Analytic theory of a wind-driven sea

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A self-sustained analytic theory of a wind-driven sea is presented. It is shown that the wave field can be separated into two ensembles: the Hasselmann sea that consists of long waves with frequency $\omega < \omega_H$, $\omega_H \sim 4 - 5\omega_p$ (ω_p is the frequency of the spectral peak), and the Phillips sea with shorter waves. In the Hasselmann sea, which contains up to 95 % of wave energy, a resonant nonlinear interaction dominates over generation of wave energy by wind. White-cap dissipation in the Hasselmann sea in negligibly small. The resonant interaction forms a flux of energy into the Phillips sea, which plays a role of a universal sink of energy. This theory is supported by massive numerical experiments and explains the majority of pertinent experimental facts accumulated in physical oceanography.

Quasi-Conservative Hasselmann equation

It is accepted by the physical oceanography community that deep water ocean gravity surface waves are described by the Hasselmann equation. This equation is also known as the kinetic equation for waves, sometimes it is called the Bolzmann equation or energy balance equation:

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial \omega_k}{\partial \mathbf{k}} \frac{\partial \epsilon}{\partial \mathbf{r}} = S_{nl} + S_{in} + S_{diss} \tag{1}$$

The wave energy spectrum $\epsilon = \epsilon(\omega_k, \theta, r, t)$ is a function of wave frequency $\omega_k = \omega(k)$, angle θ, two-dimensional real space coordinate $r = (x, y)$, and time t. The terms S_{nl}, S_{in} and S_{diss} are the nonlinear, wind input and wave-breaking dissipation source terms. We will consider the deep water case only: the dispersion law is $\omega_k = \sqrt{gk}$, where g is the gravitational acceleration and $k = |\mathbf{k}|$ is the absolute value of the vector wavenumber $\mathbf{k} = (k_x, k_y)$. Since Hasselmann's work, Eq.(1) has become the basis of operational wave forecasting models.

While the physical oceanography community agrees on the general applicability of Eq. (1), there is no consensus on universal parameterizations of the source terms S_{nl}, S_{in} and S_{diss}. For the Hasselmann sea we put $S_{diss} = 0$.

This is the quasi-conservative Hasselmann kinetic equation written for the wave action spectrum $N_{\mathbf{k}}(t)$.

$$\frac{dN}{dt} = S_{\rm nl} \qquad (2)$$

$$S_{\rm nl} = \pi g^2 \int\limits_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \left(T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right)^2 \left(N_{\mathbf{k}}N_{\mathbf{k}_2}N_{\mathbf{k}_3} + N_{\mathbf{k}_1}N_{\mathbf{k}_2}N_{\mathbf{k}_3} - N_{\mathbf{k}}N_{\mathbf{k}_1}N_{\mathbf{k}_2} - N_{\mathbf{k}}N_{\mathbf{k}_1}N_{\mathbf{k}_3}\right) \times$$

$$\times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\, \delta(\omega + \omega_1 - \omega_2 - \omega_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3$$

Coefficient $T_{kk_1k_2k_3}$ is the coupling coefficient $T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = \frac{1}{2}\left(\hat{T}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} + \hat{T}_{\mathbf{k}_3\mathbf{k}_3\mathbf{k}_1\mathbf{k}_2}\right)$

$$
\begin{aligned}
\hat{T}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = -\frac{1}{4}\frac{1}{(k_1k_2k_3k_4)^{1/4}}\Bigg\{ & \frac{1}{2}\left(k_{1+2}^2 - (\omega_1+\omega_2)^4\right) \times \left(\mathbf{k}_1\mathbf{k}_2 - k_1k_2 + \mathbf{k}_3\mathbf{k}_4 - k_3k_4\right) \\
& -\frac{1}{2}\left(k_{1-3}^2 - (\omega_1-\omega_3)^4\right) \times \left(\mathbf{k}_1\mathbf{k}_3 + k_1k_3 + \mathbf{k}_2\mathbf{k}_4 + k_2k_4\right) \\
& -\frac{1}{2}\left(k_{1-4}^2 - (\omega_1-\omega_4)^4\right) \times \left(\mathbf{k}_1\mathbf{k}_4 + k_1k_4 + \mathbf{k}_2\mathbf{k}_3 + k_2k_3\right) \\
& +\left(\frac{4(\omega_1+\omega_2)^2}{k_{1+2} - (\omega_1+\omega_2)^2} - 1\right) \times (\mathbf{k}_1\mathbf{k}_2 - k_1k_2)(\mathbf{k}_3\mathbf{k}_4 - k_3k_4) \\
& +\left(\frac{4(\omega_1-\omega_3)^2}{k_{1-3} - (\omega_1-\omega_3)^2} - 1\right) \times (\mathbf{k}_1\mathbf{k}_3 + k_1k_3)(\mathbf{k}_2\mathbf{k}_4 + k_2k_4) \\
& +\left(\frac{4(\omega_1-\omega_4)^2}{k_{1-4} - (\omega_1-\omega_4)^2} - 1\right) \times (\mathbf{k}_1\mathbf{k}_4 + k_1k_4)(\mathbf{k}_2\mathbf{k}_3 + k_2k_3)\Bigg\}
\end{aligned}
$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4; \quad \omega_1 + \omega_2 = \omega_3 + \omega_4$$

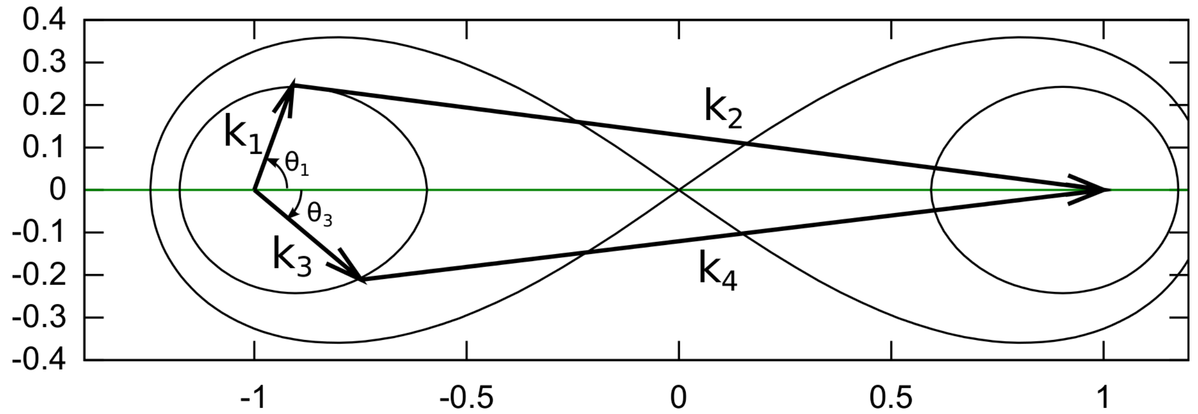


Figure 1: A wave vector quadruplet of a long-short interaction. A curve $\omega_1 + \omega_2 = \mathrm{const}$ is drawn; any two points of the curve constitute a resonant quadruplet. The θ_1 and θ_3 angles are given with respect to the vector $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$. The eight-shape figure is the Phillips curve.

Let us underline one important property of the resonant manifold $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4; \quad \omega_1 + \omega_2 = \omega_3 + \omega_4$. Suppose that the three wave vectors, k_1, k_2, k_3 are bound in length by some number $|k_i| < k_0$, $i = 1, 2, 3$. However, the last term might have a longer absolute value. In fact, in virtue of the resonant manifold we have $|k_1| < 5/4\, k_0$.

Hereafter we define $k_1 = |\mathbf{k}_1|$, $k_2 = |\mathbf{k}_2|$, etc. We have $k_1 \approx k_3 \ll k_2 \approx k_4$. After tedious algebra one may find the following asymptotic behavior for the coupling coefficient:

$$T_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)} \rightarrow \frac{1}{2} k_1^2 k_2 T_{\theta_1\theta_3}, \qquad T_{\theta_1\theta_3} = (\cos\theta_1 + \cos\theta_3)(1 + \cos(\theta_1 - \theta_3))$$

Here θ_1 is the angle between the small vector $\mathbf{k}_1$ and $\mathbf{k}_1 + \mathbf{k}_2$; the same stands for θ_3.

In the diagonal case $\theta_1 = \theta_3$, $\mathbf{k}_1 = \mathbf{k}_3$, $\mathbf{k}_2 = \mathbf{k}_4$.

$$T(\mathbf{k}_1, \mathbf{k}_2) = 2k_1^2 k_2 \cos(\theta_1)$$

Another important point is the question of conservation laws. The widely accepted opinion is that the quasi-conservative Hasselmann equation has basic conservation laws, i.e. wave action, energy and momentum:

$$N = \int N_k \, dk, \qquad E = \int \omega_k \, N_k \, dk, \qquad \mathbf{M} = \int \mathbf{k} \, N_k \, dk$$

Let us study more carefully the conservation laws. Apparently

$$\frac{dE}{dt} = \int \omega_k \, S_{nl} \, dk$$

If we boldly perform the permutation of integration order we will end up with relation

$$\frac{dE}{dt} = \pi g^2 \int |T_{kk_1k_2k_3}|^2 N_{k_1} N_{k_2} N_{k_3} (\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \, d\mathbf{k} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3 \quad (3)$$

It seems that this equation means that $dE/dt = 0$, but this would be correct only if all terms in this relation are finite and represented by convergent integrals. Now assume that N_k has asymptotic behavior $N_k \to 1/k^4$, $k \to \infty$. Then all terms will diverge logarithmically and will actually be infinite. Thus, in the presence of spectral tails, the conservation of energy and momentum fails. The asymptotic behavior $N_k \sim 1/k^4$ means that $I_k \simeq k^{-5/2}$ and $F(\omega) \simeq \omega^{-4}$. These spectra are commonly observed in the wind-driven sea in the spectral range $\omega_p < \omega < 5\omega_p$, where ω_p the is frequency of the spectral peak.

