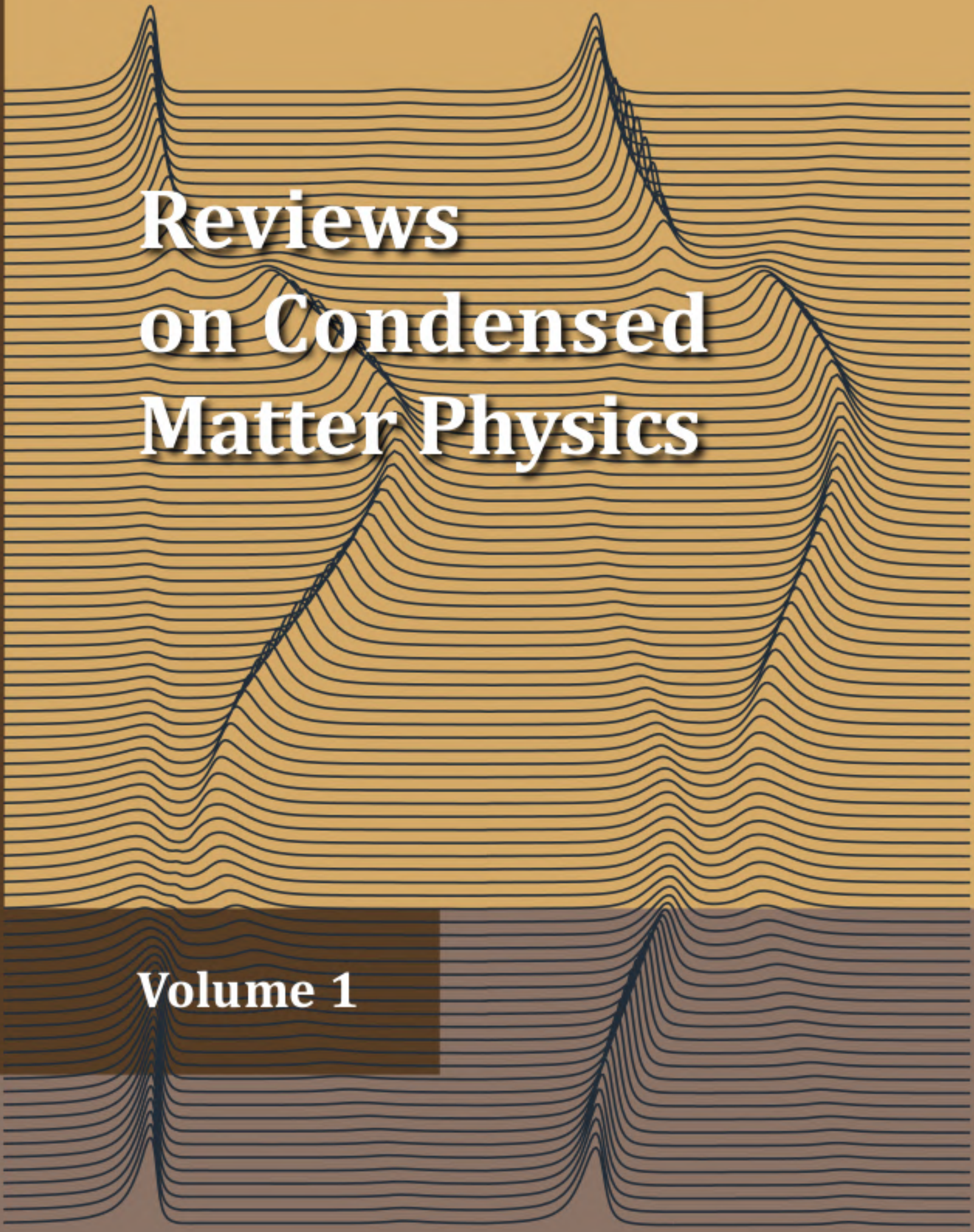
Michael V. Sadovskii



Reviews on Condensed Matter Physics

Volume 1

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В издании представлены избранные обзорные работы академика РАН М.В. Садовского.

I том охватывает обзоры, опубликованные в 1980–2000 гг. и посвященные физике сильно неупорядоченных систем: теории перехода полупроводник–металл в жидких полупроводниках, теории локализации и андерсоновского перехода металл–диэлектрик, а также анализу сверхпроводимости вблизи такого перехода.

Во II томе представлены обзоры, опубликованные в 2001–2022 гг. и, в основном, посвященные исследованию высокотемпературной сверхпроводимости. Эти работы охватывают достаточно широкий круг проблем (псевдощелевое состояние, сильные электронные корреляции), возникающих при описании сверхпроводимости и нормального состояния в высокотемпературных сверхпроводниках.

Издание может быть полезно специалистам в области физики неупорядоченных и сильнокоррелированных систем, физики высокотемпературной сверхпроводимости, студентам соответствующих специальностей, а также всем интересующимся физикой конденсированного состояния.

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PREFACE

Michael Sadovskii will be 75 on February 25th, 2023. His entire scientific career is inseparable from the Russian Academy of Sciences. He graduated from Ural State University in Sverdlovsk in 1971 and entered the postgraduate studies at the Department of Theory of the Lebedev Physical Institute in Moscow, where his scientific advisor was Leonid Keldysh. This determined the direction of his studies for many subsequent years. From 1974 to 1987 he was a research worker at the Institute of Metal Physics of the USSR Academy of Sciences, and in 1987 he enrolled at the Institute of Electrophysics of the Russian Academy of Sciences, both at his hometown Sverdlovsk (now Ekaterinburg), where he organized a Laboratory of Theoretical Physics and headed it for 30 years. At most difficult time for the Russian science (1993–2002), he was Deputy Director of the Institute of Electrophysics. From 1991 to 2010, Michael Sadovskii also worked as Professor at the Chair of Theoretical Physics of Ural State University.

M.V. Sadovskii is the author of nearly 200 scientific publications, including five monographs and a number of fundamental reviews. The main areas of his scientific activity that he started when a postgraduate student at Lebedev Institute are electron theory of disordered systems and theory of superconductivity. He proposed a number of exactly solvable models of one-dimensional electron systems, including the original model of the pseudogap state in the fluctuation region of the Peierls transition accompanied by the formation of short-range order charge density waves exhibiting non-Fermi liquid behavior. Later on, these studies became of great importance in connection with the problem of describing pseudogaps in high-temperature superconductors. He made a significant contribution to the theory of electron localization in disordered systems. He was one of the first to apply arguments to this problem based on the scaling ideas and the instanton approach. In his studies, he developed the self-consistent localization theory, which became the most important practical tool for solving localization problems in disordered systems. Together with Lev Bulaevskii he proposed for the first time a generalization of the theory of "dirty" superconductors to the case of systems with very short mean free paths near the metal–insulator localization transition. In particular, in these studies he presented a microscopic derivation of Ginzburg–Landau expansion coefficients in the region of strong disorder and predicted for the first time the possibility of realizing superconductivity in the Anderson insulator phase. He was also the first to demonstrate the importance of local density of states fluctuations near the superconductor–insulator transition, leading to a highly inhomogeneous state of superconductors in the vicinity of this transition.

In 1987, after the discovery of high-temperature superconductivity in cuprates, M.V. Sadovskii became actively engaged in the study of this new class of superconductors. Together with his students, he carried out an extensive series of works devoted to clearing up the nature of the pseudogap state in high-temperature superconductors. In particular, he generalized the one-dimensional pseudogap models proposed by him before to the two-dimensional case. In recent years, he proposed a new approach in the physics of strongly correlated electron systems that allowed the inclusion of arbitrary "external" interactions in the dynamic mean-field theory (DMFT + Sigma approximation). This approach was intensely exploited by him and his colleagues to describe the properties of the pseudogap state in high-temperature superconductors, including those obtained in calculations of the electron properties of real HTSCs, i.e., compounds based on copper oxides (the LDA + DMFT + Sigma method), and also in the analysis of the general problem of the metal–insulator transition in disordered systems with strong electron correlations (Mott–Anderson transition). This approach was also actively developed by his group to describe the region of BCS-Bose crossover in the disordered attractive Hubbard model, which is fairly pressing for a new class of high-temperature superconductors.

After the discovery in 2008 of high-temperature superconductivity in pnictides and iron chalcogenides, M.V. Sadovskii supervised the pioneering LDA and LDA + DMFT calculations of the electronic spectra of these systems that proved to be highly consistent with ARPES experiments and played the decisive role in the formation of the "standard" model of the electronic spectrum of new superconductors. Recently he started the new field of research on the limits of Eliashberg theory and the so called "Planckian relaxation" in metals.

In 1994, M.V. Sadovskii was elected a Corresponding Member, and in 2003 a Full Member of the Russian Academy of Sciences. For many years, he has been a Member of Presidium of the RAS Ural Branch and a Member of the Bureau of the Physics Department of the RAS. In 2002, he was awarded the RAS A.G. Stoletov Physics Prize. In 2016 he was honoured by first V.L. Ginzburg Gold Medal of RAS "for theoretical studies of high-temperature superconductors". In 2018 he was awarded S.V. Vonsovsky Gold Medal of the RAS Ural Branch. He actively works at the editorial boards of the leading Russian physical journals – JETP and Physics Uspekhi.

In 2013, M.V. Sadovskii was among the members of the RAS who opposed the destructive government reform of the Academy of Sciences and initiated the authoritative "July 1st Club", uniting the opponents of this reform.

In these two volumes we publish the selected reviews on condensed matter physics written by Michael Sadovskii during his career in physics. The material is presented in chronological order (Volume 1 contains reviews written in 1980–1997, while Volume 2 – reviews published in 2000–2022).

Only English versions of the review are presented, though most (but not all) of these reviews are also available in Russian. These reviews actually reflect almost all scientific interests of Michael Sadovskii and contain almost all his major results, as well as the results obtained by many other authors. We believe that the material presented is in no sense obsolete and will be of interest for many readers involved in modern studies on condensed matter theory.

Semiconductor–metal transition in liquid semiconductors

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1. INTRODUCTION

Investigations of electron transport processes in structurally disordered systems (amorphous and liquid semiconductors, and also metals) have grown into an independent and rapidly growing branch of the physics of condensed matter.

The range of physical phenomena which occur in disordered systems is at least as wide as the range in crystalline solids, and the practical importance of these systems is increasing year by year. Among disordered structures there are good metals as well as typical semiconductors and insulators. As usual, their properties are governed mainly by the structure of the electron energy spectrum. When external parameters are altered or when the system changes, this structure may be modified giving rise to changes in the various electronic properties, in particular, to semiconductor–metal or metal–semiconductor transitions. The special nature of this effect in disordered systems is associated primarily with the dominant role of disorder, which gives rise to new transition mechanisms not applicable to crystalline solids. From this point of view, investigations of the semiconductor–metal transition in liquids are particularly illuminating because they provide an opportunity to alter the structure of the system within a very wide range and thus study the role of structural changes in the formation of the electron spectrum. Accumulation of data on this topic is particularly desirable because the state of theoretical ideas on disordered systems is far from that degree of completeness which is characteristic of the theory of crystalline solids. On the other hand, investigations of this kind deal with the states of matter over a very wide range of external parameters (such as pressure and temperature) covering practically the whole range accessible at present under laboratory conditions; this extends greatly our ideas on the physics of condensed matter. The founder of investigations of noncrystalline materials and specifically of the electronic phenomena—conduction mechanisms and structure of the energy spectrum—is Abram Fedorovich Ioffe. His leading role in the establishment of this branch of physics is generally recognized and one is left to wonder at the foresight of the scientist who in the forties and fifties of the present century was able not only to formulate the fundamental problems in a new branch of physics but also to forecast (albeit in a qualitative form) the solutions of the most important problems.[1]

One of Ioffe's predictions, based on profound scientific intuition, is the hypothesis of the dominant role of the short-range structural order in the formation of the energy spectrum of electrons in noncrystalline materials and the idea of localization of electron states as the structural disorder factor increases.

Further development of the ideas of A. F. Ioffe and establishment of an electron theory of disordered systems owes much to the more recent contributions of

P. W. Anderson, N. F. Mott, I. M. Lifshits, J. M. Ziman, and others.

At present the research on the topic of interest to us is being pursued on a very wide front, including not only investigations of the semiconductor-metal (SM) transition in such noncrystalline materials as amorphous and liquid semiconductors, but also studies of impurity subsystems in crystalline matrices. Work of the latter kind is largely an independent subject and for this reason will not be considered in the present review. We shall confine ourselves mainly to investigations of electronic conduction in liquid semiconductors over a wide range of temperatures, including the transcritical state. Experimental studies, carried out mainly in the last decade owing to the development of techniques for measuring electrical properties at high pressures and temperatures[2] (4000 atm and 2000 °C), have shown that liquid semiconductors generally undergo two transformations of the electron spectrum at high temperatures: the SM transition occurs at lower temperatures and is preceded by enhancement of the interaction between electron shells of the atoms (this is a delocalization factor, operating in opposition to the structural disordering factor), and the metal-insulator (or metal-dense plasma) transition occurs at higher temperatures, in the range of critical temperatures and pressures, and is associated with a reduction in the density of the material and with an opposite relationship between the interatomic interaction and structural disordering factors. The two transitions obviously have common criteria.

Apart from the intrinsic physical interest, the subject discussed in the present paper is also of considerable importance in geophysics because it makes it possible to understand a large number of fundamental phenomena which occur in the interior of the earth. Many molten rocks are typical semiconductors, they exhibit the semiconductor-type temperature dependence of the electrical conductivity, and have high values of the thermoelectric power whose sign may vary with temperature. We can expect SM transitions at high temperatures and pressures in the earth's interior. The results of an analysis of these phenomena may be decisive in support of the hypothesis of metallization of molten matter in the earth's core and the high value of the thermoelectric power can be a powerful source of heating of certain parts of the crust and mantle.[1)]

In the present review we shall be concerned with the current status of the theory (Sec. 2) and we shall compare the theoretical and experimental results (Secs. 3–5). One should also mention that understanding of the mechanism of the SM transition requires general knowledge of the structure of the electron energy spectrum and of the special features of the electron transport phenomena in disordered systems. Consequently, we shall consider briefly these topics in Sec. 2 by way of introduction to the main theoretical and experimental data.

2. THEORETICAL MODELS OF THE SEMICONDUCTOR–METAL TRANSITION IN DISORDERED SYSTEMS

A theory of the SM transition in disordered systems should, in principle, follow the corresponding theory for crystalline solids. First of all, one should determine the structure of the electron energy spectrum of disordered systems and the changes in this structure under the action of external parameters. The ideas obtained in this way can then be used to go over to the next stage which is a description of the electron kinetics and classification of the investigated systems into metals, semiconductors, and insulators. The knowledge of possible changes in the electron spectrum should make it possible to compare directly these changes with the transitions observed experimentally. This approach is fundamentally clear, but its realization meets with considerable difficulties.

In spite of much work of theoreticians and in spite of the availability of extensive experimental data, a self-consistent theory of electrons in disordered structures is in practice still lacking. However, there are many models and ideas of a qualitative nature which make it possible to interpret the observed effects. A detailed presentation of these ideas can be found in the well-known books of Mott and Davis[3] and Mott[4], as well as in several reviews.[5-9] We shall consider briefly the main theoretical results relevant to the subject of the present review.

The main difficulty in the theory is the absence of translational symmetry because such symmetry leads to very great simplification in the case of ideal solids, in an analysis of the electronic properties associated with the Bloch theorem and with the results of the band theory. As is known, classification of solids into metals, semiconductors, and insulators is usually associated with the energy band structure. A considerable simplification of the theory arises also from the fact that it is usually possible to reduce the real many-body problem to an effective weak interaction between quasiparticles, i.e., it is possible to reduce the problem to the one- or two-particle level. In the case of disordered systems neither the former nor the latter simplification is possible. Strictly speaking, in the absence of translational symmetry we cannot introduce the concept of the band electron spectrum or the idea of effective quasiparticles. In view of the random nature of disordered systems one has to employ characteristics which are statistical averages over ensembles describing the structure of a system of one or another kind. It is then found that although usually it is not possible to specify the dispersion law for electrons in the form of a well-defined dependence of the energy on the quasimomentum $E(p)$, it is nevertheless possible to describe the electron spectrum by the density of the electron states $N(E)$ averaged over the ensemble of random configurations.[8] The work on the theory of imperfect crystals, particularly doped semiconductors and alloys, giving rise to the concept of density-of-states tails in the band gap[10-13] and the ideal of localization of electrons in such tails have been important in

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[1)] Some of these problems are discussed in Ref. 221.

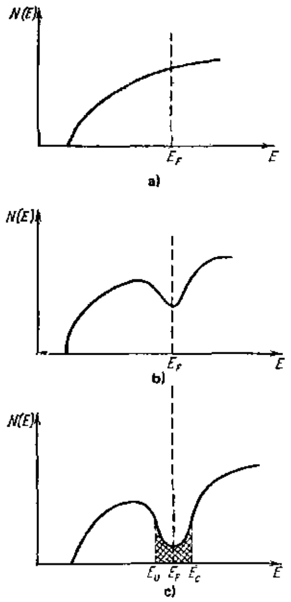

FIG. 1. Density of electron states in liquids: a) liquid metals; b) quasimetallic liquids; c) liquid semiconductors (the localized-state region is shaded).

the development of suitable representations. Mott[3,4] proposed several possible variants of transformation of the energy band spectrum of solids as a result of loss of order (for example, at the melting point). Figure 1 shows the various densities of electron states in the order of increasing importance of the short-range order or the degree of binding of electrons to the potential field of the atomic structure. The case shown in Fig. 1a corresponds to the total disappearance of the band gap and then $N(E)$ is analogous to that for free electrons and it describes, for example, liquid metals. In the case shown in Fig. 1b there is a dip in the density of states in the range of energies corresponding to the band gap in the crystalline state. This range of energies is usually called the pseudogap. The depth of this pseudogap is usually described by the ratio $g = N(E_F)/N_0(E_F)$, i.e., by the ratio of the true density of states at the Fermi level to that calculated in the free-electron approximation.

Since the pseudogap region can be regarded as the range of energies where the density-of-states tails of the "valence" and "conduction" bands overlap, the idea of localization of electrons at the band edges mentioned earlier is applied to the two-band case in Fig. 1c when the pseudogap is sufficiently pronounced.

In general, we have $N(E) \neq 0$, but we cannot exclude the possibility of a situation in which there is a true band gap, i.e., when there is a range of energies where $N(E) = 0$. However, in spite of the fact that $N(E_F) \neq 0$, in the case shown in Fig. 1c when the Fermi level lies within the region of localized states in the pseudogap, we effectively have a model of a liquid or amorphous semiconductor. Since the electron mobility at localized states bounded by the mobility thresholds E_v and E_c is much less than the mobility of electrons of energy exceeding E_c or less than E_v, at sufficiently high temperatures T the conduction process is dominated by carriers in delocalized states. The range of energies $E_c - E_v$ acts as a mobility gap[3-5] and it is responsible for the semiconductor behavior of the system in the absence of a true band gap. The conductivity then obeys a typical semiconductor equation:

$$\sigma = \sigma_0 \exp\left(-\frac{E_c - E_F}{kT}\right), \tag{1}$$

where E_F is the Fermi energy of electrons, i.e., it is governed by the thermal excitation of carriers across the pseudogap. The presence of this pseudogap gives rise to a number of special features of the transport phenomena (compared with the case of free electrons), even when there are no localized states inside the pseudogap.

The "two-band" scheme of the density of states of the type shown in Fig. 1c is a natural consequence of the approach postulating a pattern of a "faulty" crystal which has initially an energy band structure on which a random field is then superimposed. However, there is also the fundamental problem of postulating an analog of the band structure (of the type shown in Fig. 1c) for an intrinsically disordered system, such as a liquid or an amorphous body, without making any assumptions about the initial periodicity. Attempts to find the energy spectrum of disordered systems should start from two limiting cases considered in the theory of solids:[14] the tight-binding approximation or the model of almost-free electrons.

In the former case we have electrons whose wave functions are localized near individual atoms. As the atoms approach one another, their levels spread into energy bands because of the overlap of the wave functions and in general this is not related to any long-range order. Therefore, it would seem natural that in this situation there should be some kind of "band" structure governed only by the short-range order and by the overlap of the wave functions of the nearest neighbors. One would then expect to retain even the true band gap. Theoretical papers implementing these ideas in mathematical form have been published recently.[15-19] Weaire and Thorpe[15] considered a model Hamiltonian of a disordered system comprising germanium-type atoms (with covalent bonds) and electrons regarded as localized at atoms and bonds. They were able to demonstrate rigorously that, in spite of the structural disorder, a system of this kind should have a band structure with a true band gap. The valence and conduction bands are formed by electron states of the bonding and antibonding type. It is probable that this model will apply to the properties of systems such as amorphous germanium and silicon.

The Ziman group used machine calculations to show[16-19] that the density of states in amorphous Ge and Si structures is governed by the scattering of electrons by relatively small clusters (≈10 atoms) with a specific geometry of the distribution of atoms, and the density of states is found to be relatively insensitive to the distribution of the clusters themselves. The density of states has a typical pseudogap which is in full agreement with the available experimental results.

All these investigations confirm the idea first put forward by A. F. Ioffe[20] that the "band" structure of amorphous and liquid systems is governed primarily by the short-range order.

A special difficulty arises in the derivation of the density of electron states with a pseudogap using the language of almost-free electrons scattered by a weak pseudopotential of ions[2] distributed in a correlated manner in a liquid. The first attempts in this direction were made by Edwards.[21] The subsequent calculations carried out using real structure factors of liquids and model pseudopotentials of ions gave a density of states very close to that for free electrons.[22,23] Ziman[24] showed that under certain (fairly rigorous) assumptions about the higher correlation functions of ions (in particular, the four-ion function) it is possible to obtain a semiconductor-type density of states. However, there are at present no theoretical or experimental methods which would make it possible to find the higher correlation functions for a liquid.

Sadovskiĭ[25,26] showed that the use of a characteristic binary structure factor of a liquid and the assumption of a strong nonlocality of the ion pseudopotential corresponding to predominance of the backscattering makes it possible to construct a model of the electron spectrum of a liquid with a pronounced pseudogap in the energy range corresponding to the band gap of a crystal. This model is equivalent to an ensemble of ideal semiconductors whose energy gap in the spectrum varies in a random manner from one semiconductor to another.

We can summarize the current situation by saying that the evolution of the structure of the energy spectra (densities of electron states) on transition from crystalline solids to liquids and amorphous structures is now qualitatively understood but the representations employed cannot be regarded as having the same reliability as in the case of crystals and, moreover, they cannot be used to calculate the spectra of real systems.

A theoretical understanding of the kinetics of electrons in disordered systems and particularly of possible mechanisms of the SM transition, which is the subject of the present review, is in an even earlier stage of development. Before discussing the current ideas, we shall deal first with the concept of localization of electrons in a random field which is one of the main mechanisms for the transition from metallic to semiconducting type of conduction in disordered systems.

The first and still the fundamental results on localization were obtained in the well-known work of Anderson.[27] We shall consider briefly the Anderson model without attempting to be rigorous. The reader interested in details should turn to the original papers[27-36] and to reviews.[6,9]

We shall consider a three-dimensional lattice of sites distributed regularly or randomly. An electron at a site j has an energy E_j which is regarded as a stochastic variable distributed in a band of energies of width W with a certain distribution density $P\{E_j\}$. The overlap of the wave functions at neighboring sites gives rise to an interaction V_{ij} describing the transfer of an electron from one site to another. In an irregular lattice the interaction V_{ij} is also generally a stochastic quantity, but it is usual to consider the case of an ideal lattice with an energy disorder of levels at the different sites.

We shall assume that initially (at $t=0$) an electron is located at some specific site i and we shall consider the time dependence of its wave function. The main result of Anderson is that for V_{ij} which decreases with distance between the sites $|R_{ij}|$ faster than $|R_{ij}|^{-3}$ and has an average value smaller than a certain critical potential $V_c \sim W$, an electron remains localized in the vicinity of the site i in the sense that even as $t\to\infty$ the amplitude of the wave function at the site i is *finite* and decreases rapidly with distance. The physical meaning of this result is related to the circumstance that an electron can tunnel only to a site of energy which is identical with the level E_i determined by the initial conditions. However, under the above conditions the probability of finding, in the vicinity of a given site, another site with the same energy increases with distance away from this site at a rather much slower than the rate of reduction of the interaction V_{ij} with the same distance. Therefore, the probability of electron jumps is zero and there is no electron transport in the system as long as we consider the case of zero temperature.

We shall now consider a very much simplified version of the proof given by Anderson. The Hamiltonian of the Anderson model has the form

$$H=\sum_j E_j a_j^+ a_j + \sum_{ij} V_{ij} a_i^+ a_j, \tag{2}$$

where a_j^+ and a_i are the electron creation and annihilation operators at sites j and i; V_{ij} is assumed to be equal to the constant V which differs from zero only for transitions between the nearest neighbors.

It is assumed that initially an electron is at a site i whose wave function is $|i\rangle$. Subsequantly $t>0$ the wave function has the form

$$\psi(t)=\sum_i c_i(t)|i\rangle. \tag{3}$$

The quantity $p_{ii}=\lim_{t\to\infty} c_i(t)|^2$ represents the probability of finding a particle in a state $|i\rangle$ at the moment $t=\infty$. The probability p_{ii} naturally depends on the set $\{E_j\}$. If $p_{ii}\neq 0$, there are electron states localized near the site i. The nature of the electron states is governed by the one-electron Green function

$$G_{ij}(E)=\left\langle i\left|\frac{1}{E-H}\right|j\right\rangle, \tag{4}$$

representing the amplitude of the probability of a transition of an electron of energy E from a site j to a site i. In particular, p_{ii} can be expressed in terms of $G_{ii}(E)$ (Ref. 30). A renormalized series in the theory of perturbations with respect to V is obtained for this Green function. Anderson showed that the problem of

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[2] In the case of crystalline semiconductors such a description has now reached a highly advanced state.[14]

localization reduces to a study of the convergence of this series and, in view of the random nature of the quantity E_j, the convergence is understood to be in the sense of probability.[27-30] Simplifying the treatment in Ref. 31, we can say that it reduces to whether the diagonal element (4) has an isolated pole:

$$G_{ii}(E)=\frac{A_i}{E-\mathscr{E}_i(E)}, \tag{5}$$

which can be interpreted as the eigenenergy of an electron localized at the i-th site. The quantity $\mathscr{E}_i$ should be real because otherwise such a state decays with time, i.e., an electron becomes delocalized in the limit $t\to\infty$. The eigenvalue $\mathscr{E}_i$ is given by the Brillouin–Wigner perturbation theory:

$$\mathscr{E}_i=E_i+\sum_j V\frac{1}{E-E_j}V+\sum_{jj'} V\frac{1}{E-E_j}V\frac{1}{E-E_j}V+\dots . \tag{6}$$

If the series (6) (or, more precisely, a renormalized series in which the repeated sites are eliminated in the sums) converges near the real axis of E (apart from the point $\mathscr{E}_i$), a sequence of real terms converging to the real value of $\mathscr{E}_i$ is obtained along the real axis. The localization condition reduces to the requirement of convergence of a series of the type given by Eq. (6).[3]

Since E_j are random variables, the convergence of the series (6) should be considered, as mentioned above, in the statistical sense and this requires a complex analysis.[27,30] However, a simplified estimate of the convergence condition can be obtained as follows.[31] We shall consider various contributions of the order of V^{L+1}, where $L\gg 1$. Since each lattice site has Z nearest neighbors, there are in all Z^L contributions and each of them is a product of L factors of the type

$$T_i(E)=\frac{V}{E-E_i}. \tag{7}$$

If we assume that the quantities E_i in these cofactors are independent, we can find the value of the product in question by taking the average of its logarithm over the distribution of E_i:

$$\langle \ln|T_iT_{i'}\dots T_{i''}|\rangle=L\langle \ln|T|\rangle. \tag{8}$$

Then, the series (6) behaves, on the average, as a geometric progression and converges if

$$Z\exp\langle \ln|T(E)|\rangle<1. \tag{9}$$

This condition (the equation for E) defines the range of energies where electrons are localized. Anderson[27] used the following distribution of E_i:

$$P\{E_i\}=\begin{cases}\frac{1}{W} & \text{for} \quad |E_i|<\frac{1}{2}W,\\ 0 & \text{for} \quad |E_i|>\frac{1}{2}W.\end{cases} \tag{10}$$

Then,

$$\langle \ln|T(E)|\rangle=\frac{1}{W}\int_{-W/2}^{W/2} \mathrm{d}E_i \ln\left|\frac{V}{E-E_i}\right| =1-\frac{1}{2}\left[\left(1+\frac{2E}{W}\right)\ln\left|\frac{W}{2V}+\frac{E}{V}\right|+\left(1-\frac{2E}{W}\right)\ln\left|\frac{W}{2V}-\frac{E}{V}\right|\right]. \tag{11}$$

The scatter of the energy at sites W required for localization clearly depends on the energy E of the state of interest to us. We shall consider the state at the center of the band where $E=0$ (such states are most difficult to localize). Then, the condition (9) simplifies and reduces to the inequality[31]

$$\frac{W}{V}>2eZ. \tag{12}$$

For a simple cubic lattice of Eq. (8), we can introduce the band width $B=2ZV$ and obtain $W/B>e$.

Thus, if the random scatter of the energy at lattice sites is sufficiently large in the sense of the condition (12), all the electron states in a band are localized. A system is then an insulator of special type in which electron transport can take place only as a result of thermally activated jumps between local states. The metal–insulator transition caused by localization in a random field is usually called the Anderson transition. The criterion of total localization given by Eqs. (9) and (12) has been refined by many authors.[6,30,32-34] Different values for the critical scatter of the energy levels W_c are obtained depending on the approximations used in analyzing the convergence of a series of the type of (6). In order-of-magnitude estimates we can use Eq. (12) or the best numerical estimate of Anderson.[27] The fundamental point is the existence of such a critical ratio.

For a fixed value of W the localization condition can be analyzed as a function of E, which is the energy of a band electron.[30,31,35] The qualitative situation is illustrated in Fig. 2 (the localization region is shaded). The most important is the appearance of critical fields E_c and $E_{c'}$ separating the regions of localized and delocalized states. These energies are usually called the mobility thresholds. Naturally, location begins at band edges and on increase in the amplitude of the random fields W the mobility thresholds tend to move toward the center of the band, so that the localized states cover increasing parts of the spectrum.

A description of the behavior of electron states and their kinetics near the mobility thresholds is the fundamental problem in the theory of disordered systems. Anderson[29] showed that in the range of energies corresponding to localized states the electron wave functions decrease exponentially with distance over a length equal to the localization radius and as the energy of the state E tends to the localization threshold, the radius

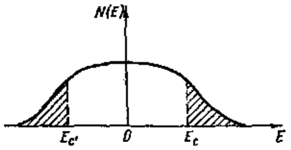

FIG. 2. Density of electron states in the Anderson model for a given ratio W/V. Here, E_c and $E_{c'}$ are the mobility thresholds.

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[3] We shall, in fact, ignore a number of finer aspects which are discussed in detail in Refs. 6, 27–30, and 32.

diverges:

$$R_{\mathrm{loc}} \sim a \left| \frac{E-E_c}{E_c} \right|^{-\nu} \text{,}^{4)} \tag{13}$$

where a is a length of the order of the interatomic distance. This behavior resembles the divergence of the correlation radius of fluctuations at a second-order phase transition point and suggests that the behavior of electron states in the critical region near the mobility threshold may be governed by scaling dependences (typical of phase transitions) with critical indices governed only by the dimensionality of space.[37] Some results on this point are obtained in Ref. 36, where it is shown that the index ν in Eq. (13) is the index of the correlation length in a specific problem of a second-order phase transition with a null-component order parameter.[38,39] The index ν is calculated there by the Wilson ε expansion method.[37] It is found that for a three-dimensional space the index is $\nu \approx 0.6$, in full agreement with the Anderson result[29] obtained by numerical analysis of the statistics of nonintersecting paths on a lattice, which governs the structure of the perturbation theory series for the Green's function (4) [see comments after Eq. (6)]. It is shown in Ref. 36 that the spatial behavior of this Green's function is identical with the corresponding behavior of the correlation function of the phase transition mentioned above. The behavior of the Green's function in question at the localization threshold demonstrates[36] that a power-law localization of wave functions in the Anderson model assumed by Thouless[40] is impossible. It should be stressed that the results of Ref. 36 apply to the *most probable* Green's function of an electron found using the Anderson approach. However, the majority of the measured physical quantities are governed by the *average* Green's functions. Attempts to apply an analogous treatment to the average correlation functions (relating to the density of states or the conductivity) have met with serious difficulties.[41-45] It has been found that the problem reduces to an analysis of an unstable field-theoretic problem and the application of perturbation theory gives nonphysical singularities reflecting probably a smooth transition from weak to strong binding.[41] The size of the transition region on the energy scale is identical with the size of the critical region in the theory of phase transitions,[41,44] but the scaling behavior of the correlation functions is not observed. Similar results are obtained in Refs. 42 and 43. An analogy between localization and a transition to a spin glass state is suggested in Ref. 45 but once again scaling behavior is not observed. The whole problem is considerably more complex than that of critical phenomena in second-order phase transitions.5)

The above analysis of localization is directly applicable to systems with energy disorder of which the only examples are probably disordered alloys. Real disordered systems usually have structural disorder (liquids, amorphous bodies). It is usual to assume that structural disorder always results in energy disorder because of unavoidable (due to the irregular distribution of atoms) fluctuations of the potential field acting on electrons. The above analogy with phase transitions suggests universality of the localization effect. Attempts to generalize the theory of Anderson have already been made.[46] With this in mind, detailed investigations have been made of electrons in a system of randomly distributed scattering centers[47-51] and fairly rigorous results have been obtained, particularly those on the behavior of the localization radius of the type described by Eq. (13). It may be that further development of methods not based directly on perturbation theory (quasiclassical approach) will help to solve the above problem of the behavior of the average correlation functions.

It is of fundamental interest to estimate the minimal metallic conductivity, which is typical of the Anderson model in the case when a system is at the threshold of localization of all the states in a band.6) Estimates of this conductivity were obtained by Mott.[3,4,52] We shall use the Kubo–Greenwood formula for the static conductivity at absolute zero:[3,4]

$$\sigma = \frac{2\pi e^2 \hbar^3 \Omega}{m^2} |D_{E_F}|^2 N^2(E_F), \tag{14}$$

where e is the electron charge, m is the electron mass, Ω is the volume of the system, and D_{E_F} is the matrix element of the momentum operator averaged over the states on the Fermi surface. The Fermi level E_F is at the midpoint of the band. It is assumed that the phases of the wave functions of electrons at neighboring localization centers (atoms) are not correlated. Then,

$$D_{E_F} \approx \sqrt{N}\,\delta, \tag{15}$$

where $N = \Omega/a^3$ is the number of sites (atoms) in a volume Ω, and the quantity δ has been estimated in various ways.[3,52] In particular, Mott[3] assumed

$$\delta \sim \frac{m}{m^*} \left(\frac{a^3}{\Omega} \right) \frac{\pi}{a}, \tag{16}$$

where m^* is the effective electron mass in a periodic lattice. Then,

$$\sigma \approx \frac{2\pi^3 e^2 \hbar^3 a}{m^{*2}} [N(E_F)]^2, \tag{17}$$

so that introducing an overlap integral $V \sim \hbar^2/2m^* a^2$ and also assuming approximately that $N(E_F) \approx 1/a^3 W$ (the notation is the same as in the above discussion of the Anderson model), we obtain

$$\sigma \approx \frac{8\pi^3 e^2}{\hbar a} \left(\frac{V}{W} \right)^2. \tag{18}$$

It follows from the best estimate of Anderson[27] that the localization of electrons in the band occurs for $(W/V_c) \approx 60$ when the lattice is characterized by $Z = 6$. Therefore, for a system with the Fermi level lying at the

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4) In the case of a transition at the center of a band we have $R_{\mathrm{loc}} \propto a \,|(W - W_c)/W_c|^{-\nu}$ for $W \geq W_c$.

5) The scaling theory of localization has been developed further in recent papers[216-220] but the problem is far from finally solved and the results obtained by different authors are quite contradictory.

6) At absolute zero.

midpoint of the band the minimal metallic conductivity is

$$\sigma_{\min} \approx 0.06 \frac{e^2}{\hbar a}. \qquad (19)$$

For $a = 4$ Å, Eq. (19) gives $\sigma_{\min} \approx 350\ \Omega^{-1} \cdot \mathrm{cm}^{-1}$.

Mott modified the estimates given by Eqs. (16)–(19) on several subsequent occasions.[4,52] In view of the scatter in the values of $(W/V)_c$ and the fairly large indeterminacy of the estimate given by Eq. (16), one should not treat the specific values too seriously. However, Eq. (19) can be expected to give the correct order of magnitude: $\sigma_{\min} \gtrsim 10^2\ \Omega^{-1} \cdot \mathrm{cm}^{-1}$. Estimates obtained using the elementary Drude formula[3,4] for the case when the mean free path is $l \sim a$ give values $\sigma \gtrsim 10^3\ \Omega^{-1} \cdot \mathrm{cm}^{-1}$.

A rigorous justification of the concept of minimal metallic conductivity is one of the main tasks of the theory. Recently, Thouless and Licciardello[53] were able to demonstrate the existence of a universal metallic conductivity minimum for two-dimensional systems. Their analysis is in fact based on the above analogy between localization and phase transitions, and on application to the two-dimensional Anderson model of an analog of the Kadanoff procedure used earlier to describe the critical region in the problem of phase transitions.[37] In the Anderson lattice a unit cell of side a is replaced with a new cell of site L containing N sites of the original lattice (Fig. 3). Periodic continuation of such a large cell in space has the effect that in the new lattice each of the N levels spreads into a band of width $2\Delta E$. The initial Anderson lattice can obviously be described as consisting of new cells of side L in each of which there are N random levels. Then, the effective binding of electrons at two levels in neighboring cells (an analog of the overlap integral V for the system of new cells) is evidently of the order of $V_N \approx (1/Z)\Delta E$ (Z is the number of the nearest neighbors) because this type of binding ensures a band of width $2\Delta E$ when the new cell is continued periodically in space. The separation between the energy levels in neighboring cells is of the order of $W_N \approx L^{-d}N^{-1}(E)$, where $N(E)$ is the density of states per unit volume (d is the number of dimensions of space) of the original lattice. If one of the N bands is considered in the new lattice, then W_N can be regarded as an analog of the statistical scatter of levels W in the original lattice. We thus have a procedure for going over from the original Anderson model characterized by the ratio W/V to a new (scale-transformed) model of the same type with an effective Hamiltonian characterized by a new ratio W_N/V_N, which is an exact analog of the scaling transformations in the theory of phase transitions.[37] The process of scale transformations can be continued along the chain:

$$V_N \to V_{N^2} \to V_{N^3} \to \dots,$$
$$W_N \to W_{N^2} \to W_{N^3} \to \dots$$

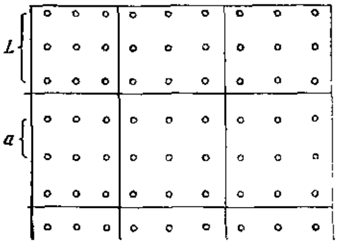

FIG. 3. Thouless and Licciardello construction. Each cell with the side La is regarded as a site in a new lattice.

by considering cells containing N^2, N^3, etc. of the original lattice sites, i.e., by increasing the sides of each new cell by the factor L. If the states of energy E are localized, then each transformation to a new scale causes V_N to decrease as $\exp(-L)$, whereas W_N decreases as $L^{-d} \propto N^{-1}$, so that V_N/W_N decreases as $\propto N \exp(-N^{1/d})$. Since for some value V/W there is a mobility threshold for any E in the original band,[30,31] there must be a maximum ratio $(V_N/W_N)_c$ for which exponential decrease of this kind still occurs. We can show[33,54] that in the case of delocalized states, we have

$$V_N \approx \hbar D(E) L^{-2}, \qquad (20)$$

where $D(E)$ is the diffusion coefficient of an electron of energy E. This relationship can be interpreted as follows. A delocalized electron in a system of cells of length L may diffuse in a time τ_N from one cell to another. The coefficient of such diffusion is $D \propto L^2/\tau_N$. Clearly, the indeterminacy of the electron energy is $V_N \propto \hbar/\tau_N$, which leads to Eq. (20). Then, $V_N/W_N \approx \hbar D(E) L^{d-2}$ and, application of the Einstein relationship between the diffusion coefficient and the conductivity

$$\sigma(E) = 2e^2 D(E) N(E), \qquad (21)$$

gives

$$\sigma_{\min} = \frac{2e^2}{\hbar} L^{2-d} \left(\frac{V_N}{W_N}\right)_c. \qquad (22)$$

This formula demonstrates the universality of the minimal metallic conductivity for $d = 2$ if $(V_N/W_N)_c$ is in this case a constant under scale transformations. Numerical analysis of this model[53,54] shows that for $d = 2$ we have $(V_N/W_N)_c = 0.12 \pm 0.003$, so that $\sigma_{\min} \approx 3 \times 10^{-5}\ \Omega^{-1} \cdot \mathrm{cm}^{-1}$, irrespective of the nature of the original lattice, confirming universality of the localization effect, similar to the universality of critical phenomena. An analogous result is not obtained for $d = 3$. In this case the minimal metallic conductivity depends on the scaling length. The Mott formula (19) follows from Eq. (22) if $(V_N/W_N) \propto N^{1/3}$, but detailed numerical calculations confirming this behavior have not yet been carried out.

It nevertheless seems natural to assume that in the region of delocalized states we have $V_N \propto L^{-2} \propto N^{-2d}$ and $W_N \propto N^{-1}$ (Ref. 33). This scaling approach is clearly not rigorous and requires special justification.[7)]

[7)] Recent results relating to this scaling description of the mobility threshold can be found in Refs. 219 and 220, where it is shown that the problem of existence of the minimal metallic conductivity is not clear even for $d = 2$.

Wegner[55] attempted to construct an analytical renormalization group scheme realizing the Thouless and Licciardello procedure. The framework of two alternative hypotheses on the behavior of the renormalization group transformations made it possible to obtain scaling behavior of the physical quantities near the mobility threshold. However, it should be stressed that explicit justification of these hypotheses (calculation of indices, etc.) may—as mentioned above—meet with difficulties associated with the inapplicability of perturbation theory. Therefore, the question of scaling at the mobility threshold remains open.

In a recent paper[56] Mott gave qualitative reasons in support of the dependence of the existence of a minimal metallic conductivity on the critical index ν of the localization radius [see Eq. (13)], because it is not possible to ensure that the phases of wave functions at neighboring sites [see Eq. (15)] are random for sufficiently small values of ν. According to these estimates, there is a minimal metallic conductivity if $\nu \geqslant 2/d$, where d is the dimensionality of space. The index $\nu \approx 0.6$ obtained for $d=3$ ($d=4-\varepsilon$) in Refs. 29 and 36 does not satisfy this condition, whereas $\nu=2/3$ obtained in Ref. 51 does satisfy it. For $d=2$ a numerical analysis[57] of a model considered in Refs. 53 and 54 is shown to give $\nu \gtrsim 1$, which satisfies the condition of existence of a minimal conductivity. These results stress the need for a correct analysis of the critical behavior near the mobility threshold. For $d=3$ the difference between the values 0.6 and 2/3 may be important but the existing theory (and experimental results!) are not sufficiently sensitive to be affected by this difference.

Assuming that the localization effect is universal, we can apply the results obtained in the Anderson model in constructing a qualitative picture of electronic phenomena in disordered systems, particularly in liquids.[3,4] It is assumed that, as in the Anderson model, regions of localized states appear near the band edges (Fig. 2) and that mobility thresholds E_v and E_c (Fig. 1c) occur if the pseudogap in the density of states of a liquid is sufficiently deep, which corresponds to a transition from a quasimetallic to a semiconducting liquid. It should be stressed that there is as yet no self-consistent theory of the transport phenomena in quasimetallic liquids. Typical conductivities of these liquids are such that the mean free path of electrons is clearly less than the interatomic distance. Then, electrons exhibit an analog of diffuse (Brownian) motion.[58,59] Mott suggested for this case the following heuristic generalization of the usual formula for the conductivity of a liquid metal:[3,4]

$$\sigma=\frac{e^2 S_F l}{12\pi^3\hbar}g^2, \tag{23}$$

where $g=N(E_F)/N_0(E_F)$, $S_F=4\pi p_F^2/\hbar^2$ is the area of the Fermi surface in the reciprocal space, l is a characteristic length resembling the mean free path or, more likely, the phase coherence length.[59] The arguments put forward by Mott in support of Eq. (23) are very qualitative and are based on generalization of the Ziman formula in the theory of liquid metals by introduction of the dependence $\propto N^2(E_F)$, typical of the Kubo–Greenwood formula (14). It is assumed that this dependence appears in a situation when $p_F l/\hbar \sim 1$ because of strong scattering. On the other hand, it is known[21] that in the case of weak scattering (liquid metals, $p_F l/\hbar \gg 1$) this dependence disappears completely. The result $\sigma \propto g^2$ was obtained by Friedman[60] in the random phase model when the phase coherent length is less than the interatomic distance. The Hall coefficient is then given by

$$R \approx \frac{C}{g}R_0, \tag{24}$$

where R_0 is the Hall coefficient in the free-electron approximation and $C \approx 0.7$. If $l \sim a$ (a is the interatomic distance) in Eq. (23), then for one electron per atom, we obtain

$$\sigma \approx \frac{e^2}{3a\hbar}g^2. \tag{25}$$

For $a=3$ Å and $g=1$, we have $\sigma \approx 1500\ \Omega^{-1}\cdot\text{cm}^{-1}$. In the case of a divalent metal, multiplication of this value by $2^{2/3}$ gives $2500\ \Omega^{-1}\cdot\text{cm}^{-1}$. We have seen above [Eq. (19)] that the transition from the scheme in Fig. 1b to the one in Fig. 1c occurs at the localization threshold for $\sigma_{\min} \approx 0.06e^2/\hbar a$.[8)] Therefore, a reasonable estimate of the depth of the pseudo gap at which localization begins and a liquid metal changes to a liquid semiconductor is $g^2 \approx 0.2$, i.e., $g_c \approx 0.45$. It should be stressed that this estimate varies depending on the estimated minimal metallic conductivity within the limits[3,4]

$$g_c \approx 0.2\text{—}0.5. \tag{26}$$

It should be noted that these ideas on the diffusive nature of the conduction process hold for microscopically homogeneous systems when the phases of the electron wave functions are uncorrelated over distances shorter than the interatomic spacing. However, real systems may contain microscopic inhomogeneities[61-66] caused, for example, by fluctuations of the density near the critical point, formation of regions with specific short-range order,[67] etc. Then, bearing in mind the ideas on the dominant role of the short-range order in the formation of the electron energy spectrum, we can imagine the appearance of regions having semiconducting and metallic spectra with the size of, for example, semiconducting regions governed by the corresponding short-range correlation radius $R_c > a$.

If the coherence length of the wave function phase is $l \ll R_c$, the electron structure of such regions can be discussed quasiclassically and it is governed by the local atomic structure. Similarly, the electron response to an external perturbation is now local. In the first approximation we can assume that the semiconducting regions are forbidden for the conduction electrons and

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[8)] It would be more correct to assume[3,4] that the estimate of $\sigma_{\min}$ includes here not the interatomic distance a but a_{E_F}, which is the distance between states with a given $\sim E_F$ and which can be greater than a.

we can introduce $C(E)$ representing the fraction of the volume of the system consisting of the regions which are allowed for electrons of energy E. We can then define $C(E)$ by[65]

$$N(E)=N_0(E)\,C(E), \qquad (27)$$

where $N(E)$ is the average density of states per unit volume of the system and $N_0(E)$ is the density of states per unit volume in a metallic region, which is usually identical with the density of states in the free-electron model. We shall now consider an electron at the Fermi level E_F. Clearly, $C(E_F)=g$ is the Mott factor. The Fermi level usually lies in the range of energies corresponding to the semiconductor gap (pseudogap) and an electron with $E\sim E_F$ experiences Bragg reflection at the boundary between metallic and semiconducting regions, its wave function decreasing rapidly within the latter region. If we ignore the tunneling across the forbidden regions, we find that the kinetics of this inhomogeneous case can be described by applying the percolation theory.[65,68-71] Then, if $C(E)<C^*$, where C^* is the critical value in the percolation theory, the system does not have a continuous path for an electron of energy E, which would pass through the metallic regions, and electrons of energy E are locked in metallic islands which are separated in space. When $C(E)=C^*$, the first path passing continuously through the whole sample appears in the system. Correspondingly, if $g<C^*$ the system is in the semiconducting state and conductivity exists only to the extent of tunneling across the forbidden regions (the conductivity vanishes in the classical percolation theory), for $g=C^*$ an infinite metallic cluster appears in the system, and for $g>C^*$ this cluster grows occupying increasingly greater portion of the system (the whole volume is occupied in the limit $g\to 1$). Thus, if $g>C^*$, we observe a continuous (percolation) semiconductor–metal (SM) transition. The critical value C^* for this type of percolation in a continuous medium lies, depending on the estimate, within the range $0.15<C<0.3$ (Refs. 72 and 73). The most reliable value is $C\approx 0.17$ for a Gaussian random potential.[73] Therefore, the criterion for the pseudogap depth obtained from the percolation theory and corresponding to the SM transition does not differ very greatly from the Mott criterion (26), but is not identical with the latter. Cohen and Jortner[65] proposed a description of the transport properties of an inhomogeneous system of this kind on the basis of the theory of an effective medium generalizing the treatment of Kirkpatrick[70] for lattices (networks) of random resistances. They provided a quantitative description of the SM transition for a large number of disordered systems in which one would expect microscopic inhomogeneities. The behavior of the electrical conductivity in the effective medium theory is shown by the curves in Fig. 4 taken from Ref. 65. It should be pointed out that the behavior of the transport characteristics of an inhomogeneous system might not be described by the simple scheme of the effective medium theory, particularly near the percolation transition itself where a special treatment is needed.[70,71,74] In a series of papers,[4,75] Mott criticized the ideas of Cohen and Jortner on inhomogeneous transport. However, his criticism is effectively based on a discussion of the Anderson model which does not allow for the appearance of strong inhomogeneities of metallic and insulating types because of the absence of correlation between energies at neighboring sites assumed in this model.[9] In fact, real systems may exhibit correlation producing a behavior of the type described above. It should be pointed out that this concept of forbidden and allowed regions may be too approximate. For example, the model of the spectrum of liquid semiconductors is considered in Ref. 26 and the theory automatically predicts an ensemble of metallic and semiconducting regions with a certain distribution of gaps in the spectrum. A description of the kinetics in terms of the percolation theory and the effective medium theory should be regarded only as the first approximation to the true situation in such a complex ensemble. The percolation SM transition is a manifestation of a specific second-order phase transition associated with the percolation phenomenon itself.[71,76] A full description of the critical percolation region in terms of the scaling pattern of this phase transition has been obtained recently.[77,78] The question arises of the relationship between the percolation and Anderson transitions. The percolation mechanism ignores the quantum-mechanical tunneling effects.[65,68]

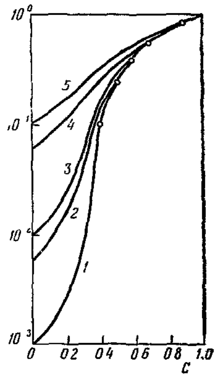

FIG. 4. Electrical conductivity of a microscopically inhomogeneous system in the effective medium theory.[65] Here, C is the fraction of the volume occupied by the allowed regions; $X=\sigma_1/\sigma_0$, where σ_0 is the conductivity of the allowed regions and σ_1 is the conductivity of the forbidden regions. 1) $X=10^{-3}$; 2) $X=6\times10^{-3}$; 3) $X=10^{-2}$; 4) $X=6\times10^{-2}$; 5) $X=10^{-1}$.

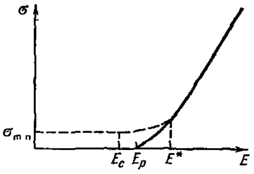

FIG. 5. Relationship between the percolation and quantum conductivities. Here, E_p is the percolation threshold, E_c is the mobility (localization) threshold, and E^* is the energy of the transition from the percolation to the tunneling mechanism.

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[9] Moreover, Mott criticizes mainly the idea of the existence of insulating regions in the classical theory of percolation. The effective medium theory deals with regions characterized by different but finite conductivities.

Classical motion of an electron in a random field is possible in the energy range $E > E_p$, where E_p is governed by the condition $C(E_p) = C^*$. Quantum-mechanical tunneling is possible in the energy range $E_c < E < E_p$, where E_c is the mobility threshold in the random field under consideration. The corresponding qualitative behavior of the conductivity as a function of the electron energy is of the kind shown in Fig. 5 (Ref. 4). At the energy $E^* = E_p + \Delta E^*$ a transition takes place from the percolation (quasiclassical) conduction mechanism to the quantum process. If ΔE^* is sufficiently small, the percolation theory can give a satisfactory description. The first attempts to estimate ΔE^* were made by Toulouse and Friedel (Refs. 79 and 80).[10]

We shall conclude this discussion by describing a transition from a liquid semiconductor to a metal in the way it can occur in a real system as temperature is increased from the melting point T_{mp} by identifying the conduction regimes which occur in the course of this transition.

1. Temperature range $T \gtrsim T_{mp}$. A typical liquid semiconductor has an energy spectrum of the type shown in Fig. 1c. Its Mott factor is $g < 0.15-0.2$. The conductivity is described by a formula of the type[3,4]

$$\sigma = \sigma_0 \exp\left(-\frac{E_c - E_F}{kT}\right) = \sigma_0 \exp\left(-\frac{E_c - E_v}{2kT}\right) \qquad (28)$$

if the Fermi level is in the middle of the pseudogap, and it is found that $\sigma_0 \sim \sigma_{min} \sim 10^2\ \Omega^{-1}\cdot\text{cm}^{-1}$. The thermoelectric power may also exhibit typical semiconducting behavior[4]

$$|\alpha| = \frac{k}{e}\left(\frac{E_c - E_v}{2kT} + A\right), \qquad (29)$$

where $A = \text{const}$. The behavior of the Hall effect usually differs considerably from that expected for semiconductors.[82]

2. $T > T_{mp}$. As temperature rises, it is most likely that the short-range order governing the energy spectrum of a liquid semiconductor is destroyed. The pseudogap fills gradually[4,65,26] and the spectrum changes to the type shown in Fig. 1c and the Mott factor becomes $0.2 < g < 1$. Then, as the factor g increases, electrons become delocalized in the pseudogap, we find that $E_c - E_v \to 0$ in the limit $g \to g_c$ [Eq. (26)] and a transition from a liquid semiconductor to a quasimetallic state takes place.[3,4] Up to the point of this transition electron transport is governed by formulas of semiconducting type with a temperature-dependent gap $E_c - E_v$. The transition point may be governed by the Mott-Anderson or by the percolation mechanism in the case of systems with pronounced structure fluctuations (spatial inhomogeneities).[4,65] If $g > g_c$, the quasimetallic state is characterized either by the homogeneous (diffusion) conduction mechanism [Eqs. (23) and (24)]:[4,60]

$$\sigma \propto g^2 \propto R^{-2}, \quad R \propto g^{-1}, \qquad (30)$$

and the thermoelectric power obeys the formula for metals[4,82,83]

$$\alpha = \frac{\pi^2}{3}\frac{k^2T}{e}\left.\frac{d\ln\sigma(E)}{dE}\right|_{E=E_F}, \qquad (31)$$

or by the percolation regime for an inhomogeneous system described by the formulas of the effective medium theory[65,70] and by the results of Levinshtein *et al.*[74] near the percolation threshold. In the quasimetallic range we typically have $(2-3)\times 10^2\ \Omega^{-1}\cdot\text{cm}^{-1} < \sigma < 10^3\ \Omega^{-1}\cdot\text{cm}^{-1}$. The selection between these two alternative (homogeneous and percolation) descriptions of the quasimetallic region of a melt can only be made experimentally.

3. Further increase in the temperature may transform the conductivity and other transport properties to the metallic regime with $\sigma \gtrsim 3\times 10^3\ \Omega^{-1}\cdot\text{cm}^{-1}$. The conductivity should then be described by the Ziman theory and the Hall effect by the model of free electrons.[7,83] The energy spectrum is then of the type shown in Fig. 1a.

4. A further increase in temperature and/or reduction in the density of the system enhances the importance of random scattering of electrons which now has the one-band spectrum of the type shown in Fig. 2. Consequently, the Anderson transition occurs when the disorder in the system becomes sufficiently strong.[4,6,27-30] The conductivity decreases to a value of the order of σ_{min} and then the conduction process changes (after localization of all the states in the band) to the hopping mechanism: $\sigma \ll \sigma_{min}$, $\ln\sigma \propto -T^{-x}$, where $x = 1/4 - 1$ (Refs. 4, 84, and 85).

Naturally, the proposed semiconductor-metal-semiconductor transition scheme for liquids is fairly approximate, in accordance with the qualitative nature of the current theory. In particular, a real system may not exhibit some of the conduction regimes described above (for example, pure metallic conduction), when an increase in importance of the random scattering causes the system to reach the maximum values of the conductivity characteristic of the diffusion regime and this is followed directly by the Anderson transition.

The most serious shortcoming of our analysis is the total neglect of possible effects of the electron-electron interaction. The dominant role of this interaction in many metal-insulator transitions in crystalline solids is well known.[4] We have not considered these effects because of the almost total lack of investigations of the role of the electron-electron interaction in disordered systems. In particular, the problem of the influence of this interaction on the Anderson localization has not been studied at all and this is also true of the influence of disorder on the mechanisms of the metal-insulator transition induced by this interaction in crystalline solids. These problems are expected to be tackled

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[10] It should be noted that the percolation transition and localization depend in different ways on the number of spatial dimensions. The critical number of dimensions which can be used to describe a transition (i.e., the number of critical indices) considered in the self-consistent field approximations is $d_c = 4$ for the localization,[36] whereas it is $d_c = 6$ for the percolation theory.[81]

soon by theoreticians.[11]

The following sections of this review will describe the results of experimental investigations of the SM transition in liquid semiconductors and metals near the critical point and will show how they compare with the theoretical ideas. We shall attempt to demonstrate to what extent these results agree or disagree with the proposed qualitative description of the transition governed by the role of disorder as given above. Naturally, any real material differs considerably from the corresponding theoretical idealizations but in discussing the experimental results we shall be forced to turn to ideas and theoretical results which have not been mentioned above. One should bear in mind that the qualitative nature of our ideas means that attempts to interpret the experimental results are also essentially qualitative, which frequently makes it difficult to select the appropriate theoretical model.

