $\omega = \omega(t)$ and $\omega(0) = \omega_0$

$$i\hbar \frac{\partial \Psi_0(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} + \frac{m\omega(t) r^2}{2} + g|\Psi_0(\mathbf{r}, t)|^2 \right) \Psi_0(\mathbf{r}, t)$$

rescaled coordinate $\boldsymbol{\rho} = \dfrac{\mathbf{r}}{b(t)}$; rescaled time $\tau(t)$

$$\Psi_0(\mathbf{r}, t) = \frac{1}{b^{3/2}(t)} \chi_0(\boldsymbol{\rho}, \tau(t)) \exp\{i\Phi(\mathbf{r}, t)\}$$

$$\Phi(\mathbf{r}, t) = \frac{mr^2}{2\hbar} \frac{\dot{b}(t)}{b(t)}$$

Dynamics of evolving condensates

$$\frac{i\hbar}{b(t)}\frac{d\tau}{dt}\frac{\partial\bar{\chi}_0}{\partial\tau} = \left\{ -\frac{\hbar^2}{2mb^3(t)}\Delta_{\boldsymbol{\rho}} + \frac{m}{2}[\ddot{b}(t) + \omega^2(t)b(t)]\rho^2 + \frac{g|\bar{\chi}_0|^2}{b^4(t)} - \frac{\mu}{b(t)}\frac{d\tau}{dt}\right\}\bar{\chi}_0$$

$$\left(\frac{\hbar\omega_0}{\mu}\right)^2 b(t) \ll 1$$

$$\frac{m}{2}[\ddot{b}(t) + \omega^2(t)b(t)]\rho^2 + \frac{g|\bar{\chi}_0|^2}{b^4(t)} - \frac{\mu}{b(t)}\frac{d\tau}{dt} = 0$$

$$\tau = \int^t \frac{dt'}{b^3(t')}$$

$$\ddot{b}(t) + \omega^2(t)b(t) = \frac{\omega_0^2}{b^4(t)}$$

$$\frac{m\omega_0^2\rho^2}{2} + g|\bar{\chi}_0|^2 - \mu = 0$$

The same as the initial equation for ψ_0 in terms of $\mathbf{r}$

$$\bar{\chi}_0 = \sqrt{\frac{\mu}{g}\left(1 - \frac{\rho^2}{R_{TF}^2}\right)} = \psi_0(\boldsymbol{\rho}) \quad \rho = \frac{r}{b(t)}$$

Dynamics of evolving condensates

$$\Psi_0(\mathbf{r},t) = \frac{1}{b^{3/2}(t)}\psi_0\left(\frac{\mathbf{r}}{b(t)}\right)\exp\left\{i\frac{mr^2}{2\hbar}\frac{\dot{b}(t)}{b(t)} - i\frac{\mu\tau(t)}{\hbar}\right\}$$

Density profile is rescaled and BEC acquires a dynamical phase

Popov/Perelomov for a particle in a harmonic potential

Kagan et al; Castin/Dum   1996 - 1997

Expansion of BEC

$\Omega(t) = 0$ at $t > 0$

$$\ddot{b}(t) = \frac{\omega_0^2}{b^4(t)}; \quad \frac{\dot{b}^2}{2} = -\frac{\omega_0^2}{3b^2(t)} + \frac{\omega_0^2}{3}$$

$b \gg 1 \Rightarrow \; b(t) = \sqrt{\frac{2}{3}}\omega_0 t;$ free expansion $(t \gg \omega_0^{-1})$

Expanding BEC radius $R(t) \propto R_{TF}\omega_0 t$

$$R_{TF} \propto \left(\frac{n_{omax} g}{m\omega_0^2}\right)^{1/2} \rightarrow \quad R(t) \propto \sqrt{\frac{n_{0max} g}{m}} t$$

BEC expands with the velocity of sound $c_s = \sqrt{\frac{n_0 g}{m}}$

Collisionless thermal gas expands with the thermal velocity $v_t = \sqrt{\frac{2T}{m}}$