Let us add a little piece of pure mathematics. Strictly speaking, even this simple equation is not correct. Permutation of integration order in multi-dimensional integrals is allowed under strict limitations that are dictated by the so-called "Fubini theorem." In our case this theorem demands that action spectra should decay fast enough at $k \to \infty$:

$$N(k) < \frac{c}{k^{25/6+\epsilon}}, \quad \epsilon > 0$$

This means that the energy spectrum $F(\omega)$ must decay faster than $\omega^{-13/3}$. In the short-scale region of a real sea we usually observe the Phillips spectrum $F(\omega) \simeq \alpha g^2/\omega^5$. Because $5 > 13/3$ the integrals are conserved.

Let us notice that this takes place in the Phillips sea, consisting of short waves ($\omega > \omega_H$), outside of the Hasselmann sea, consisting of long waves ($\omega < \omega_H \sim 5\omega_p$). The resonant nonlinear interaction throws energy and momentum from the Hasselmann sea into the Phillips sea. Thus:

$$P = -\int_0^{2\pi} d\theta \int_0^{\omega_H} \frac{d\epsilon(\omega,\theta)}{dt} d\omega, \quad R_x = -\frac{1}{g}\int_0^{2\pi} d\theta \int_0^{\omega_H} \omega \, \cos\theta \, \frac{d\epsilon(\omega,\theta)}{dt} d\omega$$

P and R_x are fluxes of energy and momentum from the Hasselmann sea into the Phillips sea. Because they are not zero, one can call equation $\frac{dN}{dt} = S_{\rm nl}$ a quasi-conservative equation. This equation is a natural model for study of the ocean swell evolution. We have solved this equation numerically and have observed a permanent loss of energy and momentum.

Kolmogorov-type spectra

Let us study isotropic solutions of the stationary quasi-conservative Hasselmann equation:

$$S_{\mathrm{nl}} = 0 \tag{4}$$

We assume that the solution of the kinetic equation is a powerlike function $N = ak^{-x}$. Then

$$S_{\mathrm{nl}} = a^3 \, g^{\frac{3}{2}} \, k^{-3x+\frac{19}{2}} \, F(x)$$

where F is a dimensionless function depending on x only.

It was shown that $F(x) = 0$ at the two points $x = 4$ and $x \simeq 23/6$ only. This is a strict mathematical theorem, which is supported by careful numerical experiments. Integrals in Eq. (4) converge if $5/2 < x < 19/4$. Function F is shown on Fig. 2.

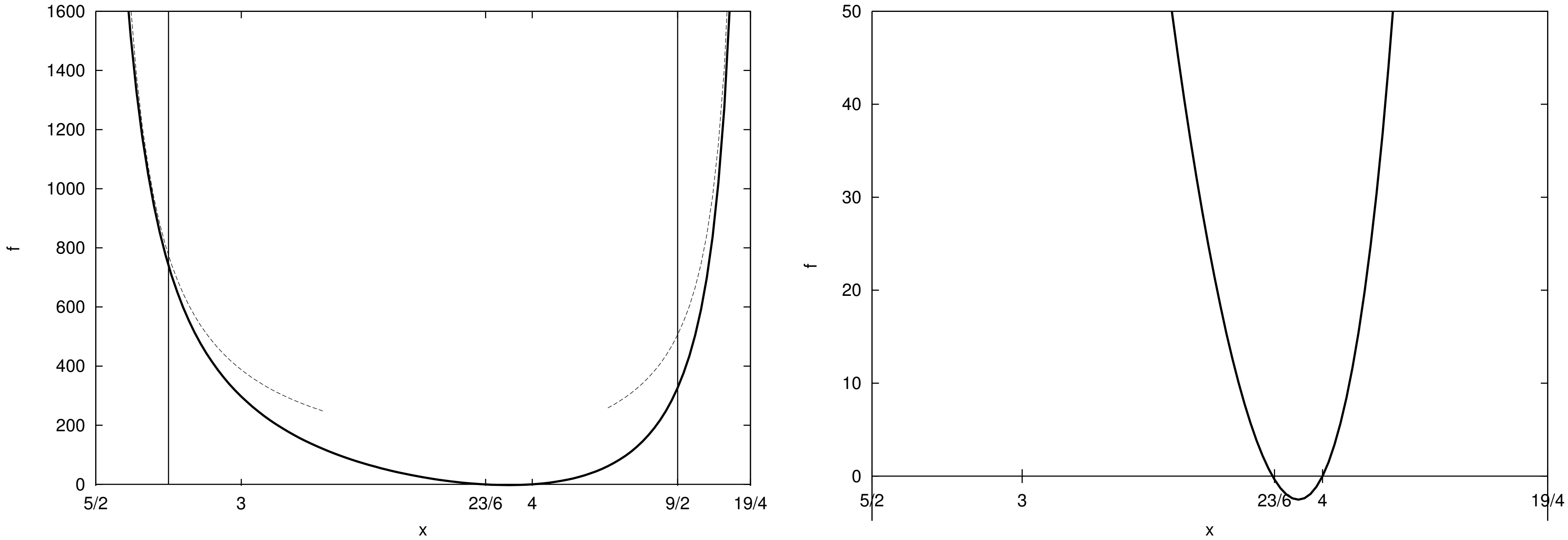


Figure 2: F function graph and its asymptotes. The second picture is the closeup of the function zeroes.

This means that the stationary kinetic equation $S_{nl} = 0$ has exactly two powerlike solutions:

$$N_k^{(1)} = c_p \frac{P_0^{1/3}}{g^{2/3}} \frac{1}{k^4}, \qquad N_k^{(2)} = c_q \frac{Q_0^{1/3}}{g^{1/2}} \frac{1}{k^{23/6}}.$$

Here P_0 is the energy flux and Q_0 is the wave action flux. The dimensionless constants c_p and c_q are defined from the first derivatives of F

$$c_p = \left(\frac{3}{2\pi\, F'(4)} \right)^{1/3}, \qquad c_q = \left(-\frac{3}{2\pi\, F'(23/6)} \right)^{1/3}$$

Our numerical calculation of the derivatives of F at $x = 4$ and $x = 23/6$ gives

$$c_p = 0.203, \quad c_q = 0.194$$

The integrated by angle energy spectra $E(\omega)$ are connected with the isotropic wave action spectra by the relation

$$F(\omega)\, d\omega = 2\pi\omega_k\, N_k\, k\, dk$$

Hence we find the following exact solutions for the stationary kinetic equation:

$$F_1(\omega) = \frac{4\pi\, c_p}{\omega^4}\, g^{4/3}\, P_0^{1/3}$$

It is a Kolmogorov-type spectrum that presumes the presence of a source of energy $P_0 = d\epsilon/dt$ at $k = 0$. This is the spectrum of "direct inverse cascade" similar to the classical Kolmogorov spectrum in the theory of turbulence in a three-dimensional incompressible fluid.

The second spectrum is the following:

$$F_2(\omega) = \frac{4\pi\, c_q}{\omega^{11/3}}\, g\, Q_0^{1/3}$$

It describes the "inverse cascade" of wave action, and can be compared with the Kolmogorov spectrum of the energy inverse cascade in the theory of turbulence in a two-dimensional incompressible fluid. The existence of solutions of the stationary kinetic equation originates from possibility of splitting S_{nl} as follows:

$$S_{nl} = F_k - \Gamma_k\, N_k,$$

$$F_k = \pi g^2 \int |T_{kk_1k_2k_3}|^2\, \delta(k{+}k_1{-}k_2{-}k_3)\, \delta(\omega_k{+}\omega_{k_1}{-}\omega_{k_2}{-}\omega_{k_3})\, N_{k_1}N_{k_2}N_{k_3}\, dk_1dk_2dk_3$$

$$\Gamma_k = \pi g^2 \int |T_{kk_1,k_2k_3}|^2\, \delta(k + k_1 - k_2 - k_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times$$
$$\times (N_{k_1}N_{k_2} + N_{k_1}N_{k_3} - N_{k_2}N_{k_3})\, dk_1dk_2dk_3$$

To outline a broader class of solutions, let us introduce the elliptic differential operator:

$$L\,f(\omega,\phi) = \left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)\,f(\omega,\phi)$$

with following parameters: $0<\omega<\infty,\;\;0<\phi<2\pi$. The equation

$$L\,G = \delta(\omega-\omega')\,\delta(\phi-\phi')$$

with boundary conditions $G|_{\omega\to 0} = 0$, $G_{\omega\to\infty}<\infty$, $G(2\pi)=G(0)$ is resolved as

$$G(\omega,\omega',\phi-\phi') = \frac{1}{4\pi}\sqrt{\omega\omega'}\sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')}\times\left[\left(\frac{\omega}{\omega'}\right)^{\Delta_n}\,\Theta(\omega'-\omega) + \left(\frac{\omega'}{\omega}\right)^{\Delta_n}\,\Theta(\omega-\omega')\right]$$

where $\Delta_n = 1/2\sqrt{1+8n^2}$. Now we define:

$$A(\omega,\phi) = \int_0^\infty d\omega'\int_0^{2\pi} d\phi'\,G(\omega,\omega',\phi-\phi')\,S_{nl}(\omega',\phi').$$

Then the kinetic equation takes the following form

$$\frac{\partial N}{\partial t} = L\,A$$

and the stationary equation is

$$L\,A = 0$$

The operator A is a regular integral operator. If we assume that

$$A = \frac{H_0}{g^4}\,\omega^{15}\,N^3,$$

then the nonlinear term S_{nl} turns into the elliptic operator:

$$S_{nl} = \frac{H_0}{g^4}\left(\frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2}\,\frac{\partial^2}{\partial \phi^2}\right)\,\omega^{15}\,N^3.$$

This is a so-called "diffusion approximation". In spite of being very simple, this approximation grasps the basic features of the wind-driven sea theory. The real case does not differ much from it, at least qualitatively.