3. SEMICONDUCTOR–METAL TRANSITION IN MOLTEN SEMICONDUCTORS AT HIGH TEMPERATURES

Experimental investigations of the electrical conductivity and some other properties of liquid semiconductors confirm, on the whole, the description of the transition to the metallic conduction on increase in temperature given in the preceding section. In the very first work of A. F. Ioffe and A. R. Regel[1] on liquid semiconductors it was shown that those materials which retain the short-range structure after melting retain also semiconducting properties in the liquid phase; the electrical conductivity of such melts increases exponentially with increase in temperature. It is clear from general considerations that the process of rise of the electrical conductivity cannot continue without any restrictions. In fact, subsequent investigations[86-90] carried out at higher temperatures have shown that the conductivity of molten semiconductors tends to saturation at a level typical of metals. However, systematic and reliable investigations at high temperatures have become possible only when the experimental methods have been developed sufficiently and techniques have become available for carrying out measurements at high neutral gas pressures in order to suppress boiling of the melts.[2] This technique has been used to investigate a wide range of melts of chalcogenide semiconductors and also of elemental materials (tellurium and selenium)[89-97] at temperatures from T_{mp} to 2000 °C.[12] Alloys belonging to the Se–Te system have been studied by the same technique also in Ref. 98.[13]

The results obtained are summarized partly in Figs. 6–10. An analysis of these results (see also Refs. 99–101) shows that at $T > T_{mp}$ the conductivity increases to a level where $\sigma(T)$ reaches a plateau. The conductivity at the plateau is $(1-3)\times 10^3\ \Omega^{-1}\cdot\mathrm{cm}^{-1}$. This typical metallic conductivity and the absence of a significant temperature dependence shows that the melts have now become metallized. This demonstrates that the SM transition has occurred. We can easily see that, in the first approximation, the experimental results are in agreement with the theoretical expectations (see preceding section). In fact, in the case of those semiconductors whose short-range order does not change drastically as a result of melting (selenium, selenium alloys, selenides, etc.) and which remain semiconducting in the liquid phase, we can identify (in accordance with Sec. 2) three regions in the functional form of $\log\sigma(1/T)$: the first corresponds to essentially semiconducting behavior with the electrical conductivity obeying an exponential law; the second (intermediate) region is characterized by a variable activation energy of conduction which eventually decreases strongly; the third region, described above, is typical of metallic conduction. We shall now analyze the experimental data in greater detail. We shall begin by considering the third region of the curves of $\sigma(T)$, i.e., the metallic state on the plateau.

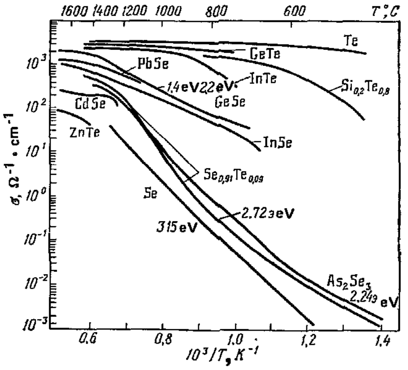

FIG. 6. Temperature dependences of the electrical conductivity of semiconductor melts.[89]

The most characteristic feature of the electrical conductivity of molten semiconductors in the plateau region is that the absolute value of σ_{sat} of all the investigated tellurides and partly of selenides is within the range 2500–3000 $\Omega^{-1}\cdot\mathrm{cm}^{-1}$ (see Table I taken from Ref. 89). This conductivity corresponds to the theoretical estimates made using the model of almost-free electrons on the assumption that only two valence electrons per atom take part in the conduction process:

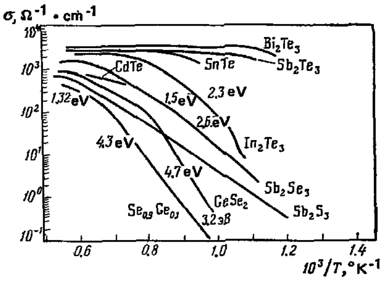

FIG. 7. Same as in Fig. 6 but taken from Ref. 90.

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[11] For lack of space we shall not discuss the first investigations on this subject, which have appeared very recently.[222-226]

[12] Liquid tellurium has been investigated close to the melting point T_{mp} at high pressures.[227]

[13] In a recent investigation[228] the Se–Te and other systems were studied at even high pressures.

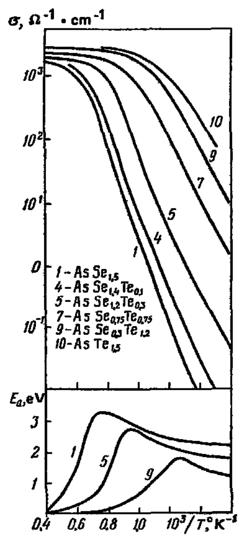

FIG. 8. Temperature dependences of the electrical conductivity and activation energy of electrical conduction of melts of glassy semiconductors.[97]

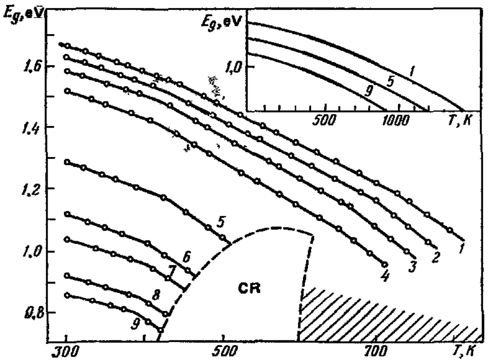

FIG. 9. Temperature dependence of the energy gap of liquid semiconductors in the As–Se–Te system over a wide range of temperatures. The thin curves in the inset are plotted using the dependence $\log\sigma = f(1/T)$. The thick curves and those with points are the results of direct optical measurements of the shift of the fundamental absorption edge with temperature (Ref. 97): 1) $AsSe_{1.5}$; 2) $AsSe_{1.49}Te_{0.01}$; 3) $AsSe_{1.45}Te_{0.05}$; 4) $AsSe_{1.4}Te_{0.1}$; 5) $AsSe_{1.2}Te_{0.3}$; 6) $AsSe_{0.9}Te_{0.6}$; 7) $AsSe_{0.75}Te_{0.75}$; 8) $AsSe_{0.45}Te_{1.05}$; 9) $AsSe_{0.3}Te_{1.2}$. The shaded region is opaque; CR is the crystallization region.

$\sigma = e^2/3\hbar a \approx 2700\ \Omega^{-1}\cdot cm^{-1}$ when the average interatomic distance is $a = 3\times10^{-8}$ cm.[14] This agreement cannot be accidental and it suggests that the upper valence subbands are destroyed by fluctuations of the potential. The electrons in these subbands form a Fermi sphere whose radius p_F is limited by the value of π/a. The Hall coefficient calculated for $p_F = \pi/a$ is approximately three times greater than the estimate obtained from the structure of the valence shells of atoms on the assumption that all the valence electrons participate in the conduction process. These ideas on the Hall coefficient are supported directly by the experimental results (see Table II for the telluride group). Thus, the available experimental data allows us to conclude that the chemical binding remains quite strong in the melts even at very high temperatures. In spite of the metallic nature of conduction, the telluride melts clearly cannot be regarded as systems of atoms with totally ionized valence shells and the free-electron model in the form applied to liquid metals does not hold for molten semiconductors even in the region of saturation of the temperature dependence of the electrical conductivity. At lower temperatures characterized by a positive temperature coefficient of the conductivity the rise in the conductivity is associated with the formation of a Fermi sphere, i.e., with an increase in the density of states at the Fermi level to values corresponding to the limiting wave vector approximately equal to π/a, i.e., $N(E_F) \leq 1\times10^{22}$ electrons $\cdot V^{-1}\cdot cm^{-3}$. The temperature dependences of the magnetic susceptibility and estimates of the paramagnetic contribution made by free carriers show (Fig. 11) that the increase in the electrical conductivity is correlated with the increase in the paramagnetic component of the magnetic susceptibility obeying the relationship $\sqrt{\sigma} \propto \chi_{para}$. Since $\chi_{para} \propto N(E_F)$, it follows that $\sigma \propto N^2(E_F)$. This last result is in full agreement with the relationship $\sigma \propto g^2$ discussed in Sec. 2.

In the case of selenides whose conductivity in the saturation region is less than that of tellurides (Table I) we can simply put forward the hypothesis that the valence shell of the chalcogen is even more stable and only a small proportion to electrons in the outer valence shell of the metal participates in the conduction process. These qualitative ideas suggest that the saturation conductivity should be between 1500 and 1000 $\Omega^{-1}\cdot cm^{-1}$, which is in agreement with the experimental values for many selenides.

We shall now consider the behavior of the curves of $\sigma(T)$ from T_{mp} to the temperature of the SM transition

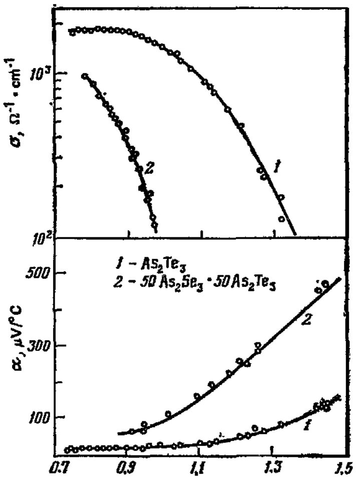

FIG. 10. Electrical conductivity and thermoelectric power of liquid As_2Te_3 (curves denoted by 1) and $As_2Te_3\cdot As_2Se_3$ (curves 2) plotted as a function of $10^3/T$ (°K⁻¹) on the basis of Ref. 107.

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[14] The formula is based on the assumption of diffusion of electrons, i.e., that the mean free path does not exceed the interatomic distance.

TABLE I.

Substance	Temperature range, °K	σ_{sat}, $\Omega^{-1}\cdot cm^{-1}$	Substance	Temperature range, °K	σ_{sat}, $\Omega^{-1}\cdot cm^{-1}$
Te	1000—1700	3300	Sb_2Te_3	1360—1700	2800
PbTe	1500—1850	2700	GeSe	1780—2000	2000
SnTe	1430—1740	2800	InSe	1850—2000	1000
CeTe	1430—1780	3000	Sb_2Se_3	1800—2000	1450
In_2Te^3	1540—1750	2500	Sb_2S_3	1880—2000	800
InTe	1450—1750	2500	PbSe	1830—2000	1300
Bi_2Te_3	950—1780	3300			

in the melts with the highest resistivity, such as selenium, alloys of selenium with tellurium, arsenic selenide, and similar melts. A characteristic feature in the graph of $\ln\sigma=f(1/T)$ observed in this range is a smooth increase in the activation energy of conduction on approach to the SM transition temperature (Fig. 8). Since the activation energy at a given temperature represents extrapolation of the temperature dependence $E_g(T)$ of the pseudogap E_g to absolute zero, we can plot this temperature dependence $E_g(T)$ (Ref. 100). We plotted the temperature dependences of the pseudogap of selenium,[89,99] ansenic selenide,[92] and $Ge_{0.1}Se_{0.9}$ (Ref. 94) using reference data on the optical width of the gap at low temperatures. The results obtained for melts of the As–Se–Te system are represented by thin lines in Fig. 9. The thick lines, as well as the curves with points in Fig. 9, are the results of direct optical measurements of the shift of the fundamental absorption edge with temperature.[97,102] Similar optical measurements carried out on liquid selenium and on glassy and liquid As_2Se_3 are reported in Refs. 99, 88, and 103–106. We can easily see from Fig. 9 that the pseudogap degrades at high temperatures and this process occurs at an increasing rate. It is important to note also that the gap disappears at those temperatures at which the conductivity reaches values close to σ_{min}. Some support for the proposed explanation of the disappearance of the pseudogap is provided also by the thermoelectric power data.[102] Figure 10 (based on Ref. 107) gives the temperature dependences of the thermoelectric power and electrical conductivity for molten chalogenides As_2Te_3 and $As_2Te_3\cdot As_2Se_3$. We can easily see that the thermoelectric power has values typical of semiconductors at temperatures $T \gtrsim T_{mp}$ and that it decreases on increase in temperature. It is important to note that the thermo-emf falls to a value of the order of $k/e\approx 80$ μV at temperatures corresponding to $\sigma=\sigma_{min}$. According to Ref. 3, the thermoelectric power of amorphous or liquid semiconductors can be described by

$$\alpha=\frac{k}{e}\left(\frac{E_F-E_v}{kT}+A\right),$$

where E_v is the valence band edge (in the case under consideration the dominant carriers are holes) and A is a constant. It is assumed in Ref. 3 that in the case of amorphous materials we have $A=1$ for carriers moving in the allowed-energy band. Thus, $\alpha\approx k/e$ applies in the case when the energy gap vanishes. It should be noted that the correlation between the thermoelectric power and electrical conductivity data is far from being as reliable or as self-evident for other materials.

The proposed explanation of the phenomena occurring in the region of the SM transition is far from unambiguous, as already pointed out in Sec. 2. It is very likely that in some materials, whose structure admits with equal probability the formation of clusters with different local structures, the SM transition is described by the formulas of the effective medium theory. An attempt to give a quantitative interpretation of the SM transition on this basis is made in Ref. 103 for melts in the In–Te system. Satisfactory agreement is

TABLE II. Electrical properties of semiconductors melts.

Semiconductor melt	σ, $\Omega^{-1}\cdot cm^{-1}$, 1) 2)	Sign of $d\sigma/dT$	R, cm^3/C	R/R_0	α, μV/deg	L/L_0 3)	References
Metallic melts of semiconductors							
Ge	15 200	—	$-3.6\cdot10^{-5}$	1.06	−1.0	0.86	1, 87, 124
Si	12 860	—				0.8	87
InSb	9 350	—	$-6.9\cdot10^{-5}$	1.4		0.94	1, 87, 124
InAs	7 000	—					
Metal-like melts 4)							
CdSb	5 200	+	$-8.3\cdot10^{-5}$	1.5	0		111
ZnSb	4 200	+	$-5.3\cdot10^{-5}$	1.2	+2.6	0.95	111
$AuTe_2$	3 460	+	$-8.3\cdot10^{-5}$	2.1			112
Bi_2Te_3	2 600	+	$-8.7\cdot10^{-5}$	2.2	+1.0	0.84	1, 87
GeTe	2 600	+	$-15\cdot10^{-5}$	3.6	+20	0.82	87, 113
CuTe	1 900	+	$-16\cdot10^{-5}$	4.0			113
Sb_2Te_3	1 850	+	$-17\cdot10^{-5}$	3.9	+10	1.03	87, 111, 125
SnTe	1 870	+	$-12\cdot10^{-5}$	2.3	+42	0.86	87, 113, 126
PbTe	1 510	+	$-14\cdot10^{-5}$		−3.7	1.1	87, 113
Te	1 700	+	$-12\cdot10^{-5}$	3,0	+26	0.97	112, 124, 127
$Te_{0,9}Se_{0,1}$	1 250	+	$-17\cdot10^{-5}$		+50		88, 128
AgTe	600	+	$-3.2\cdot10^{-4}$				117
Tl_2Te_3	500	+	$-7.4\cdot10^{-4}$	15		1.0	129
InTe	500	+	$-5.0\cdot10^{-4}$				128
In_2Te_3	250	+	$-(5-7)\cdot10^{-4}$		+10	1.2	128, 131, 132, 135
TlTe	290	+	$-13.8\cdot10^{-4}$	25		1.6	130
GaTe	250	+	$-3.2\cdot10^{-3}$				117
Ag_2S	120	+	$-2.8\cdot10^{-3}$		+190		117
Liquid semiconductors 4)							
Tl_2Te	67	+	$-1.6\cdot10^{-3}$ 5)		−150	2.8	112, 123,
GeSe	51	+	$-3.6\cdot10^{-2}$ 5)		+50		133, 137
As_2Te_3	25	+	$-5\cdot10^{-3}$		+200		88, 138
$CuSbSe_2$	16	+	$-7.3\cdot10^{-3}$		+180	7.7	138
Tl_2Se	10	+	$-2.5\cdot10^{-2}$ 5)		+240	17	120, 121, 134
Tl_2Se+0.8 ат% In	40	+	$-2.5\cdot10^{-3}$				122
$Te_{0,7}Se_{0,3}$	27	+	$-2.6\cdot10^{-3}$			290	84, 99, 128
$Te_{0,5}Se_{0,5}$	1.0	+	$-3.0\cdot10^{-2}$		+690		84, 99, 128
InSe	3.0	+	$-1.0\cdot10^{-4}$		−220	45	138
Sb_2Se_3	2.4	+	$-1.5\cdot10^{-2}$ 5)		+3	3.6	120, 139
Sb_2Se_3+2.5 ат% Sb	3.1	+	$-2.1\cdot10^{-2}$		−20		120, 139
Sb_2Se_3+5 ат%Te	3.0	+	$-2.2\cdot10^{-2}$		+36		120, 139
Sb_2S_3	$1\cdot10^{-1}$	+	$-3.5\cdot10^{-1}$ 5)		−100		114
$60As_2Te_3\cdot40As_2Se_3$	$7\cdot10^{-1}$	+	−1.0				128, 140
V_2O_5	$7\cdot10^{-2}$	+	$-3.0\cdot10^{-1}$				134
$AsTlSe_2$	$1\cdot10^{-3}$	+	-10^{-2}				140

Notes. 1) For explanations of symbols see text. 2) All the results are given for temperatures 20–50°C above the melting point. 3) Data on the Lorenz number are given in accordance with Ref. 115. 4) The selected boundaries dividing these melts into groups should be regarded as nominal. 5) Considerable changes in properties occur on deviation from the stoichiometric compostion or on addition of a third element as an impurity.

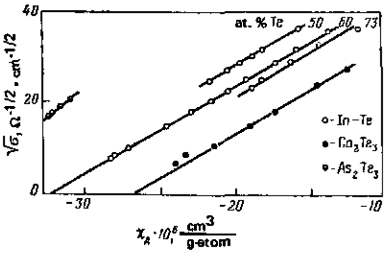

FIG. 11. Correlation between the behavior of the electrical conductivity and the paramagnetic susceptibility of In–Te and Ga_2–Te_3 alloys and of As_2Te_3 (Ref. 108).

obtained there with the experimental results but the critical values of the parameter representing the ratio of the metallic and semiconducting phases calculated on the basis of these data is higher than the theoretically permissible limit. Thus, the treatment of the SM transition on the basis of a heterogeneous model of the melt using the effective medium theory also fails to give rigorous quantitative conclusions. The problem of competition between two possible SM transition schemes based on the homogeneous and heterogeneous models of a liquid semiconductor is discussed in detail in Sec. 2. We have shown that both models predict quite similar characteristic parameters of the transition. For this reason an analysis of the electrical conductivity data, usually carried out making a number of assumptions, does not make it possible to draw definite conclusions. Convincing proof in support of the heterogeneous model and percolation nature of the transition can be obtained only as a result of careful structural investigations of melts in the intermediate region, but direct studies confirming the coexistence of two structural phases with higher and lower electrical conductivity have not yet been made.

It seems most reasonable to assume that both (homogeneous as proposed by Mott and percolation) mechanisms may coexist or predominate in different materials, depending on the characteristic structural and crystallochemical properties of a material. The percolation mechanism of conduction in the region of the metal–nonmetal transition will be discussed again in Sec. 5.

The experimental data discussed in the present section are summarized in a diagram (Fig. 12) whose theoretical description is given at the end of Sec. 2. The metal–insulator transition shown in the transcritical region will be discussed in detail in Sec. 5.

4. SEMICONDUCTOR–METAL TRANSITION AND CRITERION FOR MINIMAL METALLIC CONDUCTIVITY OF MELTS AT $T \gtrsim T_{\mathrm{mp}}$

In the preceding section we have discussed the features of the transition to metallic conduction in a liquid semiconductor on increase in temperature. Equally interesting is the transformation of the properties of molten semiconductors not on the temperature scale but on the scale of substances considered slightly above the melting point and arranged in accordance with increasing electrical conductivity. Melts of electronic semiconductors are unique objects in the sense that they include both metallic and essentially semiconducting compositions, as well as materials with intermediate properties. This makes it possible to analyze the conditions and criteria for the transition from the semiconducting to metallic conduction process. An important feature of this analysis is the fact that the degree of stability of the short-range order structure at $T \gtrsim T_{\mathrm{mp}}$ is relatively high; consequently, the level of fluctuations of the atomic potential is still low and we can assume approximately that this level is the same for a large group of melts. It follows that the variable parameter, which governs the process of localization of electron states at the Fermi level in the liquid metal–liquid semiconductor series if the interaction potential in Eq. (12), which governs the Anderson localization of electron states at the Fermi level.

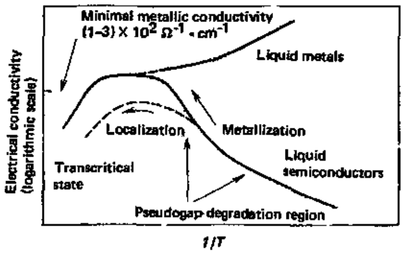

FIG. 12. Generalized temperature dependence of the electrical conductivity of molten semiconductors showing the main tendencies in changes in the properties over a wide temperature range. The dashed curve represents the probable Anderson transition.

The published experimental data for electronic melts are summarized in Table II, which gives a full list of the materials on which the Hall effect measurements have been carried out in the liquid phase. The data on the Hall effect, together with those on the electrical conductivity, thermoelectric power, and thermal conductivity make it possible to establish the main relationships governing liquid semiconductors. In the present section we shall consider the consequences which follow from Table II.

The melts in Table II are arranged in decreasing order of the absolute value of their electrical conductivity. Moreover, the melts are divided into three groups: 1) metallic; 2) metal-like; 3) semiconducting.[15]

It follows from this division that the first group consists of the melts of those semiconductors which lose their semiconducting properties at the melting point and exhibit behavior typical of liquid metals: the electrical conductivity is $(5-10)\times 10^3\ \Omega^{-1}\cdot\mathrm{cm}^{-1}$ and the temperature coefficient of the electrical conductivity is negative; the Hall coefficient R is also negative and it agrees well with the value of R_0 predicted by the theory of free electrons. It is shown in Refs. 1 and 87 that metallization caused by melting of semiconductors of the first group is due to the change in the short-range structure.

The second group of metal-like melts consists of those materials whose properties are highly contradictory: measurements of the Hall coefficient and of the thermoelectric power, an analysis of the electron component of the thermal conductivity, and the absence of any significant effect of doping with a third component all suggest that these are also metallic melts, i.e., that there is no gap at the Fermi level in the energy spectrum of these materials. On the other hand, the same melts are characterized by a positive temperature coefficient of the electrical conductivity, i.e., by semiconducting behavior. The reasons for this com-

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[15] Similar in content but different in terminology classification of semiconductor melts has been proposed earlier.[3,82,87,109,110]

plex combination of properties are discussed in Refs. 109–115. It is usual to assume at present that a positive temperature coefficient of the electrical conductivity is related not to the typical (for semiconductors) thermal-generation of carriers across a gap but to a minimum of the density-of-states function at the Fermi level (Fig. 1c) and to degradation of this minimum on increase in temperature. In other words, the Mott relationship $\sigma \propto g^2$ applies. Attempts to show this experimentally by measuring the Knight shift were reported in Ref. 116. Indirect data in support of this interpretation follow from an analysis of the Hall effect in Ref. 117. We can thus see that melts of the second group are also metals but they have certain special properties associated with a minimum of the density of states at the Fermi level. These special properties are reflected also in the adopted terminology: they are called metal-like melts.

Melts of the third group [$\sigma \lesssim (1-2)\times 10^2\ \Omega^{-1}\cdot\text{cm}^{-1}$] have a set of properties which largely resemble those of semiconductors. The temperature dependence of the electrical conductivities is clearly exponential, the thermoelectric power may reach hundreds of more of microvolts per degree, the Hall coefficient is considerably greater than the values predicted by the theory of free electrons, the electron component of the thermal conductivity is anomalously high for a degenerate electron gas, and in some cases the experimental results have to be interpreted invoking the ambipolar mechanism of heat transport.[115,118] Moreover, the doping effects are significant for some of the melts (Sb_2Se_3, GeSe, Tl_2Se). Addition of a third element makes it possible to control the sign of the thermoelectric power, as well as the values of the electrical conductivity and Hall coefficient.[114,119-123] All these properties can be explained only if we assume that the energy spectrum of the investigated semiconductor melts has an energy (mobility) gap. However, it should be pointed out that semiconducting properties of liquid semiconductors can be described only quantitatively by the standard theory. There are certain special properties, the most striking of which is the so-called p–n anomaly. It is found that the signs of the thermoelectric power and the Hall coefficient do not agree: the Hall effect is usually negative whereas the thermoelectric power is positive.[120-122] The unusual properties of liquid semiconductors can be understood qualitatively if we assume that the energy gap in the spectrum of these materials is more likely a region of localized states and not the gap in the sense used in the case of crystalline materials. The region of localized states is the mobility gap and, therefore, it acts as an activation gap in the conduction mechanism (Fig. 1c).

Our analysis of the experimental data on molten semiconductors arranged in the order of decreasing electrical conductivity thus yields an important conclusion: variation of the absolute value of the conductivity of the melts is correlated with changes in their properties. A reduction in the conductivity enhances the semiconducting properties. It follows from the data obtained that the conductivity separating melts with metallic and semiconducting properties is approximately $(1-3)\times 10^2\ \Omega^{-1}\cdot\text{cm}^{-1}$. It is worth mentioning that in analyzing the SM transition in Sec. 3 as a function of temperature we have found that a conductivity of the order of $(1-3)\times 10^2\ \Omega^{-1}\cdot\text{cm}^{-1}$ also corresponds to the point of transition from the semiconducting to the metallic mechanism.

The fact that different and independent experimental data indicate that the conductivity of $(1-3)\times 10^2\ \Omega^{-1}\cdot\text{cm}^{-1}$ separates the semiconducting and metallic states is the most important conclusion of our analysis of the experimental data in Secs. 3 and 4. Since $\sigma = (1-3)\times 10^2\ \Omega^{-1}\cdot\text{cm}^{-1}$ corresponds to $\sigma_{\min}$, which is the minimal metallic conductivity predicted for disordered systems by the Anderson–Mott criterion, we can assume that the theoretical model of the SM transition presented in Sec. 2 describes reality and that, in the first approximation, it is correct. In the next section we shall discuss a third set of experimental data showing that the concept of minimal metallic conductivity applies to the behavior of liquid metals and semiconductors at transcritical temperatures and pressures.

5. METAL–SEMICONDUCTOR TRANSITION NEAR CRITICAL POINTS OF METALS AND SEMICONDUCTORS

A. Results of experimental investigations of cesium and mercury

In the present section we shall consider the behavior of metallic and semiconducting melts at still higher temperatures at which considerable changes take place in the density of the melts and these result in weakening of the overlap of the electron shells. In the case of a metallized semiconductor we may observe localization of electron states, which is analogous to the corresponding localization in liquid metals, so that it is possible to compare such materials with the usual liquid semiconductors discussed in Sec. 4 (see Fig. 12). Continuous changes in the density of liquids can be produced at temperatures and pressures exceeding the critical values. The majority of metals and some semiconductors have far too high critical temperatures (see the reviews in Refs. 141 and 142) and calculations of the parameters of critical points and state equations for sixteen metals[143] have given values which are not accessible in steady-state experiments when a high stability of temperature and pressure is required. Recent experience has shown that the transition from the metallic to the nonmetallic state in metals and semiconductors is accompanied by effects which occur in a very narrow temperature range (10–15 °C according to Refs. 144–147) so that the attention of most investigators is currently concentrated on an increase in the precision of experimental studies of substances with critical temperatures lying below 2100 °C. Work of this kind started in the Soviet Union in the middle sixties under the leadership of Kikoin (studies of mercury) and Velikhov (studies of cesium); in the Federal German Republic similar work was performed by the group of Franck.

Among the elements of various groups in the Mendeleev periodic system the most suitable for experimental studies of this kind are cesium, rubidium, potassium, sodium, mercury, arsenic, selenium, and possibly tellurium. Transitions from the metallic to the nonmetallic state are exhibited by mercury,[148–151] cesium,[152,153] and arsenic,[154] whereas semiconductor–metal–semiconductor transitions occur in selenium (Refs. 96, 155–158).[16] All these elements exhibit a transition from the metallic to the nonmetallic state on continuous reduction in the density in the vicinity of the critical point at high temperatures and pressures.

We shall now consider the behavior of liquid metals during continuous reduction in the density in the critical temperature range. The metal–nonmetal transition in mercury and cesium has been investigated most thoroughly: the PVT dependences have been obtained for mercury[149–151,159–161] and cesium,[162–165] studies have been made of the acoustic properties of mercury,[166] specific heat of cesium,[167] and thermal conductivity of mercury (Ref. 168);[17] studies have also been made of the viscosity of mercury and cesium,[163–171] electrical conductivity of mercury[146,148–151,160,172,173,174] and cesium,[144,145,152,153,175] Hall effect in mercury[176] and cesium,[177] thermoelectric properties of cesium (Refs. 144, 145, 175, and 178)[18] and mercury,[145–147,179–181] optical properties of mercury,[182–184] and the Knight shift in mercury has been investigated.[185] These studies have given fairly comprehensive experimental data and can be used as the current illustration of the well-known work of Zel'dovich and Landau,[186] who were the first to consider various metal–nonmetal transitions and liquid–gas phase transitions.

Clearly, the nature of the metal–insulator transition and its position relative to the critical point in the liquid–vapor phase diagram can be determined if we know the equation of state of the metals and semiconductors in the region of the metal–insulator transition and have sufficiently accurate parameters of their critical points. Unfortunately, the thermodynamic properties of metals and semiconductors have not yet been studied sufficiently thoroughly near the critical points. A good agreement on the critical parameters is available only in the case of cesium. We shall use the equation of state and the critical parameters given in Ref. 163. The critical parameters of cesium are $T_{cr}=1760\pm20\,^\circ\mathrm{C}$, $P_{cr}=116\pm5$ bar, and $\rho_{cr}=0.40\pm0.02$ g/cm^3.

In the case of mercury the scatter in the critical parameters is considerably greater than the limits of the experimental errors. By way of example, we shall quote here the results of two different investigations: $P_{cr}=1510\pm20$ bar, $T_{cr}=1490\pm15\,^\circ\mathrm{C}$, $\rho_{cr}=4.2\pm0.4$ g/cm^3 (Ref. 151), and $P_{cr}=1730\pm30$ bar, $T_{cr}=1510\pm15\,^\circ\mathrm{C}$, $\rho_{cr}=5.9\pm0.2$ g/cm^3 (Ref. 159).[19] We shall use the equation of state found in Ref. 159, because this investigation was clearly carried out most carefully.

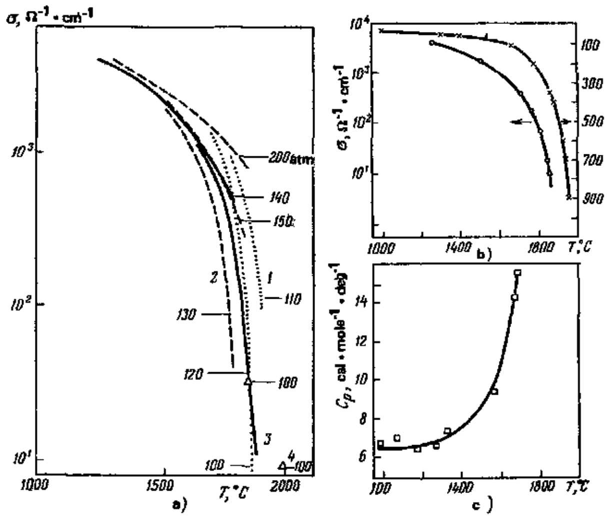

FIG. 13. Electronic and thermal properties of cesium in the region of the metal–nonmetal transition at various temperatures. a) Temperature dependences of the electrical conductivity of cesium obtained at various pressures in the region of the metal–nonmetal transition: 1) Ref. 152 (an error in the graph in Ref. 152 is corrected so that the order of the electrical conductivity agrees with the text of the paper); 2) Ref. 175; 3) Refs. 144 and 145; 4) Ref. 204. b) Temperature dependences of the electrical conductivity and thermoelectric power ($P=120$ atm) (Ref. 145). c) Specific heat at $P=120$ atm (Ref. 167).

We shall quote the experimental data on the main electrical properties of cesium (Figs. 13 and 14) and mercury (Fig. 15) at high pressures and temperatures when the density decreases continuously. In the case of mercury we shall also give the values of the Knight shift (Fig. 15e) and the velocity of sound in the region of the metal–nonmetal transition (Fig. 15f); in the case of cesium we shall discuss the specific heat data (Fig. 13c).

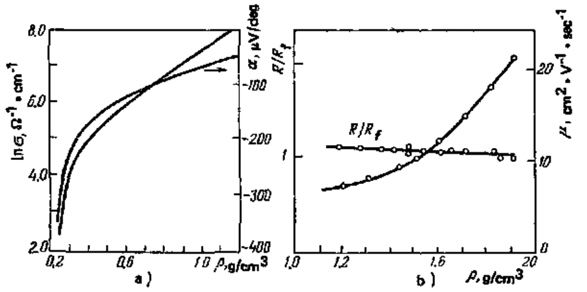

FIG. 14. Electronic properties of cesium as a function of continuous variation of the density. The Hall mobility of cesium and the ratio of the Hall coefficient to the free-electron value R_f are taken from Ref. 163 (b) and electrical conductivity and thermoelectric power are taken from Refs. 144, 145, and 152 (a).

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[16] The first data on the semiconductor–metal transition in liquid selenium were given in Refs. 96 and 158.

[17] The thermal properties of mercury and cesium were determined[167,168] before the metal–nonmetal transition.

[18] The first study[178] involving measurements and theoretical analysis of the behavior of the thermoelectric power in the region of the metal–nonmetal transition was carried out to determine the possibility of obtaining higher values of this power for the purpose of conversion of thermal energy.

[19] In a recent paper[147] a report was given of a determination of the critical pressure and temperature ($P_{cr}=1675\pm25$ bar and $T_{cr}=1495\pm5\,^\circ\mathrm{C}$). The critical temperature reported in this study agreed well with the published values, whereas the critical pressure was close to the value given in Ref. 159 but very different from the values reported in Refs. 150 and 151.

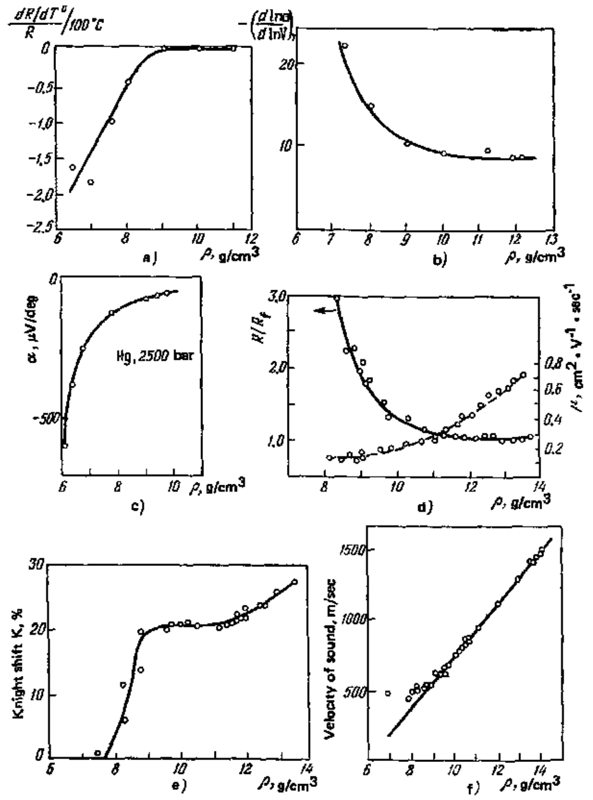

FIG. 15. Electrical and acoustic properties, and Knight shift of mercury plotted as a function of continuous reduction in the density: a) Temperature coefficient of the electrical resistance;[150] b) volume coefficient of the conductivity[173]; c) thermoelectric power[145]; d) Hall mobility and ratio of the Hall coefficient to the free-electron value[176]; e) Knight shift;[185] f) velocity of sound.[166]

The curves representing the electrical conductivity and thermoelectric power of univalent cesium and divalent mercury exhibit the following ranges characteristic of the metal–insulator transition whose theoretical aspects are considered in Sec. 2: 1) liquid metal range; 2) transition range; 3) range with properties resembling liquid semiconductors; 4) range of dense ionized gases.[20)]

1) Liquid metal range. In this range the conduction is metallic [Sec. 2.3, after Eq. (29)]. Both cesium and mercury have electrical conductivities in the range $\sigma \geqslant 2500-3000\ \Omega^{-1}\cdot\text{cm}^{-1}$ and thermoelectric powers $\alpha < -50\ \mu V/°C$. The mean free path is $L > a$. The Hall coefficient has the free-electron value $R = (nec)^{-1}$ (Figs. 13a–15d). It is clear from the figures that in the case of cesium and mercury (Figs. 14b and 15d) the Hall coefficient is a linear function of the density throughout this range and this is typical of liquid metals. In the case of cesium this is true up to ≈1 g/cm³ (Ref. 177),[21)] whereas in the case of mercury the linear dependence ends at ≈11 g/cm³ (Ref. 176).

The rapid variation in the mobility observed in this range for mercury and cesium is a consequence of a change in the mean free path of electrons. For example, in the case of cesium it is estimated in Ref. 177 using the equation of state from Ref. 163 that the mean free path should be $L = 80$ Å when the density is $\rho = 1.92$ g/cm³ and $L = 10$ Å when the density is $\rho = 1.1$ g/cm³. We recall that the average interatomic distance is $a = 7$ Å.

In the metallic range we can justifiably use the Ziman theory[83], as shown already in Sec. 2. The electrical resistivity of mercury and cesium, whose liquid structure factor was based on the hard-sphere model, were calculated in the low density range.[187-188] It would be interesting to find the maximum reduction in the density of a metal at which the Ziman electrical resistivity formula still holds well. According to calculations on the reduction in the density of mercury, the lowest density is ≈11 g/cm³ (Ref. 187).[22)]

At lower densities a good agreement between the theories and experimental results is obtained if allowance is made for the structural properties of a liquid of reduced density, which weakens screening and facilitates changes in the form factor of the pseudopotential. Several models of the pseudopotential were checked in Ref. 188 for liquid mercury with densities in the range $11.15 > \rho > 9.71$ g/cm³ at a pressure of 2000 atm assuming a reduction in the screening. Figure 16 shows the results of calculations carried out using the Ziman formula and the Harrison pseudopotential[189] and also allowing for the screening. We can see from this figure that the agreement between calculations and experiment improves considerably when the reduction in the screening is allowed for.

2) Transition range. This range is characterized by destruction of the metallic conduction and it corresponds to a situation described in Sec. 2. The electrical conductivity can then vary in the range $\sim 2500\ \Omega^{-1}\cdot\text{cm}^{-1} \geqslant \sigma \geqslant 200\ \Omega^{-1}\cdot\text{cm}^{-1}$, the thermoelectric power is $\alpha < -80\ \mu V/\text{deg}$, the mean free path L is of the order of the interatomic distance $L \sim a$, and the Mott factor is $g < 1$. In this range the Hall coefficient of mercury rises

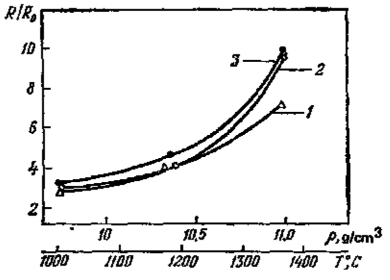

FIG. 16. Resistance of mercury at various densities and temperatures calculated and found experimentally under a pressure of 2000 atm: 1) calculation carried out using the Ziman formula and the Harrison pseudopotential;[189] 2) calculation with variable screening conditions;[188] 3) experimental results.[150]

[20)] This last region was not considered in Sec. 2.

[21)] In principle, such a dependence should be retained by cesium up to the critical region, but there are no experimental data on the range of densities from 1 g/cm³ to the critical value (0.42 g/cm³).

[22)] The minimum density was not determined for cesium. The lowest density for which calculations were made amounted to ≈1.14 g/cm³ (Ref. 188).

rapidly on reduction in the density to ~9 g/cm^3. A strong reduction in the temperature coefficient of the electrical resistivity of mercury also occurs at this density.[150,173] An increase in the Hall coefficient is usually attributed to a reduction in the mean free path to a distance of the order of the interatomic value and it corresponds to the onset of the Mott–Anderson localization. According to one of the hypotheses, this may occur on formation of a pseudogap between the S and P bands.[191]

In contrast to divalent mercury, the level corresponding to the minimum metallic conductivity of cesium lies in the critical density region where probably separation between two Hubbard bands begins.

A special feature of this region is that the onset of localization is complicated by inhomogeneities, particularly in the vicinity of the critical point where the conductivity may be governed by closed regions with a higher electrical conductivity. This possibility was first pointed out in Refs. 61–64. Thus, in accordance with Sec. 2, we are speaking here of the relationship between the Mott–Anderson and percolation transitions.

Evidence that the transition is of the percolation type can be provided by the relationship between the thermoelectric power and electrical conductivity. A comparison of these properties for mercury made using the coordinates of pressure and temperature,[145] and of density and pressure, reveals a fundamental property of the metal–nonmetal transition, namely that the thermoelectric power reaches a value corresponding to the minimum of the metallic conductivity before that minimum is reached on the conductivity scale.

By way of example, we shall consider the dependences of the thermoelectric power and conductivity of mercury at a constant pressure but at different temperatures (Fig. 17). We may assume that the thermoelectric power of this system is shunted by the more highly conducting through channels, whose number decreases on increase in the density.[145]

The acoustic properties of mercury in the region of the metal-nonmetal transition are also of interest (Fig. 15f); here, there is a slight deviation from the linear dependence of the velocity of sound on the density. It is at present not clear how such deviations are related to the metal–nonmetal transition.

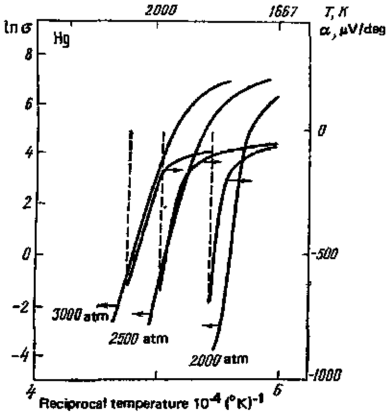

FIG. 17. Electrical and thermoelectric properties of mercury at various temperatures in the region of the metal–nonmetal transitions.[145] (Electrical conductivity data taken from Ref. 149.)

In the case of cesium information is available on the specific heat at various pressures near the metal–nonmetal transition.[167] The specific heat rises strongly on approach to the critical point (Fig. 13c). However, in contrast to nonconducting substances (for example, argon) the interval of rapid rise of C_p and C_v for cesium is wider on the T/T_{cr} scale. This may be due to the fact that the compressibility of cesium is greater than the compressibility of nonconducting substances (in particular, that of argon).

3) *Range with some properties of liquid semiconductors*. The reduction in the density beginning from the near-critical region in the case of cesium and in the range of densities below 9 g/cm^3 in the case of mercury (see Figs. 13 and 15) demonstrates the existence of a range with properties resembling those of semiconductors.

In this range the relationship between the electrical conductivity and thermoelectric power is of the type

$$\ln \sigma \sim \alpha, \tag{32}$$

which is obtained from Eqs. (28) and (29). This dependence is illustrated clearly in Fig. 18. There is an important difference in the positions of this dependence in the phase diagrams of cesium and mercury: in the case of cesium this dependence lies in the critical density region and extends to the gas densities, whereas in the case of mercury this region corresponds to densities of the liquid phase. In the case of cesium we have a transition from the metallic state to a dense plasma whose properties are briefly described in the next subsection; therefore, we shall confine our attention to the properties of mercury. The relationship (32) is valid up to the critical density region. In this region the value of the thermoelectric power falls abruptly to zero on all the isobars.[145–147,179,181] Figure 17 shows this by dashed lines. The maximum absolute values of the thermoelectric power from which it begins to fall rapidly increase on increase in the pressure.[145] The positions of the resultant "zero" values of the thermoelectric power are shown in the P–T diagram in Fig. 19. No experimental investigations have yet been made of the subsequent fate of the thermoelectric power in the plasma state.

One of the possible interpretations can be given on the basis of allowance for the contributions made to the kin-

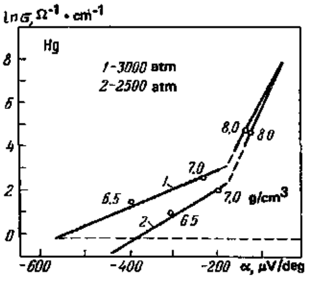

FIG. 18. Dependences of the electrical conductivity on the absolute thermoelectric power of mercury at two pressures.[145]

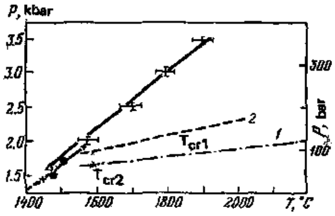

FIG. 19. Positions of points with zero thermoelectric power of mercury plotted in the $P-T$ diagram: o) results from Refs. 145 and 179; △) Ref. 174; +) Ref. 181. T_{cr1} represents the critical point of mercury according to Ref. 159; T_{cr2} is taken from Ref. 151. Curve 1 gives the calculated positions of the minima of the thermoelectric power in a mercury plasma and curve 2 gives the corresponding positions for a cesium plasma.[178]

etic properties of mercury by, on the one hand, fluctuations of the density resulting in an increase in the depth of the potential wells and, on the other, by an increase in the Fermi level of the electron gas because of increase in temperature.[179] This possibility is supported also by the calculations made in Ref. 192. The experimental data on the electrical conductivity and thermoelectric power suggest that a pseudogap does indeed appear on expansion in mercury.[191] This is demonstrated also by the data on the Hall effect,[176] while the recent measurements of the optical reflection coefficient in the frequency range $0.3\text{eV} \leq \hbar\omega \leq 3\text{eV}$ and of reduction in the density of mercury indicate (Fig. 20) that small reflection coefficients (0.2) correspond to densities below 8 g/cm^3 (Ref. 184), i.e., they lie in the same region.

Direct calculations of the electron spectrum of mercury in various hypothetical structures carried out in Ref. 193 for varying lattice constants also show that the energy gap in different structures should appear when the density becomes 8.5 g/cm^3. This value is in good agreement with the experimental data; however, further calculations[194-197] indicate that the densities at which the energy gap opens differ from structure to structure and the transition region may lie between 9.3 and 4 g/cm^3.

The uncertainty about the possible formation of a pseudogap on expansion of mercury has increased after recent measurements of the Knight shift (see Fig. 15e).[185] These results indicate the possibility that a pseudogap exists in mercury at densities $\rho > 9$ g/cm^3. The reasons for this disagreement between the experimental results are not clear.

4. *Range of dense ionized gases.* In the range of densities below the critical value a metal may undergo a transition to a dense plasma state. Calculations of the thermoelectric power and electrical conductivity have been made for this range using an ionized gas model and allowing for the reduction in the ionization potential on increase in the density because of the interaction between charged and neutral particles:[198,199,209] see also the reviews in Refs. 200 and 201. This interaction results in an exponential dependence of the electrical conductivity on the density, as demonstrated in the first paper by Vedenov[198] and in his subsequent work (Refs. 199 and 209–211).[23),24)] Moreover, allowance for this interaction may give rise to new effects in the plasma. In fact, an expression for the thermoelectric power of a dense plasma is obtained in Ref. 199:

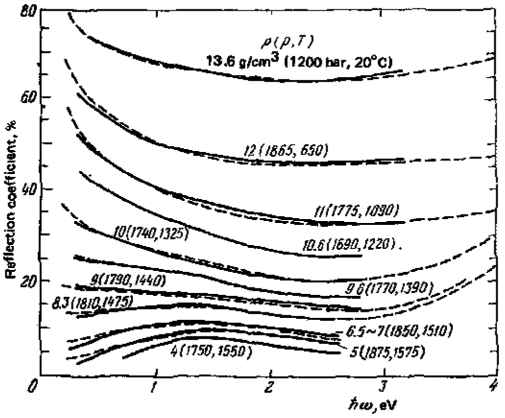

FIG. 20. Frequency dependence of the optical reflection coefficient on reduction in the density of mercury.[184]

$$\alpha \sim \frac{\Delta E}{kT}, \tag{32'}$$

where $\Delta E \sim (I_0/2)(1 - a\rho/\rho_{cr})$; $a \sim 1$; I_0 is the atomic ionization potential; ρ_{cr} is the critical density of the metal. Here, ΔE represents the change in I_0 on reduction in the density because of the interaction between charged and neutral particles. We can then expect an increase in the absolute value of the thermoelectric power since the plasma density decreases. It is clear from Eq. (32) that the thermoelectric power should exhibit a minimum at values of ΔE equal to the ionization potential of a motal atom. Lines of such minima for cesium and mercury at various pressures were predicted in Refs. 178 and 179 and were detected experimentally for cesium in Ref. 144 (Fig. 21).

The relationship between the thermoelectric power and electrical conductivity in the ionized gas range is

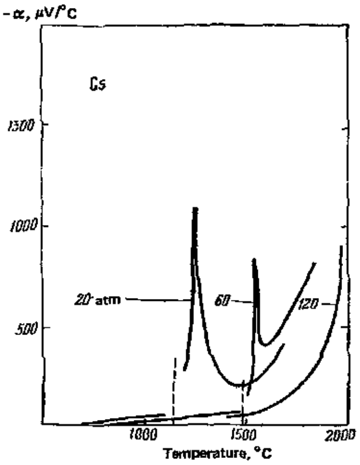

FIG. 21. Temperature dependences of the thermoelectric power of cesium at pressures below the critical point (20 and 60 atm) and above this point (120 atm), based on Ref. 144.

[23)] The results of Ref. 202 suggest that the attraction of ions via a neutral gas should induce the ionization potential.

[24)] This interaction was also partly considered in Ref. 229.

obtained in Ref. 203:

$$\alpha \sim \frac{d \ln \sigma T}{dT}. \tag{33}$$

This dependence is analogous to Eq. (32) applicable to liquid semiconductors. One of the interesting results in this range is an electrical conductivity minimum reported in Ref. 144 (Fig. 22).[25] This effect has been considered in many papers using a variety of models: allowing for the formation of ion complexes,[206] employing the fluctuation model,[207] etc.[26] The results obtained using some of these models are shown in Fig. 22. The ranges of existence of a nonideal cesium plasma were determined in Ref. 212.

A variant of the Ziman theory for the electrical properties of a dense cesium vapor was considered in Ref. 203, where a pseudopotential describing the interaction of an electron with neutral atoms was suggested. A semiconductor model, analogous to the theory of an extrinsic semiconductor with partial ionization of donor centers, was applied by Faber to cesium vapor[83] but without allowance for the reduction in the ionization potential due to the interaction between charges and neutral ions. In this range there may be a considerable contribution to the conduction process from clusters. This subject is being pursued actively[200] employing variants of the cluster model.[27]

B. Results of experimental investigations of selenium and arsenic. Se–Te system

1) Selenium and arsenic. It should be pointed out that, in contrast to metals, very little is known about liquid semiconductors near their critical temperatures and pressures. Information on the electrical conductivity obtained in two laboratories is available.[155-158] Figure 23 shows the temperature dependences of the electrical conductivity of selenium obtained at various pressures. It is clear from this figure that below 500 atm the temperature dependences exhibit a steep fall when a certain temperature (which varies with the pressure) is reached on the saturation line.[28] The 500 and 1000 atm isobars lie well above the critical point of selenium and, therefore, the results obtained apply to the region of continuous reduction in the density of selenium.[29]

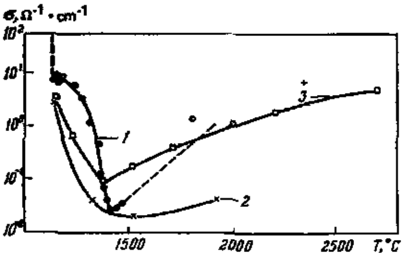

FIG. 22. Temperature dependences of the electrical conductivity of a dense cesium plasma at a pressure of 20 atm: 1) Ref. 144; o) experimental points from Ref. 204; +) point taken from Ref. 205; 2) calculated curve from Ref. 206; 3) calculated curve from Ref. 207.

Measurements of the electrical condictivity demonstrate another interesting feature, namely conductivity of the exponential dependence of the electrical conductivity at temperatures 1700–2000 °K and conductivities of the order of 200 $\Omega^{-1}\cdot\text{cm}^{-1}$, i.e., at values corresponding to the minimal metallic conductivity.

It is clear from Fig. 23 that the experimental curves demonstrate a reduction in the activation energy on increase in temperature.[83,96] This reduction in the activation energy of conduction has been interpreted as a consequence of a progressive (with temperature) reduction in the pseudogap. It is estimated that the pseudogap disappears at 1500 °K. According to the experimental data,[96] metallization occurs at $T \sim 1700$ °K. In view of the approximate nature of the extrapolation, the agreement between these metallization temperatures of selenium should be regarded as satisfactory.

The saturation region is followed by a range where the electrical conductivity falls steeply from a level corresponding to the minimum conductivity. If we assume that in this range of temperatures and pressures the selenium represents a homogeneous system, we can interpret this transition to the insulating state as an Anderson transition associated with the electron localization (Sec. 2); the electrical conductivity then changes abruptly (under experimental conditions the transition is nonabrupt because of finite temperatures) from the metallic to the hopping conduction mechanism. It would be interesting to determine the electrical conductivity corresponding to the onset of the hopping mechanism,

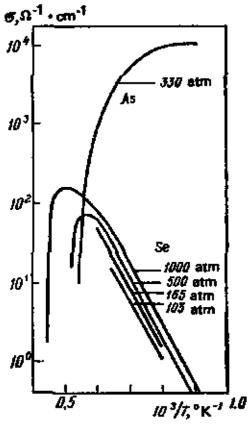

FIG. 23. Relative changes in the electrical conductivity of selenium and arsenic above the critical points.[156]

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[25] Moreover, it was reported in Ref. 208 that the electrical conductivity minimum occurred at much higher temperatures than in Ref. 144, but interpretation of this minimum encounters considerable difficulties.

[26] In one of the models it is pointed out that charged drops can appear under these conditions because of the strong attraction of atoms to ions.[208]

[27] This topic will not be considered because a recent review[230] has been devoted to the subject.

[28] The critical temperature and pressure of selenium are $T_{cr}=1766$ °K and $P_{cr}=268$ atm (Ref. 215). The critical parameters reported in Ref. 157 are $T_{cr}=1590$ °C and $P_{cr}=380$ atm, which can be seen to be different from those given in Ref. 215.

[29] The experimental data reported in Ref. 155 are in good agreement with those of Ref. 231, whereas a higher saturation value of the electrical conductivity is given in Ref. 157.

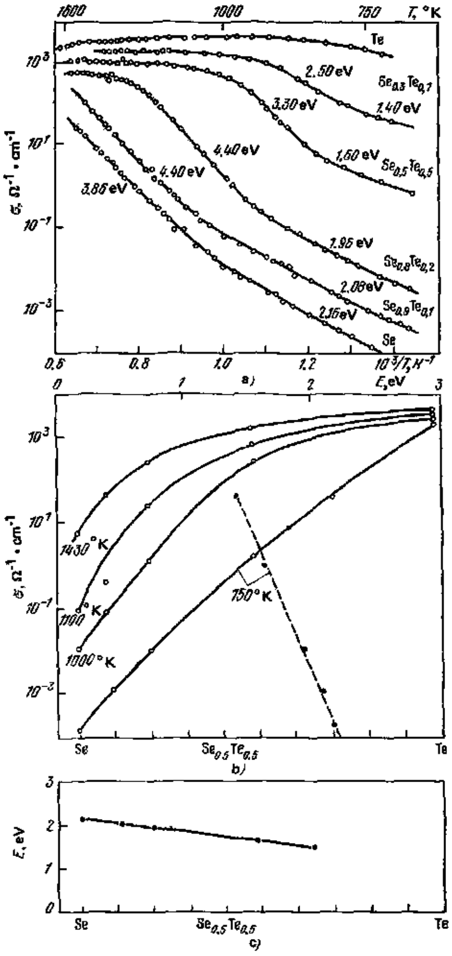

FIG. 24. Changes in the electrical conductivity of the Se–Te system with different selenium concentrations plotted as a function of temperature and composition ($P = 180$ atm).[91] a) Temperature dependences of the electrical conductivity of various compositions in the selenium–tellurium system. b) Composition dependences at a constant temperature (the dashed line shows the ratio of the energy gap to the conductivity). c) Dependence of the energy gap on the composition of the selenium–tellurium system at $T = 750$ °K.

since this would make it possible to check the correctness of theoretical estimates.[155,156]

It is interesting to compare the electrical conductivity of selenium in the electron localization region with the value for the metal which is the closest analog of selenium and which has approximately similar critical parameters. This metal is arsenic, whose electrical conductivity has been measured at pressures exceeding the critical value of 330 atm (Ref. 154).

Figure 23 shows the temperature dependences of the electrical conductivity not only of selenium but also of arsenic.[156] The steep fall of the conductivity of selenium beyond the plateau has already been explained by the Anderson localization. We can see from this figure that in the case of arsenic the steepest fall of the electrical conductivity begins from 200 $\Omega^{-1}\cdot cm^{-1}$. Up to this value the localization effect may be shunted by regions whose conductivity is equal to or greater than the minimum metallic value.

2) Se–Te *system*. We shall compare the transition from the metallic to the nonmetallic state in metals and to the metallic state in semiconductors and in ordinary semiconductor melts by considering the results of an experimental study of the electrical conductivity of the selenium–tellurium system at various temperatures and for various tellurium concentrations.[91] This investigation was carried out at a pressure of 180 atm. The results are given in Fig. 24. The metallic component here is tellurium (it has metallic conduction)[89] whose amount is reduced continuously. The system then goes over into a typically semiconducting state. The curves in Fig. 24 show some characteristic features considered above for mercury and cesium, and also for selenium at transcritical temperatures. This may be regarded as an indication of the identity of the nature of the corresponding phenomena occurring in these systems.

