Chain-solitons and their interactions

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Kadomtsev–Petviashvili equation

$$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = -3\alpha^2 u_{yy}, \quad \alpha^2 = \pm 1.$$

KP-I and KP-II equations correspond to $\alpha^2 = -1$ and $\alpha^2 = 1$, respectively.

Both of them are very different objects of mathematical study and are universal models describing weakly nonlinear waves in media with dispersion of velocity. They have numerous physical applications.

Suppose $u(x,y) \to 0$ as $|x| \to \infty$ for all y and $\int_{-\infty}^{\infty} u(x,y)\ dy = c(y)$ is finite for all y. Let us integrate KP-equation by x. Then the left part of equation vanishes, while the right hand gives

$$\frac{\partial^2}{\partial y^2} c(y) = 0 \quad \text{or} \quad c(y) = c_1 + c_2 y$$

We can rewrite KP-equation as follows

$$\frac{\partial u}{\partial t} + 6uu_x + u_{xxx} = -3\alpha^2 \int_{-\infty}^{x} u_{yy}\, dx$$

Integrating by x and changing the order of integration we obtain

$$\frac{\partial c_1}{\partial_t} = 0, \quad c_1 = const \qquad \frac{\partial c_2}{\partial t} = 0, \quad c_2 = const$$

KP equation is a Poissonian dynamical system. The Poisson bracket between two functionals P, Q is given by expression

$$\{P, Q\} = \int dy \int_{-\infty}^{\infty} \Big(\frac{\delta P}{\delta u}\frac{\partial}{\partial x}\frac{\delta Q}{\delta u} - \frac{\delta Q}{\delta u}\frac{\partial}{\partial x}\frac{\delta P}{\delta u}\Big)\, dx$$

This Poisson bracket is degenerated, constants c_1, c_2 are the Casimir constants. Fixing of these constants defines symplectic leaves. On each symplectic leaf the system is Hamiltonian.

We will not discuss the problem how c_1, c_2 should be found from the initial data and will assume that the Cazimir conditions are satisfied from the beginning.

We will study mostly the case $c_1 \neq 0$, $c_2 = 0$. Solutions for nonzero c_2 are not known so far.

Both versions of KP equation belong to a class of equations solvable by Inverse Scattering Method. The basic fact is that they admit a Lax pair, in other words, each equation can be presented as compatibility condition of an overdetermined linear system

$$\alpha\Psi_y + \frac{\partial^2\Psi}{\partial x^2} + u\Psi = 0$$

$$\Psi_t + 4\Psi_{xxx} + 6u\Psi_x + (3u_x + 3\alpha w)\Psi = 0, \quad w_x + u_y = 0.$$

For KP-I we have $\alpha = i$ and this equation is a non-stationary one-dimensional Schrödinger equation (with potential $-u$).

$\alpha = 1$ corresponds to KP-II and this equation is the heat equation with a source term. This alone reveals a substantial underlying difference of KP-I and KP-II theories.

In absence of dependence on y, KP equation reduces to KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

Now $w = 0$. One can put $\Psi = exp\,(\frac{k^2}{\alpha}y)\;\phi$ and get the standard Lax pair

$$L\,\phi = -k^2\,\phi$$

$$\phi_t + 4\,\phi_{xxx} + 6u\phi_x + 3u_x\phi = 0$$

Here "L-operator" is

$$L = \frac{\partial^2}{\partial x^2} + u$$

Now we will discuss the solitons. The KP-equation has a simple one-solitonic solution

$$u = \frac{2k^2}{cosh^2 k(x - ay - vt - x_0)}, \quad v = 4k^2 + 3\sigma^2 a^2$$

If $a = 0$, this is the one-soliton solution of the KdV equation. This soliton is upright. If $a \neq 0$ the soliton is tilted.

Notice that for all one-soliton solutions

$$c_1 = \int_{-\infty}^{\infty} u\, dx = 4k, \quad c_2 = 0$$

According to the general theory, c_1 does not depend on y.

Suppose that the solution of KP equation depends on one spatial variable $u = u(\xi, t)$, $\xi = x - ay$. Then u satisfies equation

$$u_t + 3\sigma^2 a^2 u_\xi + 6uu_\xi + u_{\xi\xi\xi} = 0$$

This is the KdV equation in the moving coordinate frame. Hence the KP equation has a trivial N-soliton solution consisting of N parallel solitons with the same tilt a.

In fact, both KP equations have a much more rich class of exact solutions which can be called ”generale soliton solution”. To give a definition of these solutions we present the solution of the Lax equation for $\alpha = i$ in the form

$$\Psi = e^{kx+\alpha k^2 y}\chi(k, x, y) \qquad i\chi_y + 2k\chi_x + \chi_{xx} + u\chi = 0$$

We will call u ”the general solitonic solution” if χ is a rational function of k. Thereafter we will consider the KP-1 equation only.

The dressing method for the KP-I equation

Assume that the function χ satisfies the following non-local $\bar{\partial}$-problem

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \chi * T = \int \chi(\eta) T(\eta, \lambda) d\eta \wedge d\bar{\eta}, \qquad \chi \to 1 \text{ as } \lambda \to \infty$$

The $\bar{\partial}$-problem is equivalent to the integral equation

$$\chi(\lambda) = 1 + \int \chi(\eta)\, K(\eta, \lambda)\, d\eta \wedge d\bar{\eta},$$

$$K(\eta, \lambda) = \frac{1}{\pi} \int \frac{T(\eta, \xi)}{\lambda - \xi} d\xi \wedge d\bar{\xi}$$

We consider the corresponding homogeneous equation

$$\Omega(\lambda) = \int \Omega(\eta)\, K(\eta,\lambda)\, d\eta \wedge d\bar{\eta}$$

Ω satisfies the same $\bar{\partial}$-problem as χ with normalization $\Omega \to 0$ as $\lambda \to \infty$. Distribution T is called the dressing kernel and is to be chosen. We require that the kernel can be expanded as an asymptotic series in $1/\lambda$ as $\lambda \to \infty$:

$$K(\eta,\lambda) = \sum_{n=1}^{\infty} \frac{I_n(\eta)}{\lambda^n}$$

This series does not necessarily converge, but the momenta

$$I_n(\eta) = \frac{1}{\pi} \int T(\eta,\xi)\, \xi^n d\xi \wedge d\bar{\xi}$$

are required to be finite.

Moreover, we require that integrals

$$I_n = \int I(\eta)\, d\eta \wedge d\overline{\eta}$$

are finite. Then solution χ of inhomogeneous $\overline{\partial}$-problem has an asymptotic expansion at infinity:

$$\chi = 1 + \frac{\chi_0}{\lambda} + \frac{\chi_1}{\lambda^2} + \cdots$$

Here $\chi_n = \int \chi(\eta, \bar{\eta})\, I_n\, (\eta, \bar{\eta})\, d\eta \wedge d\bar{\eta}, \quad n = 0, 1, 2, \ldots.$

If the dressing kernel $T(\xi, \lambda)$ has bounded support, then these conditions are trivially satisfied.