H_0 is a dimensionless tuning constant. In the kinetic equation, $N = N(\omega, \phi)$, $\epsilon(\omega, \phi) = \omega\, N(\omega, \phi)$. The elliptic operat A has the following anisotropic KZ solution

$$A = \frac{1}{2\pi\, g} \left\{ P + \omega\, Q + \frac{R_x}{\omega} \cos\phi \right\},$$

where P and R_x are fluxes of energy and momentum as $\omega \to \infty$ and Q is the flux of wave action directed to small wave numbers. In a general case, A is a nonlinear integral equation; however in the diffusion approximation the KZ solution can be found in the explicit form:

$$N(\omega, \phi) = \frac{1}{(2\pi\, H_0)^{1/3}} \frac{g}{\omega^5} \left(P + \omega\, Q + \frac{R_x}{\omega} \cos\phi \right)^{1/3}.$$

For the real sea, the stationary equation is a nonlinear integral equation which can be solved numerically only. The "toy" diffusion model allows us to find the explicit equation for the KZ-solution which grasps the main features of real solution. One can assert that the real KZ solution is

$$F(\omega) = \frac{g^{4/3} P^{1/3}}{\omega^4} R \left(\frac{\omega Q}{P}, \frac{R_x}{g\,\omega\,P}, \phi \right)$$

In the limit $P \to 0$, $R_x \to 0$ we have $R \to 4\pi\, c_p$. In the limit $R_x = 0$, $P \to 0$

$$R \to 4\pi\, c_q \left(\frac{\omega Q}{P} \right)^{1/3}$$

Q is the flux of wave action coming from the spectral area of very small scales. In the majority of physical situations one can put $Q = 0$.

From the physical viewpoint the most interesting case is $Q = 0$, for which

$$F(\omega) = \frac{g^{4/3}\, P^{1/3}}{\omega^4}\, R_0 \left(\frac{R_x}{g\,\omega\,P}, \phi \right)$$

Here R_0 is an unknown function that we believe describes the angular spreading of wave spectra. It was shown long ago that at $\omega \to 0$

$$R_0 \to 4\pi\, c_p \left(1 + \frac{\lambda\, R_x}{g\,\omega\,P} \cos\phi + \cdots \right)$$

where λ is a dimensionless constant. In the "toy" diffusion model $\lambda = 1/3$. We should stress that all KZ spectra are isotropic in the limit $\omega \to \infty$ and are very close to $F(\omega) \sim 1/\omega^4$.

Energy balance in wind-driven sea

The most painful question is: which source terms in the kinetic equation are dominant? To answer, we should present S_{nl} in the split form. After the splitting, the kinetic equation takes the following form:

$$\frac{\partial N}{\partial t} + \frac{\partial \omega_k}{\partial k}\frac{\partial N}{\partial r} = F_k - \Gamma_k\, N_k + S_{in} + S_{diss}$$

The forcing terms S_{in} and S_{dis} are not known well enough, thus it is reasonable to accept the simplest models of both terms assuming that they are proportional to the action spectrum:

$$S_{in} = \gamma_{in}(k)\, N(k), \quad S_{dis} = -\gamma_{dis}(k)\, N(k).$$

Hence

$$\gamma(k) = \gamma_{in}(k) - \gamma_{dis}(k).$$

In reality $\gamma_{dis}(k)$ depends dramatically on the overall steepness μ. So far, let us notice that the stationary balance equation can be written in the form

$$F_k - \Gamma_k \, N_k + \gamma_k \, N = 0$$

The stationary solution of kinetic equation is the following:

$$N_k = \frac{F_k}{\Gamma_k - \gamma_k}.$$

The positive solution exists if $\Gamma_k > \gamma_k$. The term Γ_k can be treated as the nonlinear damping that appears due to four-wave interaction. In the presence of nonlinear damping the dispersion relation must be renormalized

$$\omega_k \to \omega_k + \frac{1}{2} \int T_{kk_1kk_1} \, N_k \, dk + i\Gamma_k$$

The main point of the proposed theory is that the nonlinear dumping has a very powerful effect. In reality, $\Gamma_k \gg \gamma_k$. Let $k \gg k_p$. Then for Γ_k we get

$$\Gamma_k = 2\pi g^2 \int |T_{kk_1,kk_3}|^2 \, \delta(\omega_{k_1} - \omega_{k_3}) \, N_{k_1} N_{k_3} \, dk_1 dk_2.$$

$$\Gamma_\omega = 36\,\pi\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4 \cos^2\theta, \quad \mu_p^2 = \frac{g^2\,E}{\omega\,p^4}$$

It includes a huge enhancing factor: $36\pi \simeq 113.04$. For very modest value of steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06 \cdot 10^{-4} \omega \left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta.$$

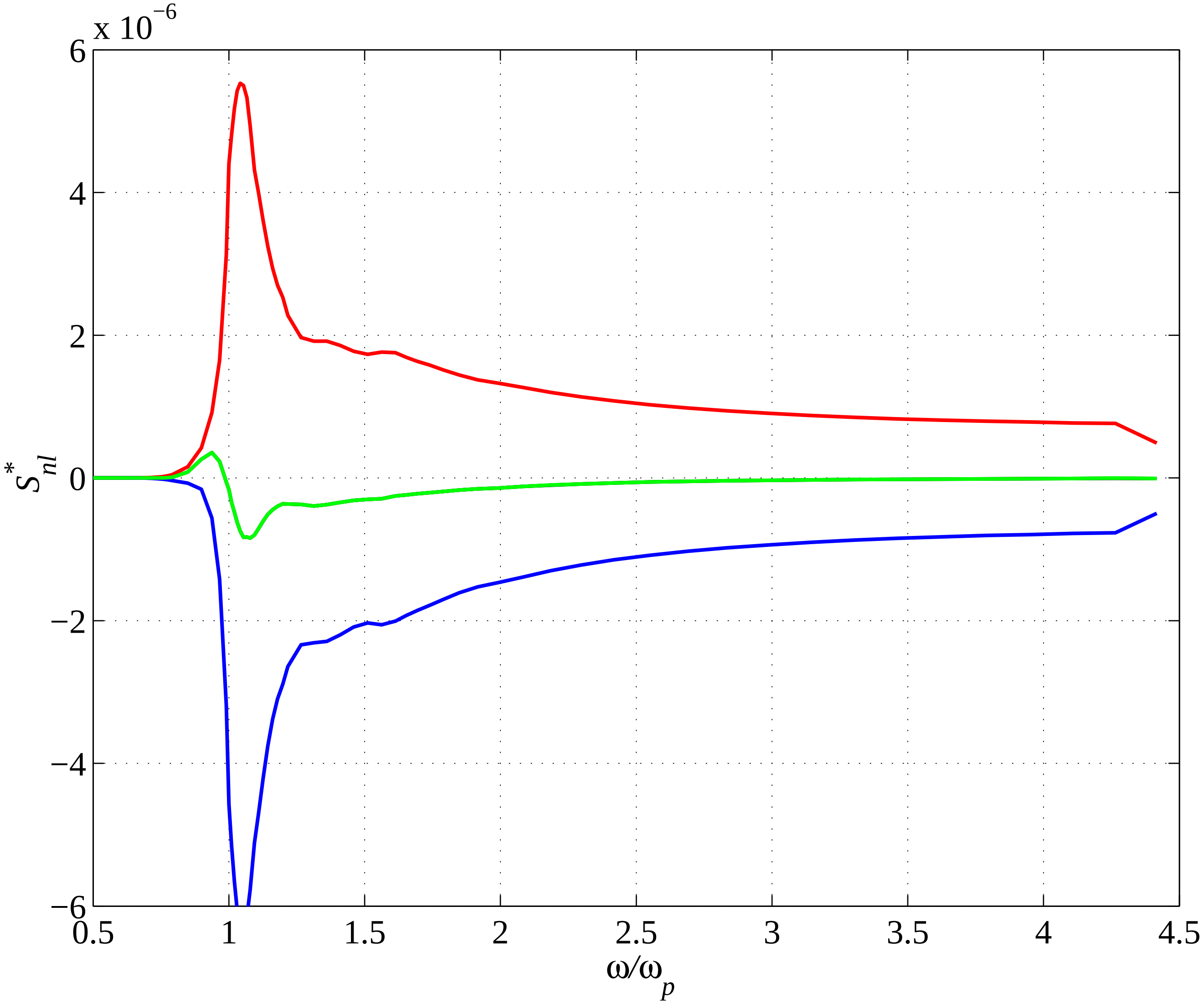


Figure 3: Split of nonlinear interaction term S_{nl} (central curve) into F_k (upper curve) and $\Gamma_k\, N_k$ (lower curve)

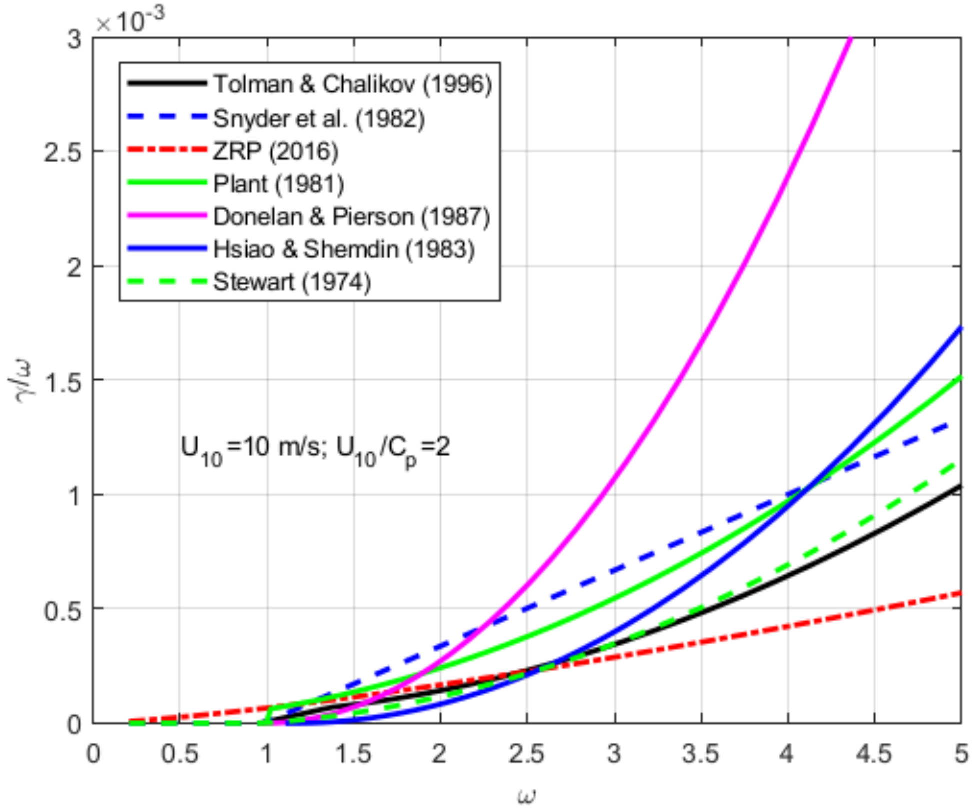


Figure 4: Dimensionless wind input for $u_{10} = 10m/sec$.