We can summarize by concluding that in the case of metal–nonmetal transitions under transcritical conditions the processes of electron localization begin in the vicinity of the electrical conductivity values corresponding to the minimum metallic conductivity and in this sense the results of these investigations are in agreement with the results reported earlier (Secs. 3 and 4) for liquid semiconductors.

6. CONCLUSIONS

We have attempted to summarize all those theoretical and experimental investigations of low-density liquid semiconductors and metals which are of direct relevance to the phenomena of localization (or delocalization) of electron states. The experimental results reviewed in the last three sections demonstrate the common features of the semiconductor–metal transition observed in independent experiments and they show that the existing experimental data largely support the qualitative description of this transition as due to the influence of disorder. In particular, it is clear that such a theoretical concept as the minimal metallic conductivity is undoubtedly a feature of the experimental results if only as a characteristic scale and it has universal meaning as the conductivity separating a metal from a semiconductor. We shall stress only this fundamental aspect and note that accurate quantitative relationships governing the minimal metallic conductivity cannot be as yet obtained theoretically or experimentally.

The general quite unsatisfactory state of the theory and experiment is manifested by the fact that selection cannot be made between the existing theoretical models. As a rule, we cannot say definitely whether in a given specific case the microscopic inhomogeneities are important (giving rise to an inhomogeneous percolation conduction regime), what is the role of the electron–electron correlations, whether the minimal metallic conductivity has any deeper meaning than a simple characteristic conductivity scale, etc. These problems can only be solved by further efforts of theoreticians and experimentalists.

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Electron localization in disordered systems: critical behavior and macroscopic manifestations

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The review deals with new results in the theory of electron localization in disordered systems and with experiments which have been stimulated by these results. An elementary scaling theory of localization and its consequences are discussed. Attention is concentrated on the phenomenon of localization in two-dimensional systems. The main experiments on the conductivity of thin metal wires and films, demonstrating qualitatively new behavior associated with localization, are discussed. The review concludes with a brief look at the problems of deriving a self-consistent scaling theory of critical behavior at a mobility edge.

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CONTENTS

INTRODUCTION

The concept of electron localization in disordered systems is central to the modern theory of such systems: the main ideas on the energy spectrum, and on the transport and other electron properties of disordered systems are based on this concept.[1,2] The concept was first formulated in the cornerstone paper of Anderson[3] and later, after a period of neglect, it was developed qualitatively by Mott, who used it to formulate the main laws of the electron theory of disordered systems.[1,2] Several good reviews[4-7] have recently been devoted to the phenomenon of localization and its fundamentals are now well known. Moreover, in spite of its importance, the problem of localization is still far from the state of a fully satisfactory solution. This applies particularly to our ideas on the behavior of electron states near what is known as the mobility edge and the associated problem of the physical properties of a system in which the electron Fermi level lies in the vicinity of a mobility edge. Difficulties in the understanding of these properties are related to the extreme mathematical difficulty of the problem under discussion and due to the fact that experimental evidence on electron localization is fairly indirect.[1,2,5]

The present review describes briefly the progress made in the last few years, both in the theoretical respect and in the formulation of new experiments. We shall concentrate on those theoretical results which relate to the "critical" behavior of a system in the vicinity of a mobility edge, using a certain analogy which clearly exists between such behavior and the usual critical behavior in the vicinity of a second-order phase transition; we shall also discuss new experiments on the conductivity of thin metallic wires and films in which the phenomenon of localization (and the corresponding critical behavior) should be manifested most clearly. The special attention currently given to two-dimensional disordered systems is due to the circumstance that practically all the devices used in modern microelectronics are examples of such systems. However, we shall ignore completely the problems and experiments relating to the localization in quasione-dimensional systems (for details see Refs. 8–11).

We shall proceed as follows. We shall begin by recalling the main ideas of the theory of localization and the relevant terminology. We shall then consider elementary scaling theory of localization based mainly on the ideas of Thouless,[4] including newest developments of his results and their applications to the description of the conductivity of thin metallic wires and two-dimensional metallic systems. Next, we shall discuss the main experiments aimed to verify these ideas. We shall conclude with a brief review of the attempts to construct a rigorous scaling theory of localization on the basis of the above-mentioned analogy with the conventional critical phenomena.

1. ANDERSON MODEL AND MINIMUM METALLIC CONDUCTIVITY

We shall recall the main results of the theory of localization. This is usually done on the basis of the Anderson model[3] in which one electron propagating in a regular lattice of sites in a d-dimensional space is considered; at each site there is a random energy level E_j (j is the number of the lattice site). It is also assumed that there is a definite amplitude of the probability V_{ij} of a transition from a jth to an ith site. It is usually postulated[3,12,13] that this transition amplitude differs from zero and is equal to a certain constant V for transitions between the nearest neighbors. We

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shall not consider the role of "nondiagonal" disorder of the matrix elements V_{ij} (for details see Ref. 14).

The Hamiltonian of the model is therefore

$$H = \sum_j E_j a_j^{+} a_j + \sum_{ij} V_{ij} a_i^{+} a_j, \tag{1.1}$$

where a_j and a_j^{+} are, respectively, the usual operators of annihilation and creation of an electron at a site j. The energy levels E_j are assumed to be distributed independently from one another at different sites. The distribution at a given site is usually assumed to be[3]

$$P(E_j) = \begin{cases} \frac{1}{W} & \text{for} \quad |E_j| < \frac{1}{2} W, \\ 0 & \text{for} \quad |E_j| > \frac{1}{2} W, \end{cases} \tag{1.2}$$

i.e., it is assumed to be uniform in a certain range of energies of width W. Qualitative results are clearly independent of the nature of the distribution $P(E_j)$ and instead of Eq. (1.2) we can use any other distribution with an effective width W.

If there is no disorder in the system ($W=0$), the solution of the problem of the electron spectrum of the Hamiltonian (1.1) is elementary. Electron states form a band of width $B=2ZV$ (for a simple cubic lattice with Z nearest neighbors) and the wave function of each of the states in a band is represented by a Bloch wave, i.e., it is shared equally by all the lattice sites. If $W \neq 0$, the situation changes and the dimensionality of space d has a considerable influence. For example, if $d=1$ (a one-dimensional disordered chain), an infinitesimally small value of W is sufficient to alter completely the nature of the electron states: they all become localized, i.e., their wave functions decrease exponentially in the coordinate space and at $T=0$ the static conductivity of the system vanishes.[8-11] Two-dimensional systems ($d=2$) represent clearly a special case and are discussed in detail below. If $d>2$, it has been established reliably[3-7,12-13] that for high values of the ratio $W/V > (W/V)$ (i.e., in the case of sufficiently strong disorder) all the electron states in a band are again localized. Typical values of this ratio are $(W/V)_c = 8$–15 (for three-dimensional lattices).[3-6]

The physical meaning of localization is fairly simple. Quantum tunneling from site to site can occur only between sites with identical (degenerate) energy levels. In a random system the probability of such degeneracy is generally small for sites sufficiently close in coordinate space and if the disorder W is sufficiently strong, this probability vanishes.[3,15,16]

If $W/V < (W/V)_c$, electron states become localized at the band edges, whereas at the center of a band the states remain delocalized. This gives rise to critical energies E_c and E_c', which separate the regions of localized and delocalized states, usually called mobility edges. This is illustrated in Fig. 1. The term "mobility edge" is due to the fact that localized states make no contribution to the static conductivity at absolute zero ($T=0$). If at $T=0$ the fermi level E_F of a many-electron system lies in the energy range corresponding to localized states, the system is an insulator: conduction is possible only at $T \neq 0$ or when electrons are excited by an alternating electromagnetic field. Conduction then takes place by the hopping mechanism.[1,7,17] However, if the Fermi level lies in the region of delocalized states, conduction is metallic. This metal–insulator transition, which occurs when the Fermi level crosses a mobility edge, is usually called the Anderson transition.

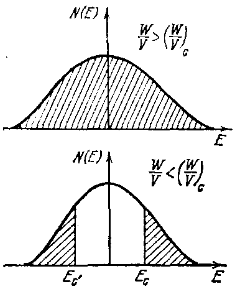

FIG. 1. Density of states in an Anderson lattice. The shaded regions are the localized states. Here, E_c and E_c' are the mobility thresholds.

One of the central problems in the theory under discussion is the nature of changes in the metallic (static, at $T=0$) conductivity that take place when the Fermi level E_F crosses a mobility edge. Possible alternative representations are shown in Fig. 2. The conductivity may abruptly vanish after reaching a certain minimum value $\sigma_{m.m}$, which denotes the minimum metallic conductivity (maximum metallic resistivity). This behavior has been suggested by Mott[1,2] on the basis of a qualitative analysis of the conduction process in the Anderson model, and also on the basis of an analysis of some experimental data. We can logically expect also a continuous decrease of the conductivity to zero and in this case $\sigma_{m.m}$ retains in a sense the meaning of a characteristic measure of the conductivity beginning from which the conductivity decreases quite rapidly to zero. This alternative model was put forward by Cohen[18] using basically an analogy between localization and percolation.[7,17] Clearly, both alternatives can have the same experimental consequences (at $T \neq 0$), but this does not solve the corresponding theoretical problem.

An estimate of the value of $\sigma_{m.m}$ can be obtained quite simply.[1,2,5] In fact, this estimate is based on the ideas put forward some time ago by Ioffe and Regel,[19] according to whom the mean free path of an electron l in a metallic system cannot be less than the interatomic distance a. It then follows from the usual Drude formula $\sigma = (ne^2/m)\tau$ (n is the electron density, τ is the mean free time between collisions, and m is the electron mass) and from the expressions relating the elec-

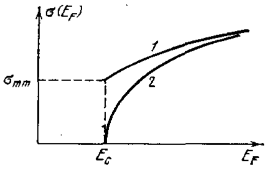

FIG. 2. Changes in the metallic conductivity when the Fermi level passes through a mobility edge: 1) according to Mott[1,2]; 2) according to Cohen.[18]

tron density and the Fermi momentum to the interatomic distance ($n \propto a^{-d}$, $p_F \propto \hbar/a$, $\tau \propto lm/p_F \propto am/p_F \propto ma^2/\hbar$) that

$$\sigma_{\mathrm{m.m}} \approx \mathrm{const} \cdot \frac{e^2}{\hbar} a^{2-d}. \tag{1.3}$$

According to Mott,[1,2,5] the constant in Eq. (1.3) is determined by the dimensionless ratio $(W/V)_c$ and lies in the range 0.025–0.06 for $d=3$, which gives $\sigma_{\mathrm{m.m}} \sim (1-5) \times 10^{-2} \Omega^{-1}\ \mathrm{cm}^{-1}$ for typical values $a \sim 2-3$ Å.

It is clear already from Eq. (1.3) that $d=2$ is a special case: the minimum metallic conductivity is governed only by the fundamental physical constants and the characteristics of a system exhibiting such conductivity affect only the fairly universal dimensionless ratio $(W/V)_c$. The question arises: is this true? Moreover, a detailed understanding of the changes occurring in the wave functions near a mobility edge is important: how does the localization radius of a wave function change: does it diverge in the limit $E \to E_c$ on the localized state side? Are the wave functions of localized states necessarily exponential or can we expect a fairly rapid (ensuring normalization to unity) power-law fall in the coordinate space? Can these problems be considered by analogy with the problem of phase transitions and is not the behavior near a mobility edge a characteristic "critical" phenomenon? Finally, what is the role of the electron–electron interaction (correlation) in the phenomenon of localization? Our review will, to a greater or lesser extent, try to answer these questions; we shall consider the current level of understanding of these problems and some of the experiments stimulated by their formulation.

2. ELEMENTARY SCALING THEORY OF LOCALIZATION

Behavior of a system in the vicinity of a mobility edge can be understood by a scaling analysis similar to that employed in the theory of critical phenomena[20,21] or, to use the modern language, by constructing some variant of transformations of the renormalization group.[22-24] The main physical idea behind this approach is the gradual transition from small cells (or scales) of a system in the coordinate space, for which the problem can be solved (at least approximately!), to larger cells (scales) which are described tentatively in terms of the same physical variables as the small-scale cells. The chain of formulas performing this transition from small to large scales is known as the transformations of the renormalization group in real space[23,24] and they are modern variants of the Kadanoff scaling transformations.[20] In the theory of critical phenomena the construction of such transformations is usually motivated[20-24] by an increase in the correlation length of fluctuations of the order parameter in the vicinity of a critical point. An analog of this phenomenon in the theory of localization is the increase in the localization radius on approach to a mobility edge from the localized-state side. However, it should be stressed that in the range of delocalized states an analog of such a diverging length is not known[1)] (just as the "order parameter" associated with the Anderson transition is not known), which limits seriously this analogy and makes it difficult to carry out a simple qualitative analysis.

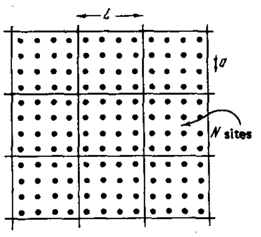

FIG. 3. Analog of the Kadanoff construction in the Anderson problem, plotted in accordance with the Licciardello–Thouless treatment.

We shall now present a variant of scaling transformations proposed for the Anderson problem by Thouless *et al.*[4,25-29] We shall consider an Anderson lattice with the period a. We shall make a transition from a cell of side a to a new cell of side L, containing N sites of the original lattice. Then, the original lattice can be described as consisting of new cells, each of which has a set of N random levels (Fig. 3). We shall consider a lattice constructed by periodic repetition of one such cell. Then any one of the N levels in a cell spreads into a band of width $2\Delta E$. The quantity ΔE is defined in Refs. 25–29 numerically by a shift of the levels on transition from the periodic to antiperiodic boundary conditions for wave functions at the boundary of the cell L. Although the lattice formed by such continuation is not identical with the scaling-transformed Anderson lattice, in a qualitative analysis we can assume that the effective coupling[2)] between electrons at two levels in neighboring cells of the Anderson lattice (i.e., an analog of the overlap integral V for a system constructed from the new cells) is of the order of

$$V_L \approx \frac{1}{Z} \Delta E, \tag{2.1}$$

since each such coupling or bond produces a band of width $2\Delta E$ when the cell is continued periodically. The average scatter of the levels in the neighboring cells is governed by the reciprocal of the density of states calculated per unit volume in the original lattice and by the cell size L:

$$W_L \approx L^{-d} N^{-1}(E). \tag{2.2}$$

If we consider one of the N energy bands in the new lattice, we can regard W_L as an analog of the parameter W of the original lattice. In this way we make the transition from the original Anderson problem with the ratio W/V to a new (scaling-transformed) problem of the same type with an "effective" Hamiltonian characterized by a new ratio W_L/V_L. The process of such

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[1)] See, however, a discussion of this question in a recent paper.[135]

[2)] The quantity $V_L \sim \Delta E$ is a measure of the mutual influence of the wave functions in neighboring cells. If the states are localized, they are practically insensitive to a change in the boundary conditions in a large cell [see Eq. (2.3)]. On the other hand, delocalized states are sensitive to changes in the boundary conditions and this results in a considerable shift ΔE of the levels[25-29] [see Eq. (2.4)].

scaling transformations can be continued along the chain $L \to 2L \to 4L \to \ldots$ or, in general, we can make the transformation $L \to bL$ going over to increasingly larger cells containing more and more of the sites in the original lattice.

If a given state of energy E is localized, then

$$V_L \propto e^{-\alpha(E)L}, \qquad (2.3)$$

where $\alpha(E) = R_{\text{loc}}^{-1}(E)$ is the reciprocal of the localization radius of a given state. Consequently, in this case the ratio V_L/W_L also decreases exponentially on increase in L. If we assume the existence of mobility edges, then for any value of E in the original band there is a maximum ratio $(V_L/W_L)_{\max}$, for which such exponential decrease is still valid. Such a localization criterion has been used[25-29] to determine numerically the positions of mobility edges in the original energy band.

In a system of cells of size $L \gg l$, where l is the mean free path, a delocalized electron can equiprobably become displaced (can diffuse) from one cell to a neighboring one[3] in a time $\tau_L \sim \hbar/V_L$. It follows from elementary transport theory that the coefficient of such diffusion is $D_L \sim L^2/\tau_L$, so that we obtain

$$V_L \approx \hbar D_L(E) L^{-2}, \qquad (2.4)$$

where $D_L(E)$ is the diffusion coefficient of an electron of energy E. Then, in the case of delocalized states, we have

$$g(L) \equiv \frac{V_L}{W_L} \approx \hbar D_L(E) N(E) L^{d-2} \approx \frac{\hbar}{2e^2}\sigma_L(E) L^{d-2}, \qquad (2.5)$$

where $\sigma(E) = 2e^2 D(E) N(E)$ is the electrical conductivity of the system expressed in terms of the diffusion coefficient and the density of states, which follows from the general Einstein relationship[30] $\sigma = e^2 D\,\mathrm{d}n/\mathrm{d}\mu$ (μ is the chemical potential) at $T = 0$. A more rigorous derivation of Eq. (2.5) is given in the original papers.[4,25-28]

It is clear from Eq. (2.5) that if $d = 2$, the universal minimum metallic conductivity does exist if scaling transformations of $(V_L/W_L)_{\max}$ give a certain universal constant. A numerical analysis carried out by Licciardello and Thouless[27,28] showed that such a universal (independent of the nature of the original lattice) constant does exist (in Ref. 27, they considered a hexagonal honeycomb lattice, as well as triangular and square lattices for $N = 64$, 100, 144, and 196). Licciardello and Thouless obtained the following expression for the metallic conductivity in the $d = 2$ case:

$$\sigma_{\text{m.m}} = \frac{2e^2}{\hbar}\left(\frac{V_L}{W_L}\right)_{\max} = (0.12 \pm 0.03)\,\frac{e^2}{\hbar} \approx 3\cdot 10^{-5}\ \Omega^{-1}. \qquad (2.6)$$

However, later numerical calculations of the same authors[29] gave somewhat unexpected results. Roughly speaking, they found that the value of $\sigma_{\text{m.m}}$ decreases as the dimensions L of the system increase. This raised doubts about the existence of $\sigma_{\text{m.m}}$ for $d = 2$ and led Licciardello and Thouless[29] to the hypothesis that in the case of two-dimensional systems there may be complete localization even in the case of infinitesimally weak disorder (by analogy with the one-dimensional case). On the other hand, the results of Ref. 31 confirmed the earlier results of Licciardello and Thouless.[27] Numerical calculations of three-dimensional systems are far too indefinite in Refs. 25–29; the fullest analysis of the three-dimensional case can be found in Ref. 32 for a diamond-type lattice.

Considerable progress in our understanding of the phenomenon of localization was made[4] by Abrahams, Anderson, Licciardello, and Ramakrishnan.[33] It was found that a simple qualitative analysis can be made of the behavior of the function $g(L)$ defined by Eq. (2.5) by postulating the simplest renormalization group equation for this function [this equation gives the change in $g(L)$ due to the transition $L \to bL$][5]

$$\frac{\mathrm{d}\ln g(L)}{\mathrm{d}\ln L} = \beta_d(g(L)). \qquad (2.7)$$

At this stage the most important assumption is that the function $\beta_d(g)$ on the right-hand side depends only on the variable $g(L)$, which is known as one-parameter scaling. Equation (2.7) is an analog of the well-known Gell-Mann–Low equation [the function $\beta_d(g)$ is an analog of the Gell-Mann–Low function] in the renormalization group used in quantum field theory.[34-36] The behavior of the function $\beta_d(g)$ is easily understood from simple physical considerations. It is clear from the definition (2.5) that $g(L) \propto \sigma_L L^{d-2}$, i.e., that this quantity is proportional to the total conductance (and not specific conductivity!) of the system (which is a cube of side L). In the case of large values of g (weak disorder, $V_L/W_L \gg 1$), we should obtain the static metallic conductivity $\sigma_{L\to\infty} = \sigma$, i.e., the following condition should be satisfied:

$$\lim_{g\to\infty} \beta_d(g) \to d - 2, \qquad (2.8)$$

since it follows from $\mathrm{d}\ln g(L)/\mathrm{d}\ln L = d - 2$ that $g(L) = \text{const}\cdot L^{d-2}$ and the constant of integration is simply $(\hbar/2e^2)\sigma$. At low values of $g(V_L/W_L \ll 1)$, we should obtain exponential localization, i.e., $g(L) = g_c(d)\exp(-\alpha L)$ [compare with Eq. (2.3)]. Then,

$$\lim_{g\to\infty} \beta_d(g) \to \ln\frac{g}{g_c(d)}, \qquad (2.9)$$

since it follows from $\ln g = \ln g_c - \alpha L$ that $\mathrm{d}\ln g(L)/\mathrm{d}\ln L = -\alpha\,\mathrm{d}L/\mathrm{d}\ln L = -\alpha L = \ln(g/g_c)$.

We shall now *assume* the existence of two "perturbation theory" expansions:

$$\beta_d(g \to 0) = \ln\left[-\frac{g}{g_c(d)}\right](1 + ag + \ldots). \qquad (2.10)$$

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[3] The quantity V_L defines, as mentioned above, the effective coupling of electrons occupying random levels in neighboring cells. This coupling or interaction results in broadening of the levels, i.e., it is responsible for the finite (of the order of τ_L) lifetime of electrons in a given cell.

[4] These authors have been called collectively as the "gang of four."

[5] Equation (2.7) is a differential variant of the postulated general scaling transformation of variables in the problem in question: $g(bL) = f(b, g(L))$. The differential form of Eq. (2.7) is most convenient for a qualitative analysis and it implies a transition from cells of side L to cells of side $bL = L + \mathrm{d}L = (1 + \lambda)L$, where $\mathrm{d}L/L = \lambda \to 0$.

$$\beta_d(g \to \infty) = d - 2 - \frac{b}{g} + \dots \quad . \tag{2.11}$$

Arguments are given in Ref. 33 to show that $a > 0, b > 0$. Using the asymptotic expressions given by Eqs. (2.8) and (2.9), and the expansion represented by Eqs. (2.10) and (2.11), we can easily deduce the behavior of $\beta_d(g)$ throughout the range of g (on the assumption that it is monotonic and continuous!). The behavior of this function is shown in Fig. 4. We can see that $\beta_d(g)$ does not have zeros for $d < 2$. If the expansion (2.11) is valid, there is no zero either for $d = 2$. If $d > 2$, the function $\beta_d(g)$ definitely has a zero, $\beta_d(g_c) = 0$, but its position cannot be found in such a simple way as above (obviously, $g_c \sim 1$). The existence of this zero implies the existence of an *unstable* fixed point for Eq. (2.7). We shall mention in this connection that in the theory of critical phenomena the scaling behavior is usually governed by stable fixed points of the renormalization group equations.[22-24] Near g_c, we can use the approximation (shown by circles in Fig. 4)

$$\beta_d(g) \approx \frac{1}{\nu} \ln \frac{g}{g_c}, \tag{2.12}$$

where $\nu < 1$, because $a > 0$ in Eq. (2.10). Then, if Eq. (2.7) is integrated beginning from $g_0 \gtrsim g_c$ $[g(L = a) = g_0$, where a is a distance of the order of the lattice period], then for $L \to \infty$ the value of σ_L will exhibit the following asymptotic behavior[6]:

$$\sigma \approx A \frac{e^2}{\hbar} \frac{g_c}{a^{d-2}} \left(\ln \frac{g_0}{g_c}\right)^{(d-2)\nu} \approx A \frac{e^2}{\hbar} \frac{g_c}{a^{d-2}} \left(\frac{g_0 - g_c}{g_c}\right)^{(d-2)\nu} \quad \text{for} \quad g_0 \gtrsim g_c, \tag{2.13}$$

where $A = \text{const} \sim 1$. Clearly, the existence of a fixed point implies the existence of a mobility edge and the behavior of $\beta_d(g)$ near absolute zero determines the critical behavior at the mobility edge. Under these assumptions, the conductivity decreases *continuously* to zero in the limit $g_0 \to g_c$ and the quantity $e^2/\hbar a^{d-2}$ (the Mott value of $\sigma_{\mathrm{m.m}}$) is simply a characteristic conductivity scale. In this sense the results of Ref. 33 confirm, to some extent, the Cohen variant. A Mott conductivity jump can be obtained at a mobility edge if the function $\beta_d(g)$ behaves as shown by the dashed curve in Fig. 4 (in the case when $d = 2$). According to the authors of Ref. 33, such behavior is not very likely because it is in conflict with the expansion (2.11). However, it should be stressed that the existence of such an expansion is an assumption made in the theory.[7]

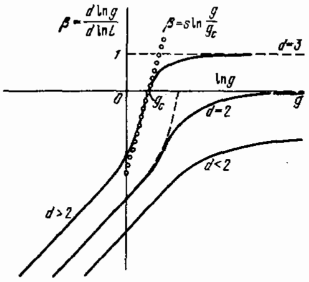

FIG. 4. Qualitative form of the Gell-Mann–Low function for various values of d. The dashed lines are used to represent the behavior needed to ensure a conductivity jump at a mobility edge in the $d = 2$ case.

Similarly, we can show that integration of Eq. (2.7) from the initial point $g_0 \lesssim g_c$ gives (for large values of L)

$$g \approx g_c \exp\left(-A' \left|\ln \frac{g}{g_c}\right|^{\nu} \frac{L}{a}\right). \tag{2.14}$$

Hence, [compare with Eq. (2.3)] it follows that

$$R_{\mathrm{loc}} \sim a \left|\frac{g - g_c}{g_c}\right|^{-\nu}, \tag{2.15}$$

i.e., ν plays the role of the critical index of the localization radius.[8]

If $d \leq 2$, it follows from Ref. 33 that $\beta_d(g) < 0$ throughout the range of g. Then, $\sigma \propto g(L \to \infty) \to 0$ for any initial conditions, there is no mobility edge, and all the states are localized (the static conductivity of an infinite system vanishes!). For $d = 1$, this agrees with the known exact results.[8-11] For $d = 2$, this is a completely new result (although suspected earlier[29]). Subject to the initial condition $g_0(L = a)$ in the range of sufficiently large values of g where the expansion (2.11) is valid (for $d = 2$ the first term of this expansion vanishes), we obtain from Eq. (2.7)

$$g(L) \approx g_0 - b \ln \frac{L}{a} \approx \frac{\hbar}{2e^2} \sigma_L, \tag{2.16}$$

i.e., the conductivity σ_L decreases on increase in L because of a *logarithmic* correction, until the correction term becomes of the same order as the main term; then, the reduction in the conductivity becomes expo-

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[6] Strictly speaking, integration in Eq. (2.7) subject to the indicated initial conditions and with Eq. (2.12) on the right-hand side gives

$$\sigma(L) = \frac{2e^2}{\hbar} \frac{g_c}{a^{d-2}} \left(\ln \frac{g_0}{g_c}\right)^{(d-2)\nu} \left\{\frac{g(L)}{g_c} \left(\ln \frac{g(L)}{g_c}\right)^{(2-d)\nu}\right\}.$$

Since because of the instability of the fixed point in the limit $L \to \infty$, we reach the range $g \gg 1$, where the true function $\beta_d(g)$ reaches the constatn value $d - 2$, the factor in the braces reduces to the constant A of Eq. (2.13) in the limit $L \to \infty$. The dependence on the initial conditions in Eq. (2.13) is governed by the function $\beta_d(g)$ linearized near the fixed point.

[7] The existence of such an expansion can be justified by considering the region of weak disorder on the basis of the conventional perturbation theory,[38-40] which becomes inapplicable for $g \sim g_c$.

[8] An attempt has been made[135] to give a physical meaning to the effective length $\xi \sim a[(g - g_c)/g_c]^{-\nu}$ also in the range $g \gtrsim g_c$. We can easily see that Eq. (2.13) can be written in the form $\sigma \approx Ae^2/\hbar\xi^{d-2}$ so that the length ξ describes the behavior of the conductivity farily close to a mobility edge in a situation when ξ becomes much larger than the interatomic distance or the mean free path in the case of inelastic scattering processes. According to Ref. 135, this length governs the scale for which the conductivity becomes "ohmic" in the sense that the resistance of a cube ($d = 3$) of side L decreases proportionally to L^{-1}. Near a mobility edge only samples of increasing dimensions ($\xi \to \infty$) can be regarded as macroscopic and in the limit $L < \xi\sigma_L$ they depend on L. It is possible that these considerations allow us to give a definite physical meaning to the diverging length in the region of delocalized states which occur in the scaling pattern. In Ref. 135, this is used to propose an original explanation of the negative temperature coefficient of the resistance, which is exhibited by many poorly conducting metallic systems.

nential. The existence of a weak logarithmic dependence in Eq. (2.16) has the effect that the behavior of the conductivity can indeed be close to that expected in the case of existence of an abrupt mobility edge. We can observe experimentally a logarithmic tendency of the "metallic" phase to become insulating (at high temperatures), followed by a fairly rapid drop of the conductivity in the vicinity of $\sigma_{\text{m.m.}}$ in accordance with Eq. (2.6). The processes which determine the effective lengths L, which govern the experimentally observed conductivity, are discussed below (Sec. 3).

The result (2.16) can be justified[33] by direct perturbation theory calculations. We have to consider (for $d=2$) a series of Feynman graphs for a two-particle Green function introduced some time ago by Langer and Neal.[37] This sequence of graphs predominates in the $d=2$ case and gives rise to a logarithmic correction in Eq. (2.16). A similar result is obtained also in Ref. 38, where summation of the Langer–Neal graphs gives the following expression for the coefficient of two-dimensional diffusion in an external field of frequency ω:

$$D \approx D_0\left(1-\frac{\hbar}{2E\tau}\ln\frac{1}{\omega\tau}\right), \tag{2.17}$$

where $D_0=v^2\tau/2$ is the usual transport formula for the two-dimensional diffusion coefficient (v is the electron velocity, E is the electron energy, and τ is the mean free time; it is assumed that $E\tau \gg 1$). We can see that in the limit $\omega\to 0$, the correction reduces the diffusion coefficient and this may indicate complete localization in a two-dimensional system. It should be stressed that Eq. (2.17) is valid as long as the correction is small compared with the main term. In the case of a finite sample in the frequency range $D_0L^{-2} \gg \omega \gg \hbar/mL^2$, Eq. (2.17) reduces to a result of the type given by Eq. (2.16) (Ref. 38). Generalization of the analysis given in Ref. 38 to the case when allowance is made for the scattering accompanied by spin flipping, and also for the effect of a weak external magnetic field,[39,40] shows that these processes suppress the logarithmic correction in Eq. (2.17), i.e., they destroy the two-dimensional localization. Allowance for the spin-orbit interaction of an electron with impurities alters the sign of the logarithmic correction to the conductivity,[41] i.e., it gives rise to a conductivity that becomes infinite in the limit $\omega\to 0$. It is at present difficult to give a quantitative interpretation of these unexpected results.9)

The above simple description of the effects near a mobility edge has not yet been proven. Moreover, after the appearance of the work of Abrahams, Anderson, Licciardello, and Ramakrishnan,[33] a numerical calculation of the conductivity was made by Lee[42] using the Anderson model and this calculation was in a sense similar to an analysis of the Kondo problem carried out by Wilson.[23] Without going into details, we shall simply point out that because of the special scheme of scaling transformations (renormalization group) in real space, Lee was able to calculate the conductivity for "computer" samples of much larger dimensions than in Refs. 25–29. In his calculations the effective number of sites was $N=256^2$! Nevertheless, these calculations reproduced almost exactly the results of earlier studies.[27,28] According to Lee,[42] the minimum metallic conductivity does exist in the $d=2$ case and the constant in Eq. (2.6) is 0.13. Moreover, Lee was able—in a sense—to calculate also the "Gell-Mann–Low function" $\beta_{d=2}(g)$. His results are given in Fig. 5. We can see that there is a clear tendency for a kink and intersection of the abscissa, and this is followed by $\beta_{d=2}(g)=0$, i.e., "nonphysical" behavior predicted and this is represented by the dashed curve in Fig. 4; this behavior ensures the existence of minimal metallic conductivity.10)

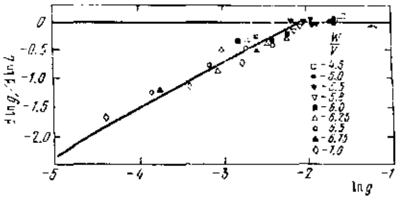

FIG. 5. Gell-Mann–Low function $\beta_{d=2}(g)$ found numerically by Lee.[42]

The question whether the one-parameter scaling scheme of Abrahams, Anderson, Licciardello, and Ramakrishnan[33] is valid is thus still open. It is equally unclear whether the renormalization group of Lee, associated with a numerical calculation, can reveal such fine logarithmic effects which result in complete localization in a two-dimensional space. In this connection, we should mention an earlier investigation,[43] where a different scheme of numerical calculations yielded a continuous metal–insulator transition for $d=2$, contradicting the existence of the minimum metallic conductivity. However, no indications were obtained there for a complete localization in a two-dimensional space. Stein and Krey[32] also found mobility edges in two-dimensional lattices but they pointed out that the precision of their calculations was clearly insufficient to detect weak logarithmic effects. In a later investigation,[136] the same authors made detailed numerical calculations of the conductivity of two-dimensional lattices and of a three-dimensional lattice of the diamond type. Their calculations were made by direct application of the Kubo formula for the conductivity expressed in terms of exact wave functions of an electron in an Anderson lattice, employing a recurrent algorithm proposed in Ref. 137. The results obtained *supported* the existence of the minimum metallic conductivity in the $d=2$ and $d=3$ cases. For $d=2$, Stein and Krey[136] obtained a universal value $\sigma_{\text{m.m.}}=(0.11\pm 0.02)e^2/\hbar$, whereas for $d=3$ they found that $\sigma_{\text{m.m.}}=(0.07\pm 0.01)e^2/\hbar a$, where a is the lattice constant. Stein and Krey[136] made an attempt to detect a logarithmic dependence of the conductivity on the dimensions of the investigated system in the quasi-metallic region assuming that $d=2$, which was postulated in Ref. 33. Such a dependence was not observed

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9) The influence of these processes was also considered in a recent study,[142] where similar results were obtained.

10) "Down with the gang of four!"

within the limits of precision of the numerical method as the dimensions of the system were increased from 50×50 to 100×100. However, this did not exclude completely the possibility that a logarithmic dependence may be observed for large systems.[11)]

Haydock[138] used a recurrent method[137] in an analytic study of the behavior of the wave functions of systems with a weak disorder. In the $d=2$ case he found that complete localization was possible for any disorder no matter how weak. Interesting suggestions were made concerning a possible physical meaning of the mobility edges obtained in the majority of numerical and analytic calculations for $d=2$. In the $d=2$ case these edges probably separated regions of exponentially localized states distributed, for example, at the band edges, from regions of localized states with a power-law attenuation of the wave functions with distance.[26] Complete localization is then retained for $d=2$ but the transition to the quasimetallic region acquires a clearer meaning, which should help in removing some of the contradictions mentioned above.

3. CONDUCTIVITY OF THIN WIRES AND FILMS

We shall now consider the experimental manifestations of the localization effect. The recent progress has been to predict a number of pronounced effects in which localization plays the dominant role and these effects differ from fairly indirect manifestations of localization considered earlier.[1,2,5] One of such pronounced effects permitting direct experimental verification is the prediction[27] of a universal minimum of metallic conductivity for two-dimensional systems of Eq. (2.6).

Another remarkable result was obtained by Thouless[44] in an analysis of the conductivity of thin metallic wires[44] (similar ideas were also put forward by Adkins).[45] This result was extremely simple: in any metallic wire with transverse dimensions much smaller than the length and with an extrinsic resistance exceeding about 10–20 kΩ all the electron states are localized. Thus, any sufficiently long metallic wire is "effectively" *an insulator* (in the meaning of the ground state, i.e., at $T=0$)! In fact, we must now return to Eq. (2.5). We have pointed out earlier that the right-hand side of Eq. (2.5) is proportional to the conductance (reciprocal of the resistance) of a finite system, i.e., the resistance of this system is

$$R_L \approx \frac{\hbar}{2e^2}\frac{W_L}{V_L}. \qquad (3.1)$$

In the case of an ordinary metallic wire we have $R_L=(1/\sigma)L/A$ (L is the length of the wire and A is its cross section), which is proportional to its length. Then, $(1/\hbar)V_L \propto D/L^2$ represents the reciprocal of the diffusion time of an electron along the whole length of the wire. Hence, we find that [$N(E)$ is the density of states per unit volume]

$$V_L \sim \hbar\frac{D}{L^2} \sim \frac{\hbar}{2e^2}\frac{\sigma}{L^2}\frac{1}{N(E)} \sim \frac{\hbar}{2e^2}\frac{\sigma}{L}A\frac{1}{LA}\frac{1}{N(E)} \sim \frac{\hbar}{2e^2}\frac{W_L}{R_L},$$

which gives Eq. (3.1). Therefore, it we select a sufficiently long wire, we can easily satisfy the condition $W_L/V_L > (V_L/W_L)^{-1}_{\max}$, i.e., we can achieve complete localization. Beginning from this length, the resistance increases exponentially with an increase in the length of the wire. The length in question is governed by the condition $R_L > \mathrm{const}\cdot\hbar/e^2$, where the constant is[44] ~2–4, i.e., $R_L > 8$–16 kΩ. This surprising result requires an explanation. One must point out the conditions when the effect can become observable. Naturally, if the temperature is sufficiently high, the localization of electron states is unimportant because scattering by phonons (and other forms of inelastic scattering) results in transitions between localized states well before an electron diffuses over a distance of the order of the localization length and "learns" that it is localized. However, cooling freezes out the inelastic scattering processes. The localization length in our case is of the order of the length of a wire whose resistance exceeds $2\hbar/e^2$. If we use the Drude formula $\sigma=(ne^2/m)\tau=(e^2/\hbar)p_F^2 l/3\pi^2$ (p_F is the Fermi momentum of an electron), it follows directly from this condition that

$$R_{\mathrm{loc}} \propto \frac{1}{\pi^2}Ap_F^2 l. \qquad (3.2)$$

An electron diffuses over this distance in a time

$$\tau_{\mathrm{diff}} \propto \frac{R_{\mathrm{loc}}^2}{D} \sim \frac{1}{\pi^4}A^2 p_F^2 \tau, \qquad (3.3)$$

where the diffusion coefficient is $D \approx (p_F^2/3m^2)\tau$. Localization of electron states becomes manifest if

$$\tau_{\mathrm{inel}} > \tau_{\mathrm{diff}}, \qquad (3.4)$$

where $1/\tau_{\mathrm{inel}}$ is the frequency of the inelastic scattering events. This frequency is usually proportional to some power of the temperature T^p (p is an integer). For example, the frequency of electron–phonon collisions in contaminated samples is proportional to T^4, whereas in the electron–electron scattering case we have proportionality to T^2 (Ref. 44). It then follows from Eqs. (3.3) and (3.4) that the temperature at which localization becomes manifest is inversely proportional to $\sqrt{A}$ in the case when scattering by phonons predominates, and inversely proportional to A when the electron–electron scattering processes are more important. Below the relevant temperature an electron may diffuse over a distance of the order of R_{loc}, but not further as long as a phonon (or another electron) does not produce a transition to a different state. Consequently, in this range of temperatures the resistance increases as a result of cooling proportionally to some reciprocal of the power of temperature: $T^{-p/2}$ (Ref. 44). Cooling to temperatures corresponding to the energy separating random electron levels results in a transition to an exponential temperature dependence of the conductivity. However, the estimates of Thouless for a wire of $A \approx 2.5\times10^{-11}$ cm² cross section, an impurity-controlled mean free path $l \approx 5\times10^{-8}$ cm, and $p_F/\hbar \approx 1.2\times10^{8}$ cm⁻¹ give $R_{\mathrm{loc}} \approx 0.012$ mm, and the condition (3.4) is already satisfied at temperatures of the order of 1 °K for reasonable estimates of the inelastic scattering frequen-

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[11)] It is pointed out in Ref. 143 that in the one-dimensional case one should treat with caution the average Kubo formula for the conductivity; further discussions of this problem are given in Refs. 144 and 145.

cies.[12)]

We shall now discuss a two-dimensional case when again complete localization occurs according to Ref. 33, i.e., when the static conductivity vanishes at $T=0$. The conductivity at $T\neq 0$ is discussed in Ref. 46. We shall again carry out a qualitative analysis in the spirit of Thouless.[44] At $T\neq 0$, an electron subject to inelastic scattering processes may diffuse over a distance[13)]

$$L_1^2 \propto D\tau_{\text{inel}} = \frac{1}{2}\, l l_{\text{inel}}, \tag{3.5}$$

where l_{inel} is the mean free path governed by inelastic processes. Over a distance given by Eq. (3.5) the coherence of electron states is lost (an electron loses information on its own state). This length can be regarded as the effective length governing the conductivity given by the two-dimensional formula (2.16) valid at finite temperatures. It is possible that some other distance is the effective length.[46] In fact, the processes of inelastic scattering result in broadening of electron levels by an amount of the order of $\hbar/\tau_{\text{inel}}$. Then, the discrete nature of the levels with $W_L \propto N^{-1}(E)$ [see Eq. (2.2) for $d=2$] is unimportant when $W_L \propto \hbar/\tau_{\text{inel}}$ and this determines the length:

$$L_2^2 \sim \frac{1}{\hbar}\,\tau_{\text{inel}} N^{-1}(E). \tag{3.6}$$

The lengths L_1 and L_2 are proportional but not equal: $(L_1/L_2)^2 \propto \sigma\hbar/e^2$. Bearing in mind that $\tau_{\text{inel}} \propto T^{-p}$, we find from both estimates that $L_{\text{eff}}^2 \propto T^{-p}$ and it follows from Eq. (2.16) that the temperature dependence of the conductivity of a two-dimensional film is[46]

$$\Delta\sigma(T) \sim \text{const}\cdot\frac{e^2}{\hbar}\ln\frac{T}{T_0}, \tag{3.7}$$

i.e., the conductivity decreases logarithmically as a result of cooling. Similar behavior follows from generalization of the work in Ref. 38 to the case of finite temperatures. Further cooling should result in a transition from the logarithmic dependence (3.7) in the "metallic" region to the exponential temperature dependence of the conductivity in the "insulating" region.

We shall now review the experiments designed deliberately to check these theoretical predictions. We shall begin by considering evidence in favor of 30 000 Ω/□ is the maximum resistance of a metallic film.[27] The first indication of the existence of such a limit was in fact obtained back in 1914 (Ref. 47). A study was then made of the resistance of platinum films as a function of their thickness, governed by the duration of evaporation. The resistance increased on reduction in the thickness (reduction in the deposition time) and in all cases did not exceed a value of about 28 000 Ω/□, which was followed by an abrupt increase to more than 10^9 Ω/□. These results were later rediscovered by Licciardello[48] and interpreted by him as supporting Eq. (2.6). Similar results were obtained more recently.[49,50] In particular, Liang *et al.*[49] studied the conductance of Bi films; the dependence of the conductance of the film thickness is plotted in Fig. 6. We can see that there is a jump in the conductance in the region of 10^{-4} Ω^{-1}. Similar results were also obtained by Anderson[50] for Au films.

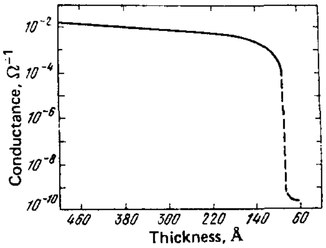

FIG. 6. Dependence of the conductivity of a Bi film on its thickness.[49]

One of the first investigations designed deliberately to check the predictions of Ref. 27 was reported by Vul *et al.*,[51] who studied conduction in thin high-conductivity layers formed at grain boundaries in germanium bicrystals.[14)] A two-dimensional network of edge dislocations formed on these boundaries and such a network represented (in germanium) a negatively charged surface of partly filled bonds with adjoining layers with p-type conduction and a thickness of a few tens of angstroms. The electrical conductance of such layers was investigated by Vul *et al.*[51] as a function of the grain misorientation angle. A transition from the metallic to the activated conduction was observed at $\sigma_{m.m} \sim 4\times 10^{-5}$ Ω^{-1}, in agreement with the estimates of Ref. 27.

The most thorough investigation of the conductance of metallic films carried out with the aim of checking the predictions of Ref. 27 was reported by Dynes, Garno, and Rowell.[52] They studied films prepared by the evaporation of Pb, Sn, Au, Al, or Cu on a substrate whose temperature was kept near 4.2 °K. The conductance was measured directly in the apparatus used to evaporate the films, which made it possible to increase gradually the film thickness until it exhibited

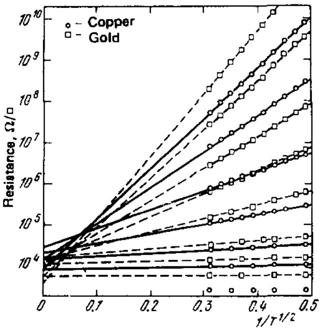

FIG. 7. Temperature dependences of the resistance of thin copper and gold films. The results demonstrate a transition from the activated to the metallic conduction near 30 000 Ω (Ref. 52).

[12)] These estimates have been found to be overoptimistic[146]: in the case of real systems the values of τ_{inel} are considerably smaller.

[13)] The length $L_1 \propto \tau_{\text{inel}}^{1/2} \propto T^{-p/2}$ determines, according to Thouless,[44] the above-mentioned temperature dependence of the resistance of a one-dimensional wire: $\propto T^{-p/2}$.

[14)] See also a later investigation reported in Ref. 147.

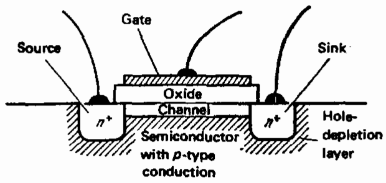

FIG. 8. Field-effect transistor with an n-type channel. An inversion layer is produced by applying an external potential to the gate electrode.

metallic conduction. The transition from the activated to the metallic conduction mechanism occurred near a resistance of 30 000 Ω/□. Typical data obtained in Ref. 52 are reproduced in Fig. 7. In the region of activated conduction the resistance of Sn films varied as $\exp(1/T)$, whereas in the case of Au and Cu films the dependence was $\exp(1/\sqrt{T})$. Dynes *et al.*[52] pointed out that films with such a high resistance could hardly be homogeneous, i.e., they probably consisted of metal "islands" linked by narrow channels or separated by tunnel barriers. The predictions of Ref. 27 are generally applicable to a homogeneous system, but according to the ideas put forward in Ref. 48, they are valid also in the case of inhomogeneous systems.

Another system which would seem to be ideally suited for investigating two-dimensional conduction is a metal–oxide–semiconductor field-effect transistor (MOSFET) with an inversion layer of carriers (Fig. 8). This layer is created by the application of an external potential to the gate electrode and the conductance of the layer is measured directly between the source and sink. Variation of the gate voltage makes it possible to alter (within wide limits) the properties of the inversion layer and, in particular, the carrier density in the layer. An important property of such a system is the fact that the carrier density can be found from electrostatic calculations and it does not have to be deduced from, for example, the Hall effect. A detailed review of the experiments and physical phenomena in inversion layers was published by Adkins,[53] and we shall not discuss them in detail. We shall simply mention that the results of such experiments generally *disagree* with the predictions of the theory of localization in general and with the conclusions of Ref. 27 in particular. The transition from the activated to the metallic conduction may occur at conductance values considerably higher than 3×10^{-5} Ω^{-1} (Fig. 9). However, results in agreement with Ref. 27 are sometimes obtained (Fig. 10). Clearly, these contradictory results arise from the fact that the role of the electron–electron correlations in inversion layers is important[48,53] and this, in particular, gives rise to phenomena of the Wigner crystallization type.[53] Therefore, it is possible that inversion layers are not a very convenient object for verifying the theory of localization, which unfortunately does not allow for the role of the electron–electron interaction.

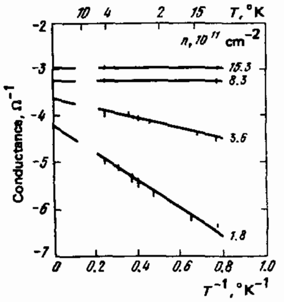

FIG. 9. Typical temperture dependences of the conductance of an inversion layer.[53] The different dependences correspond to different electron densitites in the inversion layer.

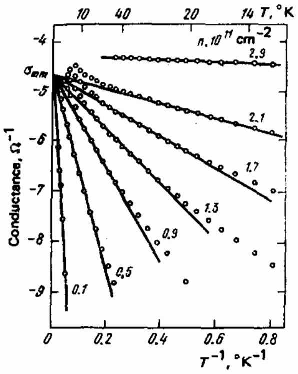

FIG. 10. Behavior of the conductance of inversion layers with low electron densities.[53]

The first experimental investigations designed deliberately to test the predictions of Thouless[44] on the conductivity of thin metallic wires have appeared recently. The very first studies[54,55] failed to observe the effect. The most convincing evidence in support of the localization in wires was obtained by Giordano, Gilson, and Prober.[56] They prepared samples by an original lithographic method (Fig. 11). A glass substrate, half-covered by a metal film [Fig. 11(a)], was bombarded with Ar$^+$ ions and this produced a step in the glass [Fig. 11(b)]; then, the metal film was removed by chemical means [Fig. 11(c)]. Next, a new metal film was evaporated [Fig. 11(d)] and then the sample was subjected again to the bombardment with Ar$^+$ ions incident at an angle such that the metal behind the step remained in the shadow [Fig. 11(d)]. This produced a thin "wire" [Fig. 11(e)]. An investigation of such wires with a scanning electron microscope showed that they were continuous strips of uniform cross section. The wires used in that investigation were made of $Au_{60}Pd_{40}$ films formed by two different evaporation methods, which yielded films with a resistivity 3.7×10^{-4} Ω.cm (and ratio of the resistances at room temperature and at 12 °K amounting to 1.03), which were referred to as "contaminated" films, as well as films with a resistivity of

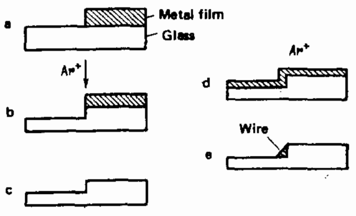

FIG. 11. Method of preparation of thin wires.[56]

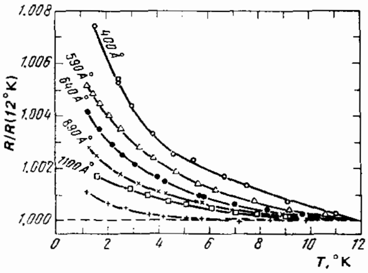

FIG. 12. Temperature dependences of the resistance of wires of different cross sections. The results given for each sample are normalized to the resistance at 10°K. The numbers above the curves give the values of $\sqrt{A}$. The lowest curve shows the behavior of a continuous film.[56]

1.0×10^{-4} Ω.cm (and a resistance ratio 1.06), which were referred to as "pure" films. The cross sections of the wires varied from 1×10^{-11} cm² to 3×10^{-10} cm². The values of the cross section A were deduced from the known lengths and resistances of the wires and the values of A found in this way were in good agreement with those expected from the known heights of the steps and film thicknesses. The $Au_{60}Pd_{40}$ films had thicknesses from 200 to 1000 Å before the Ar^+ bombardment. The resistance of the wires was from 15 to 500 kΩ and their length was from 90 to 450 μ.

Figure 12 shows the temperature dependences of the resistance of various contaminated wires with different cross sections. Figure 13 demonstrates how the resistance rises (at a fixed temperature $T = 1.5$ °K) as a function of the cross section A in the case of contaminated and pure wires. The increase in the resistance of a continuous film is subtracted here (the origin of this increase is not clear and it may even be due to two-dimensional localization).[15)] We can see that the resistance increases in accordance with the law A^{-1} and that the resistance is higher for the contaminated wires. Thus, the results obtained are in qualitative agreement with the predictions of the theory of Thouless.[44]

The temperature dependence of the resistance of a typical wire is shown in Fig. 14. Here again the effect of a continuous film is subtracted. We can see that the temperature dependence is approximately logarithmic

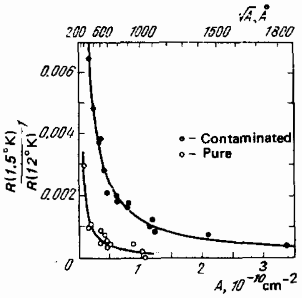

FIG. 13. Resistance plotted as a function of the cross section for contaminated (black dots) and pure (open circles) samples. The curves are proportional to A^{-1} (Ref. 56).

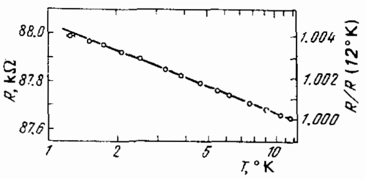

FIG. 14. Resistance plotted as a function of the logarithm of the temperature of a contaminated wire with $A = 3.6\times10^{-11}$ cm² ($\sqrt{A} = 590$ Å).[56]

although, according to Ref. 56, we cannot exclude the possibility that the dependence is of the $T^{-1/2}$ type. This behavior is not in agreement with the power-law increase in the resistance (of the T^{-2} type due to the scattering by phonons) as a result of cooling, predicted in Ref. 44. This disagreement may reflect insufficient understanding of the inelastic scattering processes and not shortcomings of the theory of localization, especially as the dependence of the effect on the geometric dimensions (cross section A) clearly supports the theory. It should be stressed that the ideas of Thouless on the power-law rise of the resistance are somewhat qualitative.[16)]

We shall now consider the results obtained in another investigation[57] designed deliberately to check the predictions of Thouless.[44] "Wires" studied in Ref. 57 were actually thin films of length L, width B, and thickness C, where $L \gg B \gg C$.[17)] Such samples should either exhibit one-dimensional behavior predicted by Thouless[44] or two-dimensional behavior predicted by Abrahams, Anderson, Licciardello, and Ramakrishnan.[33] Dolan and Osheroff[57] investigated films whose resistance was $R_{\square} < 30\,000$ Ω/□ (known as the Licciardello–Thouless limit[27]). The behavior of these films should be two-dimensional if the resistance is below this limit (metallic region) but $L_1 \propto \sqrt{D\tau_{\text{inel}}} < B$ and/or $L_2 \propto \sqrt{\tau_{\text{inel}}/\hbar}/\sqrt{N(E)} < B$ [compare with Eqs. (3.4), (3.5), and Ref. 46], i.e., under the conditions such that an electron "does not know" that it is in a "one-dimensional" wire.

Films of the composition 58 wt.% Au and 42 wt.% were prepared on substrates which were mostly made of polished sapphire (and kept at room temperature). Long and narrow strips were obtained by a modification of the lithographic method suggested in Refs. 58 and 59. The film thicknesses were $(2-4)\times10^{-7}$ cm and, therefore, they almost certainly consisted of separate metal islands, the resistance being governed by the tunneling between these islands.[52] This inhomogeneity was unimportant provided the films were homogeneous over distances of the order of R_{loc} of Eq. (3.2), as well as L_1, L_2, and B. Measurements were carried out down to temperatures of $\sim10\times10^{-3}$ °K. Nonmetallic behavior was exhibited by the temperature dependences of the

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[15)] This is confirmed by recent investigations.[146]

[16)] As pointed out earlier, simple estimates[44] clearly exaggerate the value of τ_{inel}, which explains the relative difficulty of observation of the localization effects and their small magnitude.[146]

[17)] Samples with L = 2–7 mm, B = 0.1–1 μ, and $C \approx 3\times10^{-7}$ cm were investigated.

resistance and by the nonlinear dependence of the resistance on an external electric field (voltage). These resistance dependences could be described by the following formulas:

$$R(T,\ E)=R(T_0,\ E)\left(1-S_T\ln\frac{T}{T_0}\right) \tag{3.8}$$

in the case of low fields E, and

$$R(T,\ E)=R(T,\ E_0)\left(1-S_V\ln\frac{E}{E_0}\right) \tag{3.9}$$

in the case of low values of T, where T_0 and E_0 are arbitrary normalization points, and S_T and S_V are experimentally determined parameters.

Figures 15 and 16 give the experimental results obtained for one of the samples in Ref. 57.

This behavior was interpreted in Ref. 57 in the spirit of predictions of Abrahams, Anderson, Licciardello, and Ramakrishnan,[33] i. e., as associated with complete localization in a two-dimensional system [compare with Eq. (3.7)]. It was concluded that the investigated samples which exhibited behavior of the type described by Eqs. (3.8) and (3.9) were two-dimensional. The nonlinear dependence on the field of Eq. (3.9) was explained in Ref. 46 by a mechanism involving electron heating. It should be pointed out that no nonlinear dependences on the external field were observed in Ref. 56; it was concluded there that this difference was due to inhomogeneities of the samples in Ref. 57. A surprising result reported in Ref. 57 was the saturation of the logarithmic dependence of $R(T, 0)$ at temperatures below 70×10^{-3} °K. This could not be explained theoretically on the basis of the interpretation of Abrahams *et al.*[33] The effect in question could be due to quite different factors (for example, the role of electron–electron correlations).

The results discussed above were obtained for films with a resistance of 1000–5000 Ω/□ and a width of the order of 1 μ. In the case of two samples with a resistance exceeding 10 *k*Ω/□ and several samples with a lower resistance but a width of the order of 0.1 μ the resistance $R(T, 0)$ increased exponentially with $1/T$, but saturation was again observed in the region of 70×10^{-3} °K. An exponential dependence of the localization at such very low resistances was another aspect that could not be explained from the theoretical point of view.

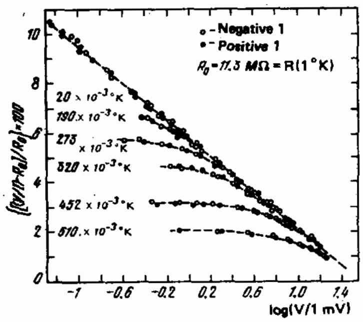

FIG. 15. Dependence of $\Delta R=[(V/I)-R_0]/R_0$ on the logarithm of an external voltage ln V plotted for different temperatures. Here, I is the current, V is the voltage, and R_0 is the resistance of a sample at T = 1°K (Ref. 57).

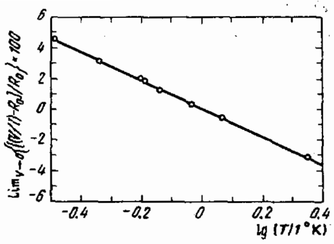

FIG. 16. Dependence of the resistance at low values of V, deduced from Fig. 15, on ln T (Ref. 57).