We consider the following differential operators:

$$D_1\chi = \frac{\partial \chi}{\partial x} + \lambda\chi, \quad D_2\chi = \frac{\partial \chi}{\partial y} + i\lambda^2\chi, \quad D_3\chi = \frac{\partial \chi}{\partial t} - 4\lambda^3\chi.$$

We require kernel T to satisfy the following equations:

$$\frac{\partial T}{\partial x} + \lambda T = \eta T, \quad \frac{\partial T}{\partial y} + i\lambda^2 T = i\eta^2 T, \quad \frac{\partial T}{\partial t} - 4\lambda^3 T = -4\eta^3 T$$

Under these conditions, the kernel can be represented as follows:

$$T(\eta,\lambda) = R(\eta,\lambda)\, e^{-\Phi(\lambda,x,y,t)+\Phi(\eta,x,y,t)}, \quad \Phi(\lambda,x,y,t) = \lambda x + i\lambda^2 y - 4\lambda^3 t.$$

$$\frac{\partial}{\partial\bar{\lambda}} D_i\chi = D_i\chi * T$$

If $P(D_1, D_2, D_3)$ is a polynomial in the operators D_i whose coefficients are functions of x, y, and t, then

$$\frac{\partial}{\partial\bar{\lambda}} P\chi = P\chi * T$$

The operator $\widehat{P} = iD_2 + D_1^2 + u$ satisfies this equation and it is clear that

$$\widehat{P}\chi \to 2\frac{\partial\chi_0}{\partial x} + u \quad \text{as} \quad \lambda \to \infty.$$

Hence, if we set

$$u = -2\frac{\partial\chi_0}{\partial x}$$

then

$$iD_2\chi + D_1^2\chi + u\chi = 0$$

In the same way χ satisfies equation

$$D_3\chi + 4D_1^3\chi + 6uD_1\chi + (3u_3 + w)\chi = 0, \quad w_x + u_y = 0$$

Let us introduce

$$\Psi = \chi\, e^{\Phi}, \qquad \Phi = \lambda\, x + i\lambda^2\, y - 4\lambda^3\, t$$

Ψ satisfies the Lax equation. Hence function u is the solution of KP-I equation.

So far we discussed complex solutions of the KP-I equation. To make a transition to real solution we must impose some restriction on the dressing function. A sufficient condition for this transition is the following reduction

$$R(\eta,\, \xi) = \bar{R}\,(-\bar{\xi}, -\bar{\eta})$$

Regular solutions

Suppose the dressing kernel $R(\eta,\xi)$ is degenerative

$$R(\eta,\xi) = -\pi \sum_{n=1}^{N} f_n\, g_n$$

Here $f_n = f_n(\eta,\bar{\eta})$, $g_n = g_n(\xi,\,\bar{\xi})$. Thereafter we denote

$$\tilde{f}_n(\eta,\bar{\eta},x,y,t) = f_n(\eta,\bar{\eta})\, e^{-\Phi(\eta,x,y,t)}$$

$$\tilde{g}_n(\xi,\tilde{\xi},x,y,t) = g_n(\xi,\bar{\xi})\, e^{\bar{\Phi}(\xi,x,y,t)}$$

Then we introduce

$$h_n(\lambda,\bar{\lambda},x,y,t) = \int \frac{\tilde{g}_n(\xi,\bar{\xi})\, e^{\Phi(\xi,x,y,t)}}{\lambda-\xi}\, d\xi \wedge d\bar{\xi}$$

Now kernel $K(\eta, \lambda)$ is also degenerative

$$K(\eta, \lambda) = -\pi \sum \tilde{f}_n(\eta, \bar{\eta})\, h_n(\lambda, \bar{\lambda})$$

We can look for solution of this equation in the form

$$\chi_n(\lambda, \bar{\lambda}) = 1 + \sum_m c_m h_m(\lambda, \bar{\lambda}), \quad c_n = c_n(x, y, t)$$

Functions $c_n(x, y, t)$ satisfy to the system of algebraic equations

$$c_n + \sum_{m=1}^{N} L_{nm}\, c_m = - < \tilde{f}_n >$$

$$< \tilde{f}_n >= \int \tilde{f}_n(\xi, \bar{\xi}, x, y, t) d_\xi \wedge d\bar{\xi}$$

$$L_{nm} = \int \tilde{f}_n(\eta,\bar{\eta}) h_m(\eta,\bar{\eta})\, d\eta \wedge d\bar{\eta} = \int \frac{\tilde{f}_n(n,\bar{n})\tilde{g_m}(\xi,\bar{\xi})}{\eta - \xi}\, d\eta \wedge d\bar{\eta}\, d\xi \wedge d\bar{\xi}$$

Now we notice that

$$h_n(\lambda,\bar{\lambda},x,y,t) \to \frac{1}{\lambda} < \tilde{g}_n > \quad \text{as} \quad \lambda \to \infty$$

$$< \tilde{g}_n >= \int \tilde{g_n}(\xi,\bar{\xi})\ e^{\Phi(\xi,x,y,t)} d\xi \wedge d\bar{\xi}$$

Now

$$\chi_0 = \sum_{n=1}^{N} c_n < \tilde{g}_n >$$

We call this class of solutions ”regular solutions” of rank N.

Let

$$\tau = det\,|\delta_{nm} + L_{nm}| \qquad \chi_o = \frac{\partial}{\partial x}\,\ln\,\tau$$

For the general complex KP-I

$$u = -2\,\frac{\partial^2}{\partial x^2}\,\ln\tau$$

In a real case this reduction implies that

$$g(\lambda, \bar\lambda) = \bar f\,(-\bar\lambda, -\lambda)$$

Now matrix L_{nm} takes form

$$L_{nm} = \int \frac{\tilde f_n(\lambda, \bar\lambda)\,\bar{\tilde f}_m(\bar\eta, \eta)}{\lambda + \bar\eta}\,d\lambda\wedge d\bar\lambda\,d\eta\wedge d\bar\eta$$

This matrix is Hermitian and is positively defined. In virtue of this fact the regular solutions are regular indeed.