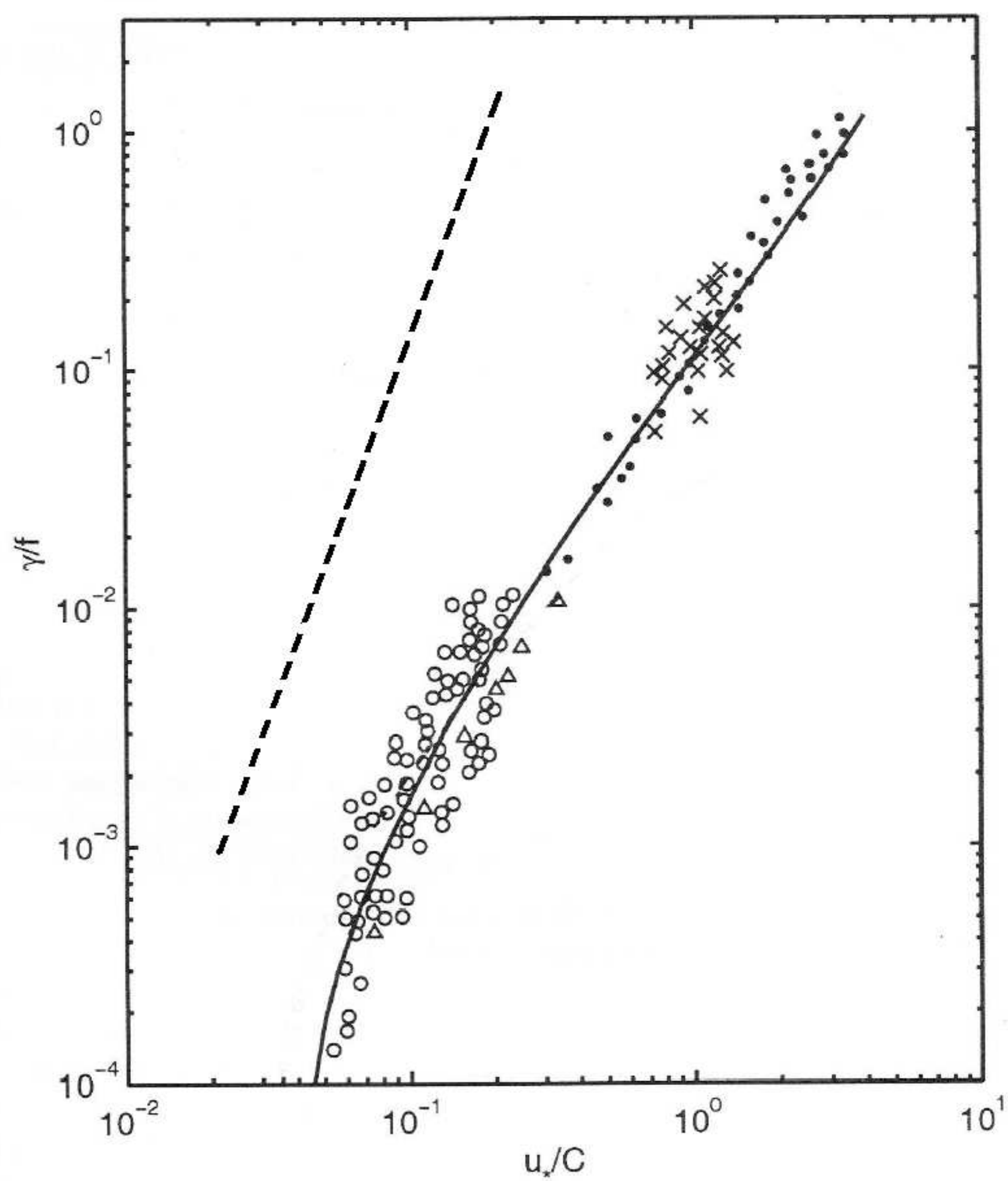


Figure 5: Comparison of experimental data for the wind-induced growth rate

We pay special attention to two models:

1. The Plant model $\gamma = 0.03\frac{\rho_a}{\rho_\omega}\,\omega\,\left(\frac{\omega\,U}{g}\right)^2\cos\phi, \qquad \cos\phi > 0$

2. ZRP model $\gamma = 0.05\frac{\rho_a}{\rho_\omega}\,\omega\,\left(\frac{\omega U}{g}\right)^{4/3}\cos^2\phi, \qquad -\pi/2 < \phi < \pi/2$

In both models $\gamma \simeq \omega^{1+s}$ is a powerlike function on frequency.

Comparison of all known models for S_{in} with the nonlinear dumping term Γ_k is presented on Fig. 5 One can see that Γ_k, at least in order of magnitude, is larger than $\gamma_{in}(k)$. This figure conspicuously demonstrates that the nonlinear wave interaction is the leading term in the energy balance of a wind-driven sea.

Experimental evidence of S_{nl} domination

As far as in the Hassenlmann sea the term S_{diss} cannot be stronger than γ_{in} (otherwise waves would not be excited), the term S_{nl} dominates over. This fact is supported by experimental data collected in a broad ranges of wind velocities: $3m/sec < U, 30m/sec$. Hereafter we will use the dimensionless duration and fetch, as well as the dimensionless frequency and energy:

$$\tau = \frac{tg}{U}, \quad \chi = \frac{xg}{U^2}, \quad \sigma = \frac{\omega U}{g}, \quad F = \frac{\epsilon g^2}{U^4}$$

Also, we introduce integral dimensionless quantities

$$\tilde{F} = \int_0^\infty F(\sigma) d\sigma, \qquad \tilde{\sigma} = \frac{1}{\tilde{F}} \int_0^\infty \sigma F(\sigma) d\sigma$$

The steepness of the main energy capacity wave can be estimated as follows

$$\mu_p \simeq \tilde{F}\,\tilde{\sigma}^4$$

During the last seven decades many experiments measuring energy spectra of a wind-driven sea and its integral characteristics were performed in laboratory, on lakes, and in the different parts of the ocean. The most significant experiments were conducted in the "fetch dominating frame," where the sea is stationary in time and the wind has the opposite direction. In these experiments, $\tilde{F}$ and $\tilde{\sigma}$ were measured as functions of fetch only: $\tilde{F} = \tilde{F}(\chi)$, $\tilde{\sigma} = \tilde{\sigma}(\chi)$. All experimenters unanimously agree that $\tilde{F}$ and $\tilde{\sigma}$ are powerlike functions

$$\tilde{F} = \epsilon_0\,\chi^p, \qquad \tilde{\sigma} = \omega_0\,\chi^{-q}$$

Exponents p, q are different in different experiments. They vary inside the following ranges

$$0.7 < p < 1.1 \qquad 0.22 < q < 0.33$$

Suppose that F obeys the stationary Hasselmann equation. After transition to dimensionless variable this equation reads

$$\frac{\cos\theta}{2\sigma}\frac{\partial F}{\partial\chi} = S_{nl} + \gamma_{in}(\sigma)\, F$$

We include in this equation the interaction with wind. Let us make a very crude estimate of the different terms in this equation. Neglecting the wind input term we come to the following balance relation

$$\frac{F}{\tilde{\sigma}\chi} \simeq \tilde{\sigma}\, F\, \mu_p^4$$

or, after cancelling F

$$\chi \tilde{F}^2\, \tilde{\sigma}^{10} \simeq 1$$

Substituting the powerlike functions into kinetic equation one can see that dependance on χ drops out if the exponents p, q are connected by the relation

$$10q - 2p = 1$$

We call it the "magic relation." In virtue of this relation

$$q = q_{th} = \frac{2p+1}{10}$$

Moreover, we can conclude that $s = \epsilon_0^{1/5}\omega_0$ is a universal constant. Comparison with numerical experiments show that

$$s = \epsilon_0^{1/5}\omega_0 \simeq 1$$

Results of experiments performed in the open sea and Lake Michigan are presented in Table 1, which represents the majority of the field experiments collected in physical oceanography for almost half of a century. Experimental data are compared with predictions of the analytic theory. According to theory, the exponents q_{chi} must coincide with the theoretically predicted value $q_{th} = 2p_\chi + 1/10$. One can see that the relative difference $\delta q \simeq \frac{1}{q_\chi}|q_\chi - q_{th}|$ does not exceed 10%. According to theory, the dimensionless quantity $s = \epsilon_0^{1/5}\omega_0$ must be a universal constant of order one.