For $T \gg n_0 g$ one has $v_T \gg c_s$

The thermal cloud flies away much faster

Cylindrical trap

$$\frac{m}{2}(\omega_\rho^2\rho^2 + \omega_z^2 z^2) + g|\psi_0|^2 - \mu = 0$$

$$\psi_0 = \sqrt{\frac{\mu}{g}\left(1 - \frac{z^2}{R_z^2} - \frac{\rho^2}{R_\rho^2}\right)}; \quad |z| \le R_z, \ \rho \le R_\rho$$

$$R_z = \left(\frac{2\mu}{m\omega_z^2}\right)^{1/2}; \quad R_\rho = \left(\frac{2\mu}{m\omega_\rho^2}\right)^{1/2}$$

Similar scaling solution for $\omega_\rho(t); \ \omega_z(t)$

$$\Psi_0(\rho, z, t) = \frac{1}{\sqrt{\mathcal{V}(t)}}\chi_0(u_\rho, u_z, \tau)\exp\{i\Phi(\rho, z, t)\}; \ u_z = \frac{z}{b_z(t)}; \ u_\rho = \frac{\rho}{b_\rho(t)}$$

$$\Phi(\rho, z, t) = \frac{m}{2\hbar}\left(\frac{\dot{b}_\rho(t)}{b_\rho(t)}\rho^2 + \frac{\dot{b}_z(t)}{b_z(t)}z^2\right); \quad \mathcal{V}(t) = b_\rho^2(t)b_z(t)$$

$$\ddot{b}_\rho(t) + \omega_\rho^2(t)b_\rho(t) = \frac{\omega_{0\rho}^2}{b_\rho(t)\mathcal{V}(t)}$$

$$\ddot{b}_z(t) + \omega_z^2(t)b_z(t) = \frac{\omega_{0z}^2}{b_z(t)\mathcal{V}(t)}$$

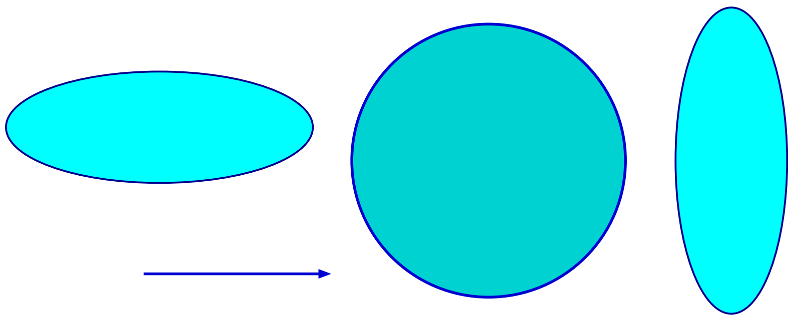
Cylindrical trap. Expansion

$$\omega_z(t) = 0; \;\; \omega_\rho(t) = 0; \;\; t \geq 0$$

$$\ddot{b}_\rho = \frac{\omega_{0\rho}^2}{b_\rho^3 b_z}; \quad \ddot{b}_z = \frac{\omega_{0z}^2}{b_z^2 b_\rho^2}$$

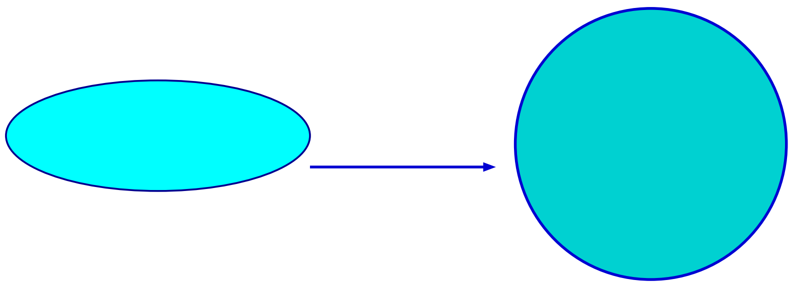
$$b_z(0) = b_\rho(0) = 1; \quad \dot{b}_z(0) = \dot{b}_\rho(0) = 0$$