The Marchenko equation

Suppose that the kernel $R(\eta,\xi)$ vanishes identically in some shifted right half-plane of ξ:

$$R(\eta,\xi) \equiv 0 \quad \text{if} \quad Re\,\xi > \xi_o > 0$$

Denote $\chi = 1 + \widetilde{\chi}$, then

$$\widetilde{\chi} = \widetilde{f} + \frac{1}{\pi}\int \frac{\widetilde{\chi}(\eta)R(\eta,\xi)\, e^{(\eta-\xi)x+i(\eta^2-\xi^2)\,y-4(\eta^3-\xi^3)\,t}}{\lambda-\xi} d\eta \wedge d\overline{\eta} \wedge d\xi \wedge d\overline{\xi}$$

where

$$\widetilde{f} = \frac{1}{\pi}\int \frac{R(\eta,\xi)\, e^{(\eta-\xi)x+i(\eta^2-\xi^2)y-4(\eta^3-\xi^3)t}}{\lambda-\eta} d\eta \wedge d\overline{\eta} \wedge d\xi \wedge d\overline{\xi}.$$

Let $\sigma > \xi_0 > 0$ and perform the Fourier transformation

$$K(x,z,y,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \chi(\lambda,x,y,t)\, e^{\lambda(\lambda-z)} d\lambda$$

Here z is a new real variable $x < z < \infty$. We apply this transformation and denote

$$F(x, z, y, t) = -\frac{1}{\pi}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{f}(\lambda, x, y, t)\, e^{\lambda(x-z)} dx$$

According to our assumption, $F(x, z) \equiv 0,\ \ z < x$ and

$$F(x, z) = -\frac{1}{\pi}\int R(\eta, \xi)\, e^{\eta x - \xi z + i(\eta^2 - \xi^2)y - 4(\lambda^3 - \eta^3)t}\, d\eta \wedge d\bar{\eta}\, d\xi \wedge d\bar{xi}$$

Now

$$K(x, x, y, t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widetilde{\chi}(\lambda, x, y, t) d\lambda$$

Since $\widetilde{\chi}$ is analytic in the right half-plane and has the asymptotic expansion $\widetilde{\chi} \to \chi_0/\lambda + \cdots$ as $\lambda \to \infty$, it follows that

$$K(x, x, y, t) = -\chi_0(x, y, t).$$

$$u = 2\frac{d}{dx}K(x,x,y,t).$$

Function F satisfies equations

$$i\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0, \quad \frac{\partial F}{\partial t} + 4\left(\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3}\right) = 0.$$

Using the Fourier transform, we obtain the Marchenko equation in the following form:

$$K(x,z,y,t) + F(x,z,y,t) + \int_{-\infty}^{x} K(x,s,y,t)F(s,z,y,t)ds = 0$$

We now pass from the complex to the real version of KP-I by requiring that $\overline{k}(x,x) = k(x,x)$. The sufficient condition for this is

$$F(x,z,y,t) = \overline{F}(z,x,y,t)$$

To construct the regular solutions of KP-I by the use of Marchenko equation one should choose

$$F = \sum_{n=1}^{N} \psi_n(x, y, t)\, \bar{\psi}(z, y, t)$$

Functions ψ_n are solutions of equations

$$i\frac{\partial \psi_n}{\partial y} + \frac{\partial^2 \psi_n}{\partial x^2} = 0 \qquad \frac{\partial \psi_n}{\partial y} + 4\frac{\partial^3 \psi_n}{\partial x^3} = 0$$

Functions ψ_n must decay at $x \to \infty$ such that the integrals

$$\int_{-\infty}^{x} |\psi_n|^2\, dx < \infty$$

Then the solution of the Marchenko equation could be found in the form

$$K(x, z, y, t) = \sum_{n=1}^{N} K_n(x, y, t)\, \bar{\psi}_n\,(z, y, t)$$

Potential u is

$$u = 2\frac{\partial}{\partial x} \sum_{n=1} K_n(x, y, t)\, \bar{\psi}_n(x, y, t)$$

Functions K_n are solutions of algebraic equations

$$K_n + \sum M_{nm}\, K_m + \psi_n = 0$$

$$M_{nm} = \int_{-\infty}^{x} \psi_n(x, y, t)\, \bar{\psi}_m\,(x, y, t)\, dx$$

Matrix M_{nm} is Hermitian and positive. We can treat $M_{nm} =< \psi_n, \psi_m >$ as a scalar product in the Hilbert space $L^2(-\infty, x)$. Hence all major minors of this matrix are Grammian determinants ant positive. As a result, all equations is uniquely resolvable.

Let us denote

$$\tilde{\tau} = det|\delta_{nm} + M_{nm}|$$

Then

$$u = +2\frac{\partial^2}{\partial x^2} \ln \tau$$

Solitons *via* dressing method

Consider a regular solution of first rank with dressing function

$$R(\eta,\xi) = -\pi\, R_0\, \delta(\eta + \bar{\lambda}_0)\, \delta(\xi - \lambda_0)$$

λ_0 is an arbitrary complex number, $R_0 > 0$ real and positive. Then τ-function is

$$\tau = 1 + \frac{R_0}{\lambda_0 + \bar{\lambda}_0}\, e^{-2S}$$

$$2S\ = (\lambda_0 + \bar{\lambda}_0)\, x + i(\lambda_0^2 - \bar{\lambda}_0{}^2)\, y - 4(\lambda_0^3 + \bar{\lambda}_0{}^3)\, t$$

Then we define

$$\lambda_0 = k(1 + \frac{ia}{2}) \qquad e^{2k\chi_0} = \frac{R_0}{2k}$$

$$u = \frac{2k^2}{\cosh^2 k(x - x_0 - ay - vt)}$$

This is exactly the tilted solution. The corresponding $\chi(\lambda)$ function is rational with a single pole at $\lambda = \lambda_0$

$$\chi = 1 + \frac{\chi_0}{\lambda - \lambda_0}$$

The corresponding Ψ-function

$$\Psi = \chi\, e^{\lambda x + i\lambda^2 y - 4\lambda^3 t}$$

is a solution of the Lax equation.

The ”simple” N-solitonic solution: a regular solution of rank N defined by N complex numbers λ_n, N real positive numbers, and dressing function

$$R = -\pi \sum_{n=1}^{N} R_n \, \delta(\eta + \bar{\lambda}_n) \, \delta(\lambda - \lambda_n)$$

Function χ is rational and has simple poles at $\lambda = \lambda_n$.

$$\chi = 1 + \sum \frac{\chi_n}{\lambda - \lambda_n}$$

The corresponding Ψ function $\Psi = \chi \, e^{\phi(\lambda)}$ is rational in the finite part of λ-plane. Its residue

$$\phi_n = \chi_n \, e^{\Phi(\lambda_n)}$$

are eigenfunctions of the non-stationary Schrodinger equation.

The τ function for simple N-solitonic solution is positive and solution has no singularities

$$\tau = \det|\delta_{nm} + \frac{R_n}{\lambda_n + \bar{\lambda}_m} e^{-\Phi(\lambda_n)+\Phi(-\bar{\lambda}_m)}| \qquad u = -2\frac{\partial^2}{\partial x^2} \ln \tau$$

If all λ_n are real, this is N-solitonic solution of the KdV equation. If they have form $\lambda_n = k(1 + \frac{ia}{2})$ (a is given), this is N-solitonic solutions of ”tilted” KdV equation.

If λ_n are arbitrary complex numbers, this is a deformation of KdV N-solitonic solution.

In the case of KP-II equation such deformation faces obstacles but in the KP-I case obstacles are absent and all deformations are possible.

However, as in the KP-II case, the KP-I equation has a lot of other more complicated solutions which we call ”general multisolitonic solutions”.

General multisolitonic solution is characterized by two numbers: rank N and order M. This is a regular solution of rank N where

$$f_N(\lambda, \bar{\lambda}) = \sum_{m=1}^{M} c_{NM}\, \delta(\lambda - \lambda_{nm})$$

Here c_{NM} is an arbitrary rectangular matrix. For simple solitonic solution $M = 1$. According to general theory, the general solitonic solution has no singularities.