	Case	$\varepsilon_0 \times 10^7$	p_χ	ω_0	q_χ	q_{th}	S
1	Wen. et al. (1989)	18.900	0.700	10.40	0.23	0.24	0.75
2	Donelan et al. (1985) var. der	8.410	0.760	11.60	0.23	0.25	0.71
3	Dobson et al. (1989) wind. int	12.700	0.750	10.68	0.24	0.25	0.71
4	Kahma & Calkoen (1992) stabl	9.300	0.760	12.00	0.24	0.25	0.75
5	Evans & Kibblewhite (1990) stra	5.900	0.786	16.27	0.28	0.26	0.92
6	Romero & Melville (2009) unstab	5.750	0.810	10.64	0.23	0.26	0.60
7	Hwang & Wang (2004)	6.191	0.811	11.86	0.24	0.26	0.68
8	Davidan (1996), U_{10} scaling	5.550	0.840	16.34	0.29	0.27	0.92
9	Evans & Kibblewhite (1990) neut	2.600	0.872	18.72	0.30	0.27	0.90
10	Black. Sea	4.410	0.890	15.14	0.28	0.28	0.81
11	Kahma & Calkoen (1992) composit	5.200	0.900	13.70	0.27	0.28	0.76
12	Kahma & Pettersson (1994)	5.300	0.930	12.66	0.28	0.29	0.70
13	Kahma & Calkoen (1992) unstab	5.400	0.940	14.20	0.28	0.29	0.79
14	Walsh, US coast (1989)	1.860	1.000	14.45	0.29	0.30	0.65
15	Mitsuyasu (1971)	1.600	1.000	21.99	0.33	0.30	0.96
16	JONSWAP (1973)	2.890	1.008	19.72	0.33	0.30	0.97
17	Donelan et al. (1992)	1.700	1.000	22.62	0.33	0.30	1.00
18	Kahma (1986)average. growth	2.000	1.000	22.00	0.33	0.30	1.01
19	Kahma (1981, 1986)rapid. growth	3.600	1.000	20.00	0.33	0.30	1.03

We solved the stationary Hasselmann equation numerically, using various models for $\gamma(\omega, \theta)$. The results are presented in the Table.

Table 1: Data of numerical experiments

Experiment	p_x	q_x	$10q - 2p$	ε_0	ω_0	$\varepsilon_0^{1/5}\omega_0$
ZRP	1	0.3	1	$2.9 \cdot 10^{-7}$	21.35	1.05
Snyder	0.7	0.23	1	$1.24 \cdot 10^{-5}$	9.04	0.94
Tolman-Chalikov	0.5	0.2	0.9	$3.2 \cdot 10^{-5}$	7.91	1.00
Hsiao-Shemdin	0.5	0.19	0.9	$2.0 \cdot 10^{-5}$	8.16	0.94
Donelan (with dissipation)	0.6	0.21	0.83	$6.1 \cdot 10^{-6}$	10.17	0.92
Donelan (without dissipation)	0.53	0.19	0.84	$2.05 \cdot 10^{-5}$	7.85	0.91
Plant	0.77	0.254	1	$2.9 \cdot 10^{-6}$	12.89	1.006
Stuart-Plant	0.5	0.21	1.1	$1.15 \cdot 10^{-5}$	9.48	0.975

Self-similaruty of wind-driven sea

Now we can answer the most "sharpest" questions: Why do both field and laboratory experiments assert that $\tilde{F} = F(\chi)$ and $\sigma = \sigma(\chi)$ are powerlike functions? Why are the exponents p, q are contained inside intervals $0.7 < p < 1.1$, $0.22 < q < 0.33$? We will discuss the Hasselmann sea only, where the Hasselmann equation is applicable. Let us consider the dimensionless kinetic equation and assume that $\gamma_{in}(\sigma)$ is a powerlike function

$$\gamma_{in}(\sigma) = \gamma_0 \, \sigma^{1+l} \cdot f(\phi)$$

One can check that this equation has the following self-similar solution

$$F = \chi^{p+q} \, G(\sigma, \chi, \phi)$$

which leads to powerlike expressions, where

$$\epsilon_0 = \int_0^{2\pi} d\phi \int_0^{\infty} G(\sigma, \phi) \, d\sigma \qquad \omega_0 = \frac{1}{\epsilon_0} \int_0^{2\pi} d\phi \int_0^{\infty} \sigma G(\sigma, \phi) \, d\sigma$$

In the self-similar solution

$$q = \frac{1}{2+l}, \quad p = \frac{8-l}{2(2+l)}$$

The function $G(\xi, \phi)$, $\xi = \sigma \chi^q$, satisfies the following equation:

$$\cos\phi[(p+q)G + q\xi\frac{\partial G}{\partial \xi}] = \tilde{S}_{nl} + \gamma_0 \xi^{1+l}\, f(\phi)\, G$$

Here $\tilde{S}_{nl}$ is a dimensionless S_{nl} and $\gamma_0 \simeq 10^{-5}$ is a dimensionless small parameter. This term can be split into income and outcome terms. Each of them dominates over S_{in}; thus near the spectral peak S_{in} can be neglected and the condition $s = \epsilon^{1/5}\,\omega_0 \sim 1$ still holds.

Now let us notice that in the ZRP model of S_{in} , $l = 4/3$. This gives $q = 0.3$, $p = 1$, in good accordance with experiments 11-19 presented in Table 1. For the Plant model, $l = 2$; this gives $q = 0.25$, $p = 0.75$, in good accordance with experiments 2-6 in Table 1. In all offered models for S_{in}, $\gamma(\sigma)/\sigma$ is a growing function, and $1 < S < 2.3$. This gives the following frames for the variation of exponents:

$$0.67 < p < 7/6, \quad 0.22 < q < 0.33$$

These frames are very close to experimentally observed results presented in the Table. The results of numerical experiments collected in the other Table show that models of S_{nl} different from the ZRP and Plant models lead to exponents outside the frames $0.7 < p < 1.1 \qquad 0.22 < q < 0.33$. This is not a weak point of theory; rather it is a weakness of the discussed models. The major prediction of the theory, the magic relation $10q - 2p = 1$, is satisfied pretty well.

In these models, $\gamma(\sigma)$ are not pure powerlike functions. However S_{in} is still a small term in the dimensionless kinetic equation, and we may seek "quasimodular solutions" such that exponents are "slow functions" of fetch $p = p(\chi)$, $q = q(\chi)$.

Critical analysis of data from field, wave tanks and numerical experiments shows that in a huge range of fetches, $10 < \chi < 10^6$ the magic relation is valid!

But it doesn't mean that all models for S_{in} are equally good. The analytic model predicts the "magic relation" between p and q as well as a relation between ϵ_0 and ω_0, but it says nothing about absolute values of these quantities. Comparing the first line of Table 1 (Wel et al experiments) with the second line in Table 2 (Snyder model prediction) we see very good qualitative coincidence but large quantitative differences. The Snyder model overestimates the rate of energy growth with fetch by almost an order of magnitude.

We cannot discuss now an extremely important question: the shape of spectra in the universal spectral range $1 < \sigma < 5$. The dimensionless kinetic equation does not preserve energy that leaks from the Hasselmann sea to the Phillips sea, forming an energy flux P. Thus it's solution must have asymptotic behavior

$$G(\xi) \to \beta \frac{P^{1/3}}{\sigma^4}$$

As far as $\gamma_0 \ll 1$, β is a small number. This implies the inevitable formation of Zakharov-Filonenko spectral tails $F(\omega) \sim 1/\omega^4$. Such tails are routinely observed in numerous field and laboratory experiments. This important subject deserves a special consideration.

Conclusions

Let us summarize the results. We claim that the majority of data obtained in field and numerical experiments can be explained in a framework of a simple model

$$\frac{d\epsilon}{dt} = S_{nl} + \gamma_{in}(\omega, \phi)\epsilon$$

Moreover, most of the facts can be explained by the assumption that $\gamma_{in}(\omega, \phi)$ is a powerlike function on frequency, $\gamma_{in}(\omega, \phi) = \gamma_0\, \omega^{1+s}\, f(\phi)$. Here $1 < s < 2.3$ and $f(\phi), \gamma_0$ are tunable. This model pertains only to the description of the Hasselmann sea, $0 < \omega < \omega_H$, $\omega_H \simeq (4-5)\omega_p$.

In fact, this model is a simplification of the widely accepted model in oceanography. What is the difference between these models? The main difference is obvious: we excluded from our consideration any mention of wave energy dissipation. This does not mean that we deny a crucial role of wave-breaking in the dynamics of ocean surface. But, from the spectral viewpoint, the wave-breaking takes place outside the Hasselmann sea. It is going into the Phillips sea, in the spectral area of short scales. This very important statement is supported by experimental data and by numerical solutions of dynamical phase-resolving equations for a free surface.

What we offer could be called "poor man's oceanography." A "poor man" refuses attempts to derive the equation for S_{in} from "first principles," but has in his possession powerful analytic and computer models to use as test beds for compatibility of models for $\gamma_{in}(\omega, \phi)$ with experimental data. The Snyder model does not pass this test and should be excluded from operational models.