The BEC cloud expands faster in the tighter confined direction



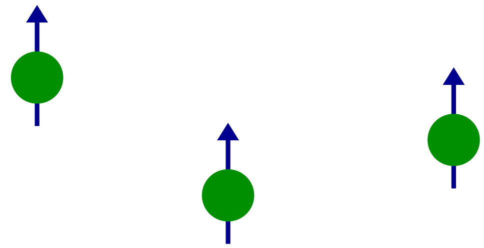
Asymmetric expansion

The collisionless thermal cloud becomes spherical

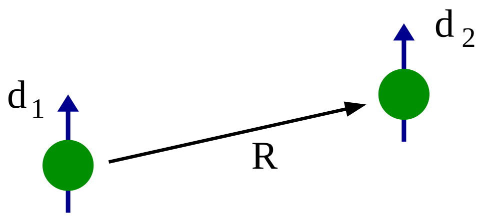


Dipolar quantum gases

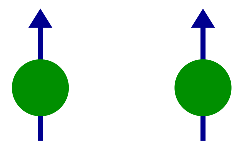
Polar molecules or atoms with a large magnetic moment



Dipole-dipole interaction $V_d = \frac{\vec{d}_1 \vec{d}_2 R^2 - 3(\vec{d}_1 \vec{R})(\vec{d}_2 \vec{R})}{R^5} \sim \frac{1}{R^3}$



long-range, anisotropic



repulsion

attraction

Different physics compared to ordinary atomic ultracold gases

Alkali-atom molecules d from $0.6\,D$ for KRb to $5.5\,D$ for LiCs

Experiments

First magnetic atom experiments

Cr ($6\mu_B$) Stuttgart; Villetaneuse

Dy ($10\mu_B$) Stanford; Stuttgart

Er ($7\mu_b$) Innsbruck

First ground-state polar molecule experiments

KRb JILA

CsRb Innsbruck

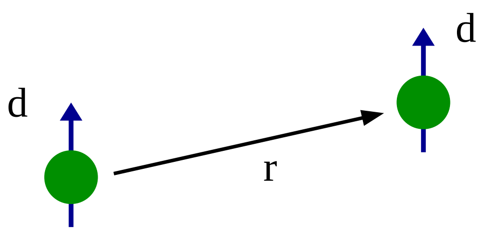
NaK and NaLi at MIT; NaK in Munich

NaRb in Hong Kong

The main initial goal $\Rightarrow$ Reveal the role of dipolar interactions

Stability diagram and the shape of a trapped cloud

Radius of the dipole-dipole interaction



$$\left(-\frac{\hbar^2}{m}\Delta + V_d(\vec{r})\right)\psi(\vec{r}) = \frac{\hbar^2 k^2}{m}\psi(\vec{r})$$

$$\frac{\hbar^2}{m r_*^2} = \frac{d^2}{r_*^3} \Rightarrow r_* \approx \frac{m d^2}{\hbar^2}$$

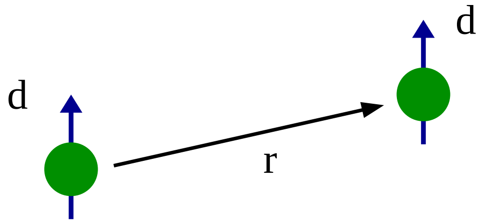
$r \gg r_*$ $\rightarrow$ free relative motion

$r_* \sim 10^6 \div 10^3 a_0$ polar molecules

$r_* \approx 50 a_0 \rightarrow$ chromium atoms

$$k r_* \ll 1 \qquad \rightarrow \underbrace{\text{Ultracold limit}}_{T \ll 1mK \ \ \text{for} \ \ \text{Cr}}$$

Scattering amplitude



$$V(\vec{r}) = \mathcal{U}(\vec{r}) + V_d(\vec{r})$$
$$f = \int \psi^*_{k_i}(\vec{r}) V(\vec{r}) e^{i\vec{k}_f \vec{r}} d^3 r$$

Ultracold limit $kr_* \ll 1$

$$V_d = 0 \Rightarrow f = g = \frac{4\pi\hbar^2}{m} a$$

What V_d does?