For all multisolitonic solutions function χ is rational. If all λ_{nm} are different, it has $N \times M$ simple poles in points $\lambda = \lambda_{nm}$.

Solitons and chain-solitons *via* Marchenko equation

Consider multisolitonic solution of rank 1 in terms of Marchenko equation

$$F(x,z,y,t) = \psi(x,y,t)\ \bar{\psi}(z,y,t), \quad \psi(x,y,t) = \sum_{n=1}^{N} c_n\ e^{\lambda_n x + i\lambda_n^2 y - 4\lambda_n^3 t}$$

The τ-function

$$\tau = 1 + \int_{-\infty}^{x} |\psi|^2\, dS \qquad u = +2\frac{\partial}{\partial x}\frac{|\psi|^2}{1 + \int_{-\infty}^{x} |\psi|^2\, dS}$$

If $M = 1$ this is the simple solution already described.

All previous versions of the dressing method missed very important class of soliton-like solutions which we call ”chain solitons”.

Note that ψ satisfies to system of Lax equations. If ψ is a solution, then $A\psi$ is solution where A is an arbitrary constant. By sending $A \to \infty$ we come to the reduced equation

$$u = 2\frac{\partial}{\partial x}\frac{|\psi|^2}{\int_{-\infty}^{x}\psi|^2\,dS} \qquad \tau = \int_{-\infty}^{x}|\psi|^2\,dS$$

We call these solutions ”chain-solitons”. Simplest ”chain-solitons” appear if $N = 2$. We start with case when $\lambda_1, \lambda_2, c_1, c_2$ are real numbers

$$\lambda_1 = k + a, \quad \lambda_2 = k - a, \quad k > a > 0 \qquad c_1 = \sqrt{2(k+a)}, \quad c_2 = -\sqrt{2(k-a)}$$

After elementary transformation, the τ-function can be derived to the form

$$\tau = 2e^{ak[x-4(k^2+3a^2)]t} \times [\cosh 2a(x - 4(3k^2 + a^2)]t - \epsilon \cos 4kay \qquad \epsilon = \sqrt{1 - \frac{a^2}{k^2}}$$

The factor outside the brackets vanishes u and we end up with the following expressions

$$\tau = \cosh 2a(x - \omega t) - \epsilon \cos 4kay, \quad \omega = 4(3k^2 + a^2)$$

$$\frac{\tau_x}{\tau} = 2a\,\frac{\sinh\, 2a(x - \omega t)}{\cosh\, 2a(x - \omega t) + \epsilon \cos\, 4ky}$$

$$u = 2\frac{d}{dx}\frac{\tau_x}{\tau} = 8a^2\,\frac{1 - \epsilon \cosh 2a(x - \omega t)\cos 4ky}{[\cos 2a(x - \omega t) + \epsilon \cos 4ky]^2}$$

This solution is the ”upright chain-soliton”, the simpleast representative of the family of chain-solitons. It has the vertical axis moving in horizontal direction with velocity ω. On this axis

$$u|_{x=\omega t} = \frac{8a^2}{1 + \epsilon \cos 4ky}$$

The chain-soliton is periodic along y. The period

$$l = \frac{2\pi}{ka} \to \infty \quad if \quad a \to o$$

The maximum and minimum values are:

$$u_{max} = \frac{8a^2}{1-\epsilon} \to 16k^2, \quad u_{min} = \frac{8a^2}{1+\epsilon} \to 4a^2, \quad a \to 0$$

$$\frac{\tau_x}{\tau} \to \pm 2a, \quad x \to \pm\infty$$

Hence

$$c_1 = \int_{-\infty}^{\infty} u\, dx = 4a > 0$$

If $a \to 0$

$$\tau \to \frac{1}{2}\frac{a^2}{k^2} + 2a^2\Big[(x-\omega t)^2 + 4k^2y^2\Big] \qquad \tau \to \frac{1}{4k^2} + (x - 12k^2t)^2 + 4k^2y^2$$

Solution u is now a rational function and is nothing but a well-known ”lump”. Thus we can treat the chain-soliton as a periodic chain of lumps. If $a \to \infty$, distance between lumps tends to infinity. The lump solution was discovered Manakov, Zakharov, Bordag, Its, Matveev in 1977. A general lump solution is tilted

$$\tau = \frac{1}{4k^2} + \Big(x - 2by - 12(k^2 - b^2)\,t\Big)^2 + 4k^2(y - 12bt)^2$$

b is an arbitrary constant. The lump moves with velocity (v_x, v_y)

$$v_x = 12(k^2 + b^2) \qquad v_y = 12b$$

Let us consider a more general one-chain-soliton solution. We put

$$\lambda_1 = a_1 + ib_1, \quad \lambda_2 = a_2 + ib_2, \quad c_1 = \sqrt{2a_1}, \quad c_2 \sqrt{2a_2}$$

$$\epsilon = \frac{2\sqrt{a_1 a_2}}{a_1 + a_2} < 1, \quad a_1 > 0, \ \ a_2 > 0$$

Then we denote

$$\Phi_k = \Phi(\lambda_k) = F_k + iG_k$$

$$F_k = a_k[(x - 2b_k y) - 4w_k t], \quad G_k = b_k(x - 4s_k t) - (a_k^2 - b_k^2)y$$

$$w_k = a_k^2 - 3b_k^2, \quad s_k = 3a_k^2 - b_k^2$$

So far, $k = 1, 2$.

Now the chain-soliton is defined by the τ-function

$$\tau = \cosh(F_1 - F_2) + \epsilon \cos(G_1 - G_2)$$

A general chain-soliton has an axis (the ”needle”) tilted to vertical axis and moving orthogonal to itself.

$$F_1 - F_2 = 0$$

$$x = \tan\theta y + wt, \quad \tan\theta = 2\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}, \quad w = \frac{a_1 w_1 - a_2 w_2}{a_1 - a_2}$$

θ is the angle with respect to vertical axis. The normal velocity of the needle is $w_n = w\cos\theta$. A periodic system of ”beads” is threaded on the needle. All beads have equal velocities described by equations

$$F_1 - F_2 = 0 \qquad G_1 - G_2 = 0$$

In the limit

$$a_2 \to a_1 \to k \qquad b_1 \to b_2 \to b$$

the beads turn to lump solitons of general type. The period between them tends to infinity.