○ Black Sea

Babanin et al., 1996

□ US coast, N.Atlantic

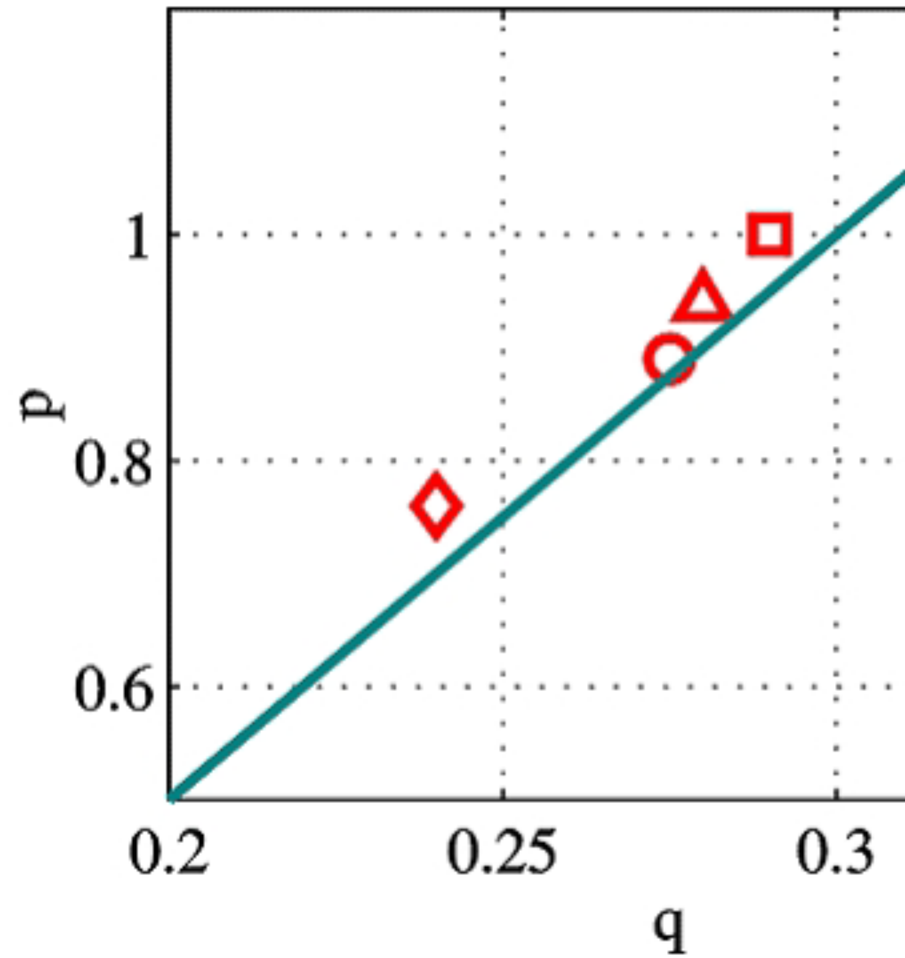
Walsh et al 1989

△ Bothnian Sea, unstable

Kahma & Calkoen 1992

◇ Bothnian Sea, stable

Kahma & Calkoen 1992

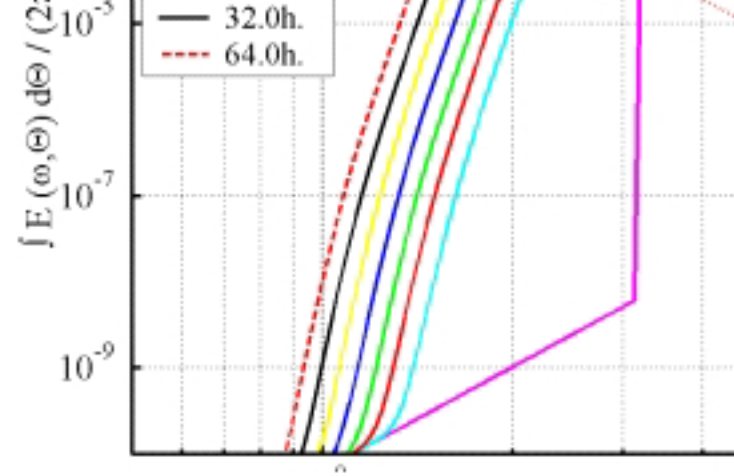


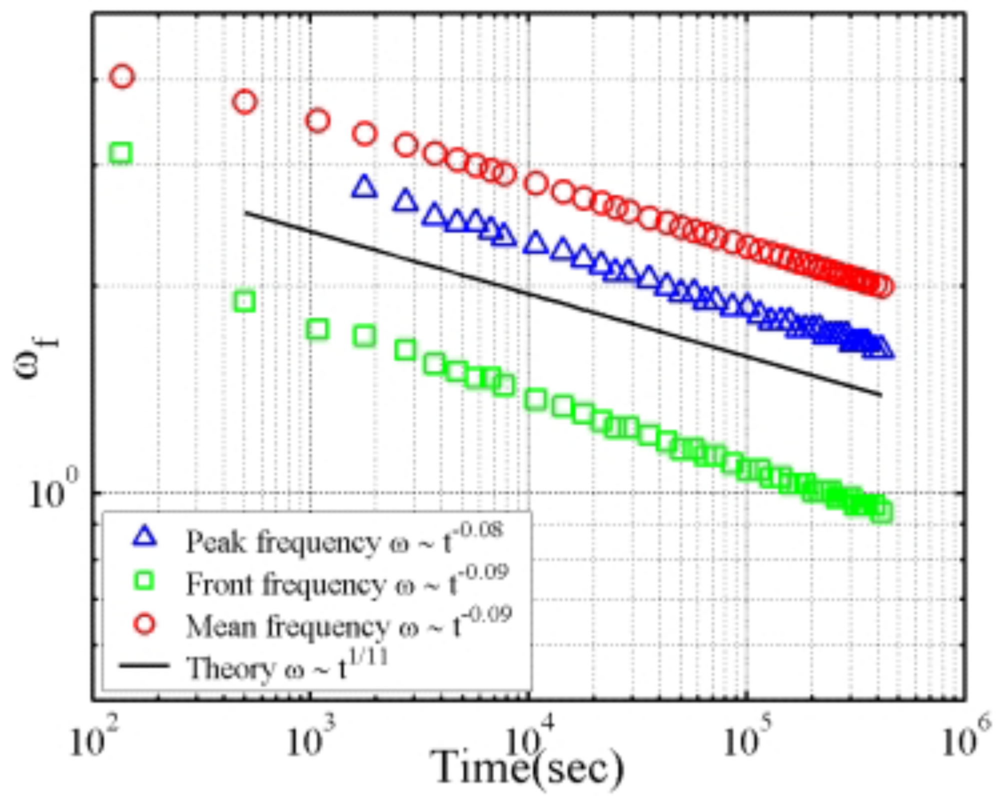
Features of self-similarity of wave spectra evolution results in simple link between exponent of energy growth ***p*** and one of frequency downshift ***q***