$$k = 0 \rightarrow g = \int \psi^*_0(\vec{r})(\mathcal{U}(\vec{r}) + V_d(\vec{r}))d^3 r = \text{const}; \qquad r \lesssim r_*$$

g may depend on d

Scattering amplitude

$$k \neq 0$$

$$f = \int \psi^*_{k_i}(\vec{r}) V(\vec{r}) e^{i\vec{k}_f \vec{r}} d^3 r$$

$$r \lesssim r_* \to \text{put } k = 0 \to g$$

$$r \gg r_* \to \psi_{k_i} = e^{i\vec{k}_i \vec{r}}$$

$$f = \int V_d(\vec{r}) e^{i\vec{q}\vec{r}} d^3 r \longrightarrow \frac{4\pi d^2}{3}(3\cos^2\theta_{qd} - 1); \; \vec{q} = \vec{k}_f - \vec{k}_i$$

$$f = g + \frac{4\pi d^2}{3}(3\cos^2\theta_{qd} - 1)$$

Dipolar BEC in free space

Uniform gas

$$H = \int d^3 \Big[\psi^\dagger(\vec{r}) \left(-\frac{\hbar^2}{2m}\Delta\right)\psi(\vec{r}) + \frac{1}{2} g \psi^\dagger(\vec{r})\psi^\dagger(\vec{r})\psi(\vec{r})\psi(\vec{r})$$
$$+\frac{1}{2}\int d^3 r' \psi^\dagger(\vec{r})\psi^\dagger(\vec{r}')V_d(\vec{r}-\vec{r}')\psi(\vec{r}')\psi(\vec{r})\Big]$$

Bogoliubov approach $\psi = \psi_0 + \delta\Psi \rightarrow$ biliniear Hamiltonian

$$H_B = \frac{N^2}{2V} g + \sum_k \Big[\frac{\hbar^2 k^2}{2m} a_k^\dagger a_k + n\left(g + \frac{4\pi d^2}{3}\left(3\cos^2\theta_k - 1\right)\right) a_k^\dagger a_k$$
$$+\frac{n}{2}\left(g + \frac{4\pi d^2}{3}\left(3\cos^2\theta_k - 1\right)\right)\left(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}\right)\Big]$$

Dipolar BEC in free space

Excitation spectrum

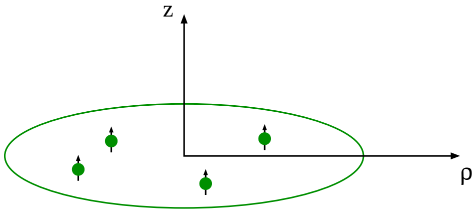
$$\epsilon_k = \sqrt{E_k^2 + 2E_k n \left(g + \frac{4\pi d^2}{3} \left(3\cos^2\theta_k - 1 \right) \right)}$$

$g > \frac{4\pi d^2}{3} \rightarrow$ dynamically stable BEC

$g < \frac{4\pi d^2}{3} \rightarrow$ complex frequencies at small k

$\cos^2\theta_k < \frac{1}{3} \rightarrow$ collapse

Trapped dipolar BEC



Cylindrical trap

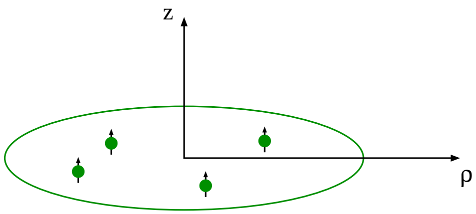
$V_h = \frac{m}{2}\left(\omega_\rho^2\rho^2 + \omega_z^2 z^2\right)$

Gross-Pitaevskii equation

$$\left[-\frac{\hbar^2}{2m}\Delta + V_h(\vec{r}) + g\psi_0^2 + \int \psi_0(\vec{r}')^2 V_d(\vec{r}-\vec{r}')d^3r'\right]\psi_0(\vec{r}) = \mu\psi_0(\vec{r})$$