Instability of chain-soliton

Chain-soliton constructed above is unstable. Consider the solution, where

$$\Psi = \sqrt{2a_1}\, e^{\Phi_1} + \sqrt{2a_2}\, e^{\Phi_2} + \sqrt{2a_3}\, e^{\Phi_3}$$

$$\Phi_i = \Phi(\lambda_i), \quad \lambda_i = a_i, \quad i = 1, 2, 3 \quad a_1 > a_2 > a_3 > 0$$
$$\Phi_k = F_k + iG_k \quad F_k = a_k(x - 4a_k t), \quad G_k = -a_k^2 y$$

Note that

$$F_k - F_l = (a_k - a_l)(x - 4w_{kl}\, t), \quad w_{kl} = a_k^2 + a_{kl} + a_l^2$$

The τ-function corresponding to the wave-function is

$$\tau = e^{2F_1}+e^{2F_2}+e^{2F_3}+2\epsilon_{12}\, e^{F_1+F_2}\cos(a_1^2-a_2^2)y+2\epsilon_{13}E^{F_1+F_3}\cos(a_1^2-a_3^2)y+2\epsilon_{23}\, e^{F_2+F_3}\cos(a_2^2-a_3^2)$$

Here

$$\epsilon_{kl} = \frac{2\sqrt{a_k a_l}}{a_k + a_l} < 1$$

Now we introduce partial τ-function

$$\tau_{ij} = \cosh(F_i - F_j) - \epsilon \cos(G_i - G_j)$$

The analysis of the general τ-function shows that

$$\tau \to \tau_{13} \quad at \quad t \to -\infty$$

$$\tau \to \tau_{12} + \tau_{23} \quad at \quad t \to +\infty$$

It means that the ”strong” chain-soliton with parameter $a_1 - a_3$ is unstable and decays to two ”more weak” chain-solitons with parameters $a_1 - a_2$ and $a_2 - a_3$.

It means that the chain-soliton is unstable.

Now we are ready to the important question: What is a long-time asymptotics of a general "vertical" chain-soliton solution of rank 1?

$$\Psi = \sum \sqrt{2a_b}\, e^{\Phi_n}$$

Here a_n is a decreasing set on real N numbers ordered as follows:

$$a_1 > a_2 \ldots > a_N$$

$$\Phi_n = a_n(x - x_n - 4a_n^2 t) + ib_n(y - y_n)$$

Here x_n, y_n are arbitrary real numbers. By application the technique developed above we end up with the following result. At $t \to -\infty$ the solution in converged to the simple strong soliton with parameter

$$\mu_{1N} = a_1 - a_N$$

This is the periodic string of lumps with period

$$L_{min} = \frac{1\pi}{a_1^2 - a_N^2}$$

When $\tau \to \infty$, this is "strong" chain-soliton that decomposes to $N - 1$ "weak" chain-solitons u_l with parameters $\mu_{l,l+1}$,$l = 1, \ldots N - 1$. The string u_l moves in the right direction with velocity

$$w_l = 4(a_l^2 + a_l a_{l+1} + a_{l+1}^2) \qquad w_1 \leq w_l \leq w_{N-1} \qquad w_1 = 4(a_1^2 + a_1 a_2 + a_2^2)$$

$$w_{N-1} = 4(a_{N-1}^2 + a_{N-1} a_N + a_N^2)$$

Each weak soliton is a periodic string of lump with period

$$L_l = \frac{2\pi}{a_l^2 - a_{l+1}^2}, \quad l = 1, \ldots N - 1$$

By choosing of complex parameters R_n one can make positions of axis of these chain-solitons as well as a position the closest to the real axis lump arbitrary. Suppose that a_1 and a_N are fixed but $N \to \infty$. Now $L_l \to \infty$ and the chain-soliton turns to the "cloud" of lumps. If $N\infty$ and a_n are chosen as arbitrary complex number covering some domain in c^2, the chain-solitonic solution becomes an expanding strip of integrable turbulence - a gas of lumps containing in the limit

$$w_1 t < x < w_{N-1} t$$

What goes on if x_n are complex numbers will be discussed later on.

Interaction of general chain-solitons

Suppose

$$\Psi = \sqrt{2a_1}\, e^{\Phi_1} + \sqrt{2a_2}\, e^{\Phi_2} + \sqrt{2a_3}\, e^{\Phi_3}$$

where Φ_i are complex numbers

$$\Phi_i = a_i + ib_i \qquad a_1 > a_2 > a_3$$

The solitonic solution describes interaction of three ”needles” defined by equations

$$F_i - F_k = 0 \qquad i \neq k$$

At the point $F_1 = F_2 = F_3$ the needles intersect and glue.

At $t \to -\infty$ survives the strongest ”needle” with parameter $a_1 - a_3$.

At $t \to +\infty$ two weakest ”needles” coexist.

Instability of plain solitons

Let us return to the regular solution of rank 1 and put

$$\phi = \sqrt{2a_1}\, e^{phi_1} + \sqrt{2a}\, e^{\phi_2}$$

where ϕ_1, ϕ_2 are defined by

$$\phi_i = a_i\lambda + ia_i^2 y - 4a_i^3 t$$

This choice defines the τ-function

$$\tau = 1 + e^{\phi_1+\bar{\phi}_1} + e^{\phi_2+\bar{\phi}_2} + \epsilon(e^{\phi_1+\bar{\phi}_2} + e^{\phi_2+\bar{\phi}_1})$$

$$\tau = 1 + e^{2F_1} + e^{2F_2} + 2\epsilon e^{F_1+F_2} \cos(G_1 - G_2)$$

Notice that

$$F_1 = a_1(x - 2b_1 y - 4w_1 t) \qquad F_2 = a_2(x - 2b_2 y - 4w_2 t)$$

$$w_1 = a_1^2 - 3b_1^2, \quad w_2 = a_2^2 - 3b_2^2$$

Then we consider the first two terms in the τ-function for what we call the ”fast” soliton

$$u_f = \frac{2a_1^2}{\cosh^2 a_1(x - 2b_1 t - 4w_1 t)}$$

and study the condition when the remaining part of τ-function could be treated as a perturbation uniformly by y vanishing at $\tau \to -\infty$. We impose on F_2 the limitation $b_2 = b_1 = b$, $a_2 < a_1$ and will consider the $\tilde{\tau}$ on the axis of the ”fast” soliton

$$x = 2by + 4w_1 t$$

Then $F_1 = 0$.

$$e^{F_2} = e^{\gamma t},\ t \to -\infty, \quad \gamma = 4a_2(a_1^2 - a_2^2)$$

This formula proves instability of an arbitrary soliton. This expression defines the growth rate of instability. It was found for vertical solitons, now we show that it is the same for arbitrary tilted solitons. The growth rate has maximum at

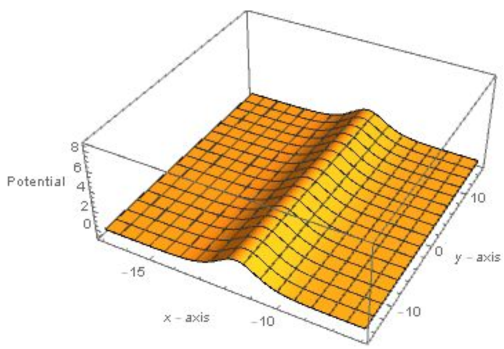
$$a_2^2 = 1/3\, a_1^2 \qquad and \qquad \gamma_{max} = 8/3\, a_1^3$$

Now we have approached to the most interesting question: What is the nonlinear stage of this instability? It is clear that there is no general answer to this question.