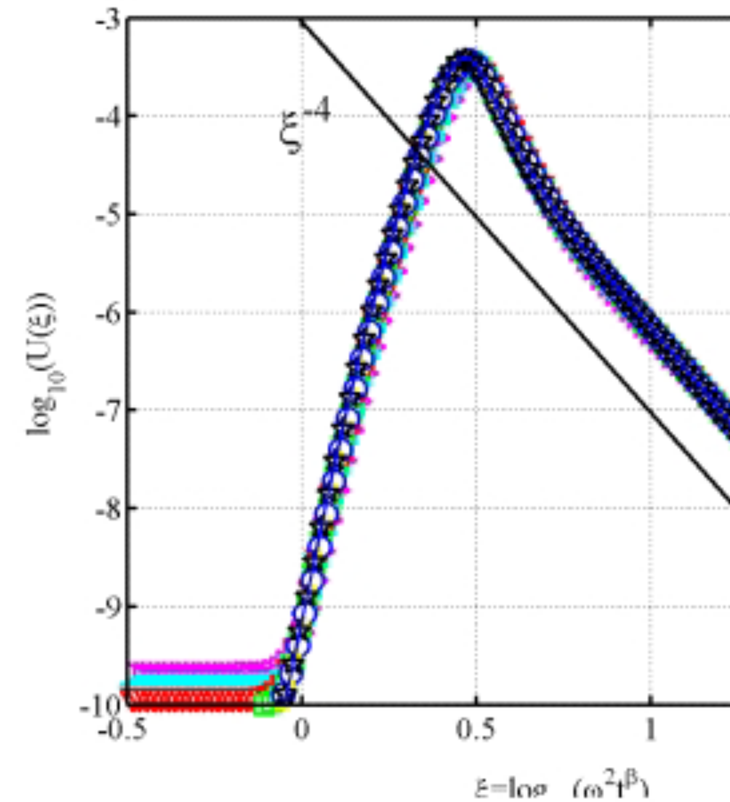
$p =$

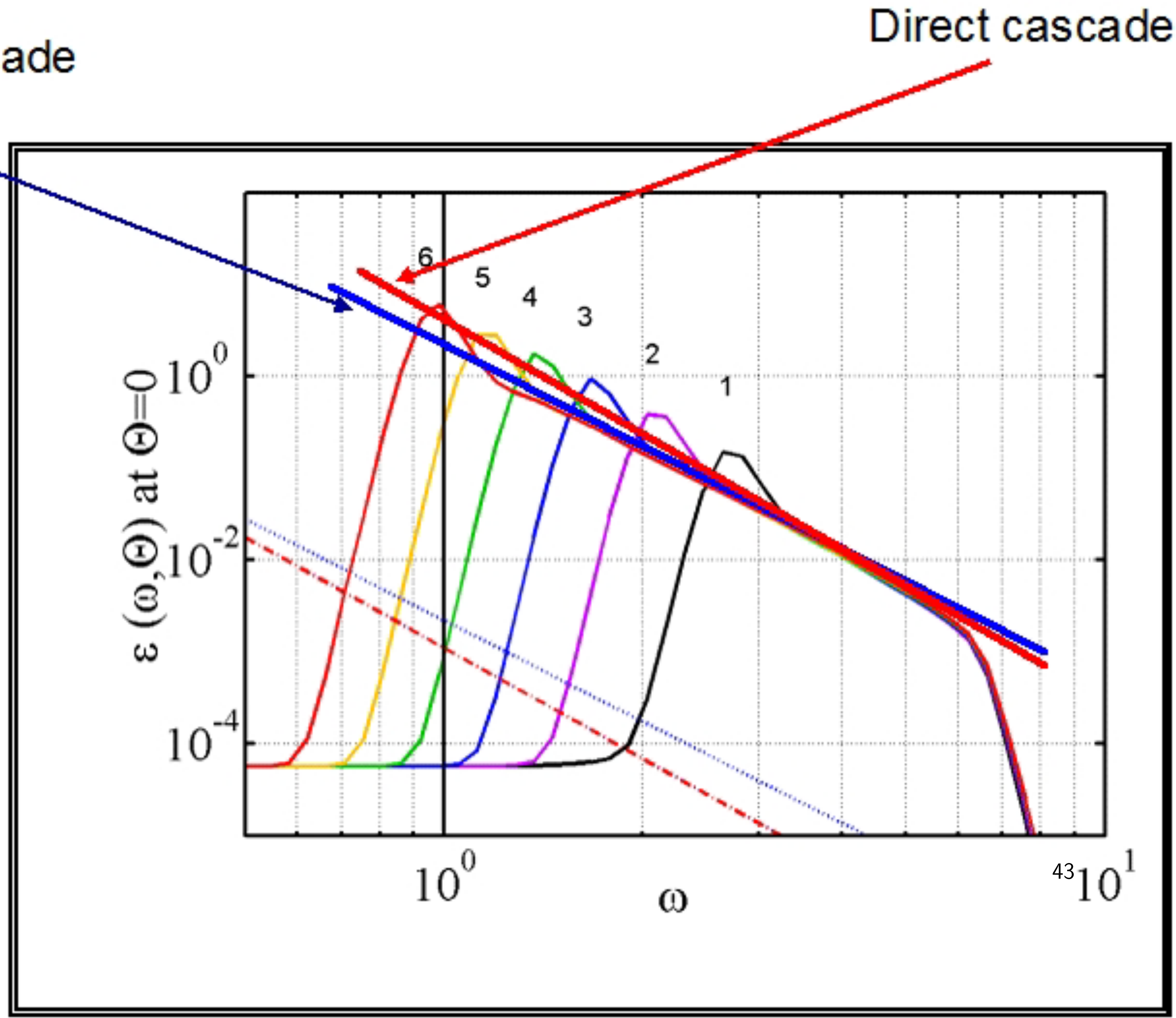
Swell

$$n = t^{2/11} U_0(\omega^2 t^{1/11}, \Theta)$$








inverse cascade
Direct cascade
6
5
4
3
2
1
10^0
10^{-2}
10^{-4}
$\varepsilon\ (\omega,\Theta)$ at $\Theta=0$
10^0
10^1
ω

Ocean Waves Generation Against the Wind: Fourier-Real Space Energy Pipelines DOES OCEAN LASER EXIST?

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SkolTech, FIAN

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Overview

Introduction

- $\frac{\partial \varepsilon}{\partial t} + \frac{\partial \omega_k}{\partial \vec{k}} \frac{\partial \varepsilon}{\partial \vec{r}} = S_{nl} + S_{in} + S_{diss}$
- $\varepsilon = \varepsilon(\vec{r}, \vec{k}, t)$
- S_{nl} - nonlinear 4-waves interaction term
- S_{in} - wind input
- S_{diss} - wave-breaking dissipation
- Basis of operational models WaveWatch, WAM

Motivation of the research

- Observation of non-stationary limited fetch regime
- Connection to SSS in homogeneous case
  $\frac{\partial \varepsilon}{\partial t} = S_{nl} + S_{in} + S_{diss}$
- Connection to SSS in stationary case
  $\frac{\partial \omega_k}{\partial \vec{k}} \frac{\partial \varepsilon}{\partial \vec{r}} = S_{nl} + S_{in} + S_{diss}$

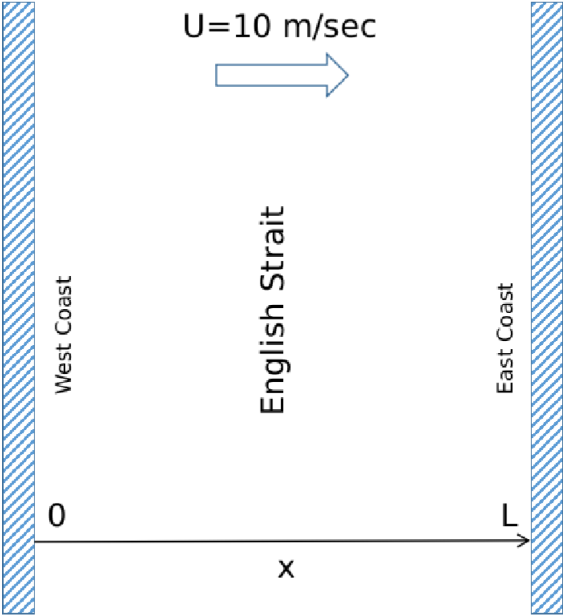
Motivation of the research

Stationary case	Non-stationary case
$\varepsilon = t^{p+q} F(\omega t^{q})$	$\varepsilon = \chi^{p+q} F(\omega \chi^{q})$
$E \sim t^{p}$ $\quad \langle \omega \rangle \sim t^{-q}$	$E \sim \chi^{p}$ $\quad \langle \omega \rangle \sim t^{-q}$
$9q - 2p = 1$	$10q - 2p = 1$
$p = 10/7$ $\quad q = 10/7$	$p = 1$ $\quad q = 3/10$
$s = 4/3$	$s = 4/3$

Problem statement

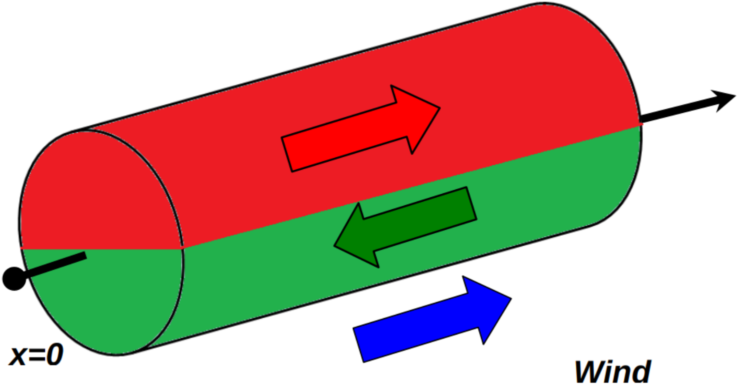
- $\frac{\partial \varepsilon}{\partial t} + \frac{1}{2}\frac{\omega_k}{k}\cos\theta\frac{\partial \varepsilon}{\partial x} = S_{nl} + S_{in} + S_{diss}$
- Exact S_{NL}
- ZRP (Zakharov, Resio, Pushkarev 2010) forcing
- Dissipation spectral tail $\sim \omega_k^{-5}$ starting from $f_{diss} = 1.1$ Hz
- Channel of 40 km width: La-Manche
- 40 points in real space, 10° angular resolution, 72 frequencies
- wind 10 m/sec blowing from France to UK

Problem statement

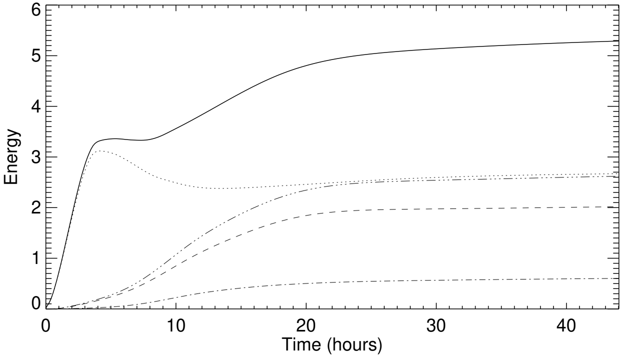


Problem statement

Problem statement

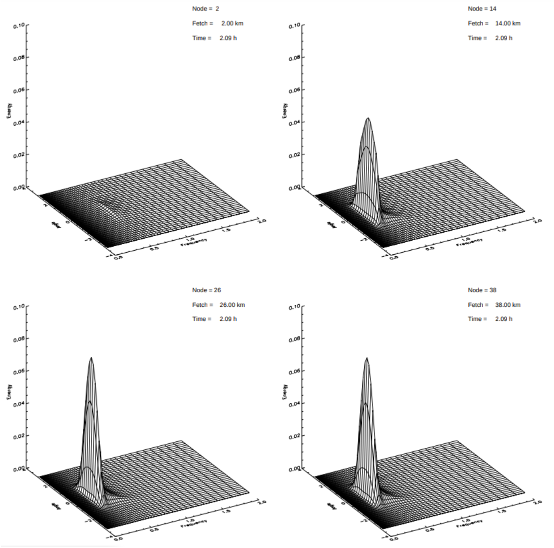


Results



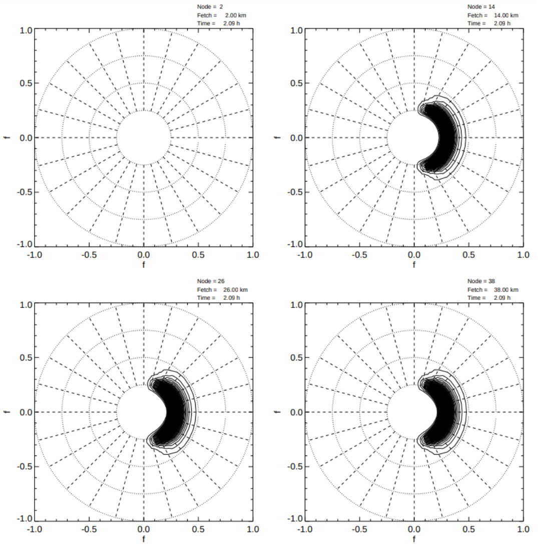
- thick solid line – total
- dotted line – in the wind direction
- dash-dotted line – normal to the wind
- dashed line – against the wind
- dotted line – not along the wind