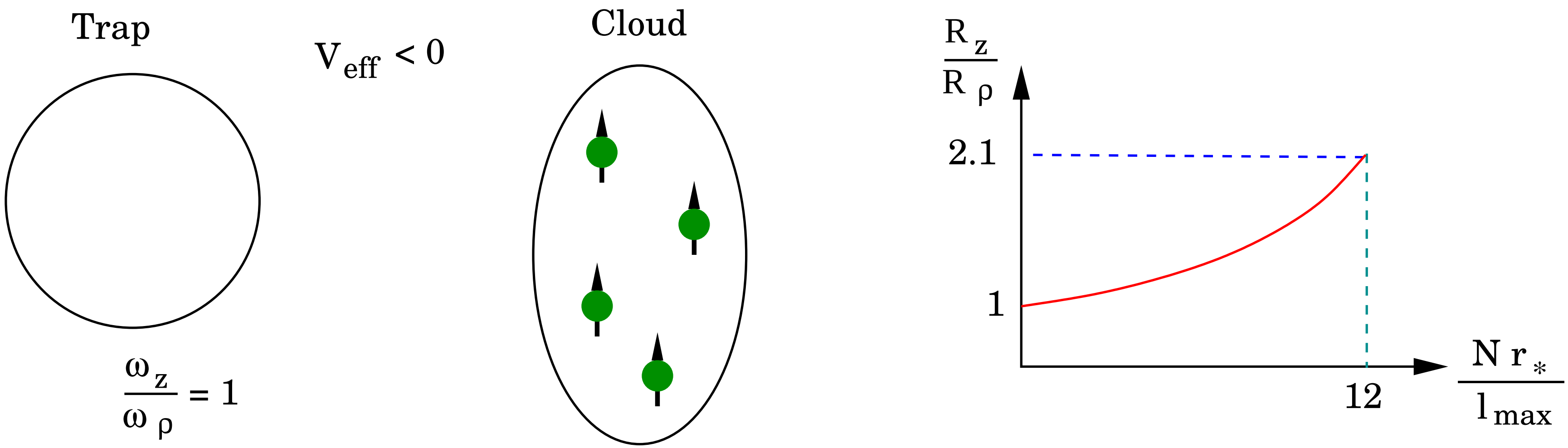
Important quantity

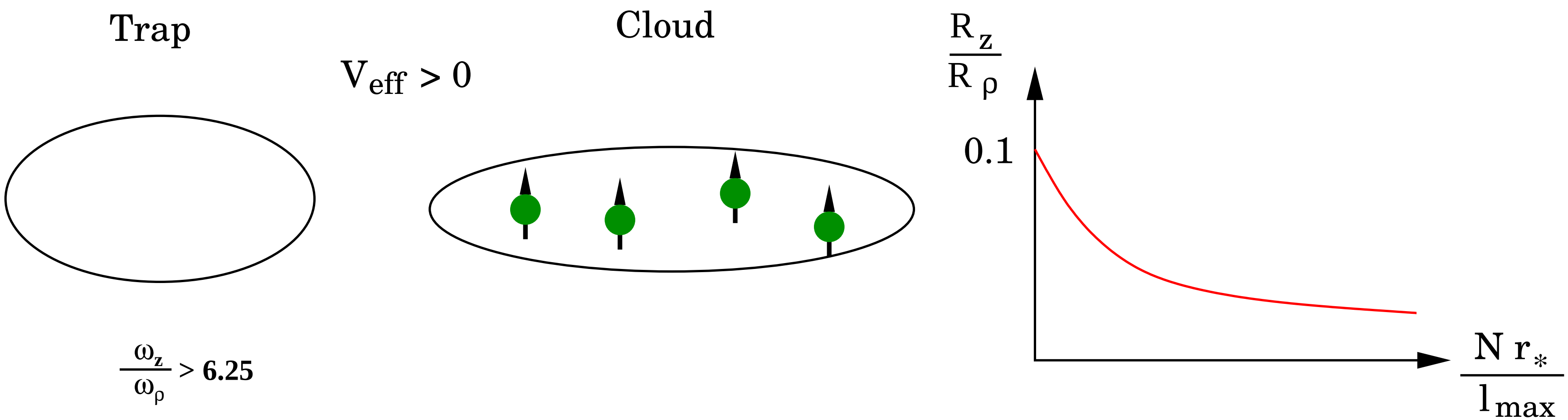
$$V_{eff} = g\int \psi_0^4(\vec{r})d^3r + \int \psi_0^2(\vec{r}')V_d(\vec{r}-\vec{r}')\psi_0^2(\vec{r})d^3r d^3r'$$



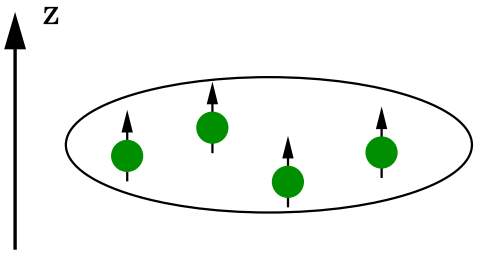
$V_{eff} > 0$ or $V_{eff} < 0$ and $|V| < \hbar\omega$
$g = 0 \rightarrow N < N_c \rightarrow$ suppressed
low k instability
(Santos et.al, 2000)