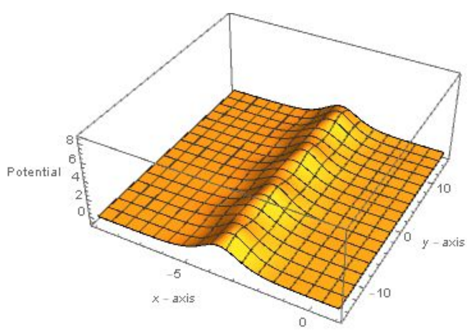
Case 1: Real Wave Vectors k1 & k2, Soliton Decay:

$$A = 1 + \frac{c_1^{\,2}}{2(c+a)} Exp[2(c+a)x - 8(c+a)^3 t] + \frac{c_2^{\,2}}{2(c-a)} Exp[2(c-a)x - 8(c-a)^3 t] \ldots$$
$$+ \frac{c_1 c_2}{c} Exp[2cx - 4((c+a)^3 + (c-a)^3)t]\, cos(4acy)$$

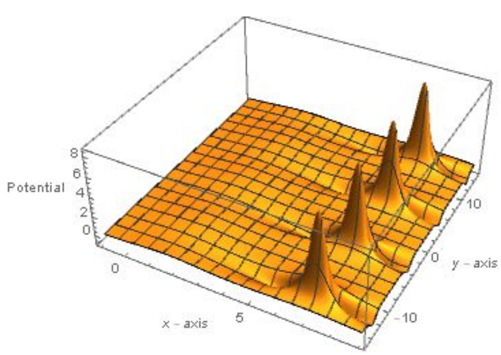
Where $c_1 = \sqrt{c+a}$, $c_2 = \sqrt{c-a}$ and define $p = \sqrt{1 - \frac{a^2}{c^2}}$
$u = 2\partial_x{}^2 ln\ A$



Values for plot: k1 = c+a = 1, k2 = c-a = ½, t = -3



Values for plot: k1 = c+a = 1, k2 = c-a = ½, t = -1



Values for plot: k1 = c+a = 1, k2 = c-a = ½, t = 1

$v_{fast} = 4(c+a)^2$
$v_{chain} = 12c^2 + 4a^2$
$v_{slow} = 4(c-a)^2$
$v_{slow} < v_{fast} < v_{chain}$

$A_{fast} = \frac{16}{9}(c+a)^2$
$A_{slow} = \frac{16}{9}(c-a)^2$
$A_{chain} = 8a^2/(1-p)$, peak amplitude at $y = (2N+1)\pi/(4ac)$, $N = 0, \pm 1, \pm 2 ...$
$p = \sqrt{1 - \frac{a^2}{c^2}}$

Case 2: Complex k1 & k2: Chain Soliton Limit:

$$A = \frac{f_1^{\,2}}{k+\bar{k}} exp[z_1 + \bar{z}_1] + \frac{f_2^{\,2}}{p+\bar{p}} exp[z_2 + \bar{z}_2] + \frac{f_1 f_2}{k+\bar{p}} exp[z_1 + \bar{z}_2] + \frac{f_1 f_2}{p+\bar{k}} exp[z_2 + \bar{z}_1]$$

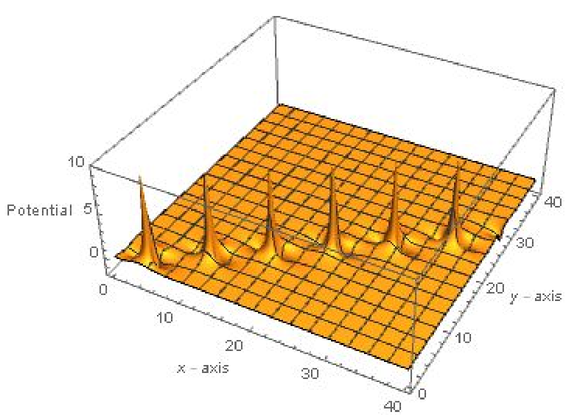
$z_1 = k\,x + ik^2 y - 4k^3 t$, and $z_2 = p\,x + ip^2 y - 4p^3 t$

And the parameters k, p are expressed in terms of a, c as

$k = c + a$, and $p = c - a$

And a, c have real and imaginary parts,

$a \equiv a_1 + ia_2$, $c \equiv c_1 + ic_2$



Values for plot: k1=1+i/4, k2= 1/2-i/4, f1=f2=1, t=0

The absolute velocity of a fixed point on a moving peak of an individual soliton in the chain is,

$$v_x = 4({a_1}^2 - {a_2}^2 + 3({c_1}^2 + {c_2}^2) + 2a_1 a_2 c_2/c_1)$$
$$v_y = 4(3c_2 + a_1 a_2/c_1)$$
$$v \equiv \{v_x,\ v_y\}$$

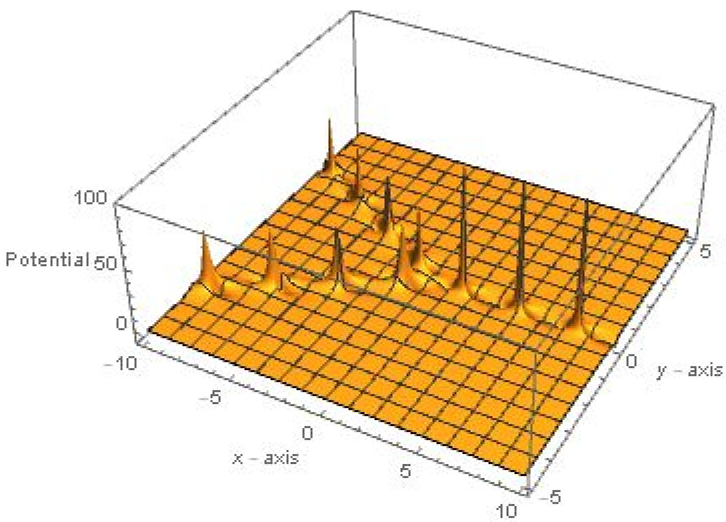
And the angle with respect to the x-axis of the chain soliton at a fixed time is,

$$\varphi = Arctan[\frac{a_1}{2(a_1 c_2 + a_2 c_1)}]$$

The distance between peaks in the chain is,

$$width = \frac{\pi\sqrt{{a_1}^2 + 4(a_1 c_2 + a_2 c_1)^2}}{2c_1({a_1}^2 + {a_2}^2)}$$

Case 3: Sum of Three Exponentials: Chain Soliton Limit:



Values for plot: k1=2.5+i, k2=2.5-i, k3=1

For sum of n exponentials,

$$\varphi = \sum_{i=1}^{n} c_i e^{z_i}$$

$$z_i = k_i x + i k_i^2 y - 4 k_i^3 t$$

$$k_i = a_i + i b_i$$

$$z_i = r_i + i q_i = [a_i x - 2 a_i b_i y - 4(a_i^3 - 3 a_i b_i^2) t] + i[b_i x + (a_i^2 - b_i^2) y - 4(3\, a_i^2 b_i - b_i^3) t]$$