There have been preliminary reports[60] of observations of the logarithmic temperature dependence of the resistance of inversion layers in field-effect transistors.[18)] According to Ref. 60, an additional analysis of the results of Dynes *et al.*[52] obtained for the "metallic" region revealed a similar logarithmic dependence. This slight rise of the resistance on cooling was not explained earlier, because the results were plotted on a logarithmic scale (of the conductance; see Fig. 7).

It is thus clear that the available experimental data indicate a qualitative agreement with the predictions of the theory of localization, but—in our opinion—these results are insufficient for reliable selection of a specific theoretical model. We cannot exclude also the possibility of a different interpretation. We should mention in this connection the work of Al'tshuler and Aronov[61] proposing an original mechanism of the increase in the resistance as a result of cooling, associated with interference between the Coulomb interaction of electrons and the impurity scattering. Generalization of the treatment given there to the $d=2$ case gives rise to corrections to the conductance, which are logarithmic functions of temperature *of the same type* as the effects of two-dimensional localization.[62,63] This demonstrates the need for further theoretical analysis and new experiments. In particular, it would be very useful to carry out measurements in a magnetic field, including those of the Hall effect in the "metallic" range of two-dimensional films. Measurements of the Hall effect in inversion layers have in fact stimulated a proposal for a new interpretation of the phenomena in such systems.[53] The negative magnetoresistance exhibited by these systems[64] has been attributed[41] to the suppression of two-dimensional localization by an external magnetic field and by the effects of the spin–orbit scattering as well as the scattering accompanied by spin flipping.[39-41] We should mention, however, a recent investigation[139] of the resistance of wires made of an amorphous W–Re alloy. These wires were produced from amorphous W–Re films of 50 Å thickness by electron-beam lithography.[140] The wires were divided into two groups in accordance with their width; from 700 to 5000 Å and 2×10^{-3} cm. A detailed investi-

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[18)] The results of experiments on inversion layers in Si were reported in detail in Ref. 141. They were very similar to the results given in Ref. 57.

gation confirmed in practice all the results of Ref. 56 and demonstrated a nonmetallic temperature dependence of the conductance of these wires[19] as well as a dependence of the conductance on the wire cross section for samples whose resistance considerably exceeded 10 $k\Omega$. Measurements in an external magnetic field gave particularly interesting results. Application of a 4×10^4 G field (either parallel or perpendicular to a wire) resulted at $T<20\,^\circ\mathrm{K}$ (i.e., in the range of temperatures where the localization effects appeared) in a fairly strong *rise* of the resistance. The effect of a perpendicular field was greater. These results supported the conclusion that an external magnetic field aids the manifestation of localization effects.[20]

4. DOES SCALING EXIST AT A MOBILITY EDGE?

The presentation of the theory in the preceding sections was fairly elementary. We will try to avoid a false impression that the theory of localization is simple by reviewing briefly attempts to construct a "consistent" scaling theory of localization in the vicinity of a mobility edge and by reviewing the problems encountered in this approach. A scaling theory is understood to be a consistent derivation and verification (or rejection) of assumptions of the elementary theory presented in Sec. 2, calculation of the critical index of the localization radius and of other critical indices describing the behavior of the correlation functions of the electron system (if such indices can be introduced at all), calculation of such functions (Green's functions), etc. In other words, it would be desirable to "raise" the scaling theory of localization from the qualitative level of the Kadanoff constructions[20] to the level of a quantitative theory and to the level of the modern theory of critical phenomena.[21-24] We shall show that this task is far from being complete and this is due to the fact that difficulties of fundamental nature are encountered.

The ideas for a consistent scaling description of a mobility edge were put forward soon after the appearance of the modern theory of critical phenomena and were formulated practically simultaneously by a number of authors.[65-72,24] There is now an extensive literature on various aspects of this problem.[73-86]

There are two alternative ways of constructing a theory of localization rigorously. One of them (used until recently to obtain all the main conclusions) is based on the pioneering work of Anderson[3] (Anderson approach). The main feature of this approach is its familiar departure from tradition involving a discussion of random configurations of the Green's function system of an electron in the model of (1.1) not averaged over an ensemble. In a sense, this approach considers *the most probable* Green's function.[15,16] This is due to the fact that a one-particle *averaged* Green's function does not (as is well known[3-6,12,15,18]) give, in principle, any information on a possible localization of electrons, but we can study localization by considering the divergence of a stochastic perturbation theory series for a nonaveraged one-particle Green's function.[3,12] The condition for convergence of this series determines, in particular, the position of the mobility edges in a band.[3-6,12,13] This condition is so far the only way of determining the mobility edges analytically (a somewhat different method proposed in Refs. 87 and 88 is related genetically to the same original Anderson approach).

Another approach to a theory of localization, put forward in the well-known paper of Edwards[89] (Edwards approach), uses Green's functions averaged over an ensemble of random configurations of a system. The advantage of this approach is its familiar "automatism" (availability of the diagram technique). Averaged Green's functions determine, in principle, all the physical quantities in the theory (this is not true of the most probable Green's function in the Anderson approach), but in order to investigate localization (static conductivity at $T=0$) we have to find the averaged two-particle Green's function of an electron.[89,90] This immediately makes the problem very complex. It has been solved only for the one-dimensional case.[10,11] Nevertheless, the traditional nature and convenience of this approach makes it a favorite with the majority of theoreticians.[21] Recent investigations[91,92] carried out using a self-consistent variant of the traditional method have yielded a number of interesting results (including complete localization in the $d=2$ case). However, some of these conclusions (such as the presence of a gap in the absorption spectrum at low frequencies) have met with objections.

The Anderson approach allows us to obtain quite simply scaling at a mobility edge[68] in the sense of the usual theory of critical phenomena.[21-24] This is because the spatial behavior of the Anderson most probable Green's function is governed entirely by the statistics of intersection-free paths in the investigated lattice,[3,12,93,94] i.e., it is a purely geometric problem. This is related to a summation of the perturbation theory series for V in the case of the Green's function of the Hamiltonian (1.1) allowing for multiple scattering processes.[3,12] The main approximation made after this summation is the restriction of the discussion to just one initial Green's function, although one of the results of the summation is an infinite number of similar series for the auxiliary Green's functions.[3,12] This is due to the assumption of identical stochastic properties of all these series. The higher approximations[3,12,13] relating to the statistical properties of a series and various ways of estimating its convergence clearly influence only the positions of the mobility edges in a band and are unimportant in our case. Consequently, a nondiagonal (with respect to the lattice sites) matrix element of

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[19] A subsequent investigation of W–Re wires in the superconducting state[148] made it possible to determine directly the value of τ_{inel}, which agreed well with the localization effect.

[20] A similar result was obtained in Ref. 146 for films, but no influence of a magnetic field on wires was detected.

[21] This duality in the theory of disordered systems is reflected in the dialogue between the "author" and "a theoretician" in the review by Éfros.[6]

a one-electron Green's function can be represented in the form[93,68]

$$G_{ij}(E)\sim\sum_{N=1}^{\infty}Z_N(\mathbf{R}_i-\mathbf{R}_j)\,K^{-N}F^N\left(E,\ \frac{W}{V}\right), \tag{4.1}$$

where $F(E, W/V)$ is the localization function determining the position of mobility edges in the band:[12,13]

$$F\left(E_c,\ \frac{W}{V}\right)=1, \tag{4.2}$$

and $Z_N(\mathbf{R}_i-\mathbf{R}_j)$ is the number of intersection-free paths consisting of N steps connecting a site i to a site j; K is the connectivity constant of the lattice $\ln K=\lim_{N\to\infty}N^{-1}\ln Z_N$, Z_N is the total number of intersection-free paths consisting of N steps. The values $Z_N(\mathbf{R}_i-\mathbf{R}_j)$ obtained as a result of computer calculations were used in Ref. 93. The main idea of Ref. 68 was to deal with Eq. (4.1) by applying an analytic theory of intersection-free paths based on the theory of critical phenomena proposed by de Gennes[95] and des Cloizeaux.[96] According to Refs. 95 and 96,

$$Z_N(\mathbf{R}_i-\mathbf{R}_j)\sim\int\limits_{c-i\infty}^{c+i\infty}\frac{d\tau}{2\pi i}e^{Na^2\tau}G(\mathbf{R}_i-\mathbf{R}_j;\ \tau), \tag{4.3}$$

i.e., this quantity is found by inverse Laplace transformation of $G(\mathbf{R}_i-\mathbf{R}_j;\tau)$ which is a correlation (Green's) function in the theory of critical phenomena, governed by the usual fluctuation-free Landau energy with an n-component order parameter[22-24]:

$$F=\frac{1}{2}\sum_{j=1}^{n}[(\nabla\Phi_j)^2+\tau\Phi_j^2]+\frac{1}{8}g_0\left(\sum_{j=1}^{n}\Phi_j^2\right)^2; \tag{4.4}$$

here, $g_0>0$, which corresponds to "repulsion" of intersection-free paths and a is the lattice constant. The statistics of intersection-free paths is derived from Eqs. (4.3)–(4.4) by going to the limit $n\to 0$ (Refs. 95 and 96). This eliminates loop graphs, which do not occur in the problem of intersection-free paths. Using Eq. (4.3), we obtain directly from Eq. (4.1)

$$\left.\begin{aligned}G_{ij}(E=0)\big|_{\frac{W}{V}\geqslant\left(\frac{W}{V}\right)_c}&\sim G\left(\mathbf{R}_i-\mathbf{R}_j;\ \frac{W-W_c}{W_c}\right),\\ G_{ij}(E)\big|_{E\sim E_c;\ \frac{W}{V}\leqslant\left(\frac{W}{V}\right)_c}&\sim G\left(\mathbf{R}_i-\mathbf{R}_j;\ \frac{E-E_c}{E_c}\right),\end{aligned}\right\} \tag{4.5}$$

i.e., the spatial behavior of the Anderson Green's function in the theory of localization is governed by the correlation function of the standard theory of critical phenomena (4.4) (with $n=0$) and the mobility edge corresponds to the phase transition point. Near the mobility edge [i.e., for $W/V\gtrsim(W/V)_c$ or for $E\lesssim E_c$] the Green's function decreases exponentially with the distance:

$$G_{ij}\propto\exp\left(-\frac{|\mathbf{R}_i-\mathbf{R}_j|}{R_{\text{loc}}}\right),\qquad|\mathbf{R}_i-\mathbf{R}_j|\gg R_{\text{loc}}, \tag{4.6}$$

where the localization radius is

$$R_{\text{loc}}\sim a\left[\frac{(W/V)-(W/V)_c}{(W/V)_c}\right]^{-\nu}, \tag{4.7}$$

for localization at the center of a band and

$$R_{\text{loc}}\sim a\left[F'\left(E_c,\ \frac{W}{V}\right)E_c\right]^{-\nu}\left(\frac{E_c-E}{E_c}\right)^{-\nu} \tag{4.8}$$

for localization in the $E\lesssim E_c$ case. The critical index of the localization radius ν is identical with the critical index of the correlation length in the theory of critical phenomena. In terms of the Wilson ε expansion[22] ($d=4-\varepsilon$, $n=0$) we have

$$\nu\approx\frac{1}{2}+\frac{\varepsilon}{16}+\frac{15\varepsilon^2}{512}+\ldots,\approx 0.592\ \text{ for }\ \varepsilon=1\ (d=3). \tag{4.9}$$

A numerical calculation of the statistics of intersection-free paths gives $\nu\approx 0.6$ (Ref. 93). For $W/V=(W/V)$ or $E=E_c$, we have

$$G_{ij}\sim|\mathbf{R}_i-\mathbf{R}_j|^{-(d-2+\eta)}, \tag{4.10}$$

where the index η obtained within the framework of the ε expansion ($d=4-\varepsilon$, $n=0$) is[22]

$$\eta\approx\frac{\varepsilon^2}{64}\left(1+\frac{17}{16}\varepsilon+\ldots\right)\approx 0.032\ \text{ for }\ \varepsilon=1\ (d=3). \tag{4.11}$$

Such very small values of η mean, in particular, that the power-law localization[68] proposed for the Anderson model in Ref. 26 is impossible. The values of the index ν can also be very important. According to Mott,[97] a conductivity jump at a mobility edge (i.e., the minimum metallic conductivity) can occur if the inequality $\nu>2/d$ is satisfied. It follows from Eq. (4.10) that $\nu<2/d$; this may mean a continuous fall of the conductivity to zero, analogous to that discussed in connection with Eq. (2.13).

Unfortunately, since the Anderson Green's function does not determine (as pointed out above) physical quantities such as the density of states or the conductivity, we cannot go far beyond the results obtained. Moreover, although the localization radius does occur in several formulas (for hopping conduction, high-frequency conduction between localized states, etc.),[1] it is not very clear whether there are direct experimental methods for determining it and the index ν; this applies even more to η. It should be mentioned only that an increase in R_{loc} on approach to a mobility edge may affect the diamagnetic susceptibility of a system.[98]

We can find directly the most important physical quantities by turning to the calculation of the averaged Green's functions (Edwards approach). By way of example of the problems encountered here we can describe the familiar model of an electron in a Gaussian random field (see, for example, Refs. 99–102), which proceeds from the usual problem of an electron in a system of randomly distributed point scatterers in the limit $\rho\to\infty$, $V\to 0$, $\rho V^2\to$ const, where ρ is the density of the scatterers and V is the scattering amplitude.[101] It is easy to formulate this model in terms of a language similar to the theory of critical phenomena,[24,67,68,70-72,75,76] or, more exactly, similar to the unstable field theory of a specific type. The main conclusions reached in these investigations largely overlap and we shall follow the treatment in Refs. 70 and 76. It is found that the averaged Green's function of an electron in a Gaussian random field (with a "white noise" correlation function) can be defined as the Green's function of a Euclidean scalar field theory with the following Lagrangian [compare with Eq. (4.4)]

$$\mathscr{L}=\frac{1}{2}\sum_{j=1}^{n}\left[\frac{1}{2m}(\nabla\Phi_j)^2-E'\Phi_j^2\right]-\frac{1}{8}\rho V^2\left(\sum_{j=1}^{n}\Phi_j^2\right)^2, \tag{4.12}$$

$$E'=E\pm i\delta,$$

where m is the mass of an electron and E is its energy. The number of field components approaches the limit

$n \to 0$, which again excludes "loop" graphs which do not occur in the Edwards diagram technique.[89] It should be pointed out that Eq. (4.12) does not contain random parameters. This is an "effective" Lagrangian, which yields directly the diagram technique for the averaged Green's functions. A Green's function is defined by the functional integral

$$G(\mathbf{r}-\mathbf{r}' | g = -\rho V^2) = -\frac{1}{Z} \lim_{n \to 0} \frac{1}{n} \sum_{j=1}^{n} \int \{\delta\Phi(\mathbf{r})\} \Phi_j(\mathbf{r}) \Phi_j(\mathbf{r}') e^{-S[\Phi]}$$

$$\equiv \lim_{n \to 0} \frac{1}{n} \sum_{j=1}^{n} \langle \Phi_j(\mathbf{r}) \Phi_j(\mathbf{r}') \rangle, \tag{4.13}$$

where

$$S[\Phi] = \int d^d r \mathscr{L}(\mathbf{r}) \tag{4.14}$$

is the action in the field theory in Eq. (4.13) and

$$Z = \int \{\delta\Phi(\mathbf{r})\} e^{-S[\Phi]}. \tag{4.15}$$

The functional integral (4.13), corresponding to the Lagrangian (4.12), diverges in the case of formal calculation because of the "incorrect" sign of the coupling constant $g = -\rho V^2 < 0$ (attraction!). This reflects the familiar instability of the ground state in field theory [Eq. (4.12)].[103] Therefore, the functional integral should be regarded as *an analytic continuation* with respect to the coupling constant from arbitrary values $g > 0$ to $g = -\rho V^2 < 0$. Analytic properties of the Green's functions in the field theory $g\Phi^4$ in the complex plane of the coupling constant are given by the following dispersion relationship applicable to this constant:[104-106]

$$G(g) = \frac{1}{\pi} \int_{-\infty}^{0} dz \frac{\Delta(z)}{z-g}, \tag{4.16}$$

where

$$\Delta(g) = \frac{1}{2i} [G(g+i\varepsilon) - G(g-i\varepsilon)] = \operatorname{Im} G(g), \tag{4.17}$$

i.e., the Green's functions in this theory are analytic with respect to the coupling constant in the complex plane with a cut along the negative part of the real axis and $\Delta(g)$ is a discontinuity at the cut (which differs from zero for $g < 0$) which in this approach governs all the principal properties of the Green's functions.

The functional integral (4.14) can be analyzed by the steepest-descent method.[108-110] This can be done by considering first the classical field equations corresponding to the Lagrangian (4.13):

$$\frac{1}{2m} \Delta\Phi_j = -E\Phi_j - \frac{1}{2} \rho V^2 \Phi_j \Big(\sum_{j=1}^{n} \Phi_j^2 \Big). \tag{4.18}$$

In the method of steepest descent we shall be interested in the solutions of Eq. (4.18) with *finite action* (4.14) (instantons).[106-110] In this problem the important solutions of Eq. (4.18) are spherically symmetric of the type

$$\Phi_j(r) = \Phi_0(r)\,\mathbf{u}, \tag{4.19}$$

where $\mathbf{u}^2 = 1$ is a unit vector in "isotopic" spin space of the $O(n)$-symmetric field theory under discussion. In this model such solutions exist for $d < 4$ if $E < 0$ (Refs. 75 and 76). These solutions have the form

$$\Phi_0(r) = \sqrt{\frac{2|E|}{\rho V^2}} \chi(t),$$
$$r = \frac{t}{\sqrt{2m|E|}}, \tag{4.20}$$

where χ and t are dimensionless variables. The explicit form of $\chi(t)$ is known only for $d = 1$ but it is easy to show that

$$\Phi_0(r) \underset{r \to \infty}{\propto} \frac{(2m|E|)^{-1/4}}{r^{(d-1)/2}} \exp[-(2m|E|)^{1/2} r], \tag{4.21}$$

$$\Phi_0(r) \underset{r \to 0}{\to} \text{const}, \quad \frac{d\Phi_0(r)}{dr}\Big|_{r=0} = 0.$$

If $E > 0$ there is only a trivial solution of Eq. (4.18) with a finite (equal to zero!) action $\Phi_0(r) \equiv 0$. Therefore, the steepest-descent method of calculation of the functional integral (4.13) reduces in this case to the conventional perturbation theory. In the space with $d = 4 - \varepsilon$ dimensions there is a dominant sequence of diagrams known as the parquet pattern.[111,112] If we analyze the problem in this parquet approximation,[70] we find that because of the incorrect sign of the coupling constant the effective vertex (a "four-leg" vertex) has an apparent pole in the theory under discussion and this pole corresponds to the inapplicability of perturbation theory in the energy range ($\hbar = 1$)

$$E \leqslant E_{sc} = \frac{1}{2ma^2} \left(\frac{u}{4-d} \right)^{/(4-d)}, \tag{4.22}$$

where $u = (m^2/2\pi^2)\rho V^2 a^{4-d}$ is a dimensionless interaction constant and a is a constant with dimensions of length (the shortest length in the problem, corresponding to the correlation radius of a random field[70,76]). The quantity E_{sc} is an exact analog of the "Ginzburg" critical range in the theory of phase transitions.[24,113] The existence of such a range in the problem of an electron in a random field was first pointed out in Ref. 65. However, in contrast to the theory of critical phenomena, the transition to this energy range does not result in weakening[22-24] but in enhancement of the effective interaction[70] (transition to the tight binding range). The perturbation theory parameter is the dimensionless ratio $(E/E_{sc})^{(d-4)/2}$ and in the limit $E \to 0$ we find that perturbation theory is no longer valid and its formal application gives nonphysical results.[70,71] This has been pointed out already in Ref. 68 and was expressed in Ref. 72 in terms of the recurrent formulas of Wilson,[22,23] because of the absence of fixed points of these equations in the problem under discussion.

In the range $E < 0$ an important role is played by nonlinear solutions (instantons) given by Eq. (4.20). The occurrence of these solutions gives rise, in the calculation of the functional integral by the steepest-descent method,[75,76] to the appearance of contributions which are nonanalytic with respect to the coupling constant and are of the type ($A_d = \text{const}$)

$$\exp(-S[\Phi_0]) = \exp\left(-A_d \frac{m^{-d/2}}{\rho V^2} |E|^{2-(d/2)} \right), \tag{4.23}$$

first obtained by Zittartz and Langer.[114] In particular, in the momentum representation, we have

$$\operatorname{Im} G(E\mathbf{p} | -\rho V^2) = C(E\mathbf{p}) \frac{1}{(\rho V^2)^{(d+1)/2}} \exp\left(-A_d \frac{m^{-d/2}}{\rho V^2} |E|^{2-(d/2)} \right), \tag{4.24}$$

where $C(E\mathrm{p})$ is some function of E and p independent of the coupling constant and the power of the coupling constant in the preexponential factor is governed by the number of "zeroth modes" of the instanton solution.[75,76] Knowing $\operatorname{Im} G(Ep\,|-\rho V^2)$, which represents the jump across a cut, we can reconstruct the real part from Eq. (4.16). The method of calculation of $C(Ep)$ based on the use of the dispersion relationship for the coupling constant (4.16) and on the correspondence with the theory of critical phenomena is proposed in Ref. 76. The form of this function is determined in Ref. 75 from dimensional considerations. If $d=1$, both methods give the result for the density of states identical with the exact solution of Halperin.[115] If $d>1$, the results for the preexponential factor given in Refs. 75 and 76 are different.[22]

The criterion of validity of the results of the (4.24) type is the large value of the argument of the exponential function, which again reduces to the condition $|E| \gg E_{sc}$ (Refs. 70, 71, 75, and 76), which should be compared with Eq. (4.22). Thus, a "Ginzburg" critical range appears also on the negative energy side. It is the range $|E| \lesssim E_{sc}$ that is of interest from the point of view of the scaling theory of localization (the mobility edge of the problem in question is located there), but an analysis of the phenomena in this range of strong (or intermediate) coupling is inaccessible to the current theory. A hypothesis of the possibility of the conventional scaling behavior of the theory in this range of energies (in spite of the "incorrect" sign of the coupling constant) is put forward in Ref. 76 on the basis of the universality of a jump across a cut in the dispersion relationship (4.16) [the same function $\Delta(z, Ep)$ governs the behavior of the correlation function in the problem of an electron in a random field corresponding to $g<0$ in Eq. (4.16) for $|E| \lesssim E_{sc}$ and the usual scaling behavior of the correlation function in the theory of critical phenomena corresponding to $g>0$ in Eq. (4.16)]. This would lead, in particular, to the usual values for the critical indices (4.9) and (4.11). Then, the derivative of the density of states with respect to the energy acquires a singularity of the type encountered in the specific heat[76] in the theory of phase transitions. However, Wegner[84] suggested that such values of the indices are in conflict with the inequalities that follow from analytic properties of the Green's functions.[23]

The problem of calculating the conductivity (two-particle averaged Green's function) requires an analysis not of Eq. (4.12) but of a different effective Lagrangian of two interacting zero-component ($n \to 0$, $m \to 0$) fields[72,75] ($E' = E + i\delta$, $E'' = E - i\delta$):

$$\mathscr{L} = \frac{1}{2} \sum_{j=1}^{n} \left[\frac{1}{2m} (\nabla \Phi_j)^2 - E' \Phi_j^2\right] + \frac{1}{2} \sum_{i=1}^{m} \left[\frac{1}{2m} (\nabla \varphi_i)^2 - E'' \varphi_i^2\right] - \frac{1}{4} \omega \left(\sum_{j=1}^{n} \Phi_j^2 - \sum_{i=1}^{m} \varphi_i^2\right) - \frac{1}{8} \rho V^2 \left(\sum_{j=1}^{n} \Phi_j^2 + \sum_{i=1}^{m} \varphi_i^2\right)^2, \tag{4.25}$$

where ω is the frequency of an external field [we are speaking here of the conductivity $\sigma(\omega)$]. This Lagrangian yields directly (for $n \to 0$, $m \to 0$) a sequence of diagrams that determine conductivity in the Edwards problem.[89,90] The expression (4.25) has the $O(n) \times O(m)$ symmetry in the isotopic spin space. In the limit $\omega \to 0$ this symmetry reduces to the $O(m+n)$ symmetry.[116] The importance of this circumstance for an electron in a random field was recently stressed by Wegner[81,82] (see below).

Unfortunately, the use of perturbation theory in the problem (4.25) gives rise, as indicated by a direct generalization of the results of Ref. 116, in the parquet approximation for $n \to 0$ and $m \to 0$ to the same nonphysical singularities in the vertex parts as in the case of the simpler problem (4.12) with the one-electron Green's function, which again indicates that perturbation theory is inapplicable. This result was obtained in Ref. 72 in terms of the recurrent Wilson formulas.

The range $E<0$, $|E| \gg E_{sc}$ for Eq. (4.25) was considered in Ref. 75, where electron localization was demonstrated for this range but the conclusions were basically preliminary. It is interesting to note that the localization is associated with the appearance of zeroth modes of the instanton solution.

We can envisage also a different approach to the problem of calculation of a two-particle averaged Green's function (conductivity) based on the use of a different effective Lagrangian and of an analogy between localization and a transition to a spin glass state.[117-119] A "fluctuating field" ("order parameter") is then of tensor (matrix) nature. Unfortunately, when variants of this approach are used, the direct link to the usual Edwards diagram technique[89] is lost, contrary to the case when the effective Lagrangians (4.12) and (4.25) are used and this link is retained. The renormalization group analysis of the tensor Lagrangian proposed in Ref. 73 demonstrated the existence of a stable fixed point in the recurrent formulas corresponding to the usual critical Wilson indices ($n=0$) of Eqs. (4.9) and (4.11), but this point is inaccessible by physical variation of the parameters of the system.

It is pointed out in Ref. 74 that the effective Lagrangian[73] does not include all the possible invariants of the corresponding tensor field. An attempt is made in Ref. 74 to relate the problem of conduction in the two-dimensional Anderson model to the critical behavior of a planar XY model.[120] In particular, a conductivity jump at a mobility edge is obtained there and for the $d=2$ case it is given by

$$\sigma_{mm} = \frac{e^2}{\hbar} \frac{K(T_c)}{2\pi} = \frac{1}{\pi^2} \frac{e^2}{\hbar} \approx 0.101 \frac{e^2}{\hbar}, \tag{4.26}$$

which is in good agreement with the Licciardello–Thouless result of Eq. (2.6) (Ref. 27). Such a jump is associated in Ref. 74 with the universal jump of the coefficient of spin-wave rigidity $K(T_c) = 2/\pi$ in the two-dimensional XY model, discovered by Nelson and Kosterlitz.[121] However, it should be pointed out that the arguments used in Ref. 74 are fairly approximate and the result (4.26) is obtained from calculated estimates. The attainment of the relevant fixed point in the XY

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[22] See also a discussion of the problem of the preexponential factor in Ref. 149.

[23] The author of the present review does not agree fully (see Ref. 150) with the arguments in Ref. 84, but the assumptions in Ref. 76 remain unproven and should be regarded as a hypothesis.

model is not demonstrated in Ref. 74. Moreover, there are some doubts about the validity of the results of Nelson and Kosterlitz (see Ref. 122). Qualitative arguments in favor of the scaling law (2.13) for the conductivity in the $d>2$ case are also given in Ref. 74.

An original approach to the problem in question not based on perturbation theory is proposed in Refs. 77 and 78. A detailed presentation of the proposed method is given in Ref. 123, where it is applied to statistics of polymer chains. In these investigations it is also pointed out that there is a region of strong coupling of the E_{sc} type given by Eq. (4.22). Moreover, inequalities are obtained there for the critical index of the conductivity in the law (2.13). These inequalities admit[77] the existence of a minimal metallic conductivity in the case of $d=2$ but not $d=3$.

Wegner[69] was the first to derive the scaling law for the conductivity (2.13). He developed an analytic scheme for transformation of the renormalization group in real space applicable to the Anderson model and analogous to the Thouless scheme.[4,25-29] Two alternative types of critical behavior are proposed in Ref. 69, but the selection between them cannot be made by the methods discussed in that paper. The scaling law (2.13) is obtained in both variants but in what is known as the inhomogeneous case the derivative of the density of states has a singularity at a mobility edge and this singularity is of the same type as that encountered in the specific heat in the theory of critical phenomena (compare with Ref. 76). In the homogeneous case the density of states has no singularity at a mobility edge; for this reason in more recent papers[81-83] Wegner prefers this variant of critical behavior. However, it should be pointed out that there are no rigorous theorems (see Ref. 4) which would forbid a weak singularity in the density of states at a mobility edge, although most authors regard the occurrence of this singularity as not very likely. A method for calculating the critical indices is not given in Ref. 69 and there is no proof that the corresponding fixed points of the equations of the renormalization group are attainable.

In recent investigations[79,80] Wegner discussed an interesting model of an electron in a random lattice with n levels at each site. For each site there are n eigenfunctions $|j\alpha\rangle$ ($\alpha=1,2,\dots,n$), where n is regarded as large and $1/n$ is the small parameter of the theory. The intention is to construct an analog of the $1/n$ expansion in the theory of critical phenomena[24] and apply it to the problem of localization.

The Hamiltonian of the model is

$$H=\frac{1}{\sqrt{n}}\sum_{\substack{ij\\ \alpha\beta}} f_{i\alpha,j\beta}|i\alpha\rangle\langle j\beta|, \tag{4.27}$$

where the matrix elements $f_{i\alpha,j\beta}$ are regarded as independent Gaussian random variables satisfying the condition of symmetry and reality: $f_{i\alpha,j\beta}=f^*_{j\alpha,j\beta}=f_{j\beta,i\alpha}$. Their average values are regarded as zero and the variance is given by

$$\langle f_{i\alpha,j\beta}f_{k\gamma,l\delta}\rangle=(\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{il}\delta_{jk}+\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ik}\delta_{jl})\,M_{i-j}. \tag{4.28}$$

The Hamiltonian (4.27) represents a generalization of the Anderson model and it includes, not only n orbitals per site, but also disorder in the transport integrals.

Since the matrix elements $f_{i\alpha,j\beta}$ are independent and have a symmetric distribution, the probability of finding a system with the specific Hamiltonian given by Eq. (4.27) is equal to the probability of finding a system with

$$H=\frac{1}{\sqrt{n}}\sum_{\substack{ij\\ \alpha\beta}} S_{i\alpha}f_{i\alpha,j\beta}S_{j\beta}|i\alpha\rangle\langle j\beta|, \tag{4.29}$$

where $S_{i\alpha}=\pm1$. If $\Psi_\alpha(j)$ is an eigenfunction of Eq. (4.27), then $S_{i\alpha}\Psi_\alpha(j)$ is an eigenfunction of Eq. (4.29). The system is invariant under local changes in the sign of the wave functions at a site. This variant of the model discussed in Refs. 80 and 81 is known as a local gauge-invariant model. The local gauge invariance imparts the following property to the Green's functions:

$$\langle G_{i\alpha,j\beta}(E)\rangle=S_{i\alpha}S_{j\beta}\langle G_{i\alpha,j\beta}(E)\rangle, \tag{4.30}$$

which shows that

$$\langle G_{i\alpha,j\beta}(E)\rangle=G(E)\,\delta_{ij}\delta_{\alpha\beta}; \tag{4.31}$$

here, the angular brackets represent averaging over random variables of the system. Thus, in this model a one-particle Green's function is of the "point" type because of random fluctuations of the phases of the wave functions. Similarly, a two-particle Green's function vanishes if the four sites defining it are not pairwise coincident.

In the limit $n=\infty$ the model has an exact solution[80] and the results of this solution are identical with the results of application of the coherent potential approximation[124] to this model. In this case *there are no mobility edges* and the density of states is

$$\frac{1}{n}N_\infty(E)=\begin{cases}\dfrac{2}{\pi E_0^2 v_0}\sqrt{E_0^2-E^2}, & |E|\leqslant E_0,\\ 0, & |E|>E_0,\end{cases} \tag{4.32}$$

where $E_0^2=4\sum_j M_{i-j}$, i.e., the density of states is proportional to the variance of the matrix elements of the Hamiltonian (4.28), and v_0 is the volume of a unit cell. The static conductivity (at $T=0$) is

$$\sigma_\infty(E)=\frac{ne^2}{\pi\, dv_0}\left(1-\frac{E^2}{E_0^2}\right)R^2\sim\frac{ne^2}{a^{d-2}}\left(1-\frac{E^2}{E_0^2}\right), \tag{4.33}$$

where $R^2=\sum_j R_{ij}^2 M_{i-j}/\sum_j M_{i-j}$ represents the radius of action of random correlations and a is the lattice constant. The last equality in Eq. (4.33) applies to the case of the "short-range" variance of the matrix elements $R\sim a$. Thus, the conductivity vanishes at a band edge.

If we begin from the exact solution for $n=\infty$, we can develop a systematic (although cumbersome) procedure for calculating the corrections in powers of $1/n$ (Ref. 80). This gives rise to an interesting but not fully understood analogy of the behavior of *two-particle* Green's functions in this model with the behavior of the longitudinal and transverse susceptibilities of an isotropic ferromagnet below the phase transition point.[80,82]

The conductivity may be represented by the following series in powers of $1/n$:

$$\frac{1}{n}\sigma = \sigma_\infty + \frac{1}{n}\sigma_1 + \ldots + \frac{1}{n_t}\sigma_t + \ldots, \tag{4.34}$$

where the coefficients σ_t are finite for $d > 2$ and diverge no faster than $(d-2)^{-t}$ in the limit $d \to 2$. If we retain in Eq. (4.34) only the most diverging terms of the order of $n^{-t}(d-2)^{-t}$ and ignore terms of the order of $n^{-t}(d-2)^{-t'}$ with $t' < t$, we find that

$$\sigma = n\sigma_\infty\,(1 - 2\gamma + O(\gamma^3)), \tag{4.35}$$

where

$$\gamma = \frac{1}{2\pi n\,[1-(E^2/E_0^2)]\,(d-2)\,R^2/a^2}. \tag{4.36}$$

This expansion is valid in the range of delocalized states [this is the zeroth approximation for Eq. (4.33)]. A mobility edge should appear at some value $\gamma = \gamma_c$, when the static conductivity vanishes; the metallic region corresponds to $\gamma < \gamma_c$. Clearly, in view of the convergence of Eq. (4.36) in the limit $d \to 2$, the latter inequality is never satisfied in the two-dimensional case. This is in agreement with the results of Ref. 33. If we assume that

$$\sigma = n\sigma_\infty\left(1 - \frac{\gamma}{\gamma_c}\right)^s, \tag{4.37}$$

we find that a comparison with Eq. (4.35) gives $\gamma_c = 1/2$ and the critical index of the conductivity is

$$s = 1. \tag{4.38}$$

If, following Refs. 33 and 69, we assume that $s = (d-2)\nu$, we find that the critical index is

$$\nu = \frac{1}{d-2}, \tag{4.39}$$

which is identical with the first term of the expansion $(d = 2 + \varepsilon)$ for an isotropic ferromagnet.[125,126] However, as pointed out earlier, the reason for this analogy is not understood.

In Refs. 81–83, Wegner draws attention to the circumstance, mentioned above in connection with the effective Lagrangian (4.25), that the frequency of an external field ω in the conductivity problem acts as a source ("field") which disturbs the $O(n+m)$ symmetry of the problem and

$$\frac{1}{4}\lim_{\substack{n\to 0\\ m\to 0}}\left(\frac{1}{n}\sum_{j=1}^{n}\langle\Phi_j^2\rangle - \frac{1}{m}\sum_{i=1}^{m}\langle\varphi_i^2\rangle\right) = \frac{1}{4}\,[G(rr, E+i\delta) - G(rr, E-i\delta)] = \frac{i\pi}{2}\,v_0 N(E), \tag{4.40}$$

i.e., the density of states acts as the "order parameter" which disturbs this symmetry. This analogy is unusual, since it is normally assumed that the density of states $N(E)$ remains finite at the mobility threshold[81,82] in contrast to the order parameter at the phase transition point. Then, assuming that the critical index β of the order parameter vanishes and employing the usual scaling relationship $\beta = (\nu/2)(d - 2 + \eta) = 0$ in the case when $\nu \neq 0$, Wegner obtains

$$\eta = 2 - d. \tag{4.41}$$

He established this analogy using a somewhat different formalism based on the matrix model[81,82] of the type employed in Refs. 73 and 74. Wegner suggested a matrix effective Lagrangian for the description of the mobility edge allowing for the above symmetry breaking and representing a generalization of the so-called nonlinear σ model. In this model there is no mobility edge for $d = 2$, in agreement with Ref. 33. Renormalization group calculations based on these models were reported in Ref. 127. These calculations give ε expansions for the critical indices $(d = 2 + \varepsilon)$ whose zeroth terms coincide with Eqs. (4.38) and (4.39) in the case of a problem symmetric relative to the orthogonal transformations of a fluctuating matrix field. In a second variant of the same model (invariant under unitary transformations) the indices are different. At present it is not clear which physical situations correspond to these two variants of critical behavior (however, see Refs. 40 and 41).[24] This is clearly associated with the fact that in the Wegner model,[81-83] as well as in earlier matrix models,[73,74] there is no direct correspondence with the standard apparatus for calculating the averaged Green's functions of an electron in a random system,[89,90] which makes it difficult to obtain specific physical results and to make comparisons with the known simple approximations to the theory. In this sense, the formalism based on the effective Lagrangians of the (4.12) and (4.25) type seems to be preferable. At present it is not clear either how the above difficulties associated with the inapplicability of perturbation theory can be avoided in the matrix formalism. It is likely that these difficulties are indeed avoided when the problem is considered near the "lower" critical dimensionality $d = 2$, whereas we have considered above—in connection with Refs. 68–73, 75–78—the vicinity of the "upper" critical dimensionality $d = 4$. It should be mentioned that in recent papers there have been some arguments to suggest that the "upper" critical dimensionality in the localization problem is $d = 8$ (Refs. 85 and 86). However, it seems to us that these conclusions are premature.

This review of various investigations of scaling at a mobility edge gives a somewhat checkered pattern in which the final and reliable results are difficult to identify. Only one thing is certain: the problem of justifying scaling at a mobility edge is much more difficult than the corresponding problem in the conventional theory of critical phenomena. In the mobility edge case we are encountering such obstacles as the need to use an unstable theory of fields, inapplicability of perturbation theory, etc. The absence of universal methods for solving problems of this kind is familiar. Therefore, the problem in question may require development of basically new approaches and methods.

We shall conclude this section by mentioning one other extremely important theoretical problem for a discussion of which we simply have no space. This is allowance for the Coulomb correlations in connection with the problem of localization or the so-called theory of Fermi glasses.[15,128-130] This problem is of fundamental importance, but so far only the first and frequently contradictory results have been obtained.[61,128-134] Clearly, the next important stage in the development of

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[24] Further developments of this subject can be found in recent interesting papers.[142,151-153]

the theory will involve this direction. However, the review of the results published so far and especially formulation of some overview of the problem would be, in our opinion, even more premature than the corresponding procedure in the case of the theory of critical behavior at a mobility edge discussed above.

CONCLUSIONS

We have discussed new results obtained recently in the theory of electron localization in disordered systems and a number of experiments stimulated by these results. In the main, the reviewed ideas on the physics of disordered systems have been developed by the application of ideas and methods taken from the theory of critical phenomena. This approach has been found to be quite successful and has given a number of new and unexpected results, such as transition of long metallic wires to the insulating state and complete localization in two-dimensional systems. From this point of view, any sufficiently long metallic wire is strictly speaking a nonmetal. Similarly, there are serious grounds for expecting nonmetallic properties in the case of two-dimensional films prepared by the evaporation of metals, although in this case the theoretical and experimental situations are still fairly contradictory. In this sense one can speak of a macroscopic manifestation of the quantum phenomenon of electron localization in disordered systems. Experimental observations of these effects are, however, only possible at very low temperatures and on very small samples. The very possibility of investigating such systems has arisen only because of the developments in microelectronics and technology of large integrated systems, which has required suitable lithographic methods. Therefore, there is no need to stress the importance of the new results from the point of view of microelectronics itself.

We have also tried to demonstrate the incompleteness of the results obtained and to stress the unresolved problems. The theory is still far from complete clarity and this applies to such fundamental problems as the very existence or otherwise of the minimum metallic conductivity. This situation is primarily due to the fact that the problem of localization and the description of behavior of electron states near a mobility edge is a much more complex problem than the corresponding tasks in the theory of phase transitions. A complete solution of these problems may require the development of new theoretical methods and formulation of subtle experiments whose importance may go beyond the problem under consideration. Although at present it is difficult to make definite forecasts, there is no doubt that a deeper understanding of the problem of electron localization in disordered systems will foster further theoretical and experimental studies of disordered systems, and practical applications of these systems.

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THEORY OF ELECTRON LOCALIZATION IN DISORDERED SYSTEMS

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Abstract

In this review article many aspects of the modern theory of electron localization in disordered systems are discussed and the general criterion for localization is formulated. This criterion is given as a requirement that a two-particle Green's function have a pole in terms of the frequency with a factorizable residue (in a momentum representation). A search for such a solution can be based on the use of a homogeneous Bethe-Salpeter equation, from which the point where the metallic phase is unstable (the mobility edge) can be determined but which does not describe the region of localized states. The self-consistent theory of localization of Vollhardt and Wölfle is extended to the space with a dimension $d>2$ and the behavior of the principal physical quantities near the mobility edge is calculated. The mobility edge is situated in the "strong-coupling" region (which diverges in the limit $d \rightarrow 2$). This region is the exact analog of the "Ginzburg critical region" in the theory of critical phenomena, in which the perturbation theory breaks down. The analytic properties of the effective field theory, for an electron in a random field, are studied in the complex plane of the coupling constant. The role of finite-action nonlinear solutions (instantons) of the classical field equations in the formation of the "tail" in the density of states is demonstrated. A method of calculating the coefficient of the exponential function of the density of states is proposed. This method is based on the use of the dispersion relation over the coupling constant and on the correspondence with the standard theory of critical phenomena. It is demonstrated that a singular (pole) contribution, in terms of the frequency, to a two-particle Green's function with a factorizable residue, which corresponds to the proposed general criterion for localization, can be determined explicitly within the framework of the instanton approach. A unified approach for the search of instabilities in the system, giving rise to the localization, is formulated. This approach is based on the use of the effective-action formalism for composite fields. The Hartree-Fock corrections, resulting from interaction between the electrons, to the density of states and thermodynamic quantities near the mobility edge are examined. The localization corrections, which are linked directly to the probability for return of an electron, are found. It is shown that these corrections correspond to the formation of a band of singly occupied states below the Fermi level. A cusp in the state density at the Fermi level, which occurs in a "dirty" metal, is shown, within the framework of the self-consistent theory of localization, to smooth out in the insulator region. The correction to the density of states at the Fermi level, however, diverges logarithmically in the entire region of localized states. The localization contribution to the polarization operator corresponding to a nonergodic behavior of the system, which accounts for the difference between the isothermal static response and the adiabatic static response, is analyzed. The isothermal static dielectric constant conserves the "metallic" behavior corresponding to the finite screening range even in the insulator phase: The "Fermi glass" screens the external static electric field.

Contents

1. Introduction

The concept of electron localization in disordered systems is central to the understanding of the modern theory of disordered systems. It is the foundation of the basic understanding of the energy spectrum and of the kinetic and other electronic properties of these systems [1, 2]. Formulated for the first time in a fundamental paper by Anderson [3], the concept of electron localization was developed qualitatively by Mott, who used it to formulate the foundations of electronic theory of disordered systems [1, 2].

The localization phenomenon has recently been discussed in many review articles and monographs [4–10] and its principal propositions are now well known. In spite of its importance, the problem of localization is, nonetheless, far from being solved satisfactorily. At issue here is principally our conceptual understanding of the behavior of electronic states near the so-called mobility edge and of the related question of the physical properties of the system, in which the Fermi level of electrons is situated near the mobility edge. The effect of electron–electron interaction, i.e., the relationship between the Mott and Anderson mechanisms for a metal-insulator transition in disordered systems, is yet to be studied extensively. The difficulty in understanding this relationship stems from the extreme mathematical complexity of this problem and from the fact that there is no direct experimental evidence for electron localization [1, 2]. In particular, it is very difficult to distinguish between disorder and electron–electron interaction in a real metal-insulator transition.

Let us summarize the principal propositions of the localization theory which is customarily based on the Anderson model [3]. In this model the electron which propagates in the regular lattice in a d-dimensional space is analyzed. Each lattice site has a random level E_j (j is the number of the lattice site in the lattice). It is assumed that there is a certain probability amplitude for the transition V_{ij} from the jth lattice site to the ith lattice site. This amplitude is usually assumed [3] to be nonvanishing and equal to a certain constant V for the transitions between the nearest neighbors. The energy levels E_j are assumed to be distributed independently at different sites and the energy distribution at a given site is usually assumed to be uniform over a certain energy interval of width W. The qualitative results do not seem to depend too strongly on these assumptions. Another useful model for analyzing the localization is the model of free electrons in a field of point scatterers randomly distributed in a space with density ρ. Each of these point scatterers has the same scattering amplitude which we will denote by V [11].

If there is no disorder in the system ($W=0$ in the Anderson model or $\rho=0$ in the free-electron model), then the problem of the electronic spectrum can be solved in a straightforward manner. In the Anderson model the electronic states form a band of width $2ZV$, where Z is the number of nearest neighbors. An infinitely wide band of free electrons is formed in a similar manner. The introduction of disorder accounts for some important changes, giving rise to a strong dependence on the dimensionality of the space d. At $d=1$ a disorder, however small, completely changes the nature of the electronic states, localizing all of them. In other words, the wave functions of these states begin to fall off exponentially in the coordinate space, while the static electrical conductivity of the system goes down to zero at $T=0$ [12–15]. The two-dimensional ($d=2$) systems are the limiting case ("lower critical dimensionality"). In these systems the electronic states presumably also become completely localized as a result of appearance of the slightest disorder. For $d>2$ all the electronic states in the band become completely localized if the ratio W/V in the Anderson model is sufficiently large—larger than a certain critical ratio $(W/V)_c$, i.e., if the disorder is appreciable. If $W/V<(W/V)_c$, the electronic states become localized at the band edges but remain delocalized at the band center (Fig. 1a). This situation gives rise to critical values of the energy $\pm E_c$, which separates the regions of the localized states from those of the extended states, customarily called mobility edges. In a model of nearly free electrons a qualitative picture of the electronic states for $d>2$ is also well known (Fig. 1b). If the Fermi level E_F is in the region of fairly high energies, the electronic states near it are just the plane waves that are slightly distorted by scattering. The importance of this scattering increases with decreasing Fermi energy toward the edge of the original band. A density of states "tail," which stems from the electron localization due to random-potential fluctuations of the scatterers [8], appears at

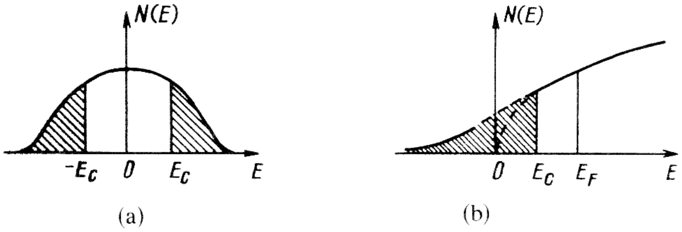

Figure 1. The density of states and location of the mobility edges in the Anderson model (a) and in the model of nearly free electrons (b). The hatched regions represent the localized states.

the band-edge and an energy E_c, which separates the delocalized states from the localized states (the mobility edge), appears near the former band edge. The term "mobility edge" is used because the localized states do not contribute to the static conductivity at zero temperature $T = 0$. If the Fermi level E_F of the many-electron system is at $T = 0$ in the energy region corresponding to the localized states, then the system is an insulator: conductivity appears only at $T \neq 0$ or when the electrons are excited by an alternating electromagnetic field. The conductivity in this case is realized by means of the hopping mechanism [1, 6]. If, however, the Fermi level is situated in the region of delocalized states, then the conductivity is metallic in nature. Such a metal-insulator transition, which occurs when the Fermi level crosses the mobility edge, is customarily called the Anderson transition.

One of the central questions raised in the theory is how the metallic state conductivity changes when the Fermi level E_F crosses the mobility edge E_c (at $T = 0$). Some possible alternatives are illustrated in Fig. 2 [1, 2, 16]. The conductivity may go down to zero discontinuously, after reaching a certain minimum value σ_{mm}—the minimum metallic conductivity. Such a behavior was suggested by Mott [1, 2] on the basis of a qualitative analysis of the conductivity in the Anderson model and on the basis of extensive experimental data. Theoretically, the conductivity can also decrease continuously down to zero, while the value σ_{mm} determines the conductivity scale for this continuous transition. This idea was initially suggested by Cohen

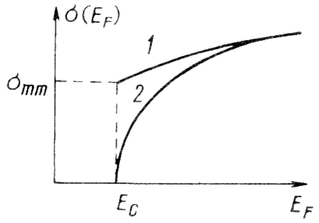

Figure 2. Variation of the metallic conductivity during the passage of the Fermi level through the mobility edge. 1—According to Mott [1, 2]; 2—according to Cohen [16].

[16], who basically used the analogy of the localization with percolation [6]. Clearly, either alternative leads (when the conductivity decreases sharply but continuously) to roughly the same experimental effects, at least at reasonably high temperatures. Highly accurate measurements at extremely low temperatures must, however, be performed before this problem can be solved experimentally. The experiments [17–19] in the temperature region of 10^{-3} K, which have recently been performed, tend to suggest that the conductivity goes continuously down to zero at the mobility edge. The current theoretical picture based on the scaling localization theory also supports this viewpoint [7].

A qualitative estimate of the typical conductivity scale near the mobility edge, as well as the estimate of its location in the band, can easily be obtained using the arguments advanced by Ioffe and Regel [20]. Their argument essentially reduces to the fact that the metallic conductivity remains in effect so long as the mean free path of an electron exceeds its wavelength. With increasing disorder, the corresponding inequality is disrupted and the nature of the conductivity changes considerably, principally because of the localization. In the case of the Anderson model with a half-filled band (one electron per site), we run across this situation when the mean free path l is of the order of the interatomic distance. Using the standard Drude equation $\sigma = ne^2\tau/m$ (n is the electron density, τ is the mean free time, and m is the electron mass) and setting $n \sim a^{-d}$ and $\tau \sim ma^2$, we thus estimate σ_{mm} to be (we will use everywhere the system of units $\hbar = 1$)

$$\sigma_{mm} \sim \mathrm{const}\,\frac{e^2}{a^{d-2}}.$$

According to Mott [1, 2], the constant in this equation, which is specified by the critical ratio $(W/V)_c$, lies within the range 0.025–0.06 when $d = 3$, giving $\sigma_{mm} \sim 1\text{–}5 \times 10^2$ mho/cm, typically for $a \sim 2\text{–}3$ Å.

In a model of nearly free electrons the Ioffe–Regel criterion indicates that the Fermi energy E_F is of the order of the reciprocal mean free time, τ^{-1}. Using the standard Born expression to estimate this time $\tau^{-1}(E) \sim \rho V^2 N(E)$, where $N(E) \sim m^{d/2} E^{(d/2)-1}$ is the density of states of the free electrons, we estimate from the condition $E_c \sim \tau^{-1}(E_c)$ the mobility edge to be at

$$E_c \sim m^{d/(4-d)} (\rho V^2)^{2/(4-d)}$$

near the band edge. This characteristic energy will have an important role in our discussion below. This energy defines the energy region [7] at the edge of the band, in which the strong scattering by disorder becomes important. Reckoning the energy from the edge of the original band, we find

$$E_c \sim V\left(\frac{W}{V}\right)^{4/(4-d)}$$

for the Anderson model by using the same procedure. In fact, we see from these estimates the prominent role of the dimensionalities of space $d = 2$ and $d = 4$ in the localization problem.

The localization theory has recently been developed extensively. The development of this theory is attributable primarily to the formulation of the basic concepts and methods of the scaling theory of localization, which are based on the use of the present-day theory of critical phenomena [7]. In the theory of critical phenomena [21–23] the scaling description of the fluctuation region of the phase transitions is generally motivated by the growth of correlation length of the fluctuations of the order parameter near the critical point. An analog of this effect in the localization theory is the increase of the localization length of the wave functions as the mobility edge is approached from the side of the localized states [24]. We should emphasize, however, that there is no analog of such diverging length in the region of extended states, just as there is no obvious order parameter associated with the Anderson transition. The various suggestions proposed in many papers (see review article [7]) are, from our viewpoint, largely not satisfactory. Immediately after the scaling model of the Anderson transition was initially suggested [24, 25], a considerable progress was, nonetheless, made in the understanding of this region. This progress was particularly rapid after Abrahams *et al.* [26] published a landmark paper, in which the scaling equations for the conductivity of a finite system, based principally on certain ideas advanced earlier by Thouless *et al.* [27–30], were formulated. This progress was particularly impressive in the analysis of two-dimensional systems [7, 9, 10], where it was possible to formulate not only totally new concepts but also to perform important experiments which have confirmed the theoretical predictions. The restrictions on the analogy with the phase transitions indicated above show, however, that the localization problem, on the whole, is much more complex than the problem of critical phenomena. Serious difficulties must yet be dealt with before this problem is finally solved.

In this review article we discuss in detail the theoretical results recently obtained in this field by the author. The main attention is focused on the discussion of the particular difficulties of the theory. The earlier theoretical studies in this field were elucidated in a review article by Sadovskiĭ [7], who also included in this study most of the experimental results, which we basically overlook in this article. The theoretical studies of other authors will be used only as the need arises. The literature cited in this article is by no means complete. An outline of this article is clear from the Contents.

2. The Electron in a Field of Random Scatterers and the Localization Theory

2.1. The General Criterion for Localization. Localization Viewed in Terms of the Bethe–Salpeter Equation

Two alternative methods can be used in a rigorous analysis of the localization phenomenon. The Anderson method, which was described in a basic paper [3] and which was used until recently to get all the principal assertions of the theory, is distinguished by its nonconventional features. The main point of this method is an analysis of the Green's functions of an electron which have not been averaged over the random configurations. In a certain sense, the *most probable single-electron Green's function* is analyzed in this method, because the *average* single-particle Green's function, as is well known [3–5], contains no information on electron localization. This problem can, nonethelesss, be studied by analyzing the convergence of the stochastic series of the perturbation theory for an unaveraged single-particle Green's function [3, 31]. In particular, the condition of convergence (convergence in probability) of this series determines the location of the mobility edge. The Anderson method can be used to solve particular problems, ranging from the construction of the scaling theory of localization [24] to the analysis of new, specific models of disorder and effects of external fields [32]. A serious drawback of this method, however, is that it allows virtually no calculation of the observable physical quantities which are defined by the correlation functions (the Green's functions) that are averaged over the ensemble of random configurations of the system. These quantities can be calculated by using the well-developed formalism and a diagram technique [11, 33]. Unfortunately, the question of how the localization manifests itself in the principal quantities, such as the average Green's functions, which are used in the standard theory, has been studied inadequately until recently. The lack of sufficiently clear analysis of this problem leads to the difficulties in the solution of the localization problem and establishing a connection between this phenomenon and the observed characteristics of the system. Clearly, the problem of localization, for example, generally differs from that of the behavior of the electrical conductivity near the mobility edge, whose solution may be considerably more difficult.

In this section we will examine, according to Refs. 34 and 35, the general condition for localization in terms of the standard formalism, keeping in mind the traditional problem encountered in the Edwards method [11]. This problem involves the noninteracting electrons which move in a field of the scatterers which are randomly distributed in space.

The Hamiltonian of the system is

$$H = \int d^d\mathbf{r}\,\psi^+(\mathbf{r})\left\{-\frac{\nabla^2}{2m} + \sum_i V(\mathbf{r}-\mathbf{R}_i)\right\}\psi(\mathbf{r}), \tag{2.1}$$

where $V(\mathbf{r}-\mathbf{R}_i)$ is the potential of the scatterer (an impurity in the crystal, for example) which is situated at the point $\mathbf{R}_i$, and $\psi^+(\mathbf{r})$ and $\psi(\mathbf{r})$ are the electron creation and annihilation operators. For generality, we will examine a d-dimensional space. Let us introduce a complete orthonormal set of exact wave functions $\varphi_\nu(\mathbf{r})$ of the Hamiltonian (2.1)

$$H\varphi_\nu(\mathbf{r}) = \epsilon_\nu\varphi_\nu(\mathbf{r}), \tag{2.2}$$

where ϵ_ν are the exact eigenvalues of the energy of an electron in a field of random scatterers. Obviously each wave function $\varphi_\nu(\mathbf{r})$ and each eigenvalue of the electron energy ϵ_ν are also functionals of the location of the scatterers $\mathbf{R}_i$ in a given random system. Given that such a dependence exists, we will not write it out explicitly in the arguments of these functions. In the absence of an external magnetic field, we can assume, without loss of generality, that the functions $\phi_\nu(\mathbf{r})$ are real [36]. However, we will write out the complex conjugation explicitly. According to Berezinskiĭ and Gor'kov [34], define the two-particle spectral density is as

$$\langle\langle\rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle = \frac{1}{N(E)}\left\langle\sum_{\nu\nu'}\varphi^*_\nu(\mathbf{r})\varphi_{\nu'}(\mathbf{r})\varphi^*_{\nu'}(\mathbf{r}')\varphi_\nu(\mathbf{r}')\right.$$
$$\left.\times\,\delta(E-\epsilon_\nu)\delta(E+\omega-\epsilon_{\nu'})\right\rangle. \tag{2.3}$$

In this equation the angular brackets denote averaging over the random configurations of the scatterers, and

$$N(E) = \left\langle\sum_\nu \varphi_\nu(\mathbf{r})\varphi^*_\nu(\mathbf{r})\delta(E-\epsilon_\nu)\right\rangle \tag{2.4}$$

is the single-electron (average) density of states. The spectral density (2.3) has the following general properties [34], which can easily be checked by invoking the conditions under which the function $\varphi_\nu(\mathbf{r})$ are complete and orthonormal:

$$\begin{aligned} &\int d^d\mathbf{r}\langle\langle\rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle = \delta(\omega) \\ &\int d\omega\langle\langle\rho_E(\mathbf{r})\rho_{E+\omega'}(\mathbf{r}')\rangle\rangle = \delta(\mathbf{r}-\mathbf{r}'). \end{aligned} \tag{2.5}$$

For a spatial Fourier transform of (2.3) we find $\langle\langle\rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{q}}$ in an analogous manner,

$$\begin{gathered} \langle\langle\rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{q}=0} = \delta(\omega) \\ \int d\omega\langle\langle\rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{q}} = 1; \qquad \langle\langle\rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{q}} \geqslant 0. \end{gathered} \tag{2.6}$$

In general, expression (2.3) has terms which correspond to the contribution of $\epsilon_\nu = \epsilon_{\nu'}$. If, however, the corresponding states of the system are extended states, then their wave functions are normalized to the total volume of the system Ω. For this reason, this contribution in (2.3), which is proportional to Ω^{-1}, vanishes in the limit of $\Omega\to\infty$. The situation changes dramatically if the states are localized. The wave functions in this case are normalized to the volume $\sim R_{\text{loc}}^d$, where R_{loc} is the localization length, and a simple estimate gives the value $\sim R_{\text{loc}}^{-d}$ for the contribution of $\epsilon_\nu = \epsilon_{\nu'}$ in (2.3). Thus a δ-function contribution in terms of the frequency ω appears in the spectral density (2.3)

$$\langle\langle\rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle = A_E(\mathbf{r}-\mathbf{r}')\delta(\omega) + \rho_E(\mathbf{r}-\mathbf{r}',\omega) \tag{2.7}$$

or in the momentum representation

$$\langle\langle\rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{q}} = A_E(\mathbf{q})\delta(\omega) + \rho_E(\mathbf{q}\omega), \tag{2.8}$$

where the second terms are regular terms in ω. The incorporation of such a singular contribution into (2.3) as a general localization criterion was suggested by Berezinskiǐ and Gor'kov [34]. It is easy to see that

$$A_E(\mathbf{r}-\mathbf{r}') = \frac{1}{N(E)}\left\langle\sum_\nu \delta(E-\epsilon_\nu)|\varphi_\nu(\mathbf{r})|^2|\varphi_\nu(\mathbf{r}')|^2\right\rangle \tag{2.9}$$

is a generalized "inverse participation ratio." The nonvanishing of this ratio was frequently regarded as the localization criterion, alternative to the well-known Economou–Cohen criterion [4, 31, 37]. Roughly speaking, this ratio is inversely proportional to the number of atomic orbitals which contribute effectively to the superposition corresponding to the state ν.