Trapped dipolar BEC





Stability problem



Dipolar BEC

$$\langle V_d \rangle = \int n_0(\vec{r'}) V_d(\vec{r'} - \vec{r}) d^3r = -d^2 \frac{\partial^2}{\partial z^2} \int \frac{n_0(\vec{r'})}{|\vec{r} - \vec{r'}|} d^3r$$

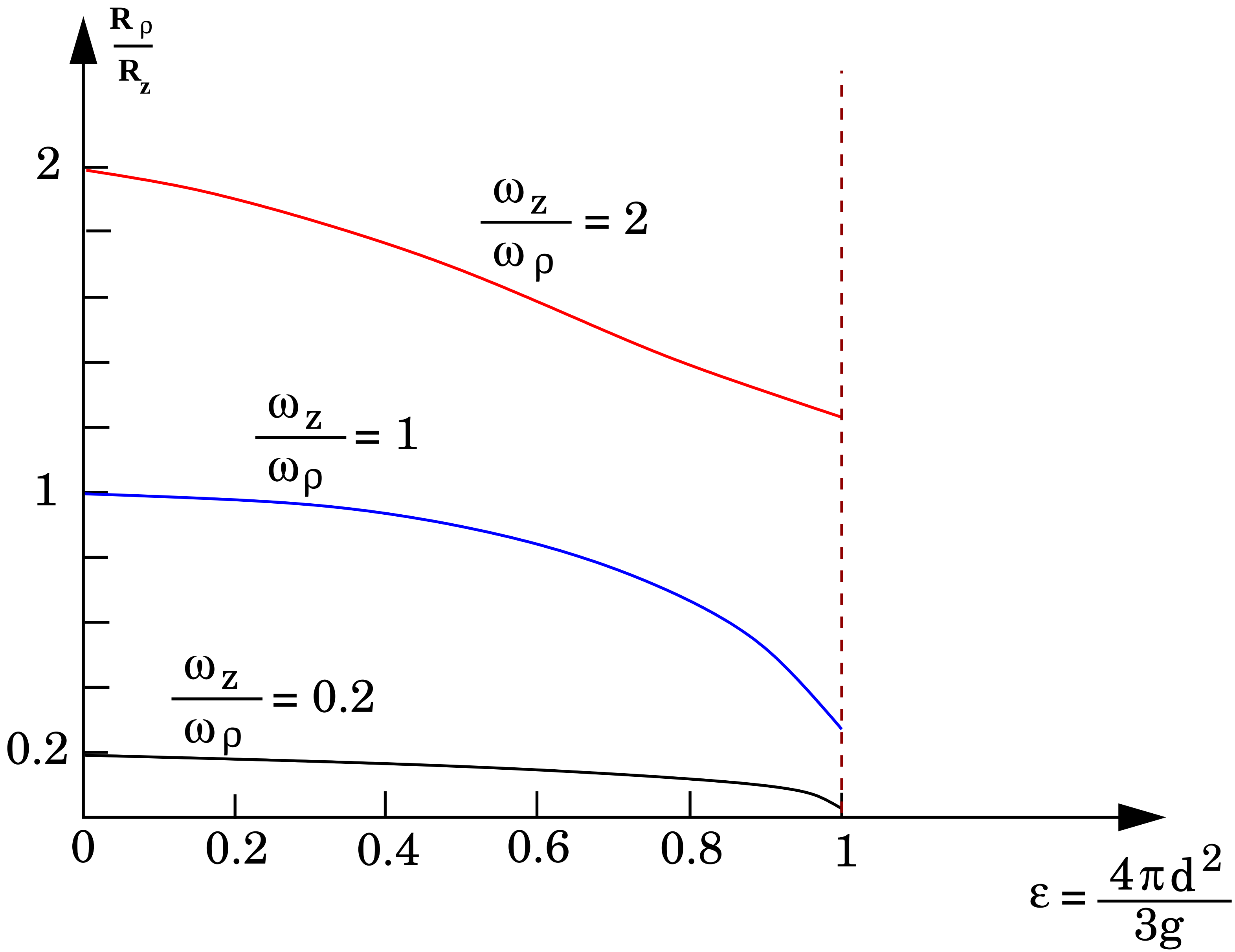
$$V_d = -d^2 \frac{\partial^2}{\partial z^2} \frac{1}{|\vec{r} - \vec{r'}|} - \frac{4\pi d^2}{3} \delta(\vec{r} - \vec{r'})$$