Results



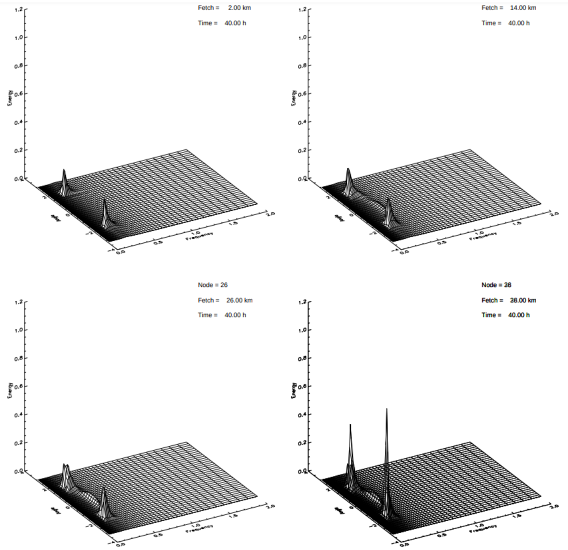
2 hours

Results



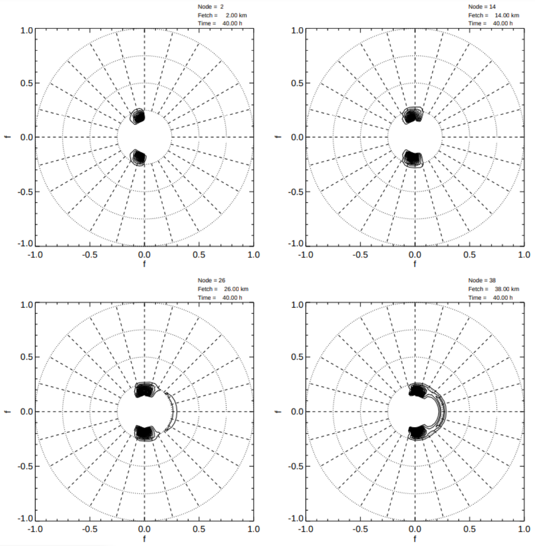
2 hours

Results



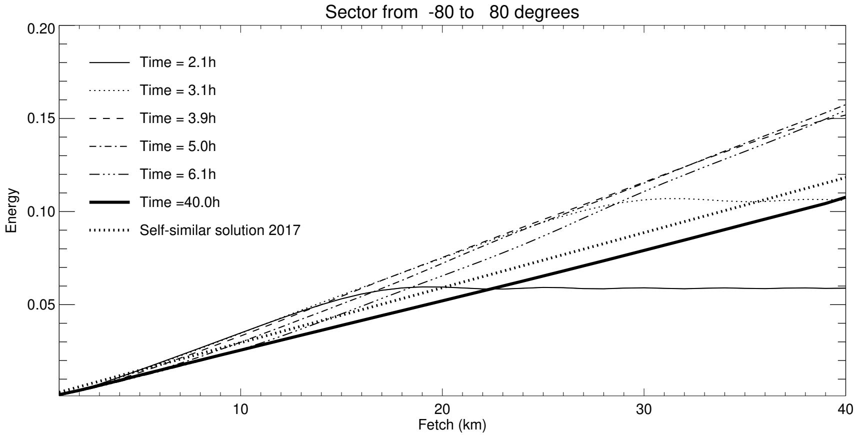
40 hours

Results



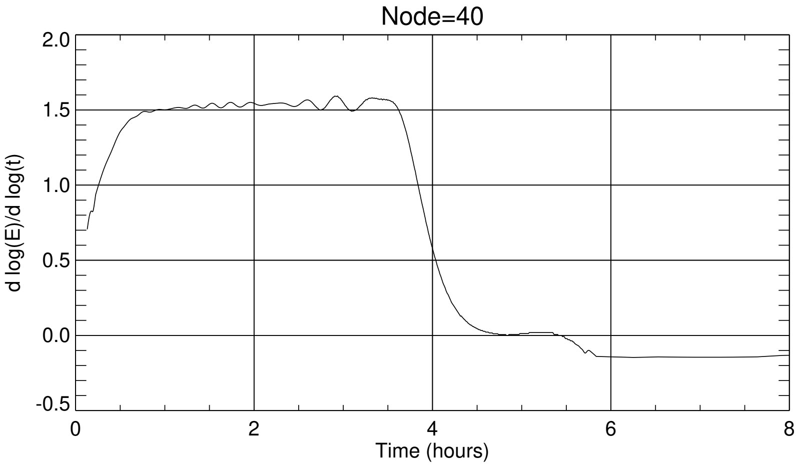
40 hours

Results

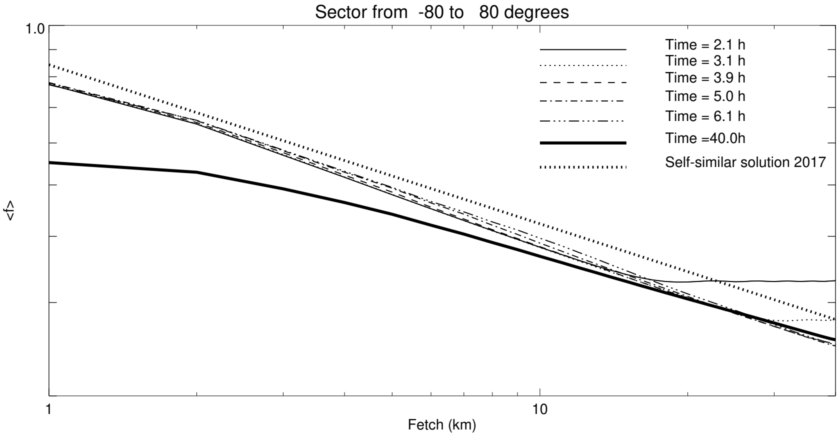


Sector from -80 to 80 degrees
Energy
0.20
0.15
0.10
0.05
Fetch (km)
10
20
30
40
Time = 2.1h
Time = 3.1h
Time = 3.9h
Time = 5.0h
Time = 6.1h
Time =40.0h
Self-similar solution 2017

Results

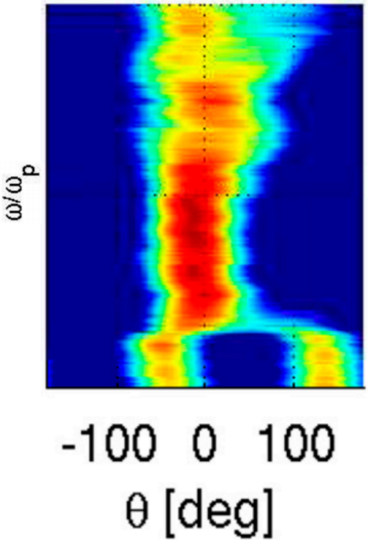


Results



Sector from -80 to 80 degrees
1.0
<f>
1
10
Fetch (km)
Time = 2.1 h
Time = 3.1 h
Time = 3.9 h
Time = 5.0 h
Time = 6.1 h
Time =40.0h
Self-similar solution 2017

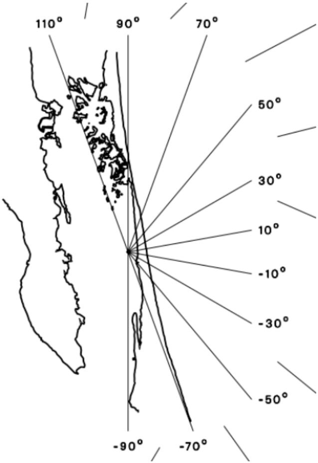
Experimental evidence

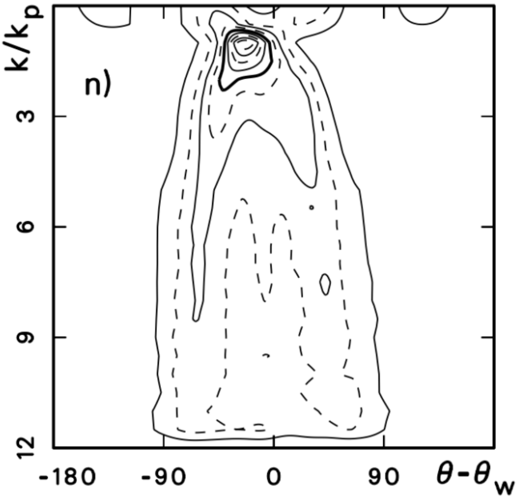


CONOCO PHILLIPS Ecofisk platform

A. Simanesew et al., 2017

Experimental evidence

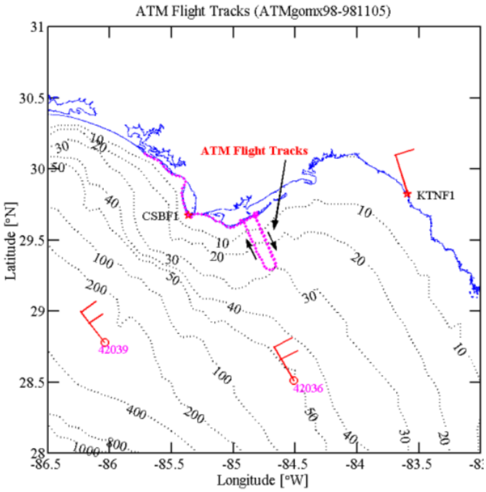




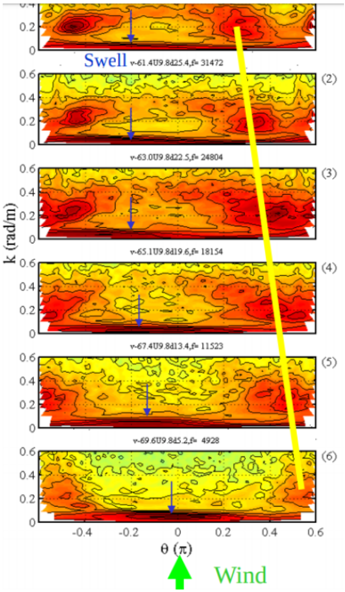
Outer Banks, Duck, NC

C. Long, D. Resio, 2008

Experimental evidence



Mexican Gulf
Paul Hwang et al, 2000



Conclusions

Nonlinear **O**cean **W**aves **A**mplifier

NOWA

Conclusions

1. Wave turbulence splits into 2 regimes in space and time:
   - Initial dual self-similar
   - Subsequent mix of self-similar wind sea and quazi-monochromatic waves orthogonal to the wind
2. Initial self-similar regime is self-similar threshold-like propagation
3. Subsequent regime works as Nonlinear Ocean Waves Amplifier (NOWA)
4. The system asymptotically evolves into stationary mixed state of wind sea and quasi-monochromatic waves orthogonal to the wind waves, slating at universal 15° closer to the origination shore
5. Laser-like radiation is apparently the attractor of complex nonlinear wave system
6. The obtained results are applicable to half-open ocean