Then the general solution is,

$$A = 1 + \sum_{i=1}^{n} c_i^2 e^{2r_i}/(2\, a_i) + \sum_{i=1}^{n} \sum_{j=1}^{i-1} 2 c_i c_j f_{ij} e^{r_i + r_j} [(a_i + a_j) cos(q_i - q_j) + (b_i - b_j) sin(q_i - q_j)]$$
$$f_{ij} = 1/[(b_i - b_j)^2 + (a_i + a_j)^2]$$

For real valued wave vectors k,

$b_i = 0$, and fixing $c_i = \sqrt{2a_i}$, with $a_1 > a_2 > a_3 > 0$

Then,

$$A = 1 + e^{2r_1} + e^{2r_2} + e^{2r_3} + 4\frac{\sqrt{a_1 a_2}}{(a_1+a_2)^2} e^{r_1 + r_2} cos(q_2 - q_1) + 4\frac{\sqrt{a_1 a_3}}{(a_1+a_3)^2} e^{r_1 + r_3} cos(q_3 - q_1) + 4\frac{\sqrt{a_2 a_3}}{(a_2+a_3)^2} e^{r_2 + r_3} cos(q_3 - q_2)$$
$$r_i = a_i x - 4 a_i^3 t\,, \quad q_i = a_i^2 y$$

The speed and amplitude at t<<0,

$$v_{-\infty} = 4 a_1^2\,, \quad A_{-\infty} = 16 a_1^3 c_1^2 / (c_1^2 + 2a_1)^2 = 2 a_1^2$$

And the speed and amplitude of the slow wave for t>>0,

$$v_{+\infty} = 4 a_3^2\,, \quad A_{+\infty} = 16 a_3^3 c_3^2 / (c_3^2 + 2a_3)^2 = 2 a_3^2$$

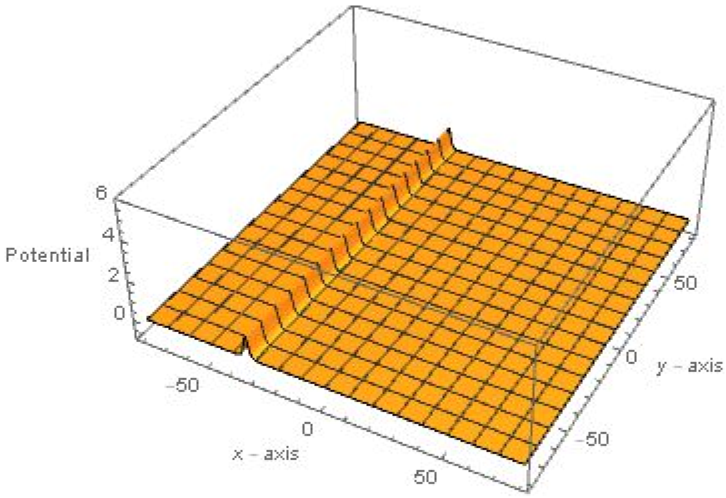
The distance between chain soliton peaks are,

$$w_{fast} = 2\pi/(a_1^2 - a_2^2)$$
$$w_{slow} = 2\pi/(a_2^2 - a_3^2)$$

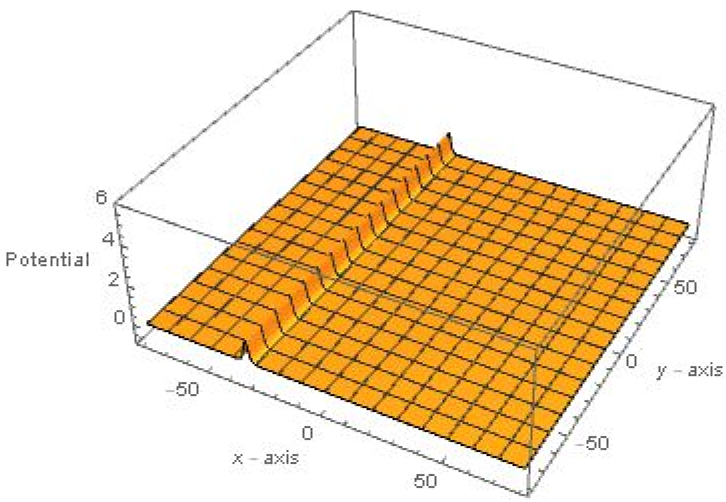
The chain soliton speeds and peak amplitudes are,

$$v_{fastchain} = 4(a_1^2 + a_1 a_2 + a_2^2)\,, \qquad A_{fastchain} = 2(a_1 + a_2)(\sqrt{a_1} + \sqrt{a_2}\,)^2$$

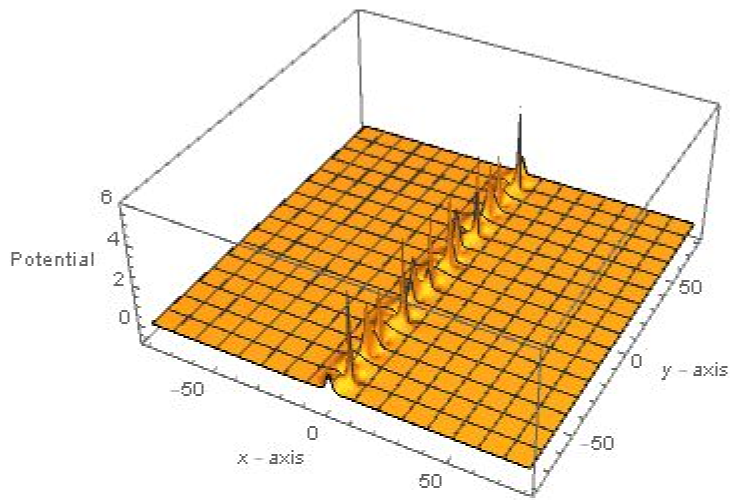
$v_{slowchain} = 4(a_2{}^2 + a_2 a_3 + a_3{}^2)$, $A_{slowchain} = 2(a_2 + a_3)(\sqrt{a_2} + \sqrt{a_3}\,)^2$