Large $N \Rightarrow$ Thomas-Fermi BEC

$$n_0 = n_{0\,max} \left(1 - \frac{z^2}{R_z^2} - \frac{\rho^2}{R_\rho^2}\right)$$ Eberlein et. al (2005)

$g > \frac{4\pi d^2}{3} \rightarrow$ stable at any N

Example



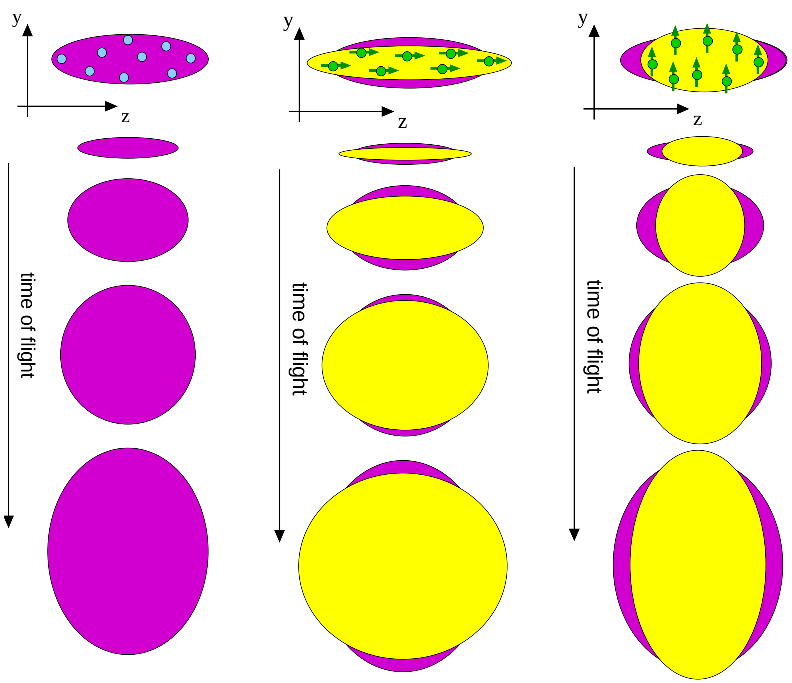
Experiment with Cr

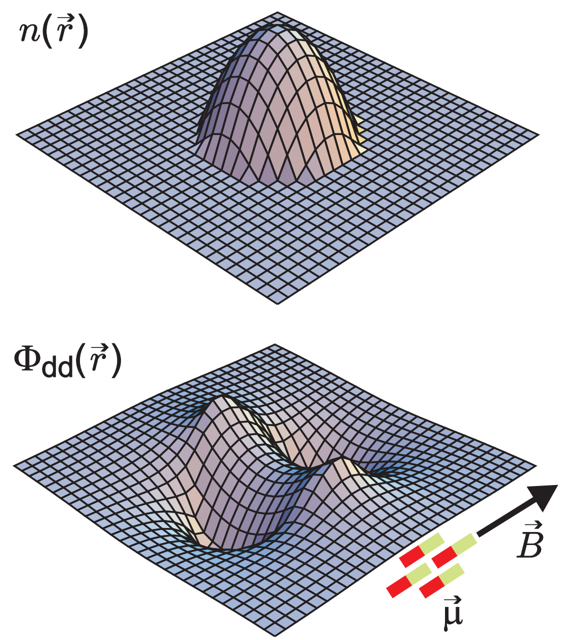
$$g > \frac{4\pi d^2}{3}$$

$(\mu = 6\mu_B!)$

(T. Pfau, Stuttgart) BEC $(n \sim 10^{14} cm^{-3})$

effect of the dipole-dipole interaction (small)





Dipolar droplets

Free space (Baillie et al, 2016)

$$H = -\int d^3r\psi_0\frac{\hbar^2}{2m}\Delta\psi_0 + \frac{g}{2}\int d^3r|\psi_0|^4$$
$$+\frac{1}{2}\int d^3rd^3r'V_d(\mathbf{r}-\mathbf{r}')|\psi_0(\mathbf{r})\psi_0(\mathbf{r}')|^2 + \frac{2}{5}\gamma_{QF}\int d^3r|\psi_0|^5$$