The $\delta(\omega)$ singularity in the spectral density of (2.7) and (2.8), which is a manifestation of *nonergodicity* of the system [38], corresponds to the appearance of *time-independent* contributions in the corresponding correlation functions. Such contributions in this instance appear in the retarded density–density Green's function, for example [34]. It is important, however, that the $\delta(\omega)$ anomaly of the spectral density does not contribute to the *commutative* Green's functions, which determine the linear response in the Kubo theory [38, 39]. These anomalies account for the difference between the adiabatic static response and the isothermal static response of the system [39, 40]. The connection between the localization and the nonergodic behavior was, in fact, pointed out by Anderson in his first paper [3]. Later we will return to the discussion of the physical manifestations of this important effect.

From the general properties of (2.5) and (2.6) we find in the limit $\mathbf{q}\to 0$ (Ref. 34)

$$\langle\langle\rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{q}} \approx \{1 - R^2_{\mathrm{loc}}(E)\mathbf{q}^2\}\delta(\omega) + \mathbf{q}^2\rho_E(\omega), \tag{2.10}$$

where

$$\begin{aligned} R^2_{\mathrm{loc}}(E) &= \int_{-\infty}^{\infty} d\omega\rho_E(\omega) \\ &= \frac{1}{2dN(E)}\int d^d r r^2 \left\langle \sum_\nu \delta(E-\epsilon_\nu)|\varphi_\nu(\mathbf{r})|^2|\varphi_\nu(0)|^2\right\rangle. \end{aligned} \tag{2.11}$$

We can assume that this expression defines the localization length.

A delocalization corresponds to a spreading out of the δ-function singularity in (2.8) for the finite values of $\mathbf{q}$. The simplest example of such spreading, which accounts for the diffusional behavior, is

$$\langle\langle\rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{q}} = \frac{1}{\pi}\frac{D_E\mathbf{q}^2}{\omega^2 + (D_E\mathbf{q}^2)^2}. \tag{2.12}$$

As was emphasized in Ref. 34, this example is only a special case among the different possible versions of the mathematical behavior of $\langle\langle \rho_E \rho_{E+\omega} \rangle\rangle_{\mathbf{q}}$ in the limit $\mathbf{q} \to 0$.

It is easy to see [34] that the conductivity of the system is

$$\begin{aligned}\sigma(\omega) &= \lim_{\mathbf{q}\to 0} \left(-\frac{i\omega}{\mathbf{q}^2} \right) e^2 \chi^R(\mathbf{q}\omega) \\ &= \lim_{\mathbf{q}\to 0} \frac{i\omega}{\mathbf{q}^2} e^2 \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\omega' N(E) \frac{f(E) - f(E+\omega')}{\omega - \omega' + i\delta} \langle\langle \rho_E \rho_{E+\omega'} \rangle\rangle_{\mathbf{q}},\end{aligned} \tag{2.13}$$

where $\chi^R(\mathbf{q}\omega)$ is the retarded response function of the density–density type, which is determined by the corresponding commutative Green's function [34], and e is the electronic charge. Using (2.8), we see that $A_E(\mathbf{q})$ does not contribute to the conductivity, consistent with the general property mentioned above. Using (2.10), we find

$$\begin{aligned}\sigma(\omega) &\simeq -i\omega e^2 \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\omega' N(E)\omega' \left(-\frac{\partial f(E)}{\partial E} \right) \frac{\rho_E(\omega')}{\omega - \omega' + i\delta} \\ \operatorname{Re} \sigma(\omega) &\simeq \pi e^2 \omega^2 N(E_F) \rho_{E_F}(\omega),\end{aligned} \tag{2.14}$$

so that at low frequencies the behavior of Re $\sigma(\omega)$ is determined entirely by the $\rho_E(\omega)$ function or by the "regular" part of the spectral density (2.8). Thus, in such a general analysis, strictly speaking, only an indirect correlation [through the sum rules such as (2.5) and (2.6), for example] can be established between the "localization signal" [the quantity $A_E(\mathbf{q})$] and the frequency dependence of the conductivity. In the diffusion (metallic) regime, the use of (2.12) yields

$$\begin{aligned}\sigma(\omega) &\simeq \frac{e^2}{i\omega} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\omega' N(E) \left(-\frac{\partial f(E)}{\partial E} \right) \frac{1}{\pi} \frac{D_E \omega'}{\omega - \omega' + i\delta} \longrightarrow \\ &\xrightarrow[\omega \to 0]{} e^2 N(E_F) D_{E_F},\end{aligned} \tag{2.15}$$

i.e., we obtain the standard result (the Einstein relation) for the metallic conductivity of the noninteracting electrons, and the quantity D_E is the diffusion coefficient of the electron with energy E.

Since $A_E(\mathbf{q})$ in (2.8) "signals" the appearance of localized states with energy E, it would be useful [35] to determine it from the standard formalism (Green's functions). Using the standard unaveraged retarded (R) and advanced (A) single-electron Green's functions

$$G^{R,A}(\mathbf{r}\mathbf{r}'E) = \sum_{\nu} \frac{\varphi_\nu(\mathbf{r})\varphi^*_\nu(\mathbf{r}')}{E - \epsilon_\nu \pm i\delta} \tag{2.16}$$

and the definition (2.3), we immediately obtain

$$\begin{aligned}\langle\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle &= \frac{1}{\pi^2 N(E)} \langle \operatorname{Im} G^{R,A}(\mathbf{r}\mathbf{r}'E+\omega) \operatorname{Im} G^{R,A}(\mathbf{r}'\mathbf{r}E)\rangle \\ &= \frac{1}{2\pi^2 N(E)} \operatorname{Re}\{\langle G^R(\mathbf{r}\mathbf{r}'E+\omega) G^A(\mathbf{r}'\mathbf{r}E)\rangle \\ &\quad - \langle G^{R,A}(\mathbf{r}\mathbf{r}'E+\omega) G^{R,A}(\mathbf{r}'\mathbf{r}E)\rangle\}\end{aligned} \tag{2.17}$$

or we obtain the following expression in the momentum representation:

$$\langle\langle \rho_E \rho_{E+\omega}\rangle\rangle_{\mathbf{q}} = \frac{1}{\pi N(E)} \operatorname{Im}\{\phi_E^{RA}(\omega\mathbf{q}) - \phi_E^{RR}(\omega\mathbf{q})\}. \tag{2.18}$$

In this expression we have introduced, for brevity, the following notation for the two-particle Green's function [41]:

$$\phi_E^{RA(R)}(\mathbf{q}\omega) = -\frac{1}{2\pi i} \sum_{\mathbf{p}\mathbf{p}'} \langle G^R(\mathbf{p}_+\mathbf{p}'_+ E+\omega) G^{A(R)}(\mathbf{p}'_-\mathbf{p}_- E)\rangle, \tag{2.19}$$

where $\mathbf{p}_\pm = \mathbf{p} \pm \frac{1}{2}\mathbf{q}$. For small values of $\mathbf{q}$ and ω the functions $\phi_E^{RR}(\mathbf{q}\omega)$ and $\phi_E^{AA}(\mathbf{q}\omega)$ are non singular functions [40, 41]. It is clear, therefore, that the δ-function component in (2.7) and (2.8), which corresponds to the appearance of localized states, may arise only from the first term in (2.18). Using in (2.8) the formal transition from ω to $\omega + i\delta$ ($\delta \to 0^+$), i.e., "smearing out" the δ-function and comparing it with (2.18), we immediately obtain an expression for the amplitude of $A_E(\mathbf{q})$ in front of the δ-function in (2.8), which is written in terms of the two-particle Green's function

$$\begin{aligned} A_E(\mathbf{q}) &= \lim_{\delta\to 0} \frac{1}{N(E)}\, \delta \operatorname{Im} \phi_E^{RA}(\omega + i\delta\mathbf{q})\Big|_{\omega=0} \\ &= \frac{1}{2\pi N(E)} \lim_{\delta\to 0} \delta \sum_{\mathbf{p}\mathbf{p}'} \operatorname{Re}\langle G^R(\mathbf{p}_+\mathbf{p}'_+ E + i\delta)\, G^A(\mathbf{p}'_-\mathbf{p}_- E - i\delta)\rangle. \end{aligned} \tag{2.20}$$

Switching to the coordinate representation, we find

$$\begin{aligned} A_E(\mathbf{r}-\mathbf{r}') &= \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')} A_E(\mathbf{q}) \\ &= \frac{1}{2\pi N(E)} \lim_{\delta\to 0} \delta\langle |G(\mathbf{r}\mathbf{r}' E + i\delta)|^2\rangle. \end{aligned} \tag{2.21}$$

It would be useful to introduce the quantity

$$\begin{aligned} A_E &= A_E(\mathbf{r}-\mathbf{r}')\Big|_{\mathbf{r}=\mathbf{r}'} = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, A_E(\mathbf{q}) \\ &= \frac{1}{2\pi N(E)} \lim_{\delta\to 0} \delta\langle |G(\mathbf{r}\mathbf{r} E + i\delta)|^2\rangle \\ &= \frac{1}{2\pi N(E)} \lim_{\delta\to 0} \sum_{\mathbf{p}\mathbf{p}'\mathbf{q}} \operatorname{Re}\langle G^R(\mathbf{p}_+\mathbf{p}'_+ E + i\delta)\, G^A(\mathbf{p}'_-\mathbf{p}_- E - i\delta)\rangle. \end{aligned} \tag{2.22}$$

In the region of localized states ($E < E_c$ "below" the mobility edge) we have $A_E > 0$. On the other hand, if we assume that the Green's function corresponds physically to the probability amplitude of the transition, we find that (2.22) is proportional to the average *probability of return* of the electron with a given energy E to the starting point in an infinite time [31]. The condition under which (2.22) is nonvanishing is therefore none other than the averaged version of

Economou–Cohen localization criterion [31]. The Economou–Cohen localization criterion in this sense is equivalent to the Berezinskiĭ–Gor'kov localization criterion [35].

As we have seen in (2.10), at $\mathbf{q}=0$ we have the general property $A_E(\mathbf{q}=0)=1$. We easily see that this property is equivalent to the condition

$$\phi_E^{RA}(0\omega)=-\frac{N(E)}{\omega}. \tag{2.23}$$

Such a divergence in the limit $\omega\to 0$ is a consequence of the conservation of the number of particles (of the equation of continuity) [41, 42].

To determine the value of $\Phi_E^{RA}(\mathbf{q}\omega)$ (2.19) we must be able to calculate the two-particle Green's function:

$$\phi_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega)=-\frac{1}{2\pi i}\langle G^R(\mathbf{p}_+\mathbf{p}'_+E+\omega)G^A(\mathbf{p}'_-\mathbf{p}_-E)\rangle. \tag{2.24}$$

Within the framework of Edwards' diagram technique, this function, as we know [11, 33, 41, 42], is determined by the Bethe–Salpeter integral equation (see Fig. 3):

$$\begin{aligned}\phi_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega)=G^R(E+\omega\mathbf{p}_+)G^A(E\mathbf{p}_-)\biggl\{&-\frac{1}{2\pi i}\,\delta(\mathbf{p}-\mathbf{p}')\\ &+\sum_{\mathbf{p}''}U_{\mathbf{pp}''}^E(\mathbf{q}\omega)\phi_{\mathbf{p}''\mathbf{p}'}^{RA}(E\mathbf{q}\omega)\biggr\},\end{aligned} \tag{2.25}$$

where $G^{R,A}(E\mathbf{p})$ is the dressed, average, retarded (advanced), single-electron Green's function, and the irreducible vertex $U_{\mathbf{pp}'}^E(\mathbf{q}\omega)$ is determined by the sum of all the diagrams that cannot be cut along two lines (the advanced and the retarded line).

Let us see [35] whether a solution of Eq. (2.25) leads to a two-particle Green's function that contains some features corresponding to a localization. We assume that $\Phi_{\mathbf{pp}''}^{RA}(E\mathbf{q}\omega)$ has a pole contribution in the energy region in which the system possesses localized states:

$$\phi_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega)=-\frac{\psi_{\mathbf{p}}^{\mathbf{q}}(E)\psi_{\mathbf{p}'}^{-\mathbf{q}}(E)}{\omega+i\delta}+\tilde{\phi}_{\mathbf{pp}'}^{RA}(E\mathbf{q}\omega), \tag{2.26}$$

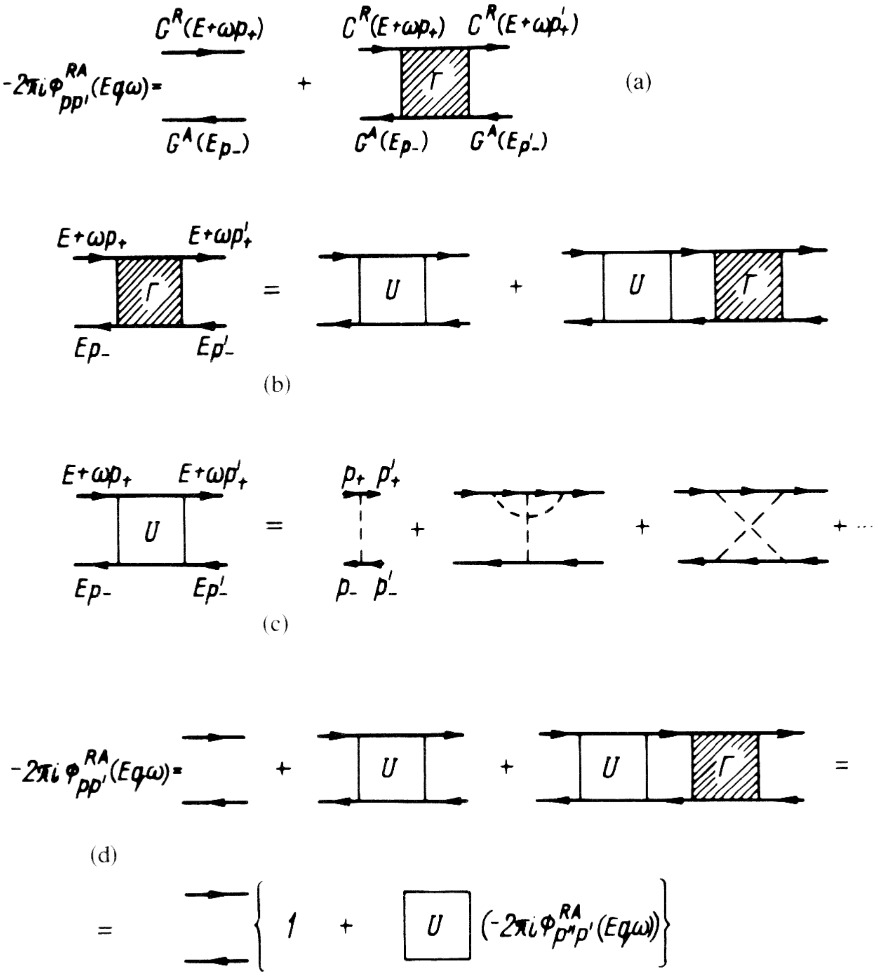

Figure 3. (a) Graphic definition of two-particle Green's function (2.24). (b) Equation for a complete vertex $\Gamma^E_{\mathbf{pp}'}(\mathbf{q}\omega)$. (c) Typical diagrams for an irreducible vertex $U^E_{\mathbf{pp}'}(\mathbf{q}\omega)$. (d) Bethe–Salpeter equation (2.25). The dashed curve denotes the "interaction" $U_0(\mathbf{p}-\mathbf{p}')=\rho|V(\mathbf{p}-\mathbf{p}')|^2$, where ρ is the density of the scatterers, and $V(\mathbf{p}-\mathbf{p}')$ is the Fourier transform of the potential of a single scattering center.

where $\tilde{\Phi}^{RA}_{\mathbf{pp}'}(E\mathbf{q}\omega)$ is the regular part, and the factorization of the residue at the pole (in the momentum representation) is assumed in analogy with the problem of the bound state. We will see later that this assumption may in a certain sense be justified.

From (2.19) and (2.26) we find

$$\phi_E^{RA}(\mathbf{q}\omega) = -\frac{\chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E)}{\omega + i\delta} + \sum_{\mathbf{pp}'} \tilde{\phi}^{RA}_{\mathbf{pp}'}(E\mathbf{q}\omega). \tag{2.27}$$

In this equation

$$\chi_{\mathbf{q}}(E) = \sum_{\mathbf{p}} \psi^{\mathbf{q}}_{\mathbf{p}}(E). \tag{2.28}$$

It then follows from (2.20) that

$$A_E(\mathbf{q}) = \frac{1}{N(E)} \chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E). \tag{2.29}$$

It is easy to see that $\chi^*_{\mathbf{q}}(E) = \chi_{-\mathbf{q}}(E)$, and $A_E(\mathbf{q}) > 0$. The general property of $A_E(\mathbf{q} = 0) = 1$ reveals the normalization condition,

$$\chi_0(E) = \sqrt{N(E)} \tag{2.30}$$

and from the asymptotic behavior of (2.10) we find

$$\chi_{\pm\mathbf{q}}(E) \approx \sqrt{N(E)}\,\{1 \pm iR_{\mathrm{loc}}(E)\mathbf{q}\}^{-1}; \quad \mathbf{q} \to 0. \tag{2.31}$$

For the "probability of return" (2.22) it follows from these equations that

$$A_E = \frac{1}{N(E)} \sum_{\mathbf{q}} \chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E). \tag{2.32}$$

After the substitution of (2.26) into (2.25) the pole term is dominant (in the limit $\omega \to 0$) and we obtain a *homogeneous* Bethe–Salpeter integral equation for $\psi^{\mathbf{q}}_{\mathbf{p}}(E)$,

$$\psi^{\mathbf{q}}_{\mathbf{p}}(E) = G^R(E\mathbf{p}_+)G^A(E\mathbf{p}_-) \sum_{\mathbf{p}'} U^E_{\mathbf{pp}'}(\mathbf{q}\omega = 0)\psi^{\mathbf{q}}_{\mathbf{p}'}(E). \tag{2.33}$$

The localization would correspond to a nontrivial solution $\psi^{\mathbf{q}}_{\mathbf{p}}(E) \neq 0$ of Eq. (2.33), which would remain nonvanishing in the entire energy region $E \leq E_c$, where E_c is the mobility edge. It may turn out, however, that we can use Eq. (2.33) only to determine the location of the mobility edge E_c as a point on the energy axis at which we find the first nontrivial solution of (2.33). To describe the region $E < E_c$, however, may require going beyond the scope of the perturbation theory, which defines the structure of the Bethe–Salpeter equation. Equation (2.33) obviously cannot be analyzed in its general form. Some approximations must therefore be used for the functions $G^{R,A}(E\mathbf{p})$ and $U^{E}_{\mathbf{pp}'}(\mathbf{q}\omega = 0)$. We will use the simplest approximation for $G^{R,A}(E\mathbf{p})$, which is valid in the region in which the perturbation theory is applicable [11, 33, 42]:

$$G^{R,A}(E\mathbf{p}) = \frac{1}{E - \dfrac{\mathbf{p}^2}{2m} \pm i\gamma(E)}, \tag{2.34}$$

where

$$\gamma(E) = \pi\rho V^2 N(E) \tag{2.35}$$

is the "Born" frequency of the collisions of the electrons with the impurities which we will assume, for simplicity, to be point like. This result, as we know, is obtained by summing those diagrams in Fig. 4a

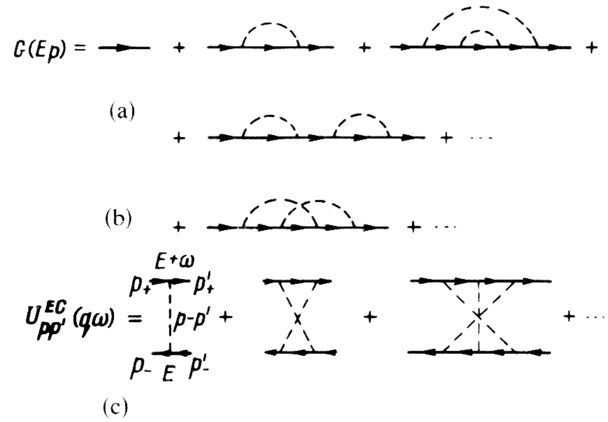

Figure 4. (a) Diagrams which determine the single-electron Green's function. (b). Langer–Neal diagrams for the irreducible vertex of the Bethe–Salpeter equation (cooperon).

which do not have any interesting "interaction" lines, i.e., diagrams similar to those shown in Fig. 4b. This approximation can be used if the inequality $E \gg \gamma(E)$ holds. As was noted by Gor'kov *et al.* [43] for the first time, the localization is linked, at least in the limit $d \to 2$, with a specific sequence of diagrams for $U^{E}_{\mathbf{pp}'}(\mathbf{q}\omega)$ that have a "maximum number of intersections". These diagrams are illustrated in Fig. 4b. Langer and Neal [44] studied this diagram sequence more than a decade earlier. This sequence can easily be summed by using Eq. (2.34). We thus find a "cooperon" propagator for small values of ω and $\mathbf{p} + \mathbf{p}'$ (Refs. 41 and 43),

$$U^{EC}_{\mathbf{pp}'}(\mathbf{q}\omega) = \frac{2\gamma(E)\rho V^2}{D_0^E(\mathbf{p}+\mathbf{p}')^2 - i\omega}, \tag{2.36}$$

where

$$D_0^E = \frac{E}{md\gamma(E)} \tag{2.37}$$

is a classical diffusion coefficient. In addition to the aforementioned condition under which the perturbation theory is applicable, we see that the condition for the applicability of the diffusion approximation $|\mathbf{p}+\mathbf{p}'| \ll l^{-1}$, where $l \sim \sqrt{E/m}\,\gamma^{-1}(E)$ is the mean free path, must also be satisfied in order for expression (2.36) to be valid. This condition is equivalent to $|\mathbf{p}+\mathbf{p}'| \ll \sqrt{2mE} \simeq \mathbf{p}_F$ ($\mathbf{p}_F$ is the Fermi momentum) near the mobility edge, where, as we will see below (see also the Introduction), $E \sim \gamma(E)$.

Using (2.34) and (2.36), we can finally write Eq. (2.33) as follows:

$$\left\{E - \frac{1}{2m}\left(\mathbf{p} + \frac{1}{2}\mathbf{q}\right)^2 + i\gamma(E)\right\}\left\{E - \frac{1}{2m}\left(\mathbf{p} - \frac{1}{2}\mathbf{q}\right)^2 - i\gamma(E)\right\}\psi^{\mathbf{q}}_{\mathbf{p}}(E)$$

$$= \lambda(E)\int \frac{d^d\mathbf{p}'}{(2\pi)^d}\,\frac{\psi^{\mathbf{q}}_{\mathbf{p}'}(E)}{(\mathbf{p}+\mathbf{p}')^2}, \tag{2.38}$$

where $\lambda(E) = 2dm\gamma^2(E)(\rho V^2/E)$. Switching to dimensionless variables $\mathbf{p} \to \mathbf{p}/\sqrt{2mE}$, we can write this equation as follows:

$$\psi^{\mathbf{q}}_{-\mathbf{p}}(E) = \frac{\lambda_E}{\left\{1 - \left(\mathbf{p} - \frac{1}{2}\mathbf{q}\right)^2 + \frac{i\gamma}{E}\right\}\left\{1 - \left(\mathbf{p} + \frac{1}{2}\mathbf{q}\right)^2 - \frac{i\gamma}{E}\right\}}\int d^d\mathbf{p}'\,\frac{\psi^{\mathbf{q}}_{\mathbf{p}'}(E)}{|\mathbf{p}-\mathbf{p}'|^2}. \tag{2.39}$$

where $\lambda_E = 4m^2/(2\pi)^d(2mE)^{(d/2)-3}\lambda(E)$. The integration in (2.39) is over d-dimensional sphere of radius $\sim p_0/\sqrt{2mE}$, where $p_0 \sim a^{-1}$, and a is the "minimum length" in the system.

The homogeneous integral equation in (2.39) resembles a similar equation in the Wick–Cutkosky relativistic model [45–47], which describes the bound states of the scalar particles produced as a result of massless quantum exchange. We can easily transform to a symmetrized equation

$$\tilde{\psi}^{\mathbf{q}}_{-\mathbf{p}}(E) = \lambda_E \int d^d\mathbf{p}' K^E_{\mathbf{q}}(\mathbf{p}, \mathbf{p}')\tilde{\psi}^{\mathbf{q}}_{\mathbf{p}'}(E), \tag{2.40}$$

where

$$\begin{gathered}\tilde{\psi}^{\mathbf{q}}_{-\mathbf{p}}(E) = \sqrt{R^{-1}_{\mathbf{q}}(\mathbf{p})}\ \psi^{\mathbf{q}}_{-\mathbf{p}}(E) \\ R_{\mathbf{q}}(\mathbf{p}) = \left\{1 - \left(\mathbf{p} + \frac{1}{2}\mathbf{q}\right)^2 + \frac{i\gamma}{E}\right\}^{-1} \left\{1 - \left(\mathbf{p} - \frac{1}{2}\mathbf{q}\right)^2 - \frac{i\gamma}{E}\right\}^{-1} \\ R_{\mathbf{q}}(\mathbf{p}) = R^*_{\mathbf{q}}(-\mathbf{p}); \qquad R_{\mathbf{q}}(\mathbf{p}) = R^*_{-\mathbf{q}}(\mathbf{p}).\end{gathered} \tag{2.41}$$

Here

$$K^E_{\mathbf{q}}(\mathbf{p}, \mathbf{p}') = R^{1/2}_{\mathbf{q}}(\mathbf{p})R^{1/2}_{\mathbf{q}}(-\mathbf{p}')\frac{1}{|\mathbf{p} - \mathbf{p}'|^2} \tag{2.42}$$

is a symmetrical (Hermitian) positive kernel [48]:

$$\begin{gathered}K^E_{\mathbf{q}}(\mathbf{p}, \mathbf{p}') = K^{*E}_{\mathbf{q}}(\mathbf{p}', \mathbf{p}); \qquad K^E_{-\mathbf{q}}(\mathbf{p}, \mathbf{p}') = K^{*E}_{\mathbf{q}}(\mathbf{p}, \mathbf{p}') \\ K^E_{\mathbf{o}}(\mathbf{p}, \mathbf{p}') > 0.\end{gathered} \tag{2.43}$$

It is clear that

$$|K^E_{\mathbf{q}}(\mathbf{p}, \mathbf{p}')| < \frac{E^2}{\gamma^2}\frac{1}{|\mathbf{p} - \mathbf{p}'|^2}. \tag{2.44}$$

We see that at $2 < d < 4$ this equation is an equation with a weak singularity [48] and at $d \geqslant 4$ we have the Fredholm equation. The integral kernel which we are considering, being a fully continuous operator, has a finite or denumerable eigenvalue spectrum which lies on the segment of the real axis. The length of this segment is determined by the norm of the integral operator $\|K\|$ which in this case is given by

$$\|K\psi\|^2 \equiv \left\{ \int d^d\mathbf{p}' K_{\mathbf{q}}^{E}(\mathbf{p}, \mathbf{p}')\psi_{\mathbf{p}'}^{\mathbf{q}}(E) \right\}^2 \leqslant \|K\|^2 \left\{ \int d^d\mathbf{p}' \psi_{\mathbf{p}'}^{\mathbf{q}}(E) \right\}^2. \tag{2.45}$$

This inequality is valid for any quadratically integrable function $\psi_{\mathbf{p}}^{\mathbf{q}}(E)$. The integrability of $\psi_{\mathbf{p}}^{\mathbf{q}}(E)$ is necessary for the solution of (2.39) to define $\chi_{\mathbf{q}}(E)$ from (2.28). It has also been established elsewhere [48] that the first eigenvalue of the kernel under consideration is a simple, positive eigenvalue, while the corresponding eigenfunction is a positive definite eigenfunction. We see from (2.44) that the norm of the operator under consideration is

$$\|K\| < \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{E^2}{\gamma^2} \frac{D_\Omega^{d-2}}{d-2}, \tag{2.46}$$

where D_Ω is the diameter of the integration range. It may seem that $D_\Omega \sim p_0/\sqrt{2mE}$. We easily see, however, that a factor such as $\{(1-\mathbf{p}'^2)^2 + \gamma^2/E^2\}^{-1/2}$, which at $\gamma \ll E$ has a sharp peak when $\mathbf{p}' \simeq 1$, whose width is $\sim \gamma/E$. The cutoff therefore occurs when $\mathbf{p}' \sim 1$ (i.e., $\mathbf{p}' \sim \sqrt{2mE}$ in dimensional variables), so that $D_\Omega \sim 1$. We can infer from this fact that the equation we are considering *does not have* any nontrivial solutions for

$$\lambda_E < \left\{ \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{d-2} \frac{E^2}{\gamma^2} \right\}^{-1}. \tag{2.47}$$

We see that there are no nontrivial solutions in the energy region given by

$$E > \left(\frac{A_d}{d-2} \right)^{2/(4-d)} E_{sc}. \tag{2.48}$$

In this inequality we have introduced the characteristic energy (compare with the Introduction) (Refs. 7 and 35):

$$E_{sc} = m^{d/(4-d)}(\rho V^2)^{2/(4-d)}, \tag{2.49}$$

where $A_d = 2^{1-d/2}\pi^{-d/2}d/\Gamma(d/2)$. As we have already seen in the Introduction, the condition $E \sim E_{sc}$, which is equivalent to $\gamma(E) \sim E$, define the limit of applicability of the perturbation theory. Since this choice of diagrams is not valid in the "strong-coupling" region $E \lesssim E_{sc}$, we must sum all the perturbation-theory diagrams. We see from (2.48) that the threshold energy E_c, where the first nontrivial solution of Eq. (2.38) appears, at $d = 3$ falls directly into the strong-coupling region $E < E_{sc} = m^3(\rho V^2)^2$. The energy region, where there is clearly no solution, goes to infinity in the limit $d \rightarrow 2$. This means that the first eigenvalue of Eq. (2.38), which is identified with the mobility edge, also goes to infinity in the limit $d \rightarrow 2$: $E_c \rightarrow \infty$, consistent with the proposition that a complete localization occurs at $d = 2$ [26]. Since $E_c \gg E_{sc}$, the mobility edge may seem, in the limit $d \rightarrow 2$, to be in the region where the perturbation theory is valid. We will see below that this is not true, since the condition under which the perturbation theory is valid in the limit $d \rightarrow 2$ must be defined more accurately. In fact, it coincides with inequality (2.48).

This analysis shows qualitatively that the homogeneous Bethe–Salpeter equation of the type in (2.33), which is based on the perturbation theory, cannot correctly describe the energy region corresponding to the localized states, if only because of the discrete nature of its spectrum in the region $E > E_c$. This discrete nature of the spectrum will probably be found even in more complex approximations. It is clear, however, that the location of the mobility edge, the "instability" point of the metallic phase, where appears the first nontrivial solution $\psi_{\mathbf{p}}^{\mathbf{q}}(E) \neq 0$ can be determined from this equation by some approximation for $G^{R,A}(E\mathbf{p})$ and $U_{\mathbf{pp}'}^{E}(\mathbf{q}\omega)$. We will give later a complete justification of this viewpoint.

2.2. *Self-Consistent Theory of Localization*

An important step in the evolution of the localization theory, which made it possible to calculate many physical quantities within the framework of the standard formalism of the average Green's functions, was, in our view, the development of the so-called self-consistent theory of localization. The original version of this theory was proposed by Götze [49–51]. A significant contribution to the development of this theory is attributed to Vollhardt and Wölfe [41, 52, 53] who were responsible for the diagram formulation of this theory, based on a self-consistent generalization of the formalism of the Bethe–Salpeter equation. The main advantage of this method is its simplicity and a certain automatism allowing further generalization.

Using the following relation from (2.25)

$$G^R(E+\omega\mathbf{p}_+)G^A(E\mathbf{p}_-) = -\frac{\Delta G_{\mathbf{p}}}{\omega - \frac{1}{m}\mathbf{p}\cdot\mathbf{q} - \Sigma^R(E+\omega\mathbf{p}_+) + \Sigma^A(E\mathbf{p}_-)}, \tag{2.50}$$

where

$$\Delta G_{\mathbf{p}} = G^R(E+\omega\mathbf{p}_+) - G^A(E\mathbf{p}_-) \tag{2.51}$$

and $\Sigma^{R,A}(E\mathbf{p})$ is the self-energy of the corresponding Green's functions, we rewrite the Bethe–Salpeter equation in the form

$$\begin{aligned}\left\{\omega - \frac{1}{m}\mathbf{p}\cdot\mathbf{q} - \Sigma^R(E+\omega\mathbf{p}_+) + \Sigma^A(E\mathbf{p}_-)\right\}\phi^{RA}_{\mathbf{pp}'}(E\mathbf{q}\omega) \\ = \Delta G_{\mathbf{p}}\left\{\frac{1}{2\pi i}\,\delta(\mathbf{p}-\mathbf{p}') - \sum_{\mathbf{p}''} U^E_{\mathbf{pp}''}(\mathbf{q}\omega)\phi^{RA}_{\mathbf{p}''\mathbf{p}'}(E\mathbf{q}\omega)\right\}.\end{aligned} \tag{2.52}$$

Bearing in mind that we are dealing with an electronic system at $T=0$, we will set $E=E_F$ at the outset. Summing both parts of (2.52) over $\mathbf{p}$ and $\mathbf{p}'$, and using the exact Ward identity [41]

$$\Sigma^R(E+\omega\mathbf{p}_+) - \Sigma^A(E\mathbf{p}_-) = \sum_{\mathbf{p}'} U^E_{\mathbf{pp}'}(\mathbf{q}\omega)\Delta G_{\mathbf{p}'}, \tag{2.53}$$

we find

$$\omega\phi^{RA}_{E_F}(\mathbf{q}\omega) - \tilde{\phi}^{RA}_{E_F}(\mathbf{q}\omega) = -N(E_F), \tag{2.54}$$

where

$$\tilde{\phi}^{RA}_{E_F}(\mathbf{q}\omega) = \sum_{\mathbf{p}\mathbf{p}'} \frac{1}{m} (\mathbf{p}\cdot\mathbf{q}) \phi^{RA}_{\mathbf{p}\mathbf{p}'}(E_F\mathbf{q}\omega). \tag{2.55}$$

Analogously, multiplying (2.52) by $1/m(\mathbf{p}\cdot\mathbf{q})$ and summing again over $\mathbf{p}$, making use of (2.53) and the approximate representation [41]

$$\sum_{\mathbf{p}'} \phi^{RA}_{\mathbf{p}\mathbf{p}'}(E_F\mathbf{q}\omega) \approx -\frac{\Delta G_{\mathbf{p}}}{2\pi i N(E_F)} \times \sum_{\mathbf{p}'\mathbf{p}''} \left\{ 1 + \frac{d}{\mathbf{p}_F^2\mathbf{q}^2} (\mathbf{p}\cdot\mathbf{q})(\mathbf{p}''\cdot\mathbf{q}) \right\} \phi^{RA}_{\mathbf{p}''\mathbf{p}'}(E_F\mathbf{q}\omega), \tag{2.56}$$

in which we have left only the first two terms of the expansion in the corresponding angular variables,* we obtain the following expression, after several transformations:

$$\{\omega + M_{E_F}(\mathbf{q}\omega)\}\tilde{\phi}^{RA}_{E_F}(\mathbf{q}\omega) - \frac{2E_F}{dm}\mathbf{q}^2\phi^{RA}_{E_F}(\mathbf{q}\omega) = 0. \tag{2.57}$$

In this expression we have introduced a current relaxation kernel [41],

$$M_{E_F}(\mathbf{q}\omega) = 2i\gamma(E_F) + \frac{id}{2\pi N(E_F)\mathbf{p}_F^2\mathbf{q}^2} \times \sum_{\mathbf{p}\mathbf{p}'} (\mathbf{p}\cdot\mathbf{q})\Delta G_{\mathbf{p}} U^{E_F}_{\mathbf{p}\mathbf{p}'}(\mathbf{q}\omega)\Delta G_{\mathbf{p}'}(\mathbf{p}'\cdot\mathbf{q}). \tag{2.58}$$

In (2.58) we have used the simplest approximation (2.34) in writing out the explicit expressions for $\Sigma^{R,A}(E\mathbf{p})$. We can thus switch from the exact Bethe–Salpeter equation for $\phi^{RA}_{\mathbf{p}\mathbf{p}'}(E_F\mathbf{q}\omega)$ to an approximate but *closed* system of equations (2.54) and (2.57), whose solution immediately gives the function $\phi^{RA}_{E_F}(\mathbf{q}\omega)$, which, according to the analysis in the preceding section, determine all the relevant characteristics of the system,

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*The approximate validity (in the limits $\omega \to 0$ and $\mathbf{q} \to 0$) of (2.56) can easily be verified by summing both parts over $\mathbf{p}$, after multiplying them by $(\mathbf{p}\cdot\mathbf{q})$.

$$\phi^{RA}_{E_F}(\mathbf{q}\omega) = -N(E_F)\frac{\omega + M_{E_F}(\mathbf{q}\omega)}{\omega^2 + \omega M_{E_F}(\mathbf{q}\omega) - \dfrac{2E_F}{dm}\mathbf{q}^2}. \tag{2.59}$$

Let us examine how we can derive the localization phenomenon from these relations. We see that under the conditions of existence of the limit

$$R^2_{\mathbf{q}}(E_F) = -\frac{2E_F}{md}\lim_{\omega\to 0}\frac{1}{\omega M_{E_F}(\mathbf{q}\omega)} > 0 \tag{2.60}$$

we can extract from (2.59) a singular contribution of the type shown in (2.27)

$$\begin{aligned}\phi^{RA}_{E_F}(\mathbf{q}\omega) &\underset{\omega\to 0}{\approx} -\frac{N(E_F)}{\omega}\frac{1}{1 - \dfrac{2E_F}{md}\dfrac{\mathbf{q}^2}{\omega M_{E_F}(\mathbf{q}\omega)}} \\ &= -\frac{N(E_F)}{\omega}\frac{1}{1 + R^2_{\mathbf{q}}(E_F)\mathbf{q}^2} = -\frac{1}{\omega}\chi_{\mathbf{q}}(E_F)\chi_{-\mathbf{q}}(E_F).\end{aligned} \tag{2.61}$$

where

$$\chi_{\pm\mathbf{q}}(E_F) = \frac{\sqrt{N(E_F)}}{1 + i\mathbf{q}R_{\mathbf{q}}(E_F)}. \tag{2.62}$$

These expressions satisfy all the general requirements of (2.29)–(2.31), in particular

$$\begin{aligned}A_{E_F}(\mathbf{q}) &= \frac{1}{N(E_F)}\chi_{\mathbf{q}}(E_F)\chi_{-\mathbf{q}}(E_F) = \frac{1}{1 + R^2_{\mathbf{q}}(E_F)\mathbf{q}^2} \\ &\underset{\mathbf{q}\to 0}{\approx} 1 - R^2_{\text{loc}}(E_F)\mathbf{q}^2,\end{aligned} \tag{2.63}$$

where the localization length is expressed as follows:

$$R^2_{\text{loc}}(E_F) = -\frac{2E_F}{md}\lim_{\omega\to 0}\frac{1}{\omega M_{E_F}(0\omega)}. \tag{2.64}$$

It is also useful to introduce the characteristic frequency [41]

$$\omega_0^2(E_F) = -\lim_{\omega \to 0} \omega M_{E_F}(0\omega) > 0, \tag{2.65}$$

so that

$$R_{\text{loc}}(E_F) = \sqrt{\frac{2E_F}{md}}\, \omega_0^{-1}(E_F). \tag{2.66}$$

We see that the localization is equivalent to the requirement that the $\lim_{\omega \to 0} \omega M_{E_F}(0\omega)$ be finite, i.e., that $R_{\text{loc}}(E_F)$ or $\omega_0(E_F)$ be finite. In other words, it is equivalent to the requirement that the current relaxation kernel (2.58) be *divergent*: $\operatorname{Re} M_{E_F}(0\omega) = -\omega_0^2(E_F)/\omega$ in the limit $\omega \to 0$. Such a behavior of the relaxation kernel was regarded as the localization criterion in Refs. 41 and 49–53. We have derived it here from slightly more general considerations based on the localization criterion suggested above.

It is easy to see [41] that for small values of ω and $\mathbf{q}$ the retarded density–density response function $\chi^R(\mathbf{q}\omega)$ is given by

$$\chi^R(\mathbf{q}\omega) = \omega\phi_{E_F}^{RA}(\mathbf{q}\omega) + N(E_F) + 0(\omega, \mathbf{q}^2). \tag{2.67}$$

We then find from (2.59)

$$\chi^R(\mathbf{q}\omega) = -N(E_F)\frac{\dfrac{2E_F}{md}\mathbf{q}^2}{\omega^2 + \omega M_{E_F}(\mathbf{q}\omega) - \dfrac{2E_F}{md}\mathbf{q}^2}. \tag{2.68}$$

Ignoring ω^2 in the denominators of (2.59) and (2.68), we can write for small values of ω

$$\phi_{E_F}^{RA}(\mathbf{q}\omega) = -N(E_F)\frac{1}{\omega + iD_{E_F}(\mathbf{q}\omega)\mathbf{q}^2}, \tag{2.69}$$

$$\chi^R(\mathbf{q}\omega) = N(E_F)\frac{iD_{E_F}(\mathbf{q}\omega)\mathbf{q}^2}{\omega + iD_{E_F}(\mathbf{q}\omega)\mathbf{q}^2}. \tag{2.70}$$

In these expressions we have introduced, by definition, the *generalized* diffusion coefficient [41]

$$D_{E_F}(\mathbf{q}\omega) = i\frac{2E_F}{dm}\frac{1}{M_{E_F}(\mathbf{q}\omega)}. \tag{2.71}$$

We can then find from (2.13) and (2.68) a general expression for the conductivity

$$\sigma(\omega) = \frac{ne^2}{m}\frac{i}{\omega + M_{E_F}(0\omega)} \xrightarrow[\omega\to 0]{} e^2 D_{E_F}(00)N(E_F). \tag{2.72}$$

In the first part of this expression we assumed that $n/N(E_F) = 2E_F/d$, where n is the electron density. We see, in particular, that the usual metallic behavior (nonvanishing static conductivity at $T=0$) corresponds to $M_{E_F}(0\omega\to 0) = i/\tau_{E_F}$, where τ_{E_F} serves as the generalized mean free time.

This discussion leads to the following behavior of the current relaxation kernel for $\mathbf{q}=0$ and $\omega\to 0$:

$$M_{E_F}(0\omega) \simeq \begin{cases} \dfrac{i}{\tau_{E_F}}; & E_F > E_c \quad \text{(Metal)} \\ \dfrac{i}{\tau_{E_F}} - \dfrac{\omega_0^2(E_F)}{\omega}; & E_F < E_c \quad \text{(Insulator)}. \end{cases} \tag{2.73}$$

Incorporating (2.73) into (2.59), using relation (2.18) and ignoring for the present the component Im $\phi_E^{RR}(\mathbf{q}\omega)$, which is nonsingular in the limits $\omega\to 0$ and $\mathbf{q}=0$, we find the explicit expression for the Berezinskii–Gor'kov spectral density

$$\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_{\mathbf{p}} = \begin{cases} \dfrac{1}{\pi}\dfrac{D_{E_F}\mathbf{q}^2}{\omega^2 + (D_{E_F}\mathbf{q}^2)^2}; & E_F > E_c \\ A_{E_F}(\mathbf{q})\delta(\omega) + \dfrac{1}{\pi}\dfrac{D_{E_F}\mathbf{q}^2}{\omega^2 + [\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^2}; & E_F < E_c, \end{cases} \tag{2.74}$$

where [compare with Eq. (2.63)]

$$A_{E_F}(\mathbf{q}) = \frac{\omega_0^2(E_F)\tau_{E_F}}{\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2} = \frac{1}{1 + R_{\mathrm{loc}}^2(E_F)\mathbf{q}^2}. \tag{2.74$'$}$$

In this expression we have introduced, by definition, the *renormalized* diffusion coefficient

$$D_{E_F} = \frac{2E_F}{dm}\,\tau_{E_F} = \frac{1}{d}\,V_F^2\tau_{E_F}, \tag{2.75}$$

where v_F is the velocity of the electron at the Fermi surface.

Our task now is to find the behavior of the type of (2.73), from the *solution* of the general equations of the microscopic theory. After making use of approximation (2.34) for single-electron Green's functions, the current relaxation kernel $M_{E_F}(\mathbf{q}\omega)$ will be determined entirely by the approximation for the vertex $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$ in (2.58). The Vollhardt–Wolfle approximation [41] is based on the use of expression (2.36) for $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$, which is determined by the summation of the Langer–Neal diagrams (Fig. 4b). A transformation of the second term in (2.58) gives rise to the following contribution to $M_{E_F}(0\omega)$:

$$\delta M_{E_F}^{c}(0\omega) = -2\rho V^2 \sum_k \frac{1}{\omega + iD_0^{E_F}k^2}. \tag{2.76}$$

Expression (2.76), however, falls short of achieving a localization. The basic idea behind the self-consistent theory of localization involves the replacement of the classical diffusion coefficient in the denominator of (2.76) by the generalized diffusion coefficient (2.71), which in turn is defined in terms of the current relaxation kernel [49–53]. As a result, we obtain the *self-consistent* equation for the kernal $M_{E_F}(0\omega)$ (Refs. 41 and 53)

$$\begin{aligned} M_{E_F}(\omega) &= 2i\gamma(E_F)\left\{1 + \frac{1}{\pi N(E_F)} \sum_{|\mathbf{k}|<k_0} \frac{1}{-i\omega + D_{E_F}(\omega)k^2}\right\}, \\ D_{E_F}(\omega) &= i\,\frac{2E_F}{dm}\,\frac{1}{M_{E_F}(\omega)}. \end{aligned} \tag{2.77}$$

We will discuss in detail below how the choice of the cutoff momentum k_0 in (2.77) is made. The most rigorous derivation of Eq. (2.77), based on a general diagram analysis, in which the less singular components for $d \geqslant 2$ were discarded in the limit $\omega \to 0$, was made by Wölfle and Vollhardt [53]. We will analyze below several simplest corrections to (2.77).

Equation (2.77) was initially analyzed [41, 52] only for the two-dimensional case. The approximation based on the summation of the Langer–Neal diagrams was specially adapted for this case, since these diagrams lead to the contributions which are predominant in the limit $d \to 2$ (see the discussion below). We will analyze its solution, however, for $d > 2$, which was carried out independently in Refs. 53 and 54. In this case the results describing the location of the mobility edge in the band, as well as the behavior of all the principal physical quantities near it turn out, as we will see below, to be entirely justifiable and presumably qualitatively correct.

We see from the definition of $M_{E_F}(\mathbf{q}\omega)$ in (2.58) that since $\Delta G_{\mathbf{p}} \sim \operatorname{Im} G^R(E\mathbf{p}) \sim \delta(E_F - p^2/2m)$ [for $\gamma(E_F) \ll E_F$], the total momentum $\mathbf{k} = \mathbf{p} + \mathbf{p}'$ in $U^{E_F}_{\mathbf{pp}'}(\mathbf{q}\omega)$, which is defined by expression (2.36), being an integration momentum in (2.77) varies (in terms of the modulus) from zero to the momentum of order $2p_F$. On the other hand, expression (2.36) is, as was noted above, valid only for $|\mathbf{p} + \mathbf{p}'| < l^{-1}$, where $l \sim v_F \gamma^{-1}$ is the mean free path of an electron. We can therefore clearly see that the cutoff momentum in (2.77) is

$$k_0 \sim \operatorname{Min}\{p_F, l^{-1}\} \sim \operatorname{Min}\{\sqrt{2mE_F};\, m^{(d+1)/2} \rho V^2 E_F^{(d-3)/2}\}. \tag{2.78}$$

It is evident that for $2 < d < 4$ the cutoff momentum is defined, in order of magnitude, by the Fermi momentum in the limit $E_F \to 0$ (the Fermi energy decreases toward the band edge) (Ref. 54; see also Refs. 49–51):

$$k_0 = x_0 p_F = x_0 \sqrt{2mE_F}, \tag{2.79}$$

where $x_0 = \text{const} \sim 1$–$2$. Such a choice of the cutoff is, in our opinion, unambiguous; however, the viewpoint of the authors of Refs. 41, 52, and 53 is not entirely clear in this regard. However, since $E_F \sim \gamma(E_F)$, we have $p_F \sim l^{-1}$ near the mobility edge, as was noted above, so that both choices are equivalent.

Switching in (2.77) to a dimensionless integration variable, we can write this equation in a form that is convenient for calculation

$$M_{E_F}(\omega) = 2i\gamma(E_F) + d\lambda x_0^{d-2} M_{E_F}(\omega) \int_0^1 dy\, y^{d-1} \frac{1}{y^2 - \dfrac{d\omega}{4x_0^2 E_F^2} M_{E_F}(\omega)}, \tag{2.80}$$

where

$$\lambda = \frac{\gamma(E_F)}{\pi E_F} = \left(\frac{m}{2\pi}\right)^{d/2} \frac{E_F^{(d/2)-1}}{\Gamma\left(\frac{d}{2}\right)} \rho V^2$$

$$= \frac{1}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \left(\frac{E_F}{E_{sc}}\right)^{(d-4)/2} \tag{2.81}$$

is the dimensionless "coupling constant" of the theory in which we used the explicit form of the density of states of the free electrons,

$$N(E) = \left(\frac{m}{2\pi}\right)^{d/2} \frac{E^{(d/2)-1}}{\Gamma\left(\frac{d}{2}\right)},$$

as well as definition (2.49). The normal condition under which the perturbation theory is applicable implies that the inequality $\lambda \ll 1$ holds, i.e., $E_F \gg E_{sc}$ (for $d < 4$).

Assuming that $\omega = 0$ in (2.80) and analyzing the metallic regime in which, according to (2.73), Re $M_{E_F}(\omega = 0) = 0$ and Im $M_{E_F}(\omega = 0) = \tau_{E_F}^{-1}$, we find

$$\frac{i}{\tau_{E_F}} = 2i\gamma(E_F) + \frac{d}{d-2} \lambda x_0^{d-2} \frac{i}{\tau_{E_F}},$$

so that

$$\tau_{E_F} = \frac{1}{2\gamma(E_F)} \left\{ 1 - \frac{d}{d-2} \lambda x_0^{d-2} \right\}. \tag{2.82}$$

We then find from (2.72) the static conductivity

$$\sigma=\frac{ne^2}{m}\frac{1}{2\gamma(E_F)}\left\{1-\left(\frac{E_c}{E_F}\right)^{(4-d)/2}\right\};\quad 2<d<4, \tag{2.83}$$

where

$$E_c=\left\{\frac{d}{d-2}\frac{x_0^{d-2}}{\Gamma\left(\frac{d}{2}\right)}(2\pi)^{-d/2}\right\}^{2/(4-d)}E_{sc}. \tag{2.84}$$

The energy E_{sc}, which is defined in (2.49), again appears in this case. We see from (2.83) that E_c serves as the mobility edge. We find that at $E_F \gtrsim E_c$

$$\sigma\approx\frac{ne^2}{m}\frac{1}{2\gamma(E_F)}\left(\frac{4-d}{2}\right)\left(\frac{E_F-E_c}{E_c}\right);\quad 2<d<4, \tag{2.85}$$

so that the conductivity vanishes at the mobility edge linearly in E_F-E_c. The result in (2.84) coincides, to within a constant, with the estimate of E_c in (2.48), which is based on a different line of reasoning. For $d=3$ the threshold energy lies in the "strong-coupling" region $E_{sc}=m^3(\rho V^2)^2$, where our choice of diagrams is generally not correct [7, 35] and all the perturbation-theory diagrams must be taken into account. We clearly see this from the fact that the requirement $\lambda \ll 1$, according to (2.81), is equivalent to the condition $E \gg E_{sc}$. In the limit $d\rightarrow 2\, E_c\rightarrow\infty$, consistent with the total localization in the case of arbitrarily small ($\lambda \ll 1$) disorder. The applicability of the perturbation theory in the limit $d\rightarrow 2$ will be discussed in greater detail below.

It is pertinent to analyze now the role of some diagrams which were not considered above. The irreducible vertex $U^{E_F}_{\mathbf{pp}'}(\mathbf{q}\omega)$ in (2.58) can be represented in a more complex form by supplementing the "cooperon" contribution (2.36) (Fig. 4b) by the "diffusion" contribution, i.e., by the diagrams (Fig. 5a) that contain the "diffusion" propagator [41, 42] (Fig. 5b, $\mathbf{q}\rightarrow 0$ and $\omega\rightarrow 0$):

$$\Gamma^{E_F D}_{\mathbf{pp}'}(\mathbf{q}\omega)=\frac{2\gamma(E_F)\rho V^2}{-i\omega+D_0^{E_F}\mathbf{q}^2}. \tag{2.86}$$

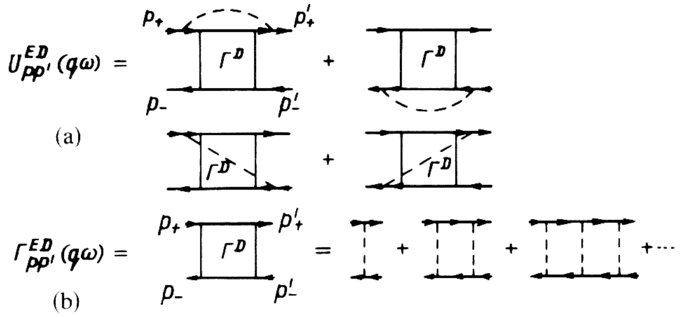


Figure 5. (a) A simplest "diffusion" contribution to the irreducible vertex $U^{E}_{\mathbf{pp}'}(\mathbf{q}\omega)$. (b) Diagrams for a diffusion "propagator" ("ladder" approximation).

The corresponding contribution to $M_{E_F}(\mathbf{q}\omega)$ in (2.58) is [41]

$$\delta M^D_{E_F}(\mathbf{q}\omega) = -\frac{d}{\mathbf{p}_F^2}\rho V^2 \sum_{\mathbf{k}} \frac{\mathbf{q}^2 + \mathbf{q}^{-2}(k\cdot\mathbf{q})}{\omega + iD_0^{E_F}k^2}. \tag{2.87}$$

Using this contribution and performing the self-consistency operation, we obtain the following equation for $M_{E_F}(\omega)$, instead of (2.77):

$$\begin{aligned} M_{E_F}(\omega) &= 2i\gamma(E_F)\left\{1 + \frac{1}{\pi N(E_F)} \sum_{|\mathbf{k}|<k_0} \frac{1 + k^2/2\mathbf{p}_F^2}{-i\omega + D_{E_F}(\omega)k^2}\right\} \\ D_{E_F}(\omega) &= i\frac{2E_F}{dm}\frac{1}{M_{E_F}(\omega)}. \end{aligned} \tag{2.88}$$

By analogy with (2.82), we then find for $\omega = 0$

$$\tau_{E_F} = \frac{1}{2\gamma(E_F)}\left\{1 - \left[\frac{d}{d-2} + \frac{x_0^2}{2}\right] x_0^{d-2}\lambda\right\} \tag{2.89}$$

in the metallic region. We see that the use of the diagrams in Fig. 5a changes the *constant* in (2.82). The cooperon contribution is dominant in the limit $d \to 2$, whereas at $d = 3$ the diffusion contribution is *of the same order of magnitude* as the cooperon contribution. The mobility edge shifts slightly: $d/d-2$ is replaced by

$d/d-2+x_0^2/2$ in (2.84). The "critical behavior," however, remains the same: conductivity goes down to zero in accordance with (2.85). We will therefore confine ourselves to approximation (2.77), which we feel "works" for $d \to 2$. At $d=3$, *all* the perturbation-theory diagrams must evidently be summed, both for $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$ and for a single-electron Green's function, which is extremely important. The mobility edge $E_c \to \infty$ for $d \to 2$, and it may seem that we could restrict ourselves to the simplest approximation (2.34) in the case of a single-electron Green's function, since the condition $E_c \gg \gamma(E_c)$ in this case is satisfied. Actually the situation is, as we will see below, more complex even for $d \to 2$.