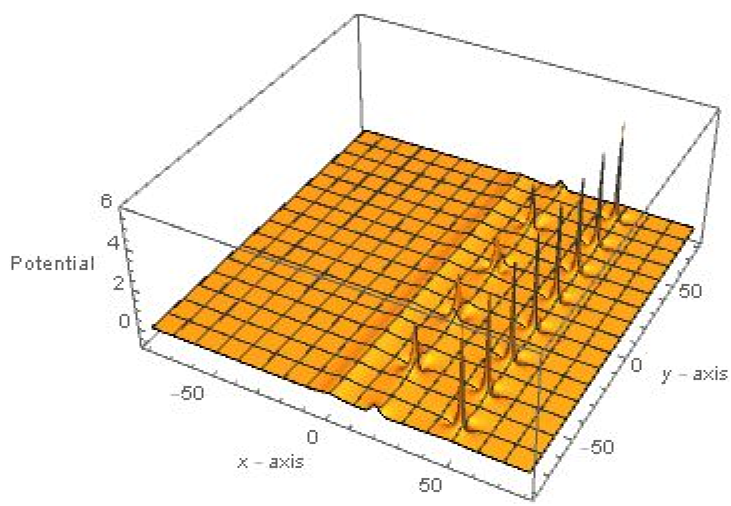
k1=¾, k2=½, k3=¼ , time=-15 to 15



k1=¾, k2=½, k3=¼ , time=-15



k1=¾, k2=½, k3=¼ , time=0



k1=¾, k2=½, k3=¼ , time=10

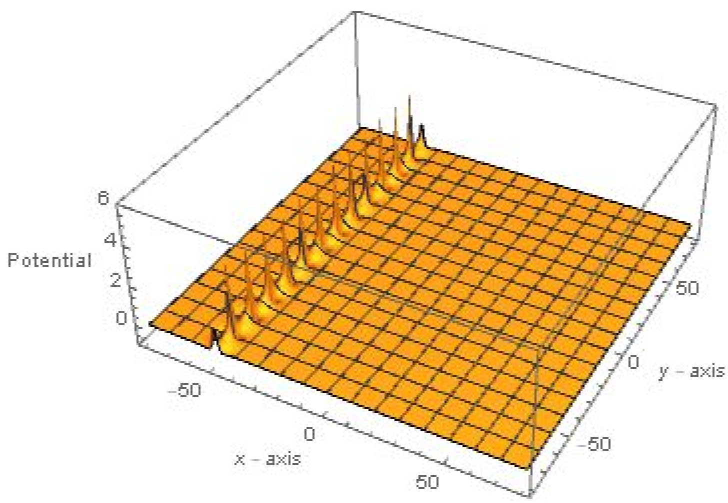
Chain soliton limit:

$$A = e^{2r_1} + e^{2r_2} + e^{2r_3} + 4\frac{\sqrt{a_1 a_2}}{(a_1+a_2)^2} e^{r_1+r_2} cos(q_2 - q_1) + 4\frac{\sqrt{a_1 a_3}}{(a_1+a_3)^2} e^{r_1+r_3} cos(q_3 - q_1) + 4\frac{\sqrt{a_2 a_3}}{(a_2+a_3)^2} e^{r_2+r_3} cos(q_3 - q_2)$$

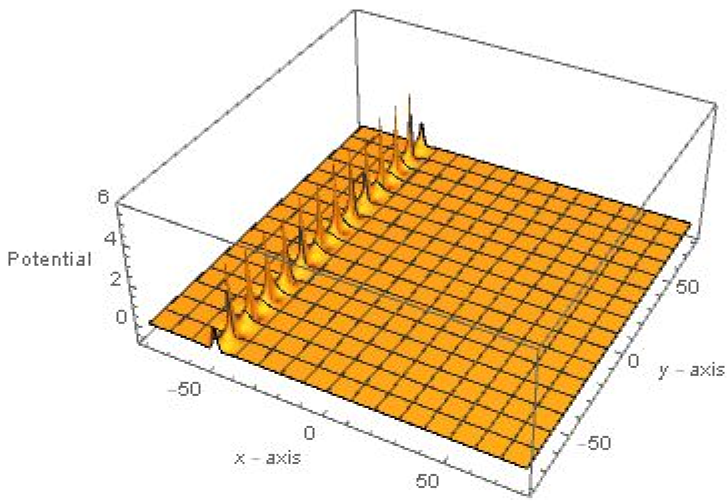
The chain soliton widths are,

$w_{-\infty} = 2\pi/(a_1{}^2 - a_3{}^2)$
$w_{fast} = 2\pi/(a_1{}^2 - a_2{}^2)$
$w_{slow} = 2\pi/(a_2{}^2 - a_3{}^2)$

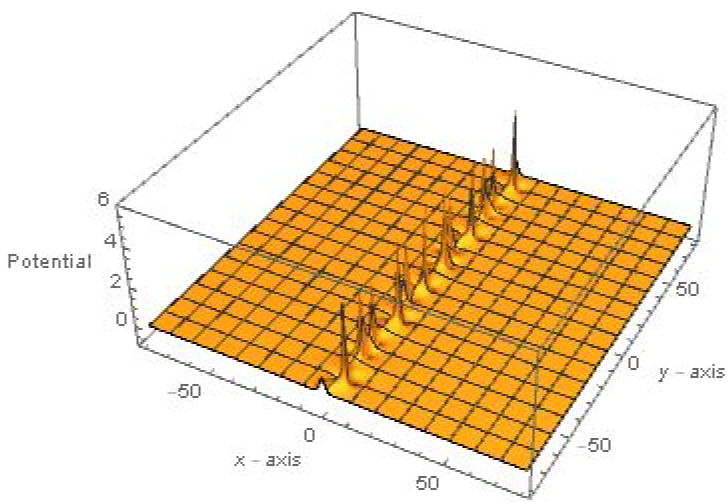
$v_{chain-\infty} = 4(a_1{}^2 + a_1 a_3 + a_3{}^2)$, $A_{chain-\infty} = 2(a_1 + a_3)(\sqrt{a_1} + \sqrt{a_3})^2$
$v_{fastchain} = 4(a_1{}^2 + a_1 a_2 + a_2{}^2)$, $A_{fastchain} = 2(a_1 + a_2)(\sqrt{a_1} + \sqrt{a_2})^2$
$v_{slowchain} = 4(a_2{}^2 + a_2 a_3 + a_3{}^2)$, $A_{slowchain} = 2(a_2 + a_3)(\sqrt{a_2} + \sqrt{a_3})^2$



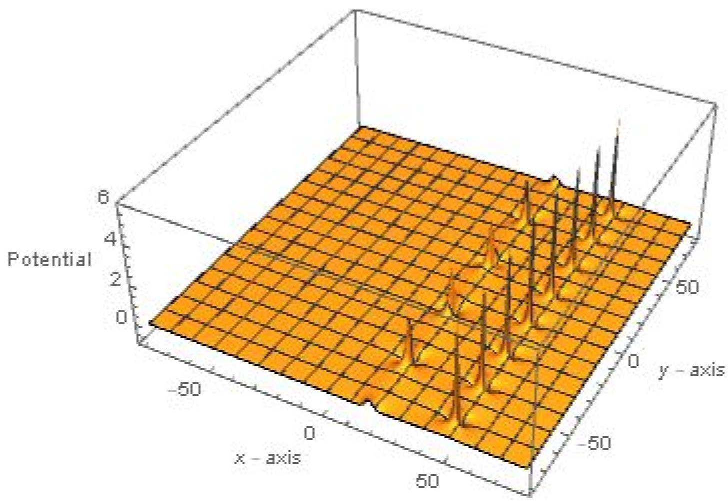
k1=¾, k2=½, k3=¼ , time=-15 to 15



k1=¾, k2=½, k3=¼ , time=-15



k1=¾, k2=½, k3=¼ , time=0



k1=¾, k2=½, k3=¼ , time=10