$\gamma_{QF} \rightarrow$ quantum fluctuations

Originates from the terms

$$g\langle\int\psi'^{\dagger}(\mathbf{r})\psi'(\mathbf{r})\psi_0^2(\mathbf{r})d^3r\rangle + \frac{g}{2}\langle[\int\psi'^2(\mathbf{r})\psi_0^2d^3r + \int\psi'^{\dagger 2}(\mathbf{r})\psi_0^2(\mathbf{r})d^3r]\rangle$$

in the energy functional

$$\gamma_{QF} = \frac{32}{3}g\sqrt{\frac{a^3}{\pi}}\left(1+\frac{3}{2}\epsilon_{dd}^2\right)$$

$a \rightarrow$ 3D scattering length for short-range interaction

$$g = \frac{4\pi\hbar^2}{m}a; \quad \epsilon_{dd} = \frac{a_{dd}}{a}; \quad a_{dd} = \frac{md^2}{12\hbar^2}$$

Dipolar droplets

Trivial solution $\psi_0 = \sqrt{n}$ ($E = 0$)

Gaussian Ansatz

$$\psi_0(\mathbf{r}) = \sqrt{\frac{8N}{\pi^{3/2}\sigma_\rho^2\sigma_z}} \exp\left\{-2\left(\frac{\rho^2}{\sigma_\rho^2} + \frac{z^2}{\sigma_z^2}\right)\right\}$$

$$E(\sigma_\rho, \sigma_z) = \frac{\hbar^2 N}{m}\left(\frac{2}{\sigma_\rho^2} + \frac{1}{\sigma_z^2}\right) + \frac{8N^2 a_{dd}}{\sqrt{2\pi}\sigma_\rho^2\sigma_z}\left[\epsilon_{dd}^{-1} - f\left(\frac{\sigma_\rho}{\sigma_z}\right)\right]$$

$$c\frac{\hbar^2}{m}\,\frac{1 + 3\epsilon_{dd}^2 N^{5/2}/2}{\sigma_\rho^3\sigma_z^{3/2}}a^{5/2}; \quad c \approx 13.8$$

$$f(x) = \frac{1+2x^2}{1-x^2} - \frac{3x^2}{(1-x^2)^{3/2}}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}$$

Droplets elongated in the direction of the dipoles

$\sigma_z \gg \sigma_\rho$ and $f \simeq 1$

Dipolar droplets

Minimize E

$$\sigma_\rho^2 \sigma_z \propto N; \qquad E \propto -N$$

$n \simeq N/\sigma_\rho^2 \sigma_z \rightarrow$ does not depend on N. Liquid droplets

Increasing $N \Rightarrow$ E the same for 2 or more droplets far from each other

Structure of f, kinetic energy, trapping potential $\rightarrow$

N above a critical value leads to 2 or more droplets

Remarkable experiments in the group of T. Pfau (last 2 years, Stuttgart)

Dipolar droplets

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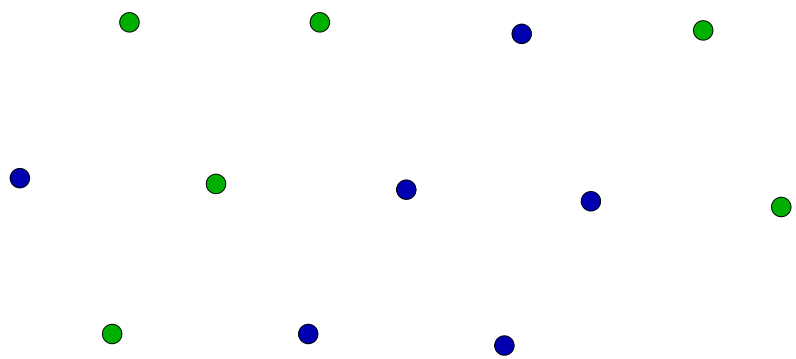

Numerical calculations: Wachtler/Santos; Saito (2016)

Distance between droplets $\rightarrow$ several μm. Not a supersolid

Goal $\Rightarrow$ Decrease the interdroplet distance and obtain a supersolid