The use of approximation (2.36) for $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$, along with approximation (2.34) for $G^{R,A}(E\mathbf{p})$, we might note, is generally in contradiction with Ward's identity (2.53). The use of "crossed" diagrams for $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$ implies that the same diagrams must also be used for $\Sigma^{R,A}(E\mathbf{p})$ [see the derivation of (2.53) in Ref. 41]. At the same time, several examples of exactly solvable models [55–57], in which it was possible to sum the crossed diagrams (such as those in Fig. 4b) for $\Sigma^{R,A}(E\mathbf{p})$ show that the single-electron Green's function obtained in this manner has a structure radically different from that given by the simplest approximation (2.34). We also clearly see this from the fact that (2.34) does not give a correct single-electron state density such as the one shown in Fig. 1b. Specifically, Eq. (2.34) does not account for the "tail" in the density of states, which appears in the perturbation theory [55, 56] only when the crossed diagrams are taken into account (see the discussion below).

All these factors can, in our opinion, change the critical behavior such as that in (2.83), whereas the estimated location of the mobility edge (2.84) apparently remains the same (to within a constant) even in a more precise theory.

Turning our attention now to the region of localized states ($E_F < E_c$), we will seek a solution of Eq. (2.80) in the form of the second expression in (2.73) and find an equation, which determines $\omega_0^2(E_F)$, from the real part of (2.80) in the limit $\omega \to 0$,

$$1 = d\lambda x_0^{d-2} \int_0^1 dy\, y^{d-1} \frac{1}{y^2 + \dfrac{d\omega_0^2(E_F)}{4(x_0 E_F)^2}}. \qquad (2.90)$$

Analogously, we find an equation for τ_{E_F} in the localization region from the imaginary part of (2.80) in the limit $\omega \to 0$

$$1 - 2\gamma(E_F)\tau_{E_F} = d\lambda x_0^{d-2} \int_0^1 dy\, y^{d+1} \frac{1}{\left[y^2 + \dfrac{d\omega_0^2(E_F)}{4(x_0 E_F)^2} \right]^2}. \tag{2.91}$$

Using (for $\omega_0^2 \to 0$) a simplest estimate of the integral in (2.90), we find

$$1 \simeq d\lambda x_0^{d-2} \int_{[(d\omega_0^2(E_F))/(4x_0^2E_F^2)]^{1/2}}^{1} dy\, y^{d-3}$$
$$= \begin{cases} d\lambda x_0^{d-2} \dfrac{1}{d-2} \left\{ 1 - \left(\dfrac{d\omega_0^2(E_F)}{4x_0^2E_F^2} \right)^{(d-2)/2} \right\}; & 2 < d < 4 \\ \lambda \ln \dfrac{2x_0^2E_F}{\omega_0^2(E_F)}; & d = 2. \end{cases} \tag{2.92}$$

Using (2.84), we then find

$$\omega_0^2(E_F) = \begin{cases} \dfrac{4}{d} x_0^2 E_F^2 \left\{ 1 - \left(\dfrac{E_F}{E_c} \right)^{(4-d)/2} \right\}^{2/(d-2)}; & 2 < d < 4 \\ 2x_0^2E_F^2 \exp\left[-\dfrac{1}{\lambda} \right]; & d = 2. \end{cases} \tag{2.93}$$

The location of the mobility edge is given by the condition $\omega_0^2(E_F = E_c) = 0$. The integral in (2.90) can also be calculated exactly [54], then the first expression in (2.93) is multiplied by

$$\left[\frac{d}{d-2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(2 - \frac{d}{2}\right) \right]^{-2/(d-2)} \quad (\sim 1 \text{ for } d = 3),$$

while the second expression remains the same [53, 54]. Using the "unity representation" (2.90) in (2.91), we find a relationship between τ_{E_F} and $\omega_0^2(E_F)$:

$$2\gamma(E_F)\tau_{E_F} = d\lambda x_0^{d-2}\frac{d\omega_0^2(E_F)}{4x_0^2E_F^2}\int_0^1 dy\,\frac{y^{d-1}}{\left[y^2+\dfrac{d\omega_0^2(E_F)}{4x_0^2E_F^2}\right]^2}$$

$$\approx d\lambda x_0^{d-2}\frac{1}{d-4}\left\{\frac{d\omega_0^2(E_F)}{4x_0^2E_F^2}-\left(\frac{d\omega_0^2(E_F)}{4x_0^2E_F^2}\right)^{(d-2)/2}\right\};\quad d<4, \tag{2.94}$$

so that as $\omega_0^2(E_F)\to 0$, i.e., near the mobility edge we have

$$2\gamma(E_F)\tau_{E_F}\approx\begin{cases}\dfrac{d}{4-d}\lambda x_0^{d-2}\left(\dfrac{d\omega_0^2(E_F)}{4x_0^2E_F^2}\right)^{(d-2)/2}; & 2<d<4\\[2ex] \lambda\left\{1-\dfrac{\omega_0^2(E_F)}{2x_0^2E_F^2}\right\}; & d=2.\end{cases} \tag{2.95}$$

We find from (2.66) and (2.93) the localization radius

$$R_{\text{loc}}(E_F)=\frac{1}{x_0\sqrt{2mE_F}}\left\{1-\left(\frac{E_F}{E_c}\right)^{(4-d)/2}\right\}^{-1/(d-2)}$$

$$\sim\frac{1}{\sqrt{2mE_F}}\left|\frac{E_F-E_c}{E_c}\right|^{-\nu};\quad E_F\lesssim E_c\quad (2<d<4), \tag{2.96}$$

where the critical exponent of the localization radius is

$$\nu=\frac{1}{d-2}. \tag{2.97}$$

We see from (2.97) and (2.85) that this theory satisfies the Wegner scaling law for the critical exponent of the conductivity [25]:

$$t=(d-2)\nu. \tag{2.98}$$

The values of the critical exponents obtained here coincide with those obtained in the main approximation of the $\epsilon = d-2$-expansion in the elementary scaling theory of localization [26, 58]. As was shown by Vollhardt and Wolfle [53], the basic equations of this scaling theory can be derived directly from the equations of the self-consistent theory [53]. We feel, however, that these values of the critical exponents should not be taken too seriously, since they were determined by going beyond the limits of the perturbation theory, which was used as the basis for derivation of the basic equations of the theory.

Let us examine the dielectric constant as an example of a physical quantity calculated in the localized region. Using the relationship between the dielectric constant and the density–density response function [59]

$$\epsilon(\mathbf{q}\omega) = 1 - \frac{4\pi e^2}{\mathbf{q}^2}\chi^R(\mathbf{q}\omega), \tag{2.99}$$

we find from (2.68), (2.93), and (2.96)

$$\epsilon(0\omega \to 0) = 1 + \frac{\omega_p^2}{\omega_0^2(E_F)} = 1 + \kappa_D^2 R_{\mathrm{loc}}^2(E_F) \sim \left|\frac{E_F - E_c}{E_c}\right|^{-2\nu}, \tag{2.100}$$

where $\omega_p^2 = 4\pi n e^2/m$ is the square of the plasma frequency, and $\kappa_D^2 = 4\pi e^2 N(E_F)$ is the inverse square of the screening length of the metal. Expression (2.100) gives the divergence of the dielectric constants as the insulator-metal transition is approached.

For $d = 2$ we find from (2.66) and (2.93) (Ref. 41)

$$R_{\mathrm{loc}}(E_F) = \frac{1}{x_0\sqrt{2mE_F}} \exp\left\{\frac{\pi E_F}{m\rho V^2}\right\}. \tag{2.101}$$

Thus the localization length in this case is exponentially large for $E \gtrsim m\rho V^2$, and we have a "weak localization." We feel it worthwhile to give detailed results on the frequency dependence of the conductivity for $d = 2$. (The equations for the frequency dependence of the conductivity for $d > 2$, which appear in this theory, may be found in Refs. 53 and 60.) A direct though slightly laborious analysis of Eq. (2.80) for $d = 2$ reveals that several frequency intervals, in which the conductivity behaves in a markedly different manner, can be singled out [54]. For extremely low frequencies $\omega \ll \lambda^{-1}\exp(-\lambda^{-1})\gamma(\lambda \ll 1)$ we find [41]

$$\sigma(\omega) \approx \frac{ne^2}{m}\frac{\gamma}{\lambda}\exp\left(\frac{2}{\lambda}\right)\frac{\omega^2}{2(x_0E_F)^4}. \tag{2.102}$$

At slightly higher frequencies $\lambda^{-1}\exp(-\lambda^{-1})\gamma \ll \omega \ll \lambda^{-2}\exp(-\lambda^{-1})\gamma$ we have

$$\sigma(\omega) \approx \frac{ne^2}{m} \exp\left(\frac{1}{\lambda}\right) \frac{\omega}{2(x_0 E_F)^2}. \tag{2.103}$$

With further increase of the frequency, we find that for $\lambda^{-2}\exp(-\lambda^{-1})\gamma \ll \omega \ll \lambda^2\gamma$ conductivity behaves in a logarithmic manner as obtained in Ref. 43:

$$\sigma(\omega) = \frac{ne^2}{m} \frac{1}{2\gamma} \left\{ 1 - \lambda \ln \frac{2\gamma}{\omega} \right\}. \tag{2.104}$$

Finally, for $\lambda^2\gamma \ll \omega \ll \gamma$ the self-consistent theory gives

$$\sigma(\omega) \approx \frac{ne^2}{m} \frac{1}{2\gamma} \left\{ 1 - \frac{\tilde{E}_c}{E_F} \right\}, \tag{2.105}$$

where $\tilde{E}_c \approx (m/\pi)\rho V^2 x^*$, with x^* defined as the biggest of the roots of the equation $x^* = \ln(x_0^2/\pi) + 2\ln x^*$ (solution exists for $x_0 > \sqrt{\pi}$). We see that the conductivity in this frequency range is essentially the same as the constant (frequency independent) metallic conductivity with the "mobility edge" $\tilde{E}_c$. This circumstance may conceivably reconcile certain discrepancies between the numerical data on two-dimensional conductivity obtained by different authors [7] and the concept of total two-dimensional localization: logarithmic corrections and the dielectric-like behavior are seen only at extremely low frequencies, although there is a broad (since λ is small) frequency range, within which a finite mobility edge $\tilde{E}_c$ can be "simulated" in the system. At $E_F \gtrsim \tilde{E}_c$, the localization length, as we have seen, also begins to increase exponentially. However, $d = 2$ plays the role of the "lower critical dimensionality."

For $d > 4$ an analysis of Eq. (2.80) gives physically meaningless results [54]. The origin of these results is clearly traceable to the fact that, according to (2.81), the perturbation-theory expansion in this model is in the parameter $(E_F/E_{sc})^{(d-4)/2}$. Accordingly, at $d < 4$ the expansion diverges in the limit $E_F \to 0$, while at $d > 4$ it diverges in the limit $E_F \to \infty$; i.e., there is an "inversion" of the strong- and weak-coupling regions on the energy axis. As a result, the conductivity vanishes, while the localization length becomes finite with *increasing* Fermi energy from the band edge [54]. At $d > 4$ such a behavior reveals a certain flaw of the model with a point interaction ("white noise" random-potential correlation) [61]. The limiting case $d = 4$ requires a careful analysis. Equation (2.80) implies that we always have a metallic behavior if the dimensionless coupling constant of the four-dimensional theory is small, $m^2 \rho V^2 \ll 1$. Equation (2.90) does not have any real solutions $\omega_0^2(E_F) > 0$. This seems to be true for any energy outside an exponentially small (over $m^2 \rho V^2 \ll 1$) vicinity of the band edge (see the analysis of $d = 4$ in Sec. 3). This situation changes if we assume that the cutoff parameter k_0 in (2.77) is defined by the effective range of the potential (or of the two-point correlation of random potentials), for example, rather than by the Fermi momentum. In this case it is possible to have $k_0 \ll p_F$. For $d < 4$ the results in this case are the same as those obtained above, but the mobility edge is given by the relation (for any $d > 2$!) (Ref. 54):

$$E_c = \frac{d}{d-2}\left(\frac{m}{2\pi}\right)^{d/2} \frac{\tilde{E}_0^{(d/2)-1}}{\Gamma\left(\frac{d}{2}\right)} \rho V^2 = \frac{d}{d-2}\frac{1}{\pi}\gamma(\tilde{E}_0). \tag{2.106}$$

where $\tilde{E}_0 = k_0^2/2m$. For $d \geqslant 4$ we find [54]

$$\sigma \simeq \frac{ne^2}{m}\frac{1}{2\gamma(E_F)}\frac{E_F - E_c}{E_c}; \quad E_F \gtrsim E_c \tag{2.107}$$

$$\omega_0^2(E_F) \simeq \frac{4}{d}\frac{d-4}{d-2}\left\{1 - \frac{E_F}{E_c}\right\}; \quad E_F \lesssim E_c . \tag{2.108}$$

Accordingly, using (2.66) we find that the critical exponent of the localization length is $\nu = \frac{1}{2}$ for $d > 4$. Thus, we can refer to $d = 4$ as the "upper critical dimensionality" of the localization theory.

We have analyzed a free electron in a field of random scattering centers. The self-consistent theory can be generalized to describe an electron with an arbitrary energy spectrum in a lattice with an Anderson-type disorder [3]. Such a generalization is relevant, first of all, in view of the availability of extensive literature on localization in the Anderson model [4–7]. The localization in the Anderson model was first analyzed within the context of the self-consistent theory by Götze [62] and Prelovšek [63]. Their analysis was based on Götze's formalism [49–51]. Kotov and Sadovskiĭ [64] performed such an analysis using the Vollhardt–Wölfle method. We will confine ourselves here to the discussion of the results of Ref. 64 for the case in which a lattice electron is described by the effective-mass approximation and the Anderson disorder is described by the Gaussian distribution with a width $\tilde{W}$. All the relevant results can then be determined, from the equations given above, by a simple substitution:

$$m \to m^* = \frac{1}{2Va^2}; \qquad \rho V^2 \to \tilde{W}^2 \Omega_0, \tag{2.109}$$

where m^* is the effective mass of the lattice electron for a strong-coupling approximation (V is the transfer integral between nearest neighbors, and a is the lattice constant), and Ω_0 is the volume per lattice site; we will restrict ourselves here to the lattices of cubic symmetry. A more general analysis was carried out by Kotov and Sadovskiĭ [64]. For the static conductivity in the metallic region, by analogy with (2.82) and (2.83), we then find

$$\sigma = \frac{e^2}{2\pi d}\,\frac{v_F^2}{\tilde{W}^2 \Omega_0}\left\{1 - \frac{d}{d-2}\,\lambda x_0^{d-2}\right\}. \tag{2.110}$$

In this expression the dimensionless "coupling constant" is analogous to that in (2.81):

$$\lambda = \left(\frac{m^*}{2\pi}\right)^{d/2} \frac{\tilde{W}^2 \Omega_0}{\Gamma\left(\frac{d}{2}\right)}\, E_F^{(d-4)/2}, \tag{2.111}$$

where E_F is the Fermi energy reckoned from the band edge. We easily see that the mobility edge is determined by

$$\left(\frac{\tilde{W}}{V}\right)^2 = \frac{d-2}{2}\,\Gamma\left(\frac{d}{2}\right)\left(\frac{m^*}{2\pi}\right)^{-d/2} \frac{x_0^{2-d}}{V^2 \Omega_0}\, E_F^{(4-d)/2}. \tag{2.112}$$

This is the condition under which the braces in (2.110) vanish. If the "disorder" is specified $(\tilde{W}/V)$, Eq. (2.112) gives $E_F = E_c$, the location of the mobility edge in the band. If the Fermi energy is specified, we find from (2.112) the critical ratio $(\tilde{W}/V)_c$, which corresponds to the electron localization at the Fermi surface. In particular, for a half-filled band $E_F = ZV$ (Z is the number of nearest neighbors in the lattice), we find

$$\left(\frac{\tilde{W}}{V}\right)_c^2 = \frac{d-2}{d}\,\Gamma\left(\frac{d}{2}\right)\left(\frac{m^* V}{2\pi}\right)^{-d/2}\frac{x_0^{2-d}}{\Omega_0}\,Z^{(4-d)/2} \tag{2.113}$$

and for $(\tilde{W}/V) \lesssim (\tilde{W}/V)_c$, for the mobility edge, we have (2.84); where for the model which we are analyzing we have

$$E_{sc} = (m^*)^{d/(4-d)}(\Omega_0)^{2/(4-d)}V^{4/(4-d)}\left(\frac{\tilde{W}}{V}\right)^{4/(4-d)} \sim V\left(\frac{\tilde{W}}{V}\right)^{4/(4-d)}. \tag{2.114}$$

In comparing these results with the literature on the Anderson model, we should bear in mind that our parameter $\tilde{W}^2$ represents the dispersion of the Gaussian energy distribution at the lattice sites. The dispersion is $W^2/12$ for a uniform distribution with a width W, used in Ref. 3. For the "Anderson" critical ratio we therefore find $(W/V)_c^2 = 12(\tilde{W}/V)_c^2$. This procedure gives only an approximate description of Anderson's disorder, since all the perturbation-theory diagrams connected with the higher-order cumulants of the Anderson random field have been ignored here. Table I gives the critical disorder for the total localization of all states in the band. The critical disorder was determined from (2.113) for various three-dimensional lattices of cubic symmetry, for two values of the dimensionless cutoff parameter x_0. Despite the obvious crudeness of the theory, we find our results to be in extremely good agreement with the results of the numerical calculations for the simple cubic lattice: $(W/V)_c \approx 15$ (Ref. 65), $(W/V)_c = 19 \pm 0.5$ (Ref. 66), and $(W/V)_c = 16 \pm 0.5$ (Ref. 67) for the "Anderson" disorder. We also find our results to be in excellent agreement with the results of a most accurate analysis of the localization, within the context of Anderson's method, performed by Licciardello and Economou [68]: $(W/V)_c \approx 14.5$. Our results are

Table 1. Critical Disorder Corresponding to the Localization of All the States in the Band for a Gaussian Energy Distribution at the Lattice Sites $(\tilde{W}/V)_c$ and for a Homogeneous Anderson Distribution $(W/V)_c$ in Cubic Symmetry Lattices.

			$\left(\frac{\tilde{W}}{V}\right)_c$	$\left(\frac{\tilde{W}}{V}\right)_c$	$\left(\frac{W}{V}\right)_c$	$\left(\frac{W}{V}\right)_c$
Lattice	Z	Ω_0	$x_0 = 1$	$x_0 = 2$	$x_0 = 1$	$x_0 = 2$
Simple cubic lattice	6	a^3	5.67	4.01	19.67	13.91
Body-centered-lattice	8	$\frac{a^3}{2}$	8.63	6.10	29.88	21.13
Face-centered-lattice	12	$\frac{a^3}{4}$	13.50	9.55	46.78	33.08

also in reasonably good agreement with the only result (known to us) of a numerical analysis of the Gaussian disorder, $(W/V)_c \simeq 7$ [69]. We know of no numerical calculations for bcc and fcc lattices.

Let us now consider the results for the hypercubic lattice in a d-dimensional space. In particular, for a static conductivity of a half-filled band we find from (2.110) $(2 < d < 4)$

$$\sigma = \sigma_{mm}\frac{\left|\frac{\tilde{W}}{V}-\left(\frac{\tilde{W}}{V}\right)_c\right|}{\left(\frac{\tilde{W}}{V}\right)_c};\quad \left(\frac{\tilde{W}}{V}\right)^2 \leqslant \left(\frac{\tilde{W}}{V}\right)_c^2$$
$$= \frac{d-2}{d}\,\Gamma\left(\frac{d}{2}\right)(4\pi)^{d/2}x_0^{2-d}Z^{(4-d)/2}, \tag{2.115}$$

where

$$\sigma_{mm} = \frac{4Z}{\pi d}\frac{e^2}{a^{d-2}}\left(\frac{V}{\tilde{W}}\right)_c^2, \tag{2.116}$$

essentially in agreement with the "minimal metallic conductivity" estimates by Mott [1, 2]. Specifically, in a model with "Anderson" disorder for $d = 3$ we would have $\sigma_{mm} \simeq 0.013 e^2/\hbar a \simeq 10^2\ \Omega^{-1}\ \mathrm{cm}^{-1}$ with $a = 3$ Å. Curiously enough, in the limit $d \to 2$ we would have $\sigma_{mm} \sim e^2/a^{d-2} 1/d-2 \to \infty$, since $(\tilde{W}/V)_c \to 0$ (2.113), which indicates a trend toward a total localization at $d = 2$.

For a characteristic frequency (2.65) in this model we find

$$\omega_0^2(E_F = ZV) \approx \frac{4}{d} Z^2 V^2 x_0^2 \left\{ 1 - \frac{\left(\frac{\tilde{W}}{V}\right)_c^2}{\left(\frac{\tilde{W}}{V}\right)^2} \right\}^{2/(d-2)}$$

$$\approx \frac{4}{d} Z^2 V^2 x_0^2 \left\{ 2 \frac{\frac{\tilde{W}}{V} - \left(\frac{\tilde{W}}{V}\right)_c}{\left(\frac{\tilde{W}}{V}\right)_c} \right\}^{2/(d-2)} \quad ; \quad \frac{\tilde{W}}{V} \gtrsim \left(\frac{\tilde{W}}{V}\right)_c ; \tag{2.117}$$

hence,

$$R_{\mathrm{loc}}(E_F = ZV) \simeq a \left\{ 2 \frac{\frac{\tilde{W}}{V} - \left(\frac{\tilde{W}}{V}\right)_c}{\left(\frac{\tilde{W}}{V}\right)_c} \right\}^{-(1/(d-2))} \tag{2.118}$$

for the localization length at the band center. We see that the critical index of the localization length is given by (2.97), even for localization at the band center. The necessary refinements resulting from calculation of the displacement of the original band edge by a random field are given in Ref. 64. The role of this displacement in the determination of the quantity $(\tilde{W}/V)_c$ is insignificant. The equations for all physical quantities in a two-dimensional Anderson model can also be easily written.

3. Field Theory Treatment of Localization

3.1. Effective Field Theory of an Electron in a Disordered System

A systematic analysis of the region near the mobility edge would be reasonable to carry out by constructing an effective field theory corresponding to the Anderson transition. This can be done by a method used in the analysis of the critical phenomena in the phase-transition theory [21–23]. The original ideas used for a scaling description of the mobility edge, which were advanced immediately after the development of the modern theory of critical phenomena, were formulated virtually at the same time by several authors [24, 25, 70–74]. A rather extensive literature on this subject is now available and several effective Lagrangians have been suggested for an electron near the mobility edge [7, 10]. We will examine the earlier field theory approach [24, 72], since it emphasizes rather than obscures, in our view, the difficulties encountered in a theoretical description of the problem of interest.

We will again consider the behavior of a free electron in a field of random scattering centers. If we confine ourselves to the Gaussian statistics of a random field produced by these centers, which is a valid approach if we are dealing with [75] the limit of $V \to 0$, $\rho \to \infty$, and $\rho V^2 \to \text{const}$, we easily see that the Edwards diagram expansion [11, 33, 42] for the average Green's function $G^{R,A}(E\mathbf{p})$ is generated by the standard series expansion in the perturbation theory [22] of the propagator (Green's function) of the scalar $O(n)$ symmetric Euclidean field theory with a Lagrangian [7],

$$L(\mathbf{r}) = \frac{1}{2} \sum_{j=1}^{n} \left\{ \frac{1}{2m} (\nabla \phi_j)^2 \underset{\delta \to 0^+}{-} (E \pm i\delta) \phi_j^2 \right\} - \frac{1}{8} \rho V^2 \left(\sum_{j=1}^{n} \phi_j^2 \right)^2 , \quad (3.1)$$

where m is the electron mass, and E is the electron energy. In this case the number of field components $n \to 0$ (at the end of the calculations), allowing the elimination of the "loop" diagrams which are missing in the Edwards diagram technique. This method, proposed by de Gennes [76] and developed by des Cloizeaux [77], was successfully used in the scaling theory of polymers with excluded volume. The use of an appropriate formalism has made it possible [24] to develop a scaling model of the mobility edge within the context of the Anderson method.

The Green's function corresponding to Lagrangian [3.1) can be formally determined by a functional integral (a minus sign was chosen in order to match this expression with the standard form of the Green's function for a free electron),

$$G(\mathbf{r}-\mathbf{r}'E) = -\frac{1}{Z}\lim_{n\to 0}\frac{1}{n}\sum_{j=1}^{n}\int \{\delta\phi(\mathbf{r})\}\phi_j(\mathbf{r})\phi_j(\mathbf{r}')\exp\{-S[\phi]\}, \quad (3.2)$$

where

$$S[\phi] = \int d^d\mathbf{r}\,L(\mathbf{r}) \quad (3.3)$$

is the action functional of the field theory (3.1), and the normalization integral is

$$Z = \int \{\delta\phi(\mathbf{r})\}\exp\{-S[\phi]\}. \quad (3.4)$$

In a formal calculation the functional integral in (3.2) diverges because of the "incorrect" sign of the coupling constant (attraction) [24]. This indicates that the ground state in the field theory (3.1) is unstable. For this reason, this functional integral should be understood as *an analytic continuation over the coupling constant* from an arbitrary constant $g > 0$ to a "physical" constant $g = -\rho V^2 < 0$. We will discuss this problem more thoroughly below. For now we will treat expression (3.2) simply as an abridged notation of the diagram rules in the perturbation theory for field theory (3.1).

We should point out an important feature of the method we are considering. In (3.1) there are no random parameters, and an averaging procedure in it has already been performed. For this reason, expression (3.1) is said to be an "effective" Lagrangian. On the other hand, such a convenience exacts its "toll": A calculation of a two-particle average Green's function requires a *different* effective Lagrangian. We saw that a solution of the localization problem generally requires knowledge of correlators like those in (2.19) and (2.24), which include single-electron Green's functions with different energy variables. We can easily write down an effective Lagrangian which immediately "generates" a diagram technique for a two-particle Green's function (see Fig. 3). In this case we need a Lagrangian of *two* interacting scalar fields ϕ (n components, $n \to 0$) and φ (m components, $m \to 0$) which has $O(n) \times O(m)$ symmetry [7, 74].

$$L(\mathbf{r})=\frac{1}{2}\sum_{j=1}^{n}\left\{\frac{1}{2m}(\nabla\phi_j)^2-(E+\omega+i\delta)\phi_j^2\right\}$$
$$+\frac{1}{2}\sum_{i=1}^{m}\left\{\frac{1}{2m}(\nabla\varphi_i)^2-(E-i\delta)\varphi_i^2\right\}$$
$$-\frac{1}{8}\rho V^2\left\{\left(\sum_{j=1}^{n}\phi_j^2\right)^2+\left(\sum_{i=1}^{m}\varphi_i^2\right)^2\right.$$
$$\left.+2\sum_{j=1}^{n}\sum_{i=1}^{m}\phi_j^2\varphi_i^2\right\};\quad \delta\to 0^+. \tag{3.5}$$

The appearance of a Langrangian of two interacting fields implies that there are certain complications arising from the transition to a two-particle Green's function. However, the principal difficulty of the theory, associated with the "incorrect" sign of the coupling constant, is nonetheless seen, already in, in (3.1).

First, we will determine at the elementary level the range of applicability of the perturbation theory, similar to that how it was done in the theory of critical phenomena during the derivation of the so-called Ginzburg criterion which determines the size of the critical region, where the order-parameter fluctuations are significant [78]. We will follow generally the procedure used in Ref. 35. The self-energy part of the Green's function in a "one-loop" approximation, which corresponds to the summation of the diagrams like those in Fig. 4a, is given by

$$\Sigma(E_0)=\rho V^2\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{E_0-\Sigma(E_0)-\dfrac{\mathbf{p}^2}{2m}}, \tag{3.6}$$

where E_0 is the "bare" energy of a free electron. Let us determine the renormalized energy E by the relation

$$E(E_0)=E_0-\operatorname{Re}\Sigma(E_0). \tag{3.7}$$

We can then rewrite (3.6) in the form

$$E_0-E(E_0)+i\operatorname{Im}\Sigma(E_0)=\rho V^2\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{E(E_0)-\dfrac{\mathbf{p}^2}{2m}-i\operatorname{Im}\Sigma(E_0)}. \tag{3.8}$$

Let us now determine in this approximation the renormalized (displaced) band edge E_{0c} (Refs. 35 and 64) from the condition under which the density of states vanishes

$$N(E_0) = \mp\frac{1}{\pi}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\,\mathrm{Im}\, G^{R,A}(E_0\mathbf{p}) \xrightarrow[E_0\to E_{0c}]{} 0. \tag{3.9}$$

we have

$$\mathrm{Im}\, G^{R,A}(E_0\mathbf{p}) = \begin{cases} \mp\pi\delta\left[E(E_0) - \dfrac{\mathbf{p}^2}{2m}\right]; & \mathrm{Im}\,\Sigma(E_0) = 0 \\ \dfrac{\mathrm{Im}\,\Sigma(E_0)}{\left[E(E_0) - \dfrac{\mathbf{p}^2}{2m}\right]^2 + [\mathrm{Im}\,\Sigma(E_0)]^2}; & \mathrm{Im}\,\Sigma(E_0) \neq 0. \end{cases} \tag{3.10}$$

To satisfy (3.9) the necessary condition $\mathrm{Im}\,\Sigma(E_0\to E_{0c})\to 0$ must hold; furthermore, we must require that

$$N(E_0\to E_{0c}) = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\,\delta\left(E(E_0) - \frac{\mathbf{p}^2}{2m}\right) \xrightarrow[E_0\to E_{0c}]{} 0, \tag{3.11}$$

which is equivalent to the condition

$$E(E_0) \xrightarrow[E_0\to E_{0c}]{} 0. \tag{3.12}$$

We can then find from (3.8) the equation which determines E_{0c},

$$E_{0c} = -\rho V^2 \int \frac{d^d\mathbf{p}}{(2\pi)^d}\,\frac{1}{\mathbf{p}^2/2m} \equiv -\rho V^2 D(0) = -\rho V^2 2mS_d\,\frac{p_0^{d-2}}{d-2}, \tag{3.13}$$

where $S_d = 2^{-(d-1)}\pi^{-d/2}1/\Gamma(d/2)$, and $p_0 = a^{-1}$ is the cutoff momentum on the order of the reciprocal minimum length in the problem determined by the correlation radius of the random potentials, i.e., the length at which the point like interaction in (3.1) becomes spread-out. Our definition of a displaced band edge differs from that of Brezin and Parisi [79] but is consistent with the definition of the displacement of the phase-transition point due to fluctuations [78]. Let us now consider $G^{-1}(E_0 p = 0) = E_0 - \operatorname{Re}\Sigma(E_0) - i \operatorname{Im}\Sigma(E_0)$ and subtracting $E_{0c} + \rho V^2 D(0) = 0$, we find

$$E_0 - E_{0c} - \operatorname{Re}\Sigma(E_0) - \rho V^2 D(0) = E_0 - \operatorname{Re}\Sigma(E_0) = E(E_0). \quad (3.14)$$

Ignoring $\operatorname{Im}\Sigma(E_0)$ in (3.8), we find

$$\operatorname{Re}\Sigma(E_0) \simeq \rho V^2 \int \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{E(E_0) - \mathbf{p}^2/2m} \equiv \rho V^2 D[E(E_0)]. \quad (3.15)$$

We then derive from (3.14) an equation which relates the renormalized energy $E(E_0)$ to the distance from the band edge in terms of the bare energy $E_0 - E_{0c'}$

$$E_0 - E_{0c} - \rho V^2 \{D[E(E_0)] - D(0)\} = E(E_0). \quad (3.16)$$

The "Ginzburg criterion" (Ref. 35) follows (by analogy with the requirement in Ref. 78) from the requirement that the following very simple equation hold:

$$E \simeq E_0 - E_{0c}. \quad (3.17)$$

This equation implies that the renormalized energy is equal to the electron energy reckoned from the displaced (in a one-loop approximation) band edge. Clearly, Eq. (3.17) holds for $(2 < d < 4)$ if

$$|E| \gg \rho V^2 |D(E) - D(0)| = \rho V^2 \pi m S_d (2mE)^{(d/2)-1} \frac{1}{\left|\sin \pi\left(\frac{d}{2} - 1\right)\right|},$$

or for

$$|E| \gg \left\{ \frac{B_d}{\left|\sin \frac{\pi}{2} d\right|} \right\}^{2/(4-d)} E_{sc}, \quad (3.18)$$

where $B_d = 2^{-d/2}\pi^{1-d/2}/\Gamma(d/2)$, and $E_{sc} = m^{d/(4-d)}(\rho V^2)^{2/(4-d)}$ [see Eq. (2.49)]. In the limit $d \to 2$ we have $|\sin(\pi/2)d| \to (\pi/2)(d-2)$, while $(2/\pi)B_d = A_d$ which is given in Eq. (1.48). We see that (3.18) is equivalent to (1.48). In the limit of $d \to 2$ condition (3.18) is more stringent than the simple requirement $E \gg \gamma(E)$ (or $E \gg E_{sc}$) which we regarded above as the condition under which the perturbation theory can be used. Omission of $\operatorname{Im}\Sigma(E_0)$ in (3.15) is fully justifiable under the conditions of (3.18). The simplest approximation (2.34), in which the variable E is taken to mean the distance from the band edge displaced by the random field, is valid only under these conditions. According to (2.48) and (2.84), the mobility edge is situated in the "Ginzburg critical region" (3.18), where the higher-orders of perturbation theory are important, even in the limit of $d \to 2$ (Ref. 35) and the "naive" hopes expressed above, for example, in determining the region where the self-consistent theory can be used are, in fact, not justifiable. The divergence of region (3.18) in the limit of $d \to 2$ is analogous to the divergence of the "Ginzburg critical region" in the problem of phase transition in two dimensions [78].

In our analysis of the field theory (3.1) we understand E [in (3.1), (3.5), etc.] everywhere to mean the electron energy (3.14)–(3.17) renormalized in a one-loop approximation. This assumption allows us to eliminate from our analysis the real part of the diagram like the one in Fig. 4a, which determines the displacement of the band edge. Formally, this can be accomplished by incorporating into (3.1) an appropriate counterterm for "mass" renormalization: $\delta E = E - E_0 = -\operatorname{Re}\Sigma(E_0)$.

We will examine the cases of $d = 2$ and 4 separately. For $d = 4$ the analysis is essentially the same as the preceding one. The displacement of the band edge, according to (3.13), is

$$E_{0c} = \rho V^2 \frac{m}{8\pi^2}\frac{p_0^2}{2} = \rho V^2 \frac{m^2}{8\pi^2}\tilde{E}_0, \tag{3.19}$$

where $\tilde{E}_0 = p_0^2/2m$ is the energy associated with the momentum cutoff. Equation (3.16) for $d = 4$ gives

$$E = E_0 - E_{0c} + 2\rho V^2 \frac{m^2}{2\pi^2} E \ln\frac{\sqrt{2mE}}{p_0}. \tag{3.20}$$

Condition $E \gg \gamma(E)$ for $d=4$ reduces to the requirement $(m^2/4\pi)\rho V^2 \ll 1$, i.e., to the requirement that the dimensionless coupling constant be small. The requirement that $E \simeq E_0 - E_{0c}$ leads, according to (3.20), to the following inequality

$$E \gg \tilde{E}_0 \exp\left\{-\frac{4\pi^2}{m^2}\frac{1}{\rho V^2}\right\}. \qquad (3.21)$$

This expression defines the "Ginzburg critical region" of the four-dimensional theory, which is exponentially small over the coupling constant.

In the case of $d=2$ the situation is more complicated. The displacement of the band edge, determined from (3.13), becomes infinite and from (3.15) we find

$$\operatorname{Re}\Sigma(E\mathbf{p}) \simeq \rho V^2 \frac{m}{2\pi} \ln\left|\frac{\tilde{E}_0}{E}\right|; \qquad E \ll \tilde{E}_0 = \frac{\mathbf{p}_0^2}{2m}. \qquad (3.22)$$

The correction to E_0 in (3.7) is appreciable everywhere at $E \ll \tilde{E}_0$, implying that formally the perturbation theory cannot be used at $d=2$. We therefore understand $d=2$ everywhere to mean $d=2+\epsilon$ when (3.13) and (3.18) are clearly defined.

The higher-order corrections to the perturbation theory in the field theory (3.1) can easily be examined in the so-called "parquet" approximation [72]. In this approximation, as we know, the theory of critical phenomena for $d=4$ or $d=4-\epsilon$ can be analyzed completely, because the "parquet" comprises the dominating sequence of diagrams for $d \sim 4$. This circumstance, by itself, does not depend on the sign of the interaction constant. We will write out the appropriate equations for the field theory (3.1), incorporating the new variable $-E=\tau>0$, and carry out an analytical continuation $\tau \to -E \mp i\delta$, where $E>0$, only at the end of the calculations. This is a convenient way of analysis, since it allows us to analyze the logarithmic integrals in the usual manner. Let us consider the simplest corrections to the initial interaction $-\rho V^2$, which are represented by the sum of the diagrams in Fig. 6 (the rules governing this diagram technique are found, for example, in the book by Shang-keng Ma [22]). The result, of course, depends on the choice of

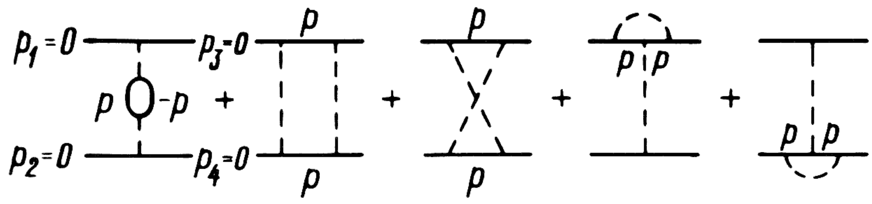


Figure 6. The first-order corrections (second-order diagrams) to the bare interaction in field theory (3.1).

momenta at the external legs. Let us assume, for definiteness, that all the external momenta vanish. We thus find

$$\Gamma(\mathbf{p}=0\tau) \simeq -\rho V^2\left\{1+\rho V^2\left(\frac{n}{2}+4\right)\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{(\mathbf{p}^2/2m+\tau)^2}+\cdots\right\}$$

$$\approx -\rho V^2\{1+\tfrac{1}{2}(n+8)us+\cdots\} \underset{n\to 0}{=} -\rho V^2\{1+4us+\cdots\}.$$

We have introduced a dimensionless interaction constant,

$$u=\frac{m^2}{2\pi^2}a^{-\epsilon}\rho V^2 \tag{3.23}$$

and a "logarithmic" variable

$$s=\begin{cases}\ln\dfrac{p_0}{\sqrt{2m\tau}}; & d=4\\[2ex] \dfrac{1}{\epsilon}\left\{\left(\dfrac{p_0}{\sqrt{2m\tau}}\right)^{\epsilon}-1\right\}; & d=4-\epsilon.\end{cases} \tag{3.24}$$

We see that the parameter us should be used to select the diagrams of perturbation theory. In the limit $\tau\to 0$, we can easily have a situation when $us \gg 1$, i.e., when the corrections are larger than the initial interaction. The procedure by which the "parquet" diagrams are summed singles out the "main logarithms," i.e., the contributions proportional to $\sim(us)^n$, and omits the contributions of the type $u^k(us)^n$ $(k \geqslant 1,\ u \ll 1)$. Since a detailed account of the "parquet" formalism is outside the scope of this review [80–82], we will simply summarize its salient point [72]. The total vertex $\Gamma(s)$, which is rendered dimensionless through multiplication by $m^2 a^{-\epsilon}/2\pi^2$, is determined in the "parquet" approximation by an integral equation of the type

$$\Gamma(s) = -u - \frac{1}{2}(n+8)\int_0^s dt\Gamma^2(t), \tag{3.25}$$

which is equivalent to the differential equation

$$\frac{d\Gamma(s)}{ds} = -\frac{1}{2}(n+8)\Gamma^2(s) \tag{3.26}$$

with the initial condition $\Gamma(s=0) = -u$. The solution of this equation yields

$$\Gamma(s) = \frac{-u}{1 - \frac{1}{2}(n+8)us} \underset{n=0}{=} \frac{-u}{1-4us}. \tag{3.27}$$

We see that in the limit $\tau \to 0$ the vertex $\Gamma(s)$ has a ghost pole at $s_c = \frac{1}{4}u$, indicating that the perturbation theory is inapplicable at $s \gtrsim u^{-1}$. In field-theory-renormalization-group terms, such a behavior corresponds to an increase of the invariant charge of the theory in the limit $s \to s_c \sim u^{-1}$, a situation which indicates that the perturbation theory does not hold [83, 84]. In the approximation under consideration, Eq. (3.26) is, in fact, the Gell-Mann–Low equation for an invariant charge. Equation (3.27) indicates that the theory in question possesses *asymptotic freedom* (the "zero-charge" situation is realized in the theory of critical phenomena). The perturbation theory is valid only at high energies and high momenta. An analytic continuation $\tau \to -E \mp i\delta$ yields

$$\Gamma(E) = -\frac{u}{1 - \frac{4u}{\epsilon}\left\{\left(-\frac{2mE}{p_0^2}\right)^{-\epsilon/2} - 1\right\}}$$

$$\simeq -\frac{u}{1 - \frac{4u}{\epsilon}\left\{\left(\frac{2m|E|}{p_0^2}\right)^{-\epsilon/2} - 1\right\} \pm iu\left(\frac{2m|E|}{p_0^2}\right)^{-\epsilon/2}\theta(E)\frac{\pi}{2}};$$

$$d = 4 - \epsilon. \tag{3.28}$$

Analogously, we have

$$\Gamma(E) = -\frac{u}{1 - 4u \ln \dfrac{p_0}{\sqrt{2m|E|}} \pm iu\dfrac{\pi}{2}\,\theta(E)}; \quad d = 4. \tag{3.29}$$

We see that the ghost pole remains in this case, although it lies on the negative part of the energy axis (where there are no electronic states in this approximation [72])

$$E \simeq -E^* = -\frac{p_0^2}{2m}\left(\frac{4u}{\epsilon}\right)^{2/\epsilon} = -\left(\frac{2}{\pi^2(4-d)}\right)^{2/(4-d)} E_{sc}; \quad d = 4 - \epsilon$$

$$E = -E^* = -\frac{p_0^2}{2m}\exp\left\{-\frac{\pi^2}{m^2}\frac{1}{\rho V^2}\right\}; \quad d = 4. \tag{3.30}$$

Thus, the meaning of Ginzburg conditions (3.18) and (3.21) is that the perturbation theory can be used to analyze the energies far from the ghost pole. The perturbation theory is not valid at $E \gtrsim E^* \sim E_{sc}(d < 4)$, since, as we can easily see from Eqs. (3.28) and (3.29),

$$|\Gamma(E \simeq +E^*)| \simeq \frac{8}{\pi\epsilon} \gg 1$$

$$|\Gamma(E = +E^*)| = \frac{2}{\pi} \sim 1, \tag{3.31}$$

so that the effective interaction becomes strong. We might note, however, that incorporation of the "imaginary" component into the denominators of (3.28) and (3.29) exceeds the acceptable accuracy, since the parquet approximation accounts only for the large logarithmic ($\sim u/\epsilon$, $\epsilon \ll 1$) contributions, whereas the imaginary contribution is of the order of $\sim u$.

Thus a physically correct description of the effective interaction in the field theory (3.1) cannot be obtained by using the parquet approximation. Nonphysical singularities appear at $E \sim E_{sc}$ during the calculation of physical quantities such as the density of states, and the energy $\sim E_{sc}$ determines the width of the "Ginzburg critical region" of our problem. In this region the perturbation theory breaks down. At $E \gg E_{sc}$ the density of states can be expanded in a series over the parameter $(E/E_{sc})^{(d-4)/2}$ [72, 73].

Let us now briefly examine the Lagrangian of two interacting fields (3.5) which is directly connected with the localization problem. In the theory of phase transitions the parquet equations were analyzed in an analogous problem in Ref. 85. In the differential form in terms of the logarithmic variable

$$S = \frac{1}{\epsilon}\left\{\left(\frac{p_0}{\max(\sqrt{2m\tau}, \sqrt{2m\tau'})}\right)^{\epsilon} - 1\right\}$$

we can write these equations for our problem in the form $[\tau = -(E+\omega) > 0,\ \tau' = -E > 0]$

$$\begin{aligned} \frac{d\Gamma_\phi}{ds} &= -\frac{1}{2}(n+8)\Gamma_\phi^2 - \frac{1}{2}m\Gamma^2, \\ \frac{d\Gamma_\varphi}{ds} &= -\frac{1}{2}(m+8)\Gamma_\varphi^2 - \frac{1}{2}n\Gamma^2, \\ \frac{d\Gamma}{ds} &= -\frac{1}{2}\{(n+2)\Gamma_\phi + (m+2)\Gamma_\varphi + 4\Gamma\}\Gamma, \end{aligned} \tag{3.32}$$

where Γ_ϕ describes the interaction ϕ^4, $\Gamma_\varphi - \varphi^4$ [4], and Γ describes the interaction of the type $\phi^2\varphi^2$, which is of main interest to us. For $m = 0$ and $n = 0$ the equations in (3.32) can be rewritten in simpler form

$$\begin{gathered} \frac{d\Gamma_\phi}{ds} = -4\Gamma_\phi^2; \qquad \frac{d\Gamma_\varphi}{ds} = -4\Gamma_\varphi^2, \\ \frac{d\Gamma}{ds} = -(\Gamma_\phi + \Gamma_\varphi)\Gamma - 2\Gamma^2. \end{gathered} \tag{3.33}$$

The equations for Γ_ϕ and Γ_φ become separate equations and coincide with (3.26) (at $n = 0$); the solution of these equations is given by (3.27):

$$\Gamma_\phi = \Gamma_\varphi = -\frac{u}{1-4us}. \tag{3.34}$$

The equation for Γ, which can then be written in the form

$$\frac{d\Gamma}{ds} = -\frac{2u\Gamma}{1-4us} + 2\Gamma^2; \quad \Gamma(0) = -u \tag{3.35}$$

reduces to a Riccati equation. The solution of this equation immediately yields

$$\Gamma = \Gamma_\phi = \Gamma_\varphi = -\frac{u}{1-4us}. \tag{3.36}$$

This result is a direct consequence of the "decoupling" of the equations in (3.32) due to the limit $n \rightarrow 0$ and $m \rightarrow 0$. This result is also evident from the fact that in the limit $\omega \rightarrow 0$ and $\delta \rightarrow 0$ Lagrangian (3.5) describes the $O(n+m)$ symmetric theory. Therefore, a naive expectation that the solution of a system of coupled equations (3.32) can lead to a behavior different from the nonphysical result like that in (3.27) is unjustifiable. The "strong-coupling" region is retained in the theory, which again indicates that the perturbation theory is basically inapplicable in the energy region of interest. The results like those in (3.33)–(3.35) were obtained by Nitzan *et al.* [74] in terms of Wilson's recurrence formulas as an indication of the absence of fixed points of Lagrangian (3.5) for the values of the parameters which correspond to the problem of an electron in a random field.

The difficulties discussed above are, in our view, the key difficulties in the description of the energy region near the mobility edge, rather than being a consequence of the use of an "illegitimate" effective Lagrangian. Any other theory must encounter these difficulties sooner or later in a systematic analysis and find a method for their solution. Unfortunately, this problem is either ignored or bypassed in most present-day studies by making use of simplest approximations (for a single-particle Green's function, for example) outside their range of applicability.

3.2. Analytic Properties over the Coupling Constant and Instantons

In the analysis of the preceding section we have, in fact, attempted to calculate in a rather naive way the functional integral of the type

(3.2). We understood this integral to mean an abbreviated notation of the diagram rules for the perturbation theory. The results we obtained show that this approach has serious flaws in the region of low energies and particularly negative energies, where the perturbation theory gives rise to the appearance of a ghost pole in the effective interaction [(3.28) and (3.29)] and other nonphysical results. It is an established fact, on the other hand, that an analysis of the behavior of an electron in a Gaussian random field of the scattering centers in the negative energy region leads to the existence of a "tail" in the density of states (Refs. 6 and 8) which is associated with the localization of electrons in the deep fluctuations of the random field. This situation typically is marked by the appearance (in the density of states) of nonanalytic contributions in terms of the coupling constant [86–89], which cannot be accounted for by the standard perturbation theory. The functional integrals such as those in (3.2) can be properly analyzed only by using the method of *analytic continuation over the coupling constant*. A corresponding method based on the calculation of the integrals by the method of steepest descent was suggested (in connection with another problem) by Langer [90]. In a field theory approach to localization this method was used for the first time by Cardy [91] and Sadovskiĭ [92]. Let us discuss the main results of these studies.

Before considering the functional integral, it would be useful to analyze the analytic continuation by using as an example an ordinary integral of the type [93–95]

$$Z(g)=\int_{-\infty}^{\infty} d\phi \exp\left\{-\frac{1}{2}\phi^2-\frac{1}{8}g\phi^4\right\}, \tag{3.37}$$

which determines the "zero-dimensional" analog of the functional integral (field theory) we are examining. We assume that $g=\lambda e^{i\theta}$, where $\lambda \in \mathrm{Re}$ and $\lambda>0$. For $\theta=0$ converging integral (3.37) can be expressed by means of a modified Bessel function

$$Z(\lambda)=\frac{1}{\sqrt{\lambda}}\exp\left(\frac{1}{4\lambda}\right)K_{1/4}\left(\frac{1}{4\lambda}\right). \tag{3.38}$$

At $\lambda \ll 1$ this expression yields $Z(\lambda \ll 1)\approx\sqrt{2}\pi$. We wish to extend

(3.37) analytically from $\theta = 0$ to $\theta = \pm\pi$, i.e., from $g = \lambda > 0$ to $g = -\lambda < 0$. We must therefore substitute the variable $\tilde{\phi} = \lambda^{1/2}\phi$ into (3.37) and find

$$Z(\lambda e^{i\theta}) = \lambda^{-1/2} \int d\tilde{\phi} \exp\left\{-\frac{1}{\lambda} S[\tilde{\phi}]\right\}, \tag{3.39}$$

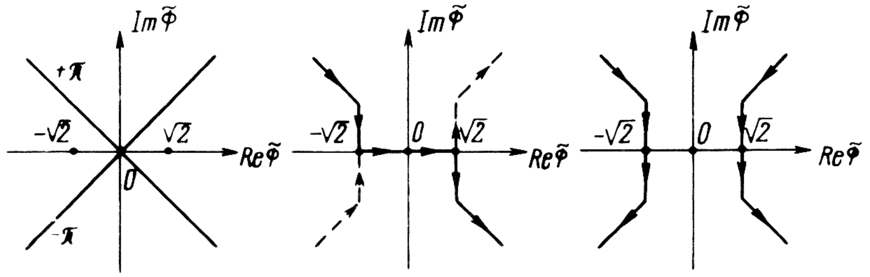
where

$$S[\tilde{\phi}] = \tfrac{1}{2}\tilde{\phi}^2 + \tfrac{1}{8} e^{i\theta}\tilde{\phi}^4. \tag{3.40}$$

At $\theta = 0$ the integration in (3.39) is over the real axis of $\tilde{\phi}$. If the value of θ changes, then the integration path should be rotated (as a result of the substitution $\tilde{\phi} \rightarrow e^{-i\theta/4}\tilde{\phi}$) through an angle $-\theta/4$ in such a way that the integral would remain converging. The integration paths corresponding to $\theta = \pm\pi$ are illustrated in Fig. 7a. Because of the presence of the factor $\lambda^{-1} (\lambda \ll 1)$ in the exponential function in (3.39), the integral can be computed by the method of steepest descent. The saddle points are determined by the condition

$$\frac{\partial S[\tilde{\phi}]}{\partial \tilde{\phi}} = \tilde{\phi} - \frac{1}{2}\tilde{\phi}^3 = 0, \tag{3.41}$$

which yields $\tilde{\phi}_0 = 0$ and $\tilde{\phi}_{1,2} = \pm\sqrt{2}$. The main contribution to $Z(e^{i\pi}\lambda)$ comes from the saddle point $\tilde{\phi}_0 = 0$, and the integration



(a) (b) (c)

Figure 7. The integration paths in the complex plane $\tilde{\phi}$ in (3.39). (a) The initial paths corresponding to $\theta = \pm\pi$; (b) the paths for calculating the main contribution, Re $Z(-\lambda)$; (c) the path for calculating the discontinuity, $\Delta Z(-\lambda) = \text{Im } Z(-\lambda)$.

path can be deformed in such a way that it would pass through this saddle point as it moves along the real axis (Fig. 7b). We find $\operatorname{Re} S[\tilde{\phi}] > \operatorname{Re} S[0]$ everywhere along this path. We then find

$$Z(e^{i\pi}\lambda) \approx \lambda^{1/2} \int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2\lambda} x^2\right) + O(\lambda) = \sqrt{2\pi} + O(\lambda). \tag{3.42}$$

Analogously we can also analyze $Z(e^{-i\pi}\lambda)$. Finally, we find

$$\operatorname{Re} Z(-\lambda) = \tfrac{1}{2}\{Z(e^{i\pi}\lambda) + Z(e^{-i\pi}\lambda)\} \approx \sqrt{2\pi}, \tag{3.43}$$

consistent with asymptotic value of (3.38) at $\lambda \ll 1$. The situation changes if we consider the *discontinuity* at the negative part of the real axis of the "coupling constant":

$$\Delta Z(-\lambda) \equiv \frac{1}{2i}\{Z(e^{i\pi}\lambda) - Z(e^{-i\pi}\lambda)\} = \operatorname{Im} Z(-\lambda). \tag{3.44}$$

The integration path for $Z(e^{i\pi}\lambda)$ *minus* the integration path for $Z(e^{-i\pi}\lambda)$ transforms into the integration paths illustrated in Fig. 7c (compare with those shown in Fig. 7b). These contours go through the "nontrivial" saddle points, $\tilde{\phi}_{1,2} = \pm\sqrt{2}$, but the "trivial" point, $\tilde{\phi}_0 = 0$, *drops out*. At all the remaining points of these contours we would again have $\operatorname{Re} S[\tilde{\phi}] > \operatorname{Re} S[\pm\sqrt{2}]$. Thus the principal contribution to $\Delta Z(-\lambda)$ comes from the "non-trivial" saddle points,

$$\begin{aligned} \Delta Z(-\lambda) &\approx -\frac{1}{i} \lambda^{-1/2} \exp\left(-\frac{1}{2\lambda}\right) \int_{-i\infty}^{+i\infty} dx \exp\left(\frac{1}{\lambda} x^2\right) + O(\lambda) \\ &= -\lambda^{-1/2} \exp\left(-\frac{1}{2\lambda}\right) \int_{-\infty}^{\infty} dz \exp\left(-\frac{1}{\lambda} z^2\right) - O(\lambda) \\ &= -\sqrt{\pi} \exp\left(-\frac{1}{2\lambda}\right) \{1 + O(\lambda)\}. \end{aligned} \tag{3.45}$$

In summary, the $Z(g)$ function, given by integral (3.37), is *an analytic function in the complex g plane with a cut along the negative part of the real axis*. The discontinuity at the cut, which is *nonanalytic* in terms of the coupling constant, *is determined by the contribution from the nontrivial saddle points.*

Let us now consider a certain function of a complex variable $G(g)$, which is analytic in the complex plane of the g variable, with a cut along the negative part of the real axis. According to the Cauchy theorem, we then find

$$G(g) = \frac{1}{2\pi i} \oint dz \, \frac{G(z)}{z-g}$$

where the contour C is illustrated in Fig. 8. In the limit $R \to \infty$, under the assumption that the integrand falls off at infinity in the complex plane rapidly enough [a situation which, incidentally, does not occur in example (3.37)!], we find a dispersion relation in terms of the coupling constant,

$$G(g) = \frac{1}{\pi} \int_{-\infty}^{0} dz \, \frac{\Delta G(z)}{z-g} = \frac{1}{\pi} \int_{-\infty}^{0} dz \, \frac{\operatorname{Im} G(z)}{z-g}, \qquad (3.46)$$

where

$$\Delta G(g) = \frac{1}{2i} \{ G(g+i\epsilon) - G(g-i\epsilon) \} = \operatorname{Im} G(g) \qquad (3.47)$$

if $g \in \mathrm{Re}$, $\epsilon \to 0^+$; $\Delta G(g) \neq 0$ if $g < 0$, and $\Delta G(g) = 0$ if $g > 0$. This dispersion relation makes it possible to reconstruct the $G(g)$ function in the entire complex g plane from the known discontinuity at the cut directed along the negative part of the real axis. The accuracy of the

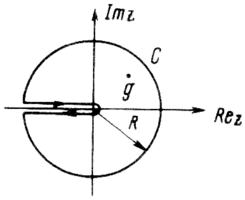


Figure 8. The path of integration which is used in the derivation of the dispersion relation over the coupling constant.

corresponding results is determined solely by the accuracy of the calculation of the discontinuity at the cut.

This analysis can be extended to the calculation of the functional integrals such as those in (3.2). These calculations involve the following general procedure.

1. We seek the "stationary points" of the classical action $S[\phi]$, defined in (3.3) as the solution of the classical equations of the field theory, $\delta S[\phi_{\mathrm{cl}}]/\delta\phi_{\mathrm{cl}} = 0$, with a *finite* action: $S[\phi_{\mathrm{cl}}] < \infty$.

2. We expand the action in powers of $\phi(\mathbf{r}) - \phi_{\mathrm{cl}}(\mathbf{r})$ and calculate the corresponding functional integrals if only in a Gaussian approximation.

Such a procedure, implemented near a trivial classical solution, $\phi_{\mathrm{cl}} = 0$, clearly leads to the conventional perturbation theory. We will see that in complete analogy with example (3.37) examined above, field theory (3.1) with the coupling constant $g = -\rho V^2 < 0$ has nontrivial solutions, $\phi_{\mathrm{cl}}(\mathbf{r}) \neq 0$, with a finite action (instantons) which determine [by analogy with (3.45)] the discontinuity of the Green's function at the cut along the negative part of the real axis in the complex plane of the coupling constant. The Green's function of the theory of interest has an essential singularity at $g = 0$ and is analytic in the complex g plane with a cut along the negative part of the real axis. This function satisfies dispersion relation (3.46).

The presence of an essential singularity in the Green's function at zero of the coupling constant was pointed out for the first time in the quantum field theory by Dyson [96], who used electrodynamics as an example to illustrate his conclusion. The described mathematical procedure, suggested for the first time by Langer [90], was used by Lipatov [97] to study the asymptotic properties of the perturbation-theory series in the quantum-field theory. The dispersion relation for the coupling constant was initially used by Bender and Wu [98] in a problem of anharmonic oscillator and by Bogomolny [99] and Dorfel *et al.* [100] in the field theory. Additional details may be found in the review articles by Kazakov and Shirkov [94] and Zinn-Justin [95].

Let us examine the nonlinear solutions with a finite action (instantons) in field theory (3.1). The corresponding classical field equations are

$$\frac{1}{2m}\nabla^2\phi_j = -E\phi_j - \frac{1}{2}\rho V^2\phi_j\Big(\sum_{j=1}^{n}\phi_j^2\Big). \tag{3.48}$$

We will seek an instanton solution of the form

$$\phi_j^{\mathrm{cl}}(\mathbf{r}) = \phi_{\mathrm{cl}}(\mathbf{r}) u_j,$$

where **u** is a unit vector in the "isotopic-spin" space of the $O(n)$-symmetric theory under consideration: $\mathbf{u}^2 = 1$. Confining ourselves to the spherically symmetric solutions, which correspond to the minimum of action [95, 101–103], we find from (3.48)

$$\frac{1}{2m}\left\{\frac{d^2\phi_{\mathrm{cl}}}{dr^2} + \frac{d-1}{r}\frac{d\phi_{\mathrm{cl}}}{dr}\right\} = -E\phi_{\mathrm{cl}} - \frac{1}{2}\rho V^2 \phi_{\mathrm{cl}}^3. \qquad (3.49)$$

The trivial solution $\phi_{\mathrm{cl}} = 0$ is self-evident. We are concerned with the nontrivial solutions of (3.49) with a finite action (instantons), i.e., such solutions that yield converging action integral (3.3). Using the results obtained by Coleman [103] and Makhankov [104], we can show that the solutions which we are seeking for this equation exist only for $d < 4$. (At $E = 0$ a conformally invariant solution is possible even for $d = 4$ [97].) Least difficult, however, is a simple qualitative analysis [92] based on a method proposed by Finkel'stein *et al.* [105] more than three decades ago. Introducing dimensionless variables

$$\phi_{\mathrm{cl}}(\mathbf{r}) = \sqrt{\frac{2|E|}{\rho V^2}}\,\chi(t); \qquad r = \frac{t}{\sqrt{2m|E|}}, \qquad (3.50)$$

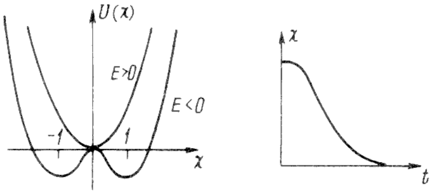
we rewrite (3.49) in the form

$$\frac{d^2\chi}{dt^2} + \frac{d-1}{t}\frac{d\chi}{dt} = \pm\chi - \chi^3, \qquad (3.51)$$

where the upper sign corresponds to $E < 0$, and the lower sign corresponds to $E > 0$. Equation (3.51) is an equation of motion of a particle of unit mass in a potential (Fig. 9a)

$$U(\chi) = \pm\tfrac{1}{2}\chi^2 + \tfrac{1}{4}\chi^4, \qquad (3.52)$$

which is subjected to a time-dependent frictional force. Clearly, we are seeking solutions that satisfy the initial conditions,



(a) (b)

Figure 9. (a) The potential energy corresponding to the equation of motion (3.51). The "particle" is moving down an incline, with friction diminishing with time as $\sim 1/t$. (b) A qualitative form minimum-action instanton solution.

$$\chi(t=0)=\text{const}; \qquad \left.\frac{d\chi}{dt}\right|_{t=0}=0. \tag{3.53}$$

The asymptotic behavior of the solutions of (3.51) at $t \gg 1$ can easily be determined by linearizing this equation near the extremal values of $U(\chi)$. An instanton with minimum action can occur only at $E<0$, and is a solution (Fig. 9b) corresponding to the motion at which a "particle" rolls down an incline of $U(\chi)$ as $t \to \infty$ and stops at the point $\chi=0$. The asymptotic form of this solution at $t \gg 1$ is

$$\chi(t) \sim t^{(1-d)/2}\exp(-t). \tag{3.54}$$

At $E>0$ there are no solutions with finite action (3.3), and for the solution given by (3.53) and (3.54), with the help of (3.50), we easily find

$$\begin{aligned} S[\phi_{\text{cl}}] &= \int d^d\mathbf{r}\,\mathscr{L}(\mathbf{r}\mid\phi_{\text{cl}}(\mathbf{r})) = \mathscr{A}_d\,\frac{m^{-d/2}}{\rho V^2}\,|E|^{2-(d/2)} \\ &= \mathscr{A}_d\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}; \qquad E_{sc}=m^{d/(4-d)}(\rho V^2)^{2/(4-d)}, \end{aligned} \tag{3.55}$$

where the constant $\mathscr{A}_d$, which depends solely on the dimensionality of the space, is determined by the dimensionless integrals of $\chi(t)$. This constant can be determined by numerically integrating (3.51) with initial conditions (3.53). Such a problem, however, is of limited value, since model (3.1), which corresponds to a random Gaussian field with a point like correlation, becomes meaningless at short ranges (small values of t), where the exact solution is now known. In a real system (in a lattice with impurities, for example) another effective Lagrangian "is operating" at small distances. This situation gives rise to a *physical* problem of the instanton core (see Ref. 106, for example, and the discussion below). Asymptotic expression (3.54) was also derived independently by Vrezin and Parisi [107].

The existence of instanton solutions in field theory (3.1) is a general property of field theories with an unstable vacuum [101, 102]. The action (3.3) in field theory (3.1) is not bounded from below for the arbitrary variations of the field $\phi(\mathbf{r})$. It has been rigorously shown [95], however, that an instanton accounts for the absolute minimum of $S[\phi_{\mathrm{cl}}]$ in the class of functions that satisfy the classical equations $\delta S[\phi_{\mathrm{cl}}]/\delta\phi_{\mathrm{cl}} = 0$.

The instanton solutions give rise, at $E < 0$, to new contributions to the single-electron Green's function; specifically, they account, as we can see, for the fact that its imaginary part is nonvanishing even at $E < 0$, which corresponds to the formation of the density of states tail [6, 8, 86–89]. We will follow below primarily the procedure used in Ref. 92 (see also Ref. 91). Given that we are interested in the effect produced by the instanton solution, we will write the field $\phi_j(\mathbf{r})$ in the form

$$\phi_j(\mathbf{r}) \approx \phi_{\mathrm{cl}}(\mathbf{r})\,u_j + \varphi_j(\mathbf{r}). \tag{3.56}$$

Having in mind the main contribution over the coupling constant (2.81) and confining ourselves to terms quadratic in $\varphi_j(\mathbf{r})$ in the action, we find a single-instanton contribution to the single-electron Green's function in the following schematic form [91]:

$$G(\mathbf{r}-\mathbf{r}' \mid -\rho V^2) \sim Z_0^{-1} \exp[-S[\phi_{\mathrm{cl}}]] \int \{\delta\phi(\mathbf{r})\}\,\phi_{\mathrm{cl}}(\mathbf{r})\,\phi_{\mathrm{cl}}(\mathbf{r}') \times \exp\left\{-\frac{1}{2}\int d^d\mathbf{r}\,\varphi_i M_{ik}\varphi_k\right\}\left\{1 + O\left[\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right]\right\}, \tag{3.57}$$

where the factor

$$\exp\{-S[\phi_{\mathrm{cl}}]\} = \exp\left\{-\mathscr{A}_d\,\frac{m^{-d/2}}{\rho V^2}\,|E|^{2-(d/2)}\right\} = \exp\left\{-\mathscr{A}_d\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\} \tag{3.58}$$

is nonanalytic in terms of the coupling constant and therefore cannot be determined from the conventional perturbation theory. Normalization integral (3.4) can be adequately described by Z_0, i.e., by a zeroth order approximation and

$$M_{ik} = M_l u_i u_k + M_T(\delta_{ik} + u_i u_k), \tag{3.59}$$

$$M_L = -\frac{1}{2m}\nabla^2 - E - \frac{3}{2}\rho V^2 \phi_{\mathrm{cl}}^2(\mathbf{r}), \tag{3.60}$$

$$M_T = -\frac{1}{2m}\nabla^2 - E - \frac{1}{2}\rho V^2 \phi_{\mathrm{cl}}^2(\mathbf{r}). \tag{3.61}$$

Introducing orthonormal sets of eigenfunctions of the operators M_L and M_T

$$M_L \psi_n^L = \lambda_n^L \psi_n^L, \tag{3.62}$$

$$M_T \psi_m^T = \lambda_m^T \psi_m^T, \tag{3.63}$$

we can expand the field $\phi(\mathbf{r})$ in these eigenfunctions

$$\boldsymbol{\phi}(\mathbf{r}) = \phi_L(\mathbf{r})\mathbf{u} + \boldsymbol{\varphi}_T(\mathbf{r}), \tag{3.64}$$

where

$$\phi_L(\mathbf{r}) = \phi_{\mathrm{cl}}(\mathbf{r}) + \sum_n \mathbf{c}_n^L \psi_n^L(\mathbf{r}), \tag{3.65}$$

$$\boldsymbol{\varphi}_T(\mathbf{r}) = \sum_m \mathbf{c}_m^T \psi_m^T(\mathbf{r}). \tag{3.66}$$

We assume that φ^T is orthogonal to $\mathbf{u}$ in the isotopic-spin space. In (3.57) we then set

$$\int \{\delta\phi(\mathbf{r})\} \ldots = \prod_n \int d\mathbf{c}_n^L \prod_m \int d\mathbf{c}_m^T \ldots. \tag{3.67}$$

However, the resulting Gaussian integral should be calculated with some caution. Let us qualitatively analyze Eqs. (3.62) and (3.63). We clearly see that these equations can be written as Schrödinger equations with the potentials

$$V_L(\mathbf{r}) = -E - \tfrac{3}{2}\rho V^2 \phi_{\mathrm{cl}}^2(\mathbf{r}) = -E - 3|E|\chi^2(\mathbf{r}), \tag{3.68}$$

$$V_T(\mathbf{r}) = -E - \tfrac{1}{2}\rho V^2 \phi_{\mathrm{cl}}^2(\mathbf{r}) = -E - |E|\chi^2(\mathbf{r}). \tag{3.69}$$

The qualitative behavior of these potentials is illustrated in Fig. 10 and the instanton solution of $\chi(\mathbf{r})$ is shown in Fig. 9b. We see that these potentials are attractive in nature ($-E > 0$ is an irrelevant constant). These potentials, and hence the eigenvalues λ_n^L and λ_m^T, do not depend on the coupling constant but are proportional to $|E|$. The structure of the spectrum can easily be understood on the basis of symmetry considerations.

A. Translational Zero Modes. If $\phi_{\mathrm{cl}}(\mathbf{r})$ is the solution of classical field equations (3.48) and (3.49), then $\phi_{\mathrm{cl}}(\mathbf{r}+\mathbf{R}_0)$ is also a solution by virtue of the translational symmetry of the problem (no condition is imposed on the location of the instanton "center" in the system). Let us consider an infinitesimal translation $\delta\mathbf{R}_0$. We then find

$$\phi_{\mathrm{cl}}(\mathbf{r}+\delta\mathbf{R}_0) \simeq \phi_{\mathrm{cl}}(\mathbf{r}) + \delta\mathbf{R}_{0\mu}\frac{\partial\phi_{\mathrm{cl}}}{\partial x_\mu}. \tag{3.70}$$

At the same time, we can treat Eq. (3.70) as a "perturbation" near the instanton solution $\phi_{\mathrm{cl}}(\mathbf{r})$ and write it as expansion (3.65)

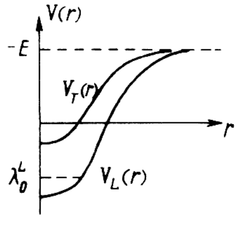


Figure 10. The potential energy in Schrödinger equations (3.62) and (3.63) expressed in qualitative form. λ_0^L is the lowest eigenvalue of the operator M_L.

$$\delta\phi_L(\mathbf{r}) \simeq \delta\mathbf{R}_{0\mu}\frac{\partial\phi_{\text{cl}}}{\partial x_\mu} = \delta\mathbf{c}_1^{L\mu}\psi_1^{L\mu}(\mathbf{r}), \tag{3.71}$$

since it becomes quite obvious after a direct substitution of $\phi_{\text{cl}}(\mathbf{r}+\mathbf{R}_0)$ into Eq. (3.48) and subsequent differentiation of it with respect to $\mathbf{R}_0$ that

$$\frac{1}{2m}\nabla^2\frac{\partial\phi_{\text{cl}}}{\partial\mathbf{R}_{0\mu}} + E\frac{\partial\phi_{\text{cl}}}{\partial\mathbf{R}_{0\mu}} + \frac{3}{2}\rho V^2\phi_{\text{cl}}^2\frac{\partial\phi_{\text{cl}}}{\partial\mathbf{R}_{0\mu}} = 0, \tag{3.72}$$

so that $\partial\phi_{\text{cl}}/\partial x_\mu$ is an eigenfunction of the operator M_L with a *zero* eigenvalue $(\lambda_1^{L\mu}=0)$. Clearly, $\psi_1^{L\mu} \sim \partial\phi_{\text{cl}}/\partial x_\mu$. Requiring that the standard normalization condition be satisfied, $\int d^d\mathbf{r}|\psi_1^{L\mu}(\mathbf{r})|^2 = 1$, we immediately find

$$\psi_1^{L\mu}(\mathbf{r}) = \left\{\int d^d\mathbf{r}\left(\frac{\partial\phi_{\text{cl}}}{\partial x_\mu}\right)^2\right\}^{-1/2}\frac{\partial\phi_{\text{cl}}}{\partial x_\mu}. \tag{3.73}$$

The translational zero mode $\psi_1^{L\mu}(\mathbf{r})$ is obviously d-fold degenerate ($\mu = 1, \ldots, d$ is the dimensionality of the space). We clearly see from the fact that there are d zero eigenvalues, $\lambda_1^{L\mu}=0$, that Schrödinger equation (3.62) has at least one *negative* eigenvalue λ_0^L: The ground state of a quantum-mechanical system cannot be degenerate. Since the translational modes are d-fold degenerate — they correspond to a "p-type" level [i.e., they are transformed according to the rotation-group representation $O(d)$ with $l=1$], we clearly see that there is only a *single* state $\psi_0^L(\mathbf{r})$ with $\lambda_0^L<0$ corresponding to $l=0$ (the s-type state) in the attractive potential $V_L(\mathbf{r})$. A more rigorous proof of the uniqueness of this state may be found in Ref. 95.

It becomes clear from (3.71) and (3.73) that integration over $dc_1^{L\mu}$ is equivalent to integration over the "collective variable" $d\mathbf{R}_{0\mu}$, i.e., instead of integrating over $dc_1^{L\mu}$ in functional measure (3.67), we can integrate over the "instanton center" $\mathbf{R}_{0\mu}$, given that the integrand in (3.57) corresponds to an *arbitrary* instanton center. A conversion to integration over $\mathbf{R}_{0\mu}$ can be realized according to a rule, which is clearly evident from (3.71) and (3.73) (Ref. 102):

$$d\mathbf{c}_1^{L\mu} \to d\mathbf{R}_{0\mu}\left\{\int d^d\mathbf{r}\left(\frac{\partial\phi_{\text{cl}}}{\partial x_\mu}\right)^2\right\}^{1/2} \tag{3.74}$$

for each one of the d translational modes.

B. Rotational Zero Modes. By complete analogy with the preceding analysis, we easily see that the existence of a classical solution of Eq. (3.48) of the type $\phi_{\rm cl}(\mathbf{r})\mathbf{u}$ implies that $\phi_{\rm cl}(\mathbf{r})\mathbf{u}'$ is also its solution, where $\mathbf{u}'$ can be obtained from $\mathbf{u}$ by an arbitrary rotation in the n-dimensional "isotopic" space. Clearly, $\phi_{\rm cl}(\mathbf{r})$ is an eigenfunction of the operator M_T in (3.63) with *zero* eigenvalue, $\lambda_0^T = 0$; $M_T\phi_{\rm cl} = 0$ coincides with (3.49). This level, which is $(n-1)$-fold degenerate [in the n-dimensional space an arbitrary rotation of the vector is determined by the $(n-1)$ angle], becomes in the limit $n \to 0$ a nondegenerate (and a ground state) level for the operator M_T. A normalized eigenfunction of the rotational zero mode is

$$\psi_0^T(\mathbf{r}) = \left\{\int d^d\mathbf{r}\,\phi_{\rm cl}^2(\mathbf{r})\right\}^{-1/2} \phi_{\rm cl}(\mathbf{r}). \tag{3.75}$$

In the case of an infinitesimal rotation $\mathbf{u} \to \mathbf{u} + \delta\mathbf{u}$ and $\mathbf{u}\cdot\delta\mathbf{u} = 0$, we would have

$$\begin{aligned}\delta\phi_T(\mathbf{r}) &\simeq \phi_{\rm cl}(\mathbf{r})\,\delta\mathbf{u} = \delta\mathbf{c}_0^T\,\psi_0^T(\mathbf{r}) \\ &= \delta\mathbf{u}\left\{\int d^d\mathbf{r}\,\phi_{\rm cl}^2(\mathbf{r})\right\}^{1/2} \psi_0^T(\mathbf{r}).\end{aligned} \tag{3.76}$$

We have incorporated (3.66) into the second relation in (3.76) and (3.75) into the third relation. We see that in functional measure (3.67) it is possible to transform from integration over $d\mathbf{c}_0^T$ to integration over the "collective coordinate" $d\mathbf{u}$ (the direction in the "istopic" space) by relating the integrand in (3.57) to this arbitrary direction specified by the vector $\mathbf{u}$. This transformation to integration over the direction of $\mathbf{u}$ is accomplished by applying the rule

$$d\mathbf{c}_{0j}^T \to d\mathbf{u}_j\left\{\int d^d\mathbf{r}\,\phi_{\rm cl}^2(\mathbf{r})\right\}^{1/2} \tag{3.77}$$

for each one of the $(n-1)$-rotational zero modes which corresponds to $j = 1, \ldots, n-1$ directions of the vector $\mathbf{u}$.

As a result, we can write the measure in functional integral (3.57) in the following way:

$$\int \{\delta\phi(\mathbf{r})\}\ldots = \int d^d\mathbf{R}_0 \left\{ \int d^d\mathbf{r} \left(\frac{\partial\phi_{\text{cl}}}{\partial x_\mu}\right)^2 \right\}^{d/2} \prod_{n\neq 1} \int d\mathbf{c}_n^L \times \int_{\mathbf{u}^2=1} d\mathbf{u} \left\{ \int d^d\mathbf{r}\,\phi_{\text{cl}}^2(\mathbf{r}) \right\}^{(n-1)/2} \prod_{m\neq 0} \int d\mathbf{c}_m^T \ldots. \tag{3.78}$$

The presence of Jacobians for transformation to the collective coordinates shown in the braces is extremely important. This heuristic derivation of the transformation from (3.67) to (3.78) is based on the work of Callan and Coleman [102].

Let us now return to the question of the *negative* eigenvalue, $\lambda_0^L < 0$, of the operator M_L. The eigenvalue-related integral over dc_0^L in (3.57) formally can be written in the form

$$\int d\mathbf{c}_0^L \exp\left\{-\frac{1}{2}\lambda_0^L(c_0^L)^2\right\} = \int d\mathbf{c}_0^L \exp\{|\lambda_0^L|(\mathbf{c}_0^L)^2\}. \tag{3.79}$$

This integral should again be thought of as an analytic continuation from positive λ_0^L to negative λ_0^L. By analogy with the analysis in the preceding section, let us turn to integration along the imaginary axis [90] and find

$$\int_{-i\infty}^{+i\infty} d\mathbf{c}_0^L e^{|\lambda_0^L|(c_0^L)^2} = \pm i \int_{-\infty}^{\infty} dx e^{-|\lambda_0^L|x^2} = \pm i \left(\frac{\pi}{|\lambda_0^L|}\right)^{1/2}. \tag{3.80}$$

We see that by analogy with the example in the preceding section, functional integral (3.57) corresponding to the "nontrivial" steepest descent solutions determines the *discontinuity* of the Green's function at the cut in the complex plane of the coupling constant along the negative part of the real axis, i.e., Im $G(\mathbf{r}-\mathbf{r}'|-\rho V^2)$. Thus (3.57) reduces to

$$\operatorname{Im} G(\mathbf{r}-\mathbf{r}'|-\rho V^2) \sim Z_0^{-1} \exp\{-S[\phi_{\text{cl}}]\} \left\{ 1 + O\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)} \right\} \times \left\{ \int d^d\mathbf{r} \left(\frac{\partial\phi_{\text{cl}}}{\partial x_\mu}\right)^2 \right\}^{d/2} \left\{ \int d^d\mathbf{r}\,\phi_{\text{cl}}^2(\mathbf{r}) \right\}^{(n-1)/2} \times \{|\text{Det}'\, M_L|\}^{-1/2} \{\text{Det}'\, M_T\}^{-(1/2)(n-1)} \times \int d^d\mathbf{R}_0 \int_{\mathbf{u}^2=1} d\mathbf{u}\,\phi_{\text{cl}}(\mathbf{r}-\mathbf{R}_0)\,\phi_{\text{cl}}(\mathbf{r}-\mathbf{R}_0), \tag{3.81}$$

where the prime on the determinants denotes that the zero eigenvalues corresponding to the translational and rotational modes have been eliminated from the products $\Pi_n \lambda_n^L$ and $\Pi_m \lambda_m^T$ that determine these determinants.

Using (3.50), we can easily obtain a dimensional estimate of the Jacobians:

$$J_L[\phi_{\text{cl}}] \equiv \int d^d\mathbf{r} \left(\frac{\partial \phi_{\text{cl}}}{\partial x_\mu}\right)^2 \sim mS[\phi_{\text{cl}}] \sim m^{1-(d/2)} \frac{|E|^{(4-d)/2}}{\rho V^2}, \tag{3.82}$$

$$J_T[\phi_{\text{cl}}] \equiv \int d^d\mathbf{r}\, \phi_{\text{cl}}^2(\mathbf{r}) \sim |E| S[\phi_{\text{cl}}] \sim m^{-d/2} \frac{|E|^{(2-d)/2}}{\rho V^2}. \tag{3.83}$$

We then immediately find from (3.81), for $n \neq 0$,

$$\begin{aligned} \operatorname{Im} G(\mathbf{r}-\mathbf{r}' \,|\, -\rho V^2) \sim{} & Z_0^{-1}\{|\text{Det}'\, M_L|\}^{-1/2}\{\text{Det}\, M_T\}^{-(1/2)(n-1)} \\ & \times m^{(d/2)[1-(d/2)]-(d/4)(n+1)} \frac{|E|^{(d/4)(4-d)+(1/4)(2-d)(n+1)}}{(\rho V^2)^{(d+n+1)/2}} \\ & \times \exp\left\{-\mathscr{A}_d \frac{m^{-d/2}}{\rho V^2} |E|^{2-(d/2)}\right\} \\ & \times \left\{1 + O\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\} \int d^d\mathbf{t}_0 \chi(\mathbf{t}-\mathbf{t}_0)\chi(\mathbf{t}'-\mathbf{t}_0). \end{aligned} \tag{3.84}$$

This expression can be simplified by means of a dimensional analysis based on the fact that the eigenvalues λ_n^L and λ_m^T are proportional to $|E|$. We then find in the limit of $n \to 0$

$$\begin{aligned} \operatorname{Im} G(\mathbf{r}-\mathbf{r}' \,|\, -\rho V^2) \sim{} & \frac{m^{(d/4)(1-d)}|E|^{(d/4)(5-d)}}{(\rho V^2)^{(d+1)/2}} \exp\left\{-\mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\} \\ & \times \int d^d\mathbf{t}_0 \chi(\mathbf{t}-\mathbf{t}_0)\chi(\mathbf{t}'-\mathbf{t}_0) \\ & \times \left\{1 + O\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\}. \end{aligned} \tag{3.85}$$

Such a result was obtained by Cardy [91]. Here, however, the dimensionless determinants must be calculated. This requires finding the spectrum of the eigenvalues of Eqs. (3.62) and (3.63) and carrying out a renormalization (at $d \geqslant 2$ the determinants reveal some divergences [107]). We will, however, use a method proposed by Sadovskiĭ [92]. This method is based on the use of the dispersion relation over the coupling constant and upon matching with the problem of phase transitions. Recalling that the eigenvalues λ_n^L and λ_m^T are independent of the coupling constants ρV^2, so that both $\mathrm{Det}' M_L$ and $\mathrm{Det}' M_T$ are also independent of it, and ignoring corrections of order $(|E|/E_{sc})^{2-(d/2)}$, we can rewrite (3.84) in the momentum representation [92, 100]

$$\operatorname{Im} G(E\mathbf{p}\,|-\rho V^2) \simeq C(E\mathbf{p})\frac{1}{(\rho V^2)^{(d+n+1)/2}} \times \exp\left\{-\mathcal{A}_d\frac{m^{-d/2}}{\rho V^2}|E|^{2-(d/2)}\right\}, \qquad (3.86)$$

where $C(E\mathbf{p})$ *does not depend* on the coupling constant. At $n = 0$, we would have

$$\operatorname{Im} G(E\mathbf{p}\,|-\rho V^2) \simeq C(E\mathbf{p})\frac{1}{(\rho V^2)^{(d+1)/2}}\exp\left\{-\frac{A(E)}{\rho V^2}\right\}. \qquad (3.87)$$

We have introduced in this expression

$$A(E) = \mathcal{A}_d m^{-d/2}|E|^{2-(d/2)}. \qquad (3.88)$$

The condition for the applicability of these equations is $S[\phi_{\text{cl}}] \gg 1$ when the method of steepest descent "works" well in the calculation of the functional integral. In other words, the condition

$$\frac{A(E)}{\rho V^2} = \mathcal{A}_d\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)} \gg 1; \quad |E| \gg E_{sc} \qquad (3.89)$$

must be satisfied. Thus the condition for the applicability of the "perturbation theory" near the instanton solution is qualitatively the same as that for the conventional perturbation theory in the region

$E > 0$. The energy region of width $\sim 2E_{sc}$ around $E = 0$ is the analog of the Ginzburg critical region in the theory of critical phenomena.

If $\operatorname{Im} G(E\mathbf{p}\,|-\rho V^2)$, i.e., the discontinuity at the cut in the complex plane of the coupling constant is known, we can determine the Green's function from dispersion integral (3.46)

$$G(E\mathbf{p}\,|\,g) = \frac{1}{\pi}\, C(E\mathbf{p}) \int_{-\infty}^{0} dz\, \frac{\exp\left\{-\dfrac{A(E)}{z}\right\}}{(z-g)(-z)^{(d+1)/2}}, \tag{3.90}$$

where g is an arbitrary coupling constant in the complex plane. For an electron in a random field we have $g \to -\rho V^2 < 0$. Integral (3.90) can be calculated

$$G(E\mathbf{p}\,|\,g) = -\frac{1}{\pi}\, C(E\mathbf{p})\, g^{-(d+1)/2} \times \exp\left\{\frac{A(E)}{g}\right\} \Gamma\left(\frac{d+1}{2}\right) \Gamma\left(\frac{1-d}{2}; \frac{A(E)}{g}\right), \tag{3.91}$$

where $\Gamma(\alpha, x) = \int_x^\infty dt\, t^{\alpha-1} e^{-t}$ is an incomplete gamma function.

It can immediately be seen from expression (3.87) that the exponential factor in the density of states is correct [8, 86–89]. We see that this factor is determined completely by the instanton solutions in the field theory (3.1). The necessity of analyzing the regions $E > 0$ and $E < 0$ from different viewpoints arises automatically, since the instantons exist only at $E < 0$. Another advantage of this method is that it allows a direct calculation of the coefficient of the exponential function $C(E\mathbf{p})$. Here we must point out [92] that at $g > 0$ Green's functions (3.90) and (3.91) describe the correlator of the *stable* field theory (the theory of the second-order phase transitions). Far from the critical region this correlator is well known [21–23]: This is the usual Ornstein–Zernike correlator. In our case, at $|E| \gg E_{sc}$ we would have

$$G(E\mathbf{p}\,|\,g > 0) \approx -\frac{1}{|E| + \mathbf{p}^2/2m}. \tag{3.92}$$

More precisely, this expression is valid if a stronger inequality of the type in (3.18) is satisfied; however, this refinement is important only in the limit $d \to 2$. We understand E everywhere to mean the renormalized energy, i.e., the distance from the displaced band edge (or the "temperature" reckoned from the fluctuation-displaced transition temperature in the theory of critical phenomena), which is calculated in a one-loop approximation. Using the asymptotic form of the incomplete gamma function: $\Gamma(\alpha, x)_{x\gg 1} \approx x^{\alpha-1}e^{-x} \times \{1 + O(x^{-1})\}$, we find from (3.91) ($|E| \gg E_{sc}$)

$$G(E\mathbf{p}\,|\,g>0) \approx -\frac{1}{\pi}\Gamma\left(\frac{d+1}{2}\right)\{A(E)\}^{-(d+1)/2}C(E\mathbf{p}). \qquad (3.93)$$

Comparing (3.92) and (3.93), we find

$$C(E\mathbf{p}) \underset{|E|\gg E_{sc}}{\approx} \frac{\pi\mathcal{A}_d^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}\, m^{-(d/4)(d+1)}\, \frac{|E|^{(d+1)[1-(d/4)]}}{|E| + \frac{\mathbf{p}^2}{2m}}. \qquad (3.94)$$

As a result, we find the following expression for the imaginary part of the Green's function of an electron, at $|E| \gg E_{sc}$,

$$\operatorname{Im} G^{R,A}(E\mathbf{p}\,|-\rho V^2) \approx \pm \frac{\pi\mathcal{A}_d}{\Gamma\left(\frac{d+1}{2}\right)} \frac{|E|^{(d+1)[1-(d/4)]}}{|E| + \frac{\mathbf{p}^2}{2m}} \frac{1}{(m^{d/2}\rho V^2)^{(d+1)/2}}$$
$$\times \exp\left\{-\frac{A(E)}{\rho V^2}\right\}\left\{1 + O\left(\left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right)\right\}. \qquad (3.95)$$

We can then calculate the density of the electronic states in the "tail" region, including the coefficient of the exponential function. We thus find ($E < 0$, $|E| \gg E_{sc}$)

$$N(E) = -\frac{1}{\pi}\int \frac{d^d\mathbf{p}}{(2\pi)^d} \operatorname{Im} G^R(E\mathbf{p}\,|-\rho V^2) \approx S_d \frac{\mathcal{A}_d^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}$$
$$\times \frac{|E|^{(d+1)[1-(d/4)]}}{(m^{d/2}\rho V^2)^{(d+1)/2}} \exp\left\{-\frac{A(E)}{\rho V^2}\right\}\int_0^{p_0} dp\,p^{d-1}\frac{1}{|E| + \frac{\mathbf{p}^2}{2m}}, \qquad (3.96)$$

where $S_d = 2^{-(d-1)}\pi^{-d/2}/\Gamma(d/2)$. At $d = 1$ we can perform the limit, $p_0 = a^{-1} \to \infty$. We find from (3.96)

$$N(E) = S_1 \frac{\pi \mathscr{A}_1}{\sqrt{2}} \frac{|E|}{\rho V^2} \exp\left\{ -\mathscr{A}_1 \frac{|E|^{3/2}}{m^{1/2} \rho V^2} \right\}. \tag{3.97}$$

The constant $\mathscr{A}_1 = \frac{4}{3}\sqrt{2}$ (Ref. 86) [at $d = 1$ Eq. (3.49) can be solved exactly [90]] is in agreement with Halperin's [108] exact result within a factor of $3/\pi$. This demonstrates the precision of the method based upon neglecting the corrections to the result (3.87) obtained by using the method of steepest descent, while performing calculations based on the dispersion relation over the coupling constant. At $d \geqslant 2$ the divergence of the integral in (3.86) is cut off at the momenta $\sim a^{-1}$ which are associated with the reciprocal radius of the random-field correlator. Our analysis is suitable for the energies $|E| \ll \tilde{E}_0 = p_0^2/2m$. For $2 < d < 4$ the density of states tail is given by

$$N(E) \underset{E_{sc} \ll |E| \ll \tilde{E}_0}{\approx} S_d (\mathscr{A}_d)^{(d+1)/2} \frac{2m}{\Gamma\left(\frac{d+1}{2}\right)} \frac{(2m\tilde{E}_0)^{(d-2)/2}}{d-2} \times \left(\frac{|E|}{E_{sc}}\right)^{(d+1)[1-(d/4)]} \exp\left\{ -\mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)} \right\}. \tag{3.98}$$

Similarly, at $d = 2$ we would have

$$N(E) \approx \text{Const} \frac{|E|^{3/2}}{m^{1/2}(\rho V^2)^{3/2}} \ln \frac{\tilde{E}_0}{|E|} \exp\left\{ -\mathscr{A}_2 \frac{|E|}{m\rho V^2} \right\}. \tag{3.99}$$

We should point out, however, that the range of applicability of Eq. (3.99) "goes off" to infinity in accordance with the inequality (3.18), which determines the range of applicability of the Ornstein–Zernike approximation.

At $|E| \gg \tilde{E}_0$ (a situation which corresponds to the assumption that the random potential correlation radius is large) the density of states tail can be determined by a semiclassical approximation [6, 55] or from a "lattice" instanton [106], which gives for any dimensionality of space a Gaussian asymptotic expression

$$N(E) \sim \exp\left\{-\frac{a^d}{\rho V^2} E^2\right\}. \tag{3.100}$$

Thus, at $d=3$ from (3.98) and Ref. 55 we find the following expression for the density of states in the tail region:

$$N(E) \underset{(E<0)}{\approx} \begin{cases} \dfrac{\mathscr{A}_3^2}{2\pi^2} \dfrac{|E|\tilde{E}_0^{1/2}}{m^{3/2}(\rho V^2)^2} \exp\left\{-\mathscr{A}_3 \dfrac{|E|^{1/2}}{m^{3/2}\rho V^2}\right\}; & E_{sc} \ll |E| \ll \tilde{E}_0 \\ \dfrac{m^{3/2}}{4\pi^2 a^3} \dfrac{\rho V^2}{E^{3/2}} \exp\left\{-\dfrac{a^3}{\rho V^2} E^2\right\}; & |E| \gg \tilde{E}_0. \end{cases} \tag{3.101}$$

The expression for the density of states can also be derived from Cardy's result (3.85). We find [91]

$$N(E) = \operatorname{Im} G^R(\mathbf{r}-\mathbf{r}\,|-\rho V^2) \sim m^{(d/4)(1-d)} \frac{|E|^{(d/4)(5-d)}}{(\rho V^2)^{(d+1)/2}} \int_0^{\infty} dt t^{d-1} \chi^2(t)$$
$$\times \exp\left\{-\mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\}. \tag{3.102}$$

This expression is in agreement with (3.98) and (3.99) with respect to the dimensionality, but the energy dependence $|E|$ turns out to be different. To understand the reasons for this difference, we will Fourier transform (3.85) to the momentum space. We then find

$$\operatorname{Im} G(E\mathbf{p}\,|-\rho V^2) \sim m^{-(d/4)(d+1)} \frac{|E|^{(d+1)[1-(d/4)]-1}}{(\rho V^2)^{(d+1)/2}} \mathscr{F}\left(\frac{p}{\sqrt{2m|E|}}\right)$$
$$\times \exp\left\{-\mathscr{A}_d \left(\frac{|E|}{E_{sc}}\right)^{2-(d/2)}\right\}, \tag{3.103}$$

where

$$\mathscr{F}(\mathbf{z}) = \int d^d(\mathbf{t}-\mathbf{t}') e^{-i\mathbf{z}(\mathbf{t}-\mathbf{t}')} \int d^d\mathbf{t}_0 \chi(\mathbf{t}-\mathbf{t}_0)\chi(\mathbf{t}'-\mathbf{t}_0). \tag{3.104}$$

A comparison of (3.103) and (3.95) shows that our approximation specifies the function $\mathscr{F}(\mathbf{z}) = (1 + z^2)^{-1}$ by matching with the Ornstein–Zernike result. Direct calculations clearly show that the use of asymptotic form (3.54) of the instanton solution of (3.104) accounts for this behavior of the $\mathscr{F}(\mathbf{z})$ function. The results of Refs. 91 and 92 for the *imaginary part of the Green's function* are therefore equivalent in the range of their applicability. Our method, however, at once yields the renormalized (finite) expression for Im $G(E\mathbf{p})$, whereas Eqs. (3.85) and (3.103) generally contain an additional dimensionless product of eigenvalues (3.62) and (3.63), which accounts for the infinite constant (which is cancelled out by the action-renormalization counterterm [95, 107]). The use of Eq. (3.103) in the calculation of the density of states gives rise to the integral

$$\int d^d\mathbf{p}\,\mathscr{F}\left(\frac{\mathbf{p}}{\sqrt{2m|E|}}\right) \sim (2m|E|)^{d/2} \int_0^{p_0/\sqrt{2m|E|}} dz z^{d-1} \frac{1}{1+z^2}$$

$$\sim m^{d/2}|E|^{(d/2)} f\left(\frac{\tilde{E}_0}{|E|}\right). \qquad (3.105)$$

If we could assume that $f(x)$ in (3.105) is a dimensionless constant, we would obtain Cardy's result (3.102) for the density of states. However *a dimensionless function* of the ratio $\tilde{E}_0/|E|$ and our results for the density of states appear in (3.105) in the "Ornstein–Zernike approximation." This situation points out to a well-known fact that the Ornstein–Zernike approximation cannot be used at short distances or for the calculation of the mean-square fluctuation [109]. The "physical cutoff" of the divergence at short distances in our case occurs at a length $\sim a$ of the random-field correlation radius, which imposes a limitation on the energy region $|E| \ll \tilde{E}_0$ mentioned above. Cardy [91] has not addressed this problem formally, since the integral $\int_0^\infty dt t^{\alpha-1}\chi^2(t)$ is finite, because the instanton contribution to the action is finite. As we have indicated above, however, the model with a point correlation of the Gaussian random field, which reduces to field theory (3.1), is not a realistic model at short distances, where the physical cutoff mechanism is always "operating." This mechanism is attributable to the fact that the random-field correlation radius in a real system is finite (in the case of a lattice, this distance is on the order of the interatomic range). There is, correspondingly, the problem involving the instanton "core" (Ref. 106), and the effective

Lagrangian and classical equations at short ranges change in such a way as to reproduce the semiclassical (Gaussian) asymptotic form of the density of states tail (3.100) at $|E| \gg \tilde{E}_0$ [6, 55, 106]. In this respect, our results for the density of states seem to correspond more closely to the physical formulation of the problem, whereas Cardy [91] analyzed a different, more formal model.

3.3. *Instantons and Localization*

It is evident from the analysis which we have carried out that localization is closely related to the appearance in the relevant energy region of nonlinear finite-action solutions (instantons) of classical equations of the effective field theory which can be associated with the problem of an electron in a random field. Let us now analyze how a two-particle Green's function is calculated and show, following Refs. 35 and 91, that the instanton solutions of the effective field theory lead to the appearance of a singular contribution of the type (2.26), which was discussed above in connection with the general criterion of localization. We will establish a direct connection between instantons and localization and verify indirectly the general relations obtained during the analysis of this criterion.

As we have indicated above, a calculation of the average two-particle Green's function in a Gaussian random field with a "white-noise" correlation (point interaction) requires an analysis of the effective field theory determined by the $O(n) \times O(m)$ symmetric ($n \to 0$, $m \to 0$) Lagrangian (3.5) of two interacting fields. We can derive from (3.5) the classical equations

$$\frac{1}{2m}\nabla^2 \phi_j = -(E+\omega)\phi_j - \frac{1}{2}\rho V^2 \phi_j \left(\sum_{j=1}^{n} \phi_j^2\right) - \frac{1}{2}\rho V^2 \phi_j \left(\sum_{i=1}^{m} \varphi_i^2\right)$$

$$\frac{1}{2m}\nabla^2 \varphi_i = -E\varphi_i - \frac{1}{2}\rho V^2 \varphi_i \left(\sum_{i=1}^{m} \varphi_i^2\right) - \frac{1}{2}\rho V^2 \varphi_i \left(\sum_{j=1}^{n} \phi_j^2\right). \tag{3.106}$$

In general, an analysis of this system of nonlinear differential equations is complex and to the best of my knowledge has not been performed by anyone. At $E < 0$ and $E + \omega > 0$, however, a simple qualitative analysis shows that Eq. (3.107) has a spherically symmetric instanton solution of the type

$$\varphi_i^{\mathrm{cl}}(\mathbf{r}) = \varphi_{\mathrm{cl}}(\mathbf{r}) e_i; \qquad \phi_j^{\mathrm{cl}}(\mathbf{r}) = 0, \tag{3.107}$$

where $\varphi_{\mathrm{cl}}(\mathbf{r})$ is again given by an equation of the type shown in (3.50)

$$\varphi_{\mathrm{cl}}(\mathbf{r}) = \sqrt{\frac{2|E|}{\rho V^2}}\, \chi(t); \qquad r = \frac{t}{\sqrt{2m|E|}}. \tag{3.108}$$

Here $\chi(t)$ is defined by (3.53) and (3.54). In Eq. (3.107) **e** is a unit (m-component) "isotopic" vector for the field φ. Although the system of equations in (3.106) probably has other finite-action solutions, we need only (3.107), which appears first on the energy scale. The two-particle Green's function is defined by the functional integral

$$\begin{aligned} &\langle \mathrm{G}^{\mathrm{R}}(\mathbf{r}\mathbf{r}'; E + \omega + i\delta)\, G^{A}(\mathbf{r}'\mathbf{r}; E - \delta)\rangle \\ &\quad = \lim_{n\to 0,\ m\to 0} Z^{-1} \int \{\delta\phi(\mathbf{r})\} \int \{\delta\varphi(\mathbf{r})\} \\ &\quad\quad \times \phi_j(\mathbf{r})\phi_j(\mathbf{r}')\varphi_i(\mathbf{r}')\varphi_i(\mathbf{r}) \exp\{-S[\phi, \varphi]\}, \end{aligned} \tag{3.109}$$

where

$$Z = \int \{\delta\phi(\mathbf{r})\} \int \{\delta\varphi(\mathbf{r})\} \exp\{-S[\phi, \varphi]\} \tag{3.110}$$

is a normalization integral, and $S[\phi, \varphi]$ is an action which corresponds to Lagrangian (3.5). By analogy with the analysis of a single-electron Green's function which we carried out above, we will analyze in (3.109) the contribution associated with the Gaussian fluctuations around a classical solution of (3.107). We then find, just as in (3.81),

$$\langle G^R(\mathbf{r}\mathbf{r}'; E+\omega+i\delta)G^R(\mathbf{r}'\mathbf{r}; E-i\delta)\rangle \sim Z_0^{-1}\exp\{-S[\varphi_{\rm cl}]\}J_L^{d/2}[\varphi_{\rm cl}]$$
$$\times J_T^{(m-1)/2}[\varphi_{\rm cl}]\int d^d\mathbf{R}_0\int_{\mathbf{e}^2=1} d\mathbf{e}\,\varphi_{\rm cl}(\mathbf{r}'-\mathbf{R}_0)\varphi_{\rm cl}(\mathbf{r}-\mathbf{R}_0)$$
$$\times\int\{\delta\phi(\mathbf{r})\}\int\{\delta\varphi(\mathbf{r})\}'\phi_j(\mathbf{r})\phi_j(\mathbf{r}')\exp\{-S_0[\phi,\varphi]\}, \tag{3.111}$$

where $S[\varphi_{\rm cl}]$ is the same as in (3.55), $J_L[\varphi_{\rm cl}]$ and $J_T[\varphi_{\rm cl}]$ were defined in (3.82) and (3.83), the normalization integral was again taken at $\rho V^2=0$, and $S[\phi,\varphi]$ denotes the action corresponding to the Gaussian fluctuations near instanton solution (3.107) [$\varphi(\mathbf{r})$ now denotes a deviation from $\varphi_{\rm cl}(\mathbf{r})$]

$$S_0[\phi,\varphi]=\int d^d\mathbf{r}\{\mathscr{L}_0(\phi)+\mathscr{L}_0(\varphi)\}, \tag{3.112}$$

where

$$\mathscr{L}_0(\phi)=\sum_i \phi_j(M_T-\omega-i\delta)\phi_j, \tag{3.113}$$

$$\mathscr{L}_0(\varphi)=\sum_{ij}\varphi_i(M_T+i\delta)(\delta_{ij}-e_ie_j)\varphi_j+\sum_{ij}\varphi_L(M_L+i\delta)e_ie_j\varphi_j. \tag{3.114}$$

Here M_L and M_T are the operators which were introduced in (3.60) and (3.61). The prime on the functional integral in (3.111) means that the zero eigenvalues of the operators M_L and M_T (zero modes), which are taken into account in the integration over the "collective" variables $\mathbf{R}_0$ and $\mathbf{e}$, have been eliminated.

Incorporating eigenfunctions (3.62) and (3.63), and calculating in (3.111) the Gaussian functional integral over the field $\phi(\mathbf{r})$, we find

$$\int\{\delta\phi(\mathbf{r})\}\phi_j(\mathbf{r})\phi_j(\mathbf{r}')\exp\{-S_0[\phi,\varphi]\}$$
$$\sim\sum_k\frac{\psi_k^T(\mathbf{r}-\mathbf{R}_0)\psi_k^T(\mathbf{r}'-\mathbf{R}_0)}{(\lambda_k^T-\omega-i\delta)^{1+(n/2)}}$$
$$\xrightarrow[n\to 0]{}\frac{\psi_0^T(\mathbf{r}-\mathbf{R}_0)\psi_0^T(\mathbf{r}'-\mathbf{R}_0)}{\omega+i\delta}+\sum_{k\neq 0}\ldots, \tag{3.115}$$

where the normalized eigenfunction of the lowest level, associated with operator M_T ($\lambda_0^T = 0$—the "rotational" zero mode) is given by (3.75). Accordingly, a singular (in the limit of $\omega \rightarrow 0$) contribution to a two-particle Green's function ($m \rightarrow 0$, $n \rightarrow 0$) arises from (3.111):

$$\langle G^R(\mathbf{r}\mathbf{r}'; E + \omega + i\delta) G^A(\mathbf{r}'\mathbf{r}; E - i\delta)\rangle \sim \frac{i}{\omega + i\delta} \exp\{-S[\varphi_{\mathrm{cl}}]\} Z_0^{-1}$$

$$\times J_L^{d/2}[\varphi_{\mathrm{cl}}] J_T^{1/2}[\varphi_{\mathrm{cl}}] \{|\mathrm{Det}'\, M_L|\}^{-1/2} \{\mathrm{Det}'\, M_T\}^{1/2}$$

$$\times \int d^d\mathbf{R}_0 \varphi_{\mathrm{cl}}^2(\mathbf{r} - \mathbf{R}_0) \varphi_{cl}^2(\mathbf{r}' - \mathbf{R}_0). \tag{3.116}$$

Here there are no zero eigenvalues in Det′ M_L and Det′ M_T. The factor i arises from a single negative eigenvalue of the operator M_L. An expression equivalent to (3.116) was initially obtained (for $\omega = 0$) by Cardy [91] (see also Ref. 110). We have based our analysis of the derivation given by Sadovskiĭ [35].

The singular contribution, we might note, is related to the existence of the "rotational zero mode," i.e., to the symmetry of the system. It is therefore justifiable to assume that this contribution does not vanish when higher-order corrections are taken into account.

The result (3.116) is in complete agreement with the form of the two-particle Green's function in the energy region corresponding to the localized states suggested in (2.26) and (2.27). In particular, if we transform in (3.116) to a Fourier representation using

$$\chi_{\mathbf{q}} = \int d^d\mathbf{r} e^{-i\mathbf{q}\mathbf{r}} \varphi_{\mathrm{cl}}^2(\mathbf{r}), \tag{3.117}$$

we find for the "localization amplitude" (2.20)

$$A_E(\mathbf{q}) \sim \chi_{\mathbf{q}} \chi_{-\mathbf{q}}, \tag{3.118}$$

consistent with (2.29). Introducing the Fourier representation of the instanton

$$\varphi_{\mathbf{q}}^{\mathrm{cl}} = \int d^d\mathbf{r} e^{-i\mathbf{q}\mathbf{r}} \varphi_{\mathrm{cl}}(\mathbf{r}), \tag{3.119}$$

we see that

$$\tilde{\chi}_{\mathbf{q}} = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\,\varphi^{\mathrm{cl}}_{\mathbf{p}}\varphi^{\mathrm{cl}}_{\mathbf{q}-\mathbf{p}}. \qquad (3.120)$$

Comparing this expression with (2.26)–(2.28), we find

$$\psi^{\mathbf{q}}_{\mathbf{p}}(E) \sim \varphi^{\mathrm{cl}}_{\mathbf{p}}(E)\varphi^{\mathrm{cl}}_{\mathbf{q}-\mathbf{p}}(E). \qquad (3.121)$$

Thus this analysis is, in fact, a justification of the form of the two-particle Green's function proposed in (2.26). Taking into account (2.21) and (2.22) and the explicit instanton contribution to the density of states which follows from (3.81)

$$N(E) \sim Z_0^{-1}\exp\{-S[\varphi_{\mathrm{cl}}]\}J_L^{d/2}[\varphi_{\mathrm{cl}}]J_T^{-1/2}[\varphi_{\mathrm{cl}}]\{|\mathrm{Det}'\,M_L|\}^{-1/2}$$
$$\times\{\mathrm{Det}'\,M_T\}^{1/2}\int d^d\mathbf{R}_0\varphi^2_{\mathrm{cl}}(\mathbf{r}-\mathbf{R}_0), \qquad (3.122)$$

we immediately find from (3.116) the following expression for $A_E(\mathbf{r}-\mathbf{r}')$, which is valid to within a dimensionless constant:

$$A_E(\mathbf{r}-\mathbf{r}') \sim \int d^d\mathbf{R}_0\varphi^2_{\mathrm{cl}}(\mathbf{r}-\mathbf{R}_0)\varphi^2_{\mathrm{cl}}(\mathbf{r}'-\mathbf{R}_0)\left\{\int d^d\mathbf{r}\varphi^2_{\mathrm{cl}}(\mathbf{r})\right\}^{-1}. \qquad (3.123)$$

For the "probability of return" A_E (2.22) we find from (3.123)

$$A_E \sim \int d^4\mathbf{r}\varphi^4_{\mathrm{cl}}(\mathbf{r})\left\{\int d^d\mathbf{r}\varphi^2_{\mathrm{cl}}(\mathbf{r})\right\}^{-1} \sim (m|E|)^{d/2}. \qquad (3.124)$$

Incorporating now asymptotic expression (3.54), we easily see that

$$A_E(\mathbf{r}-\mathbf{r}') \sim \exp\{-|\mathbf{r}-\mathbf{r}'|R^{-1}_{\mathrm{loc}}(E)\}, \qquad (3.125)$$

where

$$R_{\mathrm{loc}}(E) \sim \{2m|E|\}^{-1/2} \sim (m|E|)^{-\nu}; \quad \nu = \tfrac{1}{2}. \qquad (3.126)$$

Thus, at $|E| \gg E_{sc}$ (in the depth of the tail) we have a "classical" result for the localization length which is determined simply by the instanton "radius." We see from (3.124) and (3.126) that $A_E \sim R_{\mathrm{loc}}^{-d}$, consistent with the a simple estimate mentioned in Sec. 1.

Accordingly, an instanton method, in contrast with the analysis in Sec. 1 which is based on the use of a homogeneous Bethe–Salpeter equation, describes an entire energy region corresponding to the localized states and, in contrast with the self-consistent localization theory (Sec. 2), it describes correctly the asymptotic behavior of the tail region in the density of states. Several other results can also be obtained within the framework of this method. For example, on the basis of its modification proposed in Ref. 111, Houghton *et al.* [112] carried out a rigorous analysis of the dynamic conductivity and derived the well-known Mott law: $\sigma(\omega) \sim \omega^2(\ln \omega)^{d+1}$ (Ref. 1) for the conductivity along the localized states.

Let us consider the relationship between the two methods of determining the singular part of a two-particle Green's function—the method based on the use of a homogeneous Bethe–Salpeter equation (Sec. 1) and the instanton method. We will show that in general each description arises in a natural way as a manifestation of a *different* instability of the system within the framework of the effective-action formalism for composite fields [35].

The effective action for this system of fields ϕ and φ is a functional Γ of the "classical" (average) fields ϕ_{cl} and φ_{cl} and the corresponding Green's functions which satisfies the following variational principle [113]:

$$\frac{\delta\Gamma}{\delta\phi_{\mathrm{cl}}(\mathbf{r})} = 0; \qquad \frac{\delta\Gamma}{\delta\varphi_{\mathrm{cl}}(\mathbf{r})} = 0; \qquad \frac{\delta\Gamma}{\delta G(\mathbf{r}, \mathbf{r}')} = 0. \tag{3.127}$$

The functional can be determined from a double Legendre transformation of the generating functional of the "classical" fields and Green's functions of the field theory under consideration [113]. The effective-action formalism is especially fit for the analysis of the symmetry breaking with respect of the fields ϕ and φ and the corresponding Green's functions G. An analogous formalism in the many-body problem was analyzed by several authors [114–117].

For brevity, we will use the matrix notations

$$\Phi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}; \qquad \Phi^{+} = (\phi\varphi), \tag{3.128}$$

$$\hat{G} = \begin{bmatrix} G_{\phi\phi} & G_{\phi\varphi} \\ G_{\varphi\phi} & G_{\varphi\varphi} \end{bmatrix}; \qquad G_{\phi\varphi} = G_{\varphi\phi}. \tag{3.129}$$

Introducing the matrix (the δ symbols correspond to the isotopic field indices)

$$\hat{G}_0^{-1}(\mathbf{rr}') = \begin{bmatrix} \left\{\frac{1}{2m}\nabla^2 - (E + \omega + i\delta)\right\}\delta_{ij} & 0 \\ 0 & \left\{\frac{1}{2m}\nabla^2 - (E - i\delta)\right\}\delta_{ij} \end{bmatrix} \delta(\mathbf{r} - \mathbf{r}'), \tag{3.130}$$

we can rewrite Lagrangian (3.5) in a compact form

$$\mathscr{L}(\mathbf{r}) = \frac{1}{2}\,\mathrm{Sp}\int d^d\mathbf{r}'\Phi^{+}(\mathbf{r})\hat{G}_0^{-1}(\mathbf{rr}')\Phi(\mathbf{r}') - \frac{1}{8}\rho V^2(\mathrm{Sp}\,\Phi^{+}\Phi)^2 \tag{3.131}$$

Introducing the external sources

$$J = (J_\phi J_\varphi); \qquad \hat{K} = \begin{bmatrix} K_{\phi\phi} & K_{\phi\varphi} \\ K_{\varphi\phi} & K_{\varphi\varphi} \end{bmatrix}, \tag{3.132}$$

we can write the generating functional of the "classical" (average) fields and Green's functions as follows:

$$\begin{aligned} Z\{J, K\} &\equiv \exp\{-W(J, K)\} \\ &= \int \{\delta\Phi\} \exp\Big\{-S[\Phi] - \mathrm{Sp}\int d^d\mathbf{r} J(\mathbf{r})\Phi(\mathbf{r}) \\ &\quad - \frac{1}{2}\,\mathrm{Sp}\int d^d\mathbf{r}\int d^d\mathbf{r}'\Phi^{+}(\mathbf{r})\hat{K}(\mathbf{rr}')\Phi(\mathbf{r}')\Big\}, \end{aligned} \tag{3.133}$$

where the action is

$$S[\Phi] = \mathrm{Sp} \int d^d\mathbf{r} \int d^d\mathbf{r}' \Phi^{+}(\mathbf{r}) \hat{G}_0^{-1}(\mathbf{rr}')\Phi(\mathbf{r}') + S_{\text{int}}[\Phi]. \quad (3.134)$$

We then find

$$\frac{\delta W\{J, K\}}{\delta J_{\phi}(\mathbf{r})} = \langle \phi(\mathbf{r}) \rangle \equiv \phi_{\text{cl}}(\mathbf{r}); \qquad \frac{\delta W\{J, K\}}{\delta J_{\varphi}(\mathbf{r})} = \langle \varphi(\mathbf{r}) \rangle = \varphi_{\text{cl}}(\mathbf{r}). \quad (3.135)$$

In this expression the angular brackets denote the functional averaging. Equation (3.135) essentially defines the average ("classical") fields. Analogously, we find

$$\begin{aligned} \frac{\delta W\{J, K\}}{\delta K_{\phi\phi}(\mathbf{rr}')} &= \frac{1}{2}\{\phi_{\text{cl}}(\mathbf{r})\phi_{\text{cl}}(\mathbf{r}') + G_{\phi\phi}(\mathbf{rr}')\}, \\ \frac{\delta W\{J, K\}}{\delta K_{\varphi\varphi}(\mathbf{rr}')} &= \frac{1}{2}\{\varphi_{\text{cl}}(\mathbf{r})\varphi_{\text{cl}}(\mathbf{r}') + G_{\varphi\varphi}(\mathbf{rr}')\}, \qquad (3.136) \\ \frac{\delta W\{J, K\}}{\delta K_{\phi\varphi}(\mathbf{rr}')} &= \frac{1}{2}\{\phi_{\text{cl}}(\mathbf{r})\varphi_{\text{cl}}(\mathbf{r}') + G_{\phi\varphi}(\mathbf{rr}')\} = \frac{\delta W\{J, K\}}{\delta K_{\varphi\phi}(\mathbf{r}'\mathbf{r})}. \end{aligned}$$

The effective-action functional which we are seeking can be determined from a double Legendre transformation [113]

$$\begin{aligned} \Gamma(\Phi_{\text{cl}}, \hat{G}) = W\{J, K\} &- \mathrm{Sp} \int d^d\mathbf{r} J(\mathbf{r})\Phi_{\text{cl}}(\mathbf{r}) \\ &- \frac{1}{2}\mathrm{Sp} \int d^d\mathbf{r} \int d^d\mathbf{r}' \Phi_{\text{cl}}^{+}(\mathbf{r}) \hat{K}(\mathbf{rr}')\Phi_{\text{cl}}(\mathbf{r}') \\ &- \frac{1}{2}\int d^d\mathbf{r} \int d^d\mathbf{r}' \hat{K}(\mathbf{rr}')\hat{G}(\mathbf{r}'\mathbf{r}), \qquad (3.137) \end{aligned}$$

where the sources J and $\hat{K}$ are expressed in terms of Φ_{cl} and $\hat{G}$ with the help of Eqs. (3.135) and (3.136). We then easily see that

$$\begin{aligned}\frac{\delta\Gamma(\Phi_{\text{cl}},\hat{G})}{\delta\phi_{\text{cl}}(\mathbf{r})} &= -J_{\phi}(\mathbf{r}) - \int d^d\mathbf{r}'\, K_{\phi\phi}(\mathbf{r}\mathbf{r}')\phi_{\text{cl}}(\mathbf{r}'),\\ \frac{\delta\Gamma(\Phi_{\text{cl}},\hat{G})}{\delta\varphi_{\text{cl}}(\mathbf{r})} &= -J_{\varphi}(\mathbf{r}) - \int d^d\mathbf{r}'\, K_{\varphi\varphi}(\mathbf{r}\mathbf{r}')\varphi_{\text{cl}}(\mathbf{r}'),\end{aligned} \tag{3.138}$$

$$\begin{aligned}&\frac{\delta\Gamma(\Phi_{\text{cl}},\hat{G})}{\delta G_{\phi\phi}(\mathbf{r}\mathbf{r}')} = -\frac{1}{2}K_{\phi\phi}(\mathbf{r}\mathbf{r}'); \qquad \frac{\delta\Gamma(\Phi_{\text{cl}},\hat{G})}{\delta G_{\varphi\varphi}(\mathbf{r}\mathbf{r}')} = -\frac{1}{2}K_{\varphi\varphi}(\mathbf{r}\mathbf{r}'),\\ &\frac{\delta\Gamma(\Phi_{\text{cl}},\hat{G})}{\delta G_{\phi\varphi}(\mathbf{r}\mathbf{r}')} = -\frac{1}{2}K_{\phi\varphi}(\mathbf{r}\mathbf{r}').\end{aligned} \tag{3.139}$$

If we set the external sources J and $\hat{K}$ equal to zero, we see that the functional Γ determined in this manner satisfies variational principle (3.127). This functional is [113] the generating functional of the Green's functions which are irreducible in a two-particle channel. According to Cornwall *et al.* [113], after a straightforward extension to the two-field system, we obtain

$$\Gamma(\Phi_{\text{cl}},\hat{G}) = S[\Phi_{\text{cl}}] - \tfrac{1}{2}\operatorname{Tr}\ln\tilde{G}^{-1} - \tfrac{1}{2}T\{\tilde{G}^{-1}\hat{G} - 1\} + \Phi(\Phi_{\text{cl}},\hat{G}), \tag{3.140}$$

where Tr and ln are to be understood in the functional context [113]; specifically, Tr includes all the necessary integrations, and $\operatorname{Tr}\ln\hat{G} = \ln\operatorname{Det} G$. In Eq. (3.140) $\tilde{G}^{-1}$ denotes the reciprocal Green's function matrix in a classical field

$$\tilde{G}^{-1}(\mathbf{r},\mathbf{r}') = \begin{bmatrix} A_{ij} & B_{ij}\\ C_{ij} & D_{ij}\end{bmatrix}\delta(\mathbf{r}-\mathbf{r}'), \tag{3.141}$$

where

$$\begin{aligned}A_{ij} &= \left\{-\frac{1}{2m}\nabla^2 - (E+\omega+i\delta) - \frac{1}{2}\rho V^2(\phi^2_{\text{cl}} + \varphi^2_{\text{cl}})\right\}\delta_{ij} - \rho V^2\phi_{\text{cl}_i}\phi_{\text{cl}_j},\\ B_{ij} &= -\rho V^2\phi_{\text{cl}_j}\varphi_{\text{cl}_i}; \qquad C_{ij} = -\rho V^2\varphi_{\text{cl}_i}\phi_{\text{cl}_j},\\ D_{ij} &= \left\{-\frac{1}{2m}\nabla^2 - (E-i\delta) - \frac{1}{2}\rho V^2(\phi^2_{\text{cl}} + \varphi^2_{\text{cl}})\right\}\delta_{ij} - \rho V^2\varphi_{\text{cl}_i}\varphi_{\text{cl}_j},\end{aligned} \tag{3.142}$$

$$\phi^2_{\text{cl}} = \sum_{j=1}^{n}\phi^2_{\text{cl}_j}; \qquad \varphi^2_{\text{cl}} = \sum_{i=1}^{m}\varphi^2_{\text{cl}}.$$

The functional $\Phi(\Phi_{\text{cl}}, \hat{G})$ can be calculated in the following way [113]. In the classical action $S[\Phi]$ the field Φ must be "displaced" by the amount Φ_{cl}. The new action $S[\Phi_{\text{cl}} + \Phi]$ would then determine the new interaction vertices which depend on Φ_{cl}. The functional $\Phi(\Phi_{\text{cl}}, \hat{G})$ would then be determined by all the vacuum diagrams, which are irreducible in a two-particle channel, of the theory with the action $S[\Phi_{\text{cl}} + \Phi]$ and by the propagators which are equal to the matrix Green's functions $\hat{G}$. In other words, only those vacuum diagrams need be retained which give the irreducible self-energy diagrams after cutting any one of the lines denoting a Green's function. Because of the dependence of the vertices on $\Phi_{\text{cl}}(\mathbf{r})$, the translational invariance is generally missing. Thus the functional $\Phi(\Phi_{\text{cl}}, \hat{G})$ satisfies the condition

$$\frac{\delta\Phi}{\delta\hat{G}} = \frac{1}{2}\hat{\Sigma}, \tag{3.143}$$

and the equation

$$\frac{\delta\Gamma}{\delta\hat{G}} = \frac{1}{2}\hat{G}^{-1} - \frac{1}{2}\tilde{G}^{-1} + \frac{1}{2}\hat{\Sigma} = 0 \tag{3.144}$$

is just the matrix Dyson equation for the corresponding Green's functions. The matrix $\hat{\Sigma}$ is comprised of irreducible self-energies with "dressed" internal lines.

Let us first examine the "normal" phase, in which $\phi_{\text{cl}} = \varphi_{\text{cl}} = 0$ and only the Green's functions $G_{\phi\phi}$ and $G_{\varphi\varphi}$ are nonvanishing. In this case, Eq. (3.140) essentially reduces to

$$\Gamma(\hat{G}) = \Phi(\hat{G}) - \tfrac{1}{2}\operatorname{Tr}\ln\hat{G}^{-1} - \tfrac{1}{2}\operatorname{Tr}\{\hat{G}_0^{-1}\hat{G} - 1\}. \tag{3.145}$$

Matrix (3.141) reduces to a simple form (3.130). All the expressions are translationally invariant by virtue of $\phi_{\text{cl}} = \varphi_{\text{cl}} = 0$.

A stable system must satisfy the condition $\delta^2\Gamma > 0$ for any variations of Φ_{cl} and $\hat{G}$. Let us consider its stability relative to arbitrary variations of the *Green's functions* in the "normal" phase. We easily find

$$\frac{\delta^2\Gamma}{\delta G_{\phi\phi}\delta G_{\phi\phi}} = -\frac{1}{2}G_{\phi\phi}^{-2} + \frac{1}{2}\frac{\delta\Sigma_{\phi\phi}}{\delta G_{\phi\phi}} = -\frac{1}{2}G_{\phi\phi}^{-2} + \frac{1}{2}U_{\phi\phi\phi\phi} \tag{3.146}$$

$$\frac{\delta^2\Gamma}{\delta G_{\phi\varphi}\delta G_{\phi\varphi}}=\frac{1}{2}\frac{\delta G^{-1}_{\phi\varphi}}{\delta G_{\phi\varphi}}+\frac{1}{2}\frac{\delta\Sigma_{\phi\varphi}}{\delta G_{\phi\varphi}}=-\frac{1}{2}G^{-1}_{\phi\phi}G^{-1}_{\varphi\varphi}+\frac{1}{2}U_{\phi\varphi\phi\varphi}, \tag{3.147}$$

etc. Here U are the vertices which are irreducible in the corresponding two-particle channels. The first term in (3.147) can be determined by directly calculating the appropriate derivative and then transforming to the "normal" phase ($G_{\phi\varphi}=G_{\varphi\phi}=0$). Figure 11a is a graphic representation of the variation of the self-energies resulting from the variation of the Green's functions, which was used in the derivation of (3.146) and (3.147). Clearly, we are concerned with the stability of the system with respect to the variations $\delta G_{\phi\varphi}$. In a stable system we would have

$$\operatorname{Tr}\,\delta G_{\phi\varphi}\frac{\delta^2\Gamma}{\delta G_{\phi\varphi}\delta G_{\varphi\phi}}\,\delta G_{\phi\varphi}\geqslant 0. \tag{3.148}$$

Using $\delta G_{\phi\varphi}=G_{\phi\phi}\psi_{\phi\varphi}G_{\varphi\varphi}$ (see Fig. 11b) in (3.148), we find, with the help of (3.147), that the stability threshold of the "normal" phase

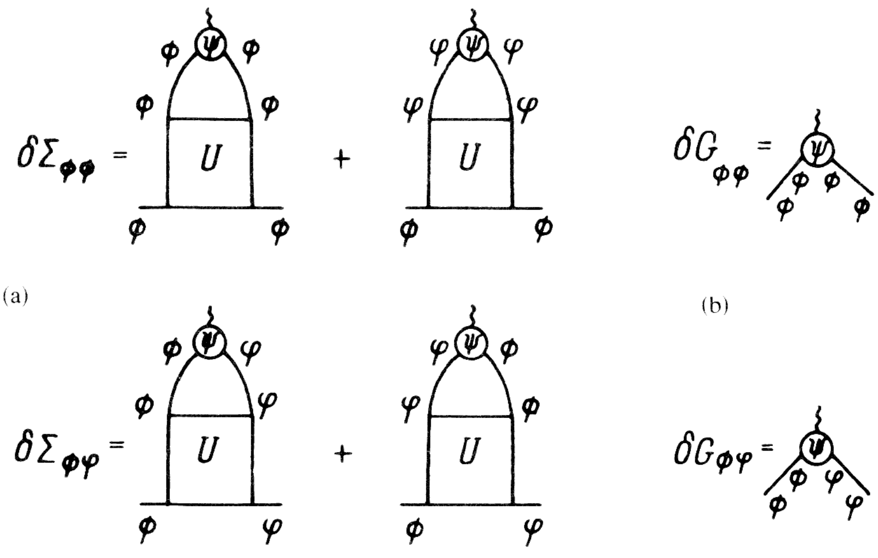


Figure 11. (a) Variation of self-energy as a result of arbitrary variation of the Green's function. (b) Graphic representation of the arbitrary variation of the Green's functions (variation of the external source, $\delta K_{\phi\varphi}\equiv\psi_{\phi\varphi}$).

relative to this variation is determined by the condition

$$\mathrm{Tr}\, G_{\varphi\varphi}\psi_{\varphi\phi}G_{\phi\phi}U_{\phi\varphi\varphi\phi}G_{\phi\phi}\psi_{\phi\varphi}G_{\varphi\varphi} - \mathrm{Tr}\, G_{\varphi\varphi}\psi_{\varphi\phi}\psi_{\phi\varphi}G_{\phi\phi} = 0 \qquad (3.149)$$

which is graphically illustrated in Fig. 12a. It is fairly clear that the stability condition is broken in the case where a nontrivial solution of a homogeneous Bethe–Salpeter equation with $\omega = 0$ appears (Fig. 12b). We have seen, in fact, using a specific example which was analyzed in Sec. 1, that (at $\omega = 0$) the operator $U_{\phi\varphi\varphi\phi}$ appearing in (3.149) is a Hermitian, and that its eigenvalues are real. The stability condition is equivalent to the requirement that the quadratic form be positive

$$\mathrm{Tr}\, \psi_{\varphi\phi}X_{\phi\varphi}\psi_{\varphi\phi} > 0, \qquad (3.150)$$

where

$$X_{\phi\varphi} = G_{\varphi\varphi}G_{\phi\phi}U_{\phi\varphi\varphi\phi}G_{\phi\phi}G_{\varphi\varphi} - G_{\varphi\varphi}G_{\phi\phi}. \qquad (3.151)$$

At $\omega = 0$, it is clear that $G_{\phi\phi} = G^*_{\varphi\varphi}$. Introducing a system of eigenfunctions of the operator $X_{\phi\varphi}$

$$X_{\phi\varphi}\psi^{\nu}_{\varphi\phi} = \lambda_{\nu}\psi^{\nu}_{\varphi\phi}; \qquad \lambda_{\nu} \in \mathrm{Re}, \qquad (3.152)$$

we find that the nontrivial solution of a homogeneous Bethe–Salpeter equation is equivalent to the existence of the eigenvalue $\lambda_0 = 0$. We then find that the choice of $\psi_{\varphi\phi} \sim \psi^{0}_{\varphi\phi}$ violates condition (3.150). A similar analysis in case of ordinary phase transitions was carried out by Morandi [117]. Accordingly, our analysis shows that the appearance of a nontrivial solution of the Bethe–Salpeter equation

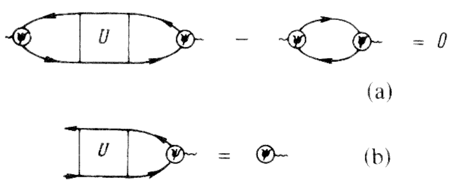


Figure 12. Graphic representation of Eq. (3.149) (a) and of the homogeneous Bethe–Salpeter equation derived from it (b).

determines the stability threshold of the "normal" phase. Here we have in mind the stability relative to the arbitrary variations $\delta G_{\phi\varphi}$. An expansion of the functional $\Gamma(\hat{G})$ in (3.145) in powers of $\delta G_{\phi\varphi} \sim \psi_{\phi\varphi}$ gives, in principle, a method of analyzing the corresponding "condensed" phase. The quantity $\psi_{\phi\varphi}$ in this case is the *order parameter*.

The first two equations in (3.127) are generalizations of the classical field equations given in (3.106). We are concerned with the case in which they acquire nontrivial solutions of the type shown in (3.107). Matrix (3.141) in this case reduces to

$$\tilde{G}^{-1}(\mathbf{r}\mathbf{r}') = \begin{bmatrix} (M_T - \omega - i\delta)\delta_{ij} & 0 \\ 0 & (M_L + i\delta)e_i e_j + (M_T + i\delta)(\delta_{ij} - e_i e_j) \end{bmatrix} \delta(\mathbf{r} - \mathbf{r}'), \tag{3.153}$$

where the operators M_L and M_T were introduced above in (3.60) and (3.61). A simplest approximation for effective action $\Gamma(\Phi_{\mathrm{cl}}, \hat{G})$ reduces to the omission of the term $\Phi(\Phi_{\mathrm{cl}}, \hat{G})$ in (3.140). In this case we see from (3.143) that $\hat{\Sigma} = 0$, i.e., $\hat{G} = \tilde{G}$. To avoid confusion we emphasize that these Green's functions are not related to the average Green's functions for an electron in a random field. Equation (3.140) then yields

$$\Gamma(\varphi_{\mathrm{cl}}) = S[\varphi_{\mathrm{cl}}] - \tfrac{1}{2}\operatorname{Tr}\ln \tilde{G}^{-1}_{\phi\phi} - \tfrac{1}{2}\operatorname{Tr}\ln \tilde{G}^{-1}_{\varphi\varphi}, \tag{3.154}$$

and the the equation $\delta\Gamma/\delta\varphi_{\mathrm{cl}} = 0$ reduces to the generalized equation for an instanton

$$-\frac{1}{2m}\nabla^2\varphi_{\mathrm{cl}} - E\varphi_{\mathrm{cl}} - \frac{1}{2}\rho V^2\varphi_{\mathrm{cl}}^3 + \frac{\delta\Gamma_1(\varphi_{\mathrm{cl}})}{\delta\varphi_{\mathrm{cl}}} = 0, \tag{3.155}$$

where we set

$$\begin{aligned}\Gamma_1(\varphi_{\mathrm{cl}}) &= -\tfrac{1}{2}\operatorname{Tr}\ln \tilde{G}^{-1}_{\phi\phi} - \tfrac{1}{2}\operatorname{Tr}\ln \tilde{G}^{-1}_{\varphi\varphi} \\ &= -\tfrac{1}{2}\operatorname{Tr}\ln \operatorname{Det} \tilde{G}^{-1}_{\phi\phi} - \tfrac{1}{2}\operatorname{Tr}\ln \operatorname{Det} \tilde{G}^{-1}_{\varphi\varphi}.\end{aligned} \tag{3.156}$$

In the last equation Det is taken over the isotopic indices, while Tr is taken over the arguments of the Green's functions. We find from (3.153)

$$\operatorname{Det} \tilde{G}^{-1} = [\operatorname{Det}(M_T - \omega - i\delta)]^n \operatorname{Det} M_L [\operatorname{Det} M_T]^{m-1}. \quad (3.157)$$

Hence, we find

$$\begin{aligned}\Gamma_1(\varphi_{\text{cl}}) &= -n \sum_k (\lambda_k^T - \omega - i\delta) - \sum_k (\lambda_k^L + i\delta) - (m-1) \sum_k (\lambda_k^T + i\delta) \\ &\underset{n\to 0,\, m\to 0}{=} -\sum_k \lambda_k^L + \sum_k \lambda_k^T. \qquad (3.158)\end{aligned}$$

In other words ($\ln \operatorname{Det} M_{L,T} = \operatorname{Tr} \ln M_{L,T}$),

$$\begin{aligned}\Gamma_1(\varphi_{\text{cl}}) = &-\frac{1}{2} \operatorname{Tr} \ln\left\{1 - \left[-\frac{\nabla^2}{2m} - E\right]^{-1} \frac{3}{2} \rho V^2 \varphi_{\text{cl}}^2\right\} \\ &+ \frac{1}{2} \operatorname{Tr} \ln\left\{1 - \left[-\frac{\nabla^2}{2m} - E\right]^{-1} \frac{1}{2} \rho V^2 \varphi_{\text{cl}}^2\right\}. \qquad (3.159)\end{aligned}$$

Expanding (3.159) in a series in powers of $\rho V^2 \varphi_{\text{cl}}^2$, we see [95] that $\Gamma_1(\varphi_{\text{cl}})$ is the result of the summation of *one-loop* corrections to the classical action. Since we are dealing here with the first-order term in $\rho V^2 \varphi_{\text{cl}}^2$, we find

$$\begin{aligned}\Gamma_1^{(1)}(\varphi_{\text{cl}}) &= -\frac{1}{2} \operatorname{Tr}\left\{\left[-\frac{\nabla^2}{2m} - E\right]^{-1} \rho V^2 \varphi_{\text{cl}}^2\right\} \\ &= \frac{1}{2} \rho V^2 \int d^d\mathbf{r}\, \varphi_{\text{cl}}^2(\mathbf{r}) \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{1}{E - \mathbf{p}^2/2m} \\ &\equiv -\frac{1}{2}\, \delta E \int d^d\mathbf{r}\, \varphi_{\text{cl}}^2(\mathbf{r}), \qquad (3.160)\end{aligned}$$

where $\delta E = E - E_0 = -\operatorname{Re} \Sigma(E)$ is the "mass" renormalization in one-loop approximation, which was discussed in Sec. 3. Thus the last term in (3.155) can be dropped if we assume at the outset (as we have done everywhere) that E is the renormalized energy in the one-loop approximation. The second-order term in the expansion of $\Gamma_1(\varphi_{\text{cl}})$

gives rise to a one-loop correction to the coupling constant, while the higher-order terms seems to be unimportant insofar as their effect on the instanton solutions is concerned. Thus the results of the instanton approach analyzed above are reproduced in this approximation. The appearance of a nontrivial solution of $\varphi_{\mathrm{cl}}(\mathbf{r})$ implies that there is a "*local*" symmetry breaking over the field φ, which is different from the symmetry breaking that occurs in ordinary phase transitions. The corresponding translational and "rotational" symmetry breaking is "masked" by integration over the collective coordinates. This integration is an essential part of the calculation of the physical correlators. The condition under which the approximation we are considering can be used reduces to inequality (3.18) which was examined above. This inequality determines the energy region in which the variable E retains the simple meaning of energy reckoned from the displaced (in the one-loop approximation) band edge, and the renormalization of the coupling constant is irrelevant. We see that the critical energy at which the instanton solutions appear ($E = 0$) falls within the energy region where the nontrivial solutions of the homogeneous Bethe–Salpeter equation (2.33) appear and in which the approximations used above break down. Thus, the effective-action formalism leads in a natural way to instabilities of the "normal" (metallic) phase linked both to the appearance of a nontrivial solution of the homogeneous Bethe–Salpeter equation and to the instanton solutions. These two types of instabilities remain independent within the framework of approximations used above. This might possibly indicate that there are *two types* of electron localization. It is clear, on the other hand, that in the higher-order approximations these instabilities can be closely related in the effective-action formalism and a complete solution of the problem of their interplay requires an actual "penetration" into the "strong-coupling" region.

The energy region $|E| \leqslant [1/(d-2)]^{2/(4-d)} E_{sc}$, $E_{sc} = m^{d/(4-d)} (\rho V^2)^{2/(4-d)}$ remains outside the limits of all the methods to solve the localization problem discussed above. At the same time, we must stress that this energy region is of principal interest from the viewpoint of constructing a scaling model of the Anderson transition, since the mobility edge belongs to this region. This difficulty, which stems from the fact that neither the conventional perturbation theory nor a one-instanton approximation can be used in this case, is a fundamental difficulty which accounts for the main difference between the localization theory and the theory of critical phenomena. A systematic solution of this problem, to the best of my knowledge, is yet to be obtained, although several attempts toward this end were discussed in my previous review article [7]. Immediately after the publication of a well-known paper by Wegner [118], considerable attention has been devoted to the development of various ways of reducing the problem of the electron in a random field, near the mobility edge, to the analysis of different variants of the nonlinear σ model [119–126]. Since different formalisms have been used, the results are not always in complete agreement with each other. These studies, nonetheless, give a general picture of the localization, equivalent to that of the elementary scaling theory [26, 58]. These attempts are therefore frequently viewed as a quantitative justification of the scaling concept in localization theory. This approach is basically an attempt to construct an effective Lagrangian for the description of the region around the mobility edge directly in terms of a two-particle Green's function, along the lines initially suggested by Aharony and Imry [127]. This theory is based to some degree on the assumption that a single-particle Green's function for an electron near a mobility edge has no singularities and can, in fact, be described correctly in the simplest approximations of the perturbation theory. The resulting critical behavior is essentially in agreement with the corresponding results of the self-consistent theory of localization (see Sec. 2), although the role of corrections in the expansion in $\epsilon = d - 2$ can generally be estimated for the critical indices like those in (2.97) and (2.98). Although the studies of the region near the mobility edge based on the formalism of nonlinear σ models have contributed to our understanding of localization, this contribution, in this author's view, is relatively small. Because of the complexity of the formalism of the nonlinear σ models, in contrast with the self-consistent theory, the physical quantities near the mobility edge cannot be calculated explicitly, and the specific results reduce solely to the determination of some critical indices. These studies, as well as the self-consistent theory of localization, in our view, basically ignore the complex problem associated with the appearance of the strong-coupling region near the mobility edge, where the *functional form* of a single-particle Green's function can change substantially (although there can conceivably be no singularities, e.g., in the density of states associated with the mobility edge). This situation, as we have seen above, also applies to the case $d = 2 + \epsilon$, where some claims to describe the region near the mobility edge in terms of the perturbation theory were made. Furthermore, the results obtained in the known studies based on the σ models correspond only to the metallic region, in which respect they lose ground even to the self-consistent theory. It has not yet been possible to incorporate into these models the study of the instanton effects, whose importance was demonstrated above. For this reason, we prefer the method based on effective Lagrangians given in (3.1) and (3.5), which makes it possible to establish in each case a connection with the known results of the standard approximations and which does not obscure the difficulties associated with the inapplicability of both the standard perturbation theory and the perturbation theory near the instanton solution (one-instanton approximation) near the mobility edge. The mobility edge can hopefully be described in terms of this formalism, although such an attempt may conceivably require the use of completely new ideas and methods which are not related to the known versions of the perturbation theory. The problem we are discussing is thus linked with most urgent problems of the modern field theory, such, for example, as the problem of quark confinement.

4. Electron–Electron Interaction in the Localization Theory

4.1. Hartree–Fock Approximation. Localization Contributions

Localization is usually studied neglecting the electron–electron interaction, although its importance in the description of the metal-insulator transitions in disordered systems has been established long ago [2]. This fact has recently been confirmed in the studies of "dirty" metals [128, 129] (see also Ref. 9) and also in the analysis of the problem of "Coulomb gap" of the Fermi level for the electron system with strongly localized states [6, 130–133]. Several studies, in which an attempt was made to analyze the effects of electron–electron interaction as the metal-insulator transition in disordered systems is approached, have recently been published [134–139]. In all of these works the authors have analyzed only the metallic region near the Anderson (or Mott) transition, without focussing any attention on the insulator part of the transition. In addition to the studies already mentioned [6, 130–139], the electron–electron interaction in the case of localized electrons was considered in connection with the early attempts to build a theory of "Fermi-glass" (Refs. 140–142). All these works have shown that correlations play an important if not the decisive role in the description of a metal-insulator transition in disordered systems. The results of these studies, on the other hand, contradict each other to a large extent, so that this problem is far from being solved. The answer to such a basic question as to whether the localization persists in systems with interaction remains unclear. The difficulties discussed above, which arise in the theoretical description of the Anderson transition even in a single-electron approximation, make the problem even less tractable. In this case it would be justifiable to analyze first the case of a weak interaction in a highly disordered system in order to identify the physical properties most strongly affected by correlations. Following principally the studies of Katsnel'son and Sadovskiĭ [143, 144], we will therefore examine the first-order corrections of the perturbation theory over interaction to the density of states and some other characteristics of the system near the Anderson transition, focusing particular attention on the region of localized states.* In this sense, we will attempt to extend and generalize the known results of Aronov and Al'tshuler [9, 128, 129] from metallic to insulating region. We assume that Anderson's single-electron problem is solved and that the concept of localization applies to a system with interaction. By making this assumption, which is central to our analysis, we can hope to justify the use of only the first-order corrections in terms of the interaction, although higher-order corrections in terms of the interaction, as we will see below, must be studied thoroughly in the immediate vicinity of the mobility edge.

If we assume that the single-electron problem is solved, we can again introduce the complete orthonormal set of exact wave functions $\varphi_\nu(\mathbf{r})$ and the corresponding exact eigenvalues ϵ_ν of the energy of an electron in a random field of a disordered system

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*Some further developments, concerning in particular the possible interplay between localization and superconductivity can be found in Ref. 157.

which we have examined in Sec. 1. These functions and energies can correspond to both the localized and delocalized states. Let us consider a causal single-electron Green's function as it is expressed in these exact eigenfunctions and, in particular, its diagonal matrix element,

$$G_{\nu\nu}(\epsilon)=\left\langle \nu\left|\frac{1}{\epsilon-H+i\delta\operatorname{sign}\epsilon}\right|\nu\right\rangle, \tag{4.1}$$

where H is a complete Hamiltonian which takes the electron–electron interaction into account, and ϵ is the energy reckoned from the Fermi level. The interaction is taken into account by introducing the corresponding self-energy [145, 146] (see also Refs. 140–142), $\Sigma_\nu(\epsilon)=\Delta_\nu(\epsilon)-i\Gamma_\nu(\epsilon)\operatorname{sign}\epsilon$,

$$G_{\nu\nu}(\epsilon)=\frac{1}{\epsilon-\epsilon_\nu-\Sigma_\nu(\epsilon)}. \tag{4.2}$$

The feasibility of introducing such a self-energy in the representation of arbitrary quantum numbers ν was discussed extensively by Migdal [145]. In accordance with the standard procedure [145, 146], let us introduce the renormalized energy $\tilde{\epsilon}_\nu$ as a solution of the equation

$$\tilde{\epsilon}_\nu-\epsilon_\nu-\Delta_\nu(\tilde{\epsilon}_\nu)=0, \tag{4.3}$$

and for $\epsilon\sim\tilde{\epsilon}_\nu$ we can write (4.2) in the form

$$G_{\nu\nu}(\epsilon)\simeq\frac{Z_\nu}{\epsilon-\tilde{\epsilon}_\nu+i\gamma_\nu\operatorname{sign}\epsilon}, \tag{4.4}$$

where

$$Z_\nu=\left\{1-\frac{\partial\Delta_\nu(\epsilon)}{\partial\epsilon}\right]^{-1}_{\epsilon=\tilde{\epsilon}_\nu};\qquad \gamma_\nu=Z_\nu\Gamma_\nu(\epsilon=\tilde{\epsilon}_\nu). \tag{4.5}$$

Let us introduce, in accordance with Ref. 146, the "self-energy" $\tilde{\Sigma}_E(\epsilon)$, which was averaged over an isoenergetic surface $E=\epsilon_\nu$ and over the configurations of the random field of a disordered system,

$$\tilde{\Sigma}_E(\epsilon) = \tilde{\Delta}_E(\epsilon) - i\tilde{\Gamma}_E(\epsilon) = \frac{1}{N_0(E)} \left\langle \sum_\nu \delta(E - \epsilon_\nu) \sum\nolimits_\nu(\epsilon) \right\rangle, \quad (4.6)$$

where $N_0(E)$ is the single-electron (average) density of states.

We will consider next a single-electron density of states, in which the interaction is taken into account. This density of states is usually given by

$$N(E) = -\frac{1}{\pi} \left\langle \sum_\nu \operatorname{Im} G^R_{\nu\nu}(E) \right\rangle. \quad (4.7)$$

Under the assumption that the interaction corrections are weak in the sense that the inequalities $\gamma_\nu \ll \epsilon_\nu \sim \tilde{\epsilon}_\nu$ are satisfied, we find in a first approximation, after some simple calculations using (4.4)–(4.6) in (4.7),

$$\frac{\delta N(E)}{N_0(E)} = \frac{N(E) - N_0(E)}{N_0(E)} \approx -\frac{\partial \tilde{\Delta}_E(\tilde{\epsilon}_\nu)}{\partial E} + \frac{\partial \tilde{\Delta}_E(\tilde{\epsilon}_\nu)}{\partial \tilde{\epsilon}_\nu}. \quad (4.8)$$

For reasons that will be clarified below, we call the quantity

$$\frac{\delta \tilde{N}(E)}{N_0(E)} = -\frac{\partial \tilde{\Delta}_E(\epsilon_\nu)}{\partial E} \quad (4.9)$$

a correction to the "thermodynamic density of states." This correction was analyzed for the first time by Abraham *et al.* [146] (see also Ref. 136). Here the different ways to define the density of states in a system with interactions are reflected (compare with Refs. 130 and 131).

Let us examine a model problem, in which the electron–electron interaction is described by a repulsive static potential with a finite range,

$$H_{\text{int}} = \frac{1}{2} \int d^d\mathbf{r} \int d^d\mathbf{r}' \sum_{\mu\nu\mu'\nu'} \varphi^*_\nu(\mathbf{r}')\varphi^*_\nu(\mathbf{r}) v(\mathbf{r} - \mathbf{r}') \\ \times \varphi_{\mu'}(\mathbf{r})\varphi_{\mu'}(\mathbf{r}') a^+_\mu a^+_\nu a_{\mu'} a_{\nu'}. \quad (4.10)$$

Analyzing the Hartree–Fock diagrams (Fig. 13), we then find

$$\begin{aligned}\Sigma^H_\mu &= \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \sum_\nu f_\nu \varphi^*_\mu(\mathbf{r}')\varphi^*_\nu(\mathbf{r})\varphi_\nu(\mathbf{r})\varphi_\mu(\mathbf{r}') \\ \Sigma^F_\mu &= -\int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \sum_\nu f_\nu \varphi^*_\mu(\mathbf{r}')\varphi^*_\nu(\mathbf{r})\varphi_\mu(\mathbf{r})\varphi_\nu(\mathbf{r}'),\end{aligned} \tag{4.11}$$

where $f_\nu = f(\epsilon_\nu)$ is the Fermi distribution function. Accordingly, we find from (4.6)

$$\begin{aligned}\tilde{\Sigma}^H_E &= \int_{-\infty}^{\infty} d\omega f(E+\omega) \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \langle\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle^H \\ \tilde{\Sigma}^F_E &= -\int_{-\infty}^{\infty} d\omega f(E+\omega) \int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}') \langle\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle^F.\end{aligned} \tag{4.12}$$

In these expressions we have introduced the spectral densities,

$$\langle\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle^H = \frac{1}{N_0(E)}\left\langle \sum_{\mu\nu} \delta(E-\epsilon_\mu)\delta(E+\omega-\epsilon_\nu)\varphi^*_\mu(\mathbf{r}')\varphi_\mu(\mathbf{r}')\varphi^*_\nu(\mathbf{r})\varphi_\nu(\mathbf{r})\right\rangle \tag{4.13}$$

$$\langle\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\rangle^F = \frac{1}{N_0(E)}\left\langle \sum_{\mu\nu} \delta(E-\epsilon_\mu)\delta(E+\omega-\epsilon_\nu)\varphi^*_\mu(\mathbf{r}')\varphi^*_\nu(\mathbf{r})\varphi_\mu(\mathbf{r})\varphi_\nu(\mathbf{r}')\right\rangle. \tag{4.14}$$

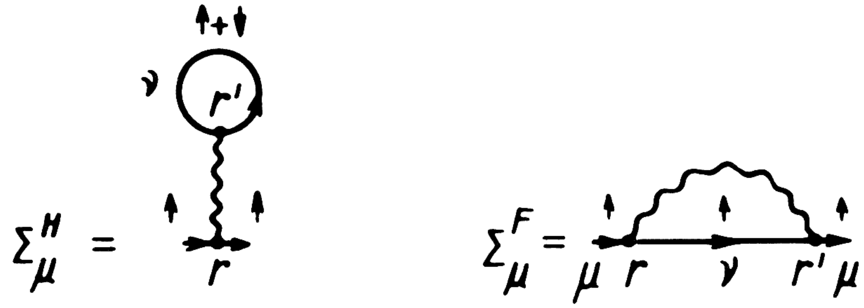

Figure 13. The Hartree–Fock diagrams for the self-energy.

Spectral density (4.14), introduced for the first time by Berezinskiĭ and Gor'kov [34], and was analyzed in Sec. 1 in connection with the general criterion of localization. Applying the same line of reasoning to spectral density (4.13) as was applied there to the contribution of $\epsilon_\mu = \epsilon_\nu$, we easily see that this spectral density acquires a δ-function in ω term in the region of localized states ($E < E_c$):

$$\begin{aligned} \langle\!\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\!\rangle^H &= A_E(\mathbf{r}-\mathbf{r}')\delta(\omega) + \rho_E^H(\omega, \mathbf{r}-\mathbf{r}') \\ \langle\!\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\!\rangle^F &= A_E(\mathbf{r}-\mathbf{r}')\delta(\omega) + \rho_E^F(\omega, \mathbf{r}-\mathbf{r}'), \end{aligned} \tag{4.15}$$

where, as in (1.9),

$$A_E(\mathbf{r}-\mathbf{r}') = \frac{1}{N_0(E)}\left\langle \sum_\mu \delta(E-\epsilon_\mu)|\varphi_\mu(\mathbf{r}')|^2|\varphi_\mu(\mathbf{r})|^2 \right\rangle > 0; \quad E < E_c \tag{4.16}$$

is the generalized inverse participation ratio.

Incorporating (4.15) into (4.12), we find the following contributions to $\tilde{\Sigma}_E$ which are associated with the appearance of localized states in the system

$$\begin{aligned} \tilde{\Sigma}^{H,F}_{E_{\text{loc}}} = \tilde{\Delta}^{H,F}_{E_{\text{loc}}} &= \pm f(E)\int d^d\mathbf{r} \int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}')A_E(\mathbf{r}-\mathbf{r}') \\ &= \pm f(E)\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(-\mathbf{q})A_E(\mathbf{q}). \end{aligned} \tag{4.17}$$

In the case of point-like interaction $\nu(\mathbf{r}-\mathbf{r}') = v_0\delta(\mathbf{r}-\mathbf{r}')$ we find

$$\tilde{\Sigma}^{H,F}_{E_{\text{loc}}} = \pm f(E)v_0\int \frac{d^d\mathbf{q}}{(2\pi)^d} A_E(\mathbf{q}) = \pm f(E)v_0 A_E, \tag{4.18}$$

where the quantity A_E (2.22) is proportional to the total probability of return of an electron to the starting point during an infinite time. We note that because of the property

$$\langle\!\langle \rho_E(\mathbf{r})\rho_{E'}(\mathbf{r})\rangle\!\rangle^H = \langle\!\langle \rho_E(\mathbf{r})\rho_{E'}(\mathbf{r})\rangle\!\rangle^F, \tag{4.19}$$

which is evident from (4.13) and (4.14), the "regular" contributions to $\tilde{\Sigma}_E^H$ and $\tilde{\Sigma}_E^F$, which are associated with $\rho_E^{H,F}(\omega,\mathbf{r}-\mathbf{r}')$ in (4.15), are also equal (and opposite in sign) in the case of point-like interaction.

In the case of spin-zero electrons the Hartree–Fock contributions in (4.18) cancel each other. After taking the spin into account, the Hartree contribution acquires an additional factor of 2 which is connected with the summing over the spin directions in the electron loop. As a result, we find a nonvanishing "localization" contribution

$$\tilde{\Sigma}_{E_{\rm loc}}^{H+F}=\tilde{\Delta}_{E_{\rm loc}}^{H+F}=f(E)v_0A_E. \tag{4.20}$$

To avoid ambiguity, we will write out the equations for the point-like interaction. We will now take into account that the main energy dependence in (4.20) is determined by the Fermi function, which varies sharply near the Fermi energy. At $E\approx E_F$ the quantity A_E may be assumed to be a constant (a smooth function of E). This assumption may generally turn out to be incorrect near the mobility edge where A_E vanishes. The corresponding critical index, strictly speaking, is not known, although we can deduce from estimates like those in (3.124) that $\partial A_E/\partial E\to 0$ also in the limit of $E\to E_c$. We then find from (4.9) and (4.20)

$$\frac{\delta\tilde{N}_{\rm loc}(E)}{N_0(E)}=-\frac{\partial\tilde{\Delta}_{E_{\rm loc}}^{H+F}}{\partial E}\approx v_0A_{E_F}\left(-\frac{\partial f(E)}{\partial E}\right). \tag{4.21}$$

In the "total" density of states determined by (4.7) and (4.8) the singular (localization) contribution (4.21) is cancelled by the second term in (4.8),

$$\begin{aligned}\frac{\partial\tilde{\Delta}_{E_{\rm loc}}^{H+F}}{\partial\tilde{\epsilon}_\nu}\simeq\frac{\partial\tilde{\Delta}_{E_{\rm loc}}^{H+F}}{\partial\epsilon_\nu}&=\frac{1}{N_0(E)}\left\langle\sum_\nu\int d^d\mathbf{r}\int d^d\mathbf{r}'v(\mathbf{r}-\mathbf{r}')\right.\\&\quad\left.\times\frac{\partial f_\nu}{\partial\epsilon_\nu}\delta(E-\epsilon_\nu)|\varphi_\nu(\mathbf{r})|^2|\varphi_\nu(\mathbf{r}')|^2\right\rangle\\&=\frac{\partial f(E)}{\partial E}\int d^d\mathbf{r}\int d^d\mathbf{r}'v(\mathbf{r}-\mathbf{r}')A_E(\mathbf{r}-\mathbf{r}')\\&=v_0A_E\frac{\partial f(E)}{\partial E}.\end{aligned} \tag{4.22}$$

The localization contribution is retained in the "thermodynamic" density of states (4.9). This circumstance, as we will see below, accounts for the peculiar behavior of several thermodynamic quantities.

To have a clearer understanding of the physical meaning of the localization contribution to $\tilde{\Sigma}_E^{H+F}$ we should note that it is actually connected with the interaction of electrons which are in the same quantum state ν. We see from Fig. 13 that the contributions from the interaction of electrons with the same spin directions (indicated by the arrows in Fig. 13) are canceled out completely in the sum of these diagrams, and $\tilde{\Sigma}_E^{H+F}$ is determined by the interaction of two electrons with opposite spins which are in the same state ν, i.e., by the effective "Hubbard"-type interaction

$$H_{\text{eff}} = \frac{1}{2}\sum_{\nu\sigma} \int d^d\mathbf{r} \int d^d\mathbf{r}'\, v(\mathbf{r}-\mathbf{r}')|\varphi_\nu(\mathbf{r})|^2|\varphi_\nu(\mathbf{r}')|^2 n_{\nu\sigma} n_{\nu-\sigma}, \tag{4.23}$$

where $n_{\nu\sigma}$ is the operator of the number of electrons in the state ν with a spin σ. Using a simplest estimate of the quantity A_E, we find $(E_F < E_c)$

$$\tilde{\Delta}_{E_{\text{loc}}}^{H+F} \sim \begin{cases} v_0\left(\dfrac{1}{R_{\text{loc}}(E)}\right)^d; & E < E_F \\ 0; & E > E_F. \end{cases} \tag{4.24}$$

Comparing our results with the familiar qualitative estimates of Mott [2, 147], we find that they correspond to the formation of the "band" of singly occupied electron states of width $\tilde{\Delta}_{E_{\text{loc}}}^{H+F}$, which appear below the Fermi level. Similar arguments were also advanced by Kamimura [148] and Berezinskiĭ and Gor'kov [149]. Clearly, the single spins at the levels of this band account for the additional contribution to the paramagnetic susceptibility. Since the number of unpaired spins in order of magnitude is

$$\mathcal{N} \approx \int_{E_F-\tilde{\Delta}_{F_{\text{loc}}}^{H+F}}^{E_F} dE N_0(E) \approx N_0(E_F)\tilde{\Delta}_{E_{\text{loc}}}^{H+F} = v_0 N_0(E_F) A_{E_F}, \tag{4.25}$$

we find the Curie law (μ_B is the Bohr magneton)

$$\delta\chi^{\mathrm{loc}} \sim \frac{1}{T}\mathcal{N}\mu_B^2 = \frac{v_0}{T} N_0(E_F) A_{E_F}\mu_B^2. \tag{4.26}$$

This result can also be easily derived from a direct analysis of the Hartree–Fock corrections to the thermodynamic potential which are determined by the diagrams shown in Fig. 14. Carrying out some straightforward calculations, we find ($h = \mu_B H$, where H is the external magnetic field)

$$\langle\delta\Omega_H\rangle = \frac{1}{2}\int d^d\mathbf{r}\int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}')\int_{-\infty}^{\infty} dE\int_{-\infty}^{\infty} dE' \times \sum_{\sigma\sigma'} f(E-\sigma h) f(E'-\sigma' h)\langle\langle\rho_E(\mathbf{r})\rho_{E'}(\mathbf{r}')\rangle\rangle^H, \tag{4.27}$$

$$\langle\delta\Omega_F\rangle = -\frac{1}{2}\int d^d\mathbf{r}\int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}')\int_{-\infty}^{\infty} dE\int_{-\infty}^{\infty} dE' \sum_{\sigma} f(E-\sigma h) \times [1-f(E'-\sigma h)]\langle\langle\rho_E(\mathbf{r})\rho_{E'}(\mathbf{r}')\rangle\rangle^F. \tag{4.28}$$

The term with unity in the square brackets in (4.28) reduces to an unessential constant which renormalizes the chemical potential,

$$\int d^d\mathbf{r}\int d^d\mathbf{r}' v(\mathbf{r}-\mathbf{r}')\left\langle\sum_{\mu\nu} f_\mu \varphi_\mu^*(\mathbf{r}')\varphi_\mu(\mathbf{r})\varphi_\nu^*(\mathbf{r}')\varphi_\nu(\mathbf{r})\right\rangle = v(0)\int d^d\mathbf{r}\left\langle\sum_\mu f_\mu|\varphi_\mu(\mathbf{r})|^2\right\rangle = v(0)\sum_\mu f_\mu = Nv(0), \tag{4.29}$$

where N is the total number of electrons. Incorporating (4.12), we then find

$$\langle\delta\Omega_{H,F}\rangle = \int_{-\infty}^{\infty} dE f(E) N_0(E)\tilde{\Sigma}_E^{H,F}. \tag{4.30}$$

Figure 14. The Hartree–Fock corrections to the thermodynamic potential.

Integrating by parts in (4.30), we get

$$\delta\Omega = \langle\delta\Omega_H\rangle + \langle\delta\Omega_F\rangle = TN_0(E_F)\int_{-\infty}^{\infty} dE\,\frac{\partial\tilde{\Delta}_E^{H+F}}{\partial E}\ln[1+e^{-E/T}]. \tag{4.31}$$

The term "thermodynamic density of states," which was introduced above in connection with (4.9), can be understood by comparison of (4.31) with the familiar expression for the thermodynamic potential of free electrons

$$\Omega = -T\int_{-\infty}^{\infty} dEN(E)\ln[1+e^{-E/T}]. \tag{4.32}$$

Calculating the magnetic susceptibility $\chi = -\langle\partial^2\Omega/\partial H^2\rangle_{H=0}$, we find, after a straightforward differentiation of (4.27) and (4.28),

$$\begin{aligned}\delta\chi_H = -4\mu_B^2\int d^d\mathbf{r}\int d^d\mathbf{r}'v(\mathbf{r}-\mathbf{r}')\int_{-\infty}^{\infty} dE\,\frac{\partial^2 f(E)}{\partial E^2}\int_{-\infty}^{\infty} dE' f(E')\\ \times\langle\!\langle\rho_E(\mathbf{r})\rho_{E'}(\mathbf{r}')\rangle\!\rangle^H,\end{aligned} \tag{4.33}$$

$$\begin{aligned}\delta\chi_F = 2\mu_B^2\int d^d\mathbf{r}\int d^d\mathbf{r}'v(\mathbf{r}-\mathbf{r}')\int_{-\infty}^{\infty} dE\int_{-\infty}^{\infty} dE'\left\{\frac{\partial^2 f(E)}{\partial E^2}f(E')\right.\\ \left.+\frac{\partial f(E)}{\partial E}\frac{\partial f(E')}{\partial E'}\right\}\langle\!\langle\rho_E(\mathbf{r})\rho_{E'}(\mathbf{r}')\rangle\!\rangle.\end{aligned} \tag{4.34}$$

Integrating by parts the first term in (4.34), we see that $\delta\chi_F = 0$ in the limit of $T\to 0$. Finally, the localization contribution to the magnetic susceptibility is

$$\begin{aligned}\delta\chi_{\text{loc}} &= -4\mu_B^2\int d^d\mathbf{r}\int d^d\mathbf{r}'v(\mathbf{r}-\mathbf{r}')\\ &\quad\times\int_{-\infty}^{\infty} dEN_0(E)A_E(\mathbf{r}-\mathbf{r}')\frac{\partial^2 f(E)}{\partial E^2}f(E)\\ &\approx 4N_0(E_F)A_{E_F}v_0\mu_B^2\int_{-\infty}^{\infty} dE\left(\frac{\partial f(E)}{\partial E}\right)^2\\ &= \frac{2\mu_B^2}{3}\frac{v_0}{T}N_0(E_F)A_{E_F};\end{aligned} \tag{4.35}$$

consistent with the estimate in (4.26). This result fully confirms the qualitative arguments for the existence of singly occupied states below the Fermi level. A result like that in (4.35) was obtained by Berezinskiĭ and Gor'kov [149] in a one-dimensional model. Clearly, the interaction of these single spins generally becomes important in the limit of $T \to 0$ [2, 147].

The singular (localization) part of the thermodynamic potential is

$$\begin{aligned}\delta\Omega_{\mathrm{loc}} &= \langle \delta\Omega_H \rangle_{\mathrm{loc}} + \langle \delta\Omega_F \rangle_{\mathrm{loc}} \\ &= \int d^d\mathbf{r} \int d^d\mathbf{r}' \int_{-\infty}^{\infty} dE v(\mathbf{r}-\mathbf{r}') A_E(\mathbf{r}-\mathbf{r}') N_0(E) f^2(E) \\ &= \int_{-\infty}^{\infty} dE \int \frac{d^d\mathbf{q}}{(2\pi)^d} v(-\mathbf{q}) A_E(\mathbf{q}) N_0(E) f^2(E). \end{aligned} \tag{4.36}$$

The corresponding contributions to the entropy and specific heat can easily be found:

$$\begin{aligned} S_{\mathrm{loc}} = -\frac{\partial}{\partial T}\delta\Omega_{\mathrm{loc}} &= -\int_{-\infty}^{\infty} dE N_0(E) \frac{\partial}{\partial T} f^2(E) \int \frac{d^d\mathbf{q}}{(2\pi)^d} v(-\mathbf{q}) A_E(\mathbf{q}) \\ &\xrightarrow[T\to 0]{} v_0 N_0(E_F) A_{E_F} \end{aligned} \tag{4.37}$$

$$C_{\mathrm{loc}} = T\frac{\partial S_{\mathrm{loc}}}{\partial T} \approx -\frac{\pi^2}{3} T v_0 \frac{\partial}{\partial E_F}\{N_0(E_F) A_{E_F}\}. \tag{4.38}$$

We see that the entropy tends toward the positive constant in the limit $T \to 0$,* and that the localization contribution to the specific heat is linked with a small ($\sim \partial A_{E_F}/\partial E_F$) correction to the thermodynamic state density, which was ignored above. Similarly, the small localization correction to the correlation contribution to the

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*This contribution is connected with the entropy of "free" spins at the singly-occupied states and disappear when some kind of spin–spin interaction leading to magnetic order as $T \to 0$ is taken into account.

compressibility is (μ is the chemical potential)

$$\delta\kappa_{\mathrm{loc}} = -\frac{\partial^2}{\partial\mu^2}\,\delta\Omega_{\mathrm{loc}} = -v_0\int_{-\infty}^{\infty} dE A_E N_0(E)\,\frac{\partial^2}{\partial\mu^2}\,f^2(E)$$

$$= v_0\,\frac{\partial}{\partial E_F}\{A_{E_F} N_0(E_F)\}. \tag{4.39}$$

It is thus quite obvious that a singular contribution (4.21) to the "thermodynamic" density of states is fully consistent with the third law of thermodynamics and that it "signals" the appearance of a band of singly occupied states below the Fermi level.

Our analysis has so far been completely general in nature. In our further analysis we have to use a particular single-electron model of the Anderson transition. We will then be able to examine the "regular" contributions to the density of states which are associated with the nonsingular terms in (4.15). We will confine ourselves solely to the Fock contribution to (4.11), since the Hartree contribution, as was pointed out in Refs. 9, 129, and 146, is small if the interaction potential falls off at a distance greater than the inverse Fermi momentum. We will see below that such estimates also apply to a "regular" contribution to (4.11) in the localization region. In the case of a point-like interaction, however, the Hartree contribution, as we have already seen from (4.19), is twice as large (taking the account of spin) as the Fock contribution, so that the results which we will obtain below should be taken with a different sign (compare also with Ref. 150). As the single-electron model of the Anderson transition, we use the self-consistent theory of localization, since we can easily derive within its framework the explicit expressions for all the relevant quantities. Specifically, we find from (4.12) the "regular" contribution to $\tilde{\Sigma}_E^F$ at $T = 0$ to be

$$\tilde{\Sigma}^F_{E_{\mathrm{reg}}} = -\frac{1}{\pi}\int_{-\infty}^{0} dE' \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q})\,\frac{D_{E_F}\mathbf{q}^2}{(E'-E)^2 + [\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^2} \tag{4.40}$$

where we have used the Berezinskiĭ–Gor'kov spectral density calculated from (2.74). Accordingly, the correction to the density of states is

$$\frac{\delta N(E)}{N_0(E)} \approx -\frac{d\tilde{\Sigma}^F_{E_{\mathrm{reg}}}}{dE} = -\frac{1}{\pi}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q})\, \frac{D_{E_F}\mathbf{q}^2}{E^2+[\omega_0^2(E_F)\tau_{E_F}+D_{E_F}\mathbf{q}^2]^2}. \tag{4.41}$$

Using, for simplicity, the point-like interaction model, and recalling that all the energies so far in this part of the article were reckoned from the Fermi level E_F, we find $(2 < d < 4)$ *

$$\frac{\delta N(E)}{N_0(E_F)} \approx \frac{v_0}{\pi}\frac{S_d}{d-2} D_{E_F}^{-d/2} \times \begin{cases} |E-E_F|^{(d-2)/2} - \tilde{E}^{(d-2)/2}; & |E-E_F| \gg \omega_0^2(E_F)\tau_{E_F} \\ \omega_0^{d-2}(E_F)\tau_{E_F}^{(d-2)/2} - \tilde{E}^{(d-2)/2}; & |E-E_F| \ll \omega_0^2(E_F)\tau_{E_F}, \end{cases} \tag{4.42}$$

where the characteristic energy $\tilde{E}$ is connected with the choice of the cutoff parameter at the upper limit of the integral over $\mathbf{q}$ in (4.41). This cutoff is necessary because the "diffusion" approximation for the integrand cannot be used at large values of the momentum. As we have done in the analysis of a similar cutoff in (2.77), we will set the cutoff momentum equal to the Fermi momentum, so that

$$\tilde{E} = D_{E_F} p_F^2, \tag{4.43}$$

An alternative approach would be to set the cutoff parameter equal to the reciprocal mean free path l^{-1}, but, as we have pointed out above, near the mobility edge we would have $l^{-1} \sim p_F$, so that both options would be equivalent in this region. According to the scaling argument advanced by Lee [136], the cutoff parameter should be set equal to about R_{loc}^{-1} near the mobility edge when $R_{\mathrm{loc}}(E_F) \gg l, p_F^{-1}$; however, such a choice of this parmaeter in the main equation of the self-consistent theory (2.77) leads to some contradictory results. Since our further analysis is essentially based on the self-consistent theory, we will use Eq. (4.43). It is easy to see that the estimate (4.42) is valid if the following condition is satisfied:

$$|E-E_F|; \qquad \omega_0^2(E_F)\tau_{E_F} \ll \tilde{E}. \tag{4.44}$$

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*Slight modifications of (4.42) associated with the frequency dependence of generalized diffusion coefficient [53, 60] can be found in Ref. 157.

In the special case of $d=2$, instead of (4.42) we get:

$$\frac{\delta N(E)}{N_0(E_F)}=\frac{v_0}{4\pi^2 D_{E_F}}\begin{cases}\ln\dfrac{|E-E_F|}{\tilde{E}}; & |E-E_F|\gg\omega_0^2(E_F)\tau_{E_F}\\[2ex] \ln\dfrac{\omega_0^2(E_F)\tau_{E_F}}{\tilde{E}}; & |E-E_F|\ll\omega_0^2(E_F)\tau_{E_F}.\end{cases}\tag{4.45}$$

For $\omega_0^2(E_F)=0$, i.e., in the metallic region, Eqs. (4.42) and (4.43) are consistent with the known results of Aronov and Al'tshuler [9, 128, 129, 136, 146], if the renormalized diffusion coefficient (2.75) is substituted for the classical diffusion coefficient. For $|E-E_F|\gg\omega_0^2(E_F)\tau_{E_F}$ this behavior also applies to the dielectric region; however, for $|E-E_F|\ll\omega_0^2(E_F)\tau_{E_F}$ a cusp in the density of states at the Fermi level of a metal flattens out, giving way to a smooth minimum.

Let us consider some specific dependences found in the self-consistent localization theory. Discarding unessential constants, we find from (2.93), (2.95), and (2.96), for $2<d<4$

$$\omega_0^2(E_F)\tau_{E_F}\sim\frac{\lambda}{4-d}\frac{E_F}{(p_F R_{\mathrm{loc}}(E_F))^d}\sim\frac{\lambda E_F}{4-d}\left|\frac{E_F-E_c}{E_c}\right|^{d\nu},\tag{4.46}$$

$$D_{E_F}\sim\frac{1}{4-d}\frac{1}{m}(p_F R_{\mathrm{loc}}(E_F))^{2-d}\underset{E_F\leqslant E_c}{\sim}\frac{1}{m}\left|\frac{E_F-E_c}{E_c}\right|^{(d-2)\nu},\tag{4.47}$$

$$\begin{aligned}\tilde{E}&\sim\frac{dE_F}{4-d}\left(\frac{d\omega_0^2(E_F)}{4E_F^2}\right)^{(d-2)/2}\\ &\sim E_F(p_F R_{\mathrm{loc}}(E_F))^{2-d}\underset{E_F\leqslant E_c}{\sim}E_F\left|\frac{E_F-E_c}{E_c}\right|^{(d-2)\nu},\end{aligned}\tag{4.48}$$

where ν is the critical index of the localization length (2.97). We see that condition (4.44) can easily be satisfied. For the correction to the density of states at the Fermi level, or more precisely, for $|E-E_F|\ll\omega_0^2(E_F)\tau_{E_F}$ we find from (4.42) and (4.46)–(4.48)

$$\frac{\delta N(E_F)}{N_0(E_F)} \approx v_0 \frac{4-d}{d-2} m^{d/2} E_F^{(d/2)-1} \{1-(p_F R_{\mathrm{loc}}(E_F))^{d-2}\}_{E_F \sim E_c}$$
$$\underset{E_F \sim E_c}{\sim} -v_0 N_0(E_F) \left| \frac{E_F - E_c}{E_c} \right|^{-(d-2)\nu}, \qquad (4.49)$$

where $N_0(E_F)$ in the last equation denotes the density of states of free electrons at the Fermi level.* A divergence of the correction to the density of states in the limit of $E_F \to E_c$ (there is a similar divergence in the metallic region) shows that our analysis breaks down in the immediate vicinity of the mobility edge. The estimates which we have made, however, are valid so long as $\delta N(E) \ll N_0(E_F)$.

For $d = 2$, we find [see Eqs. (2.93) and (2.95)]

$$\begin{aligned} \omega_0^2(E_F)\tau_{E_F} &\sim E_F \exp\left(-\frac{1}{\lambda}\right) \\ \tilde{E} &\sim E_F \left\{1 - \exp\left(-\frac{1}{\lambda}\right)\right\}, \end{aligned} \qquad (4.50)$$

so that from (4.45) we get:

$$\frac{\delta N(E)}{N_0} \sim v_0 N_0 \begin{cases} \ln \dfrac{|E-E_F|}{E_F}; & |E-E_F| \gg E_F \exp\left\{-\dfrac{\pi E_F}{\gamma}\right\} \\ -\dfrac{\pi E_F}{\gamma}; & |E-E_F| \ll E_F \exp\left\{-\dfrac{\pi E_F}{\gamma}\right\}, \end{cases} \qquad (4.51)$$

where $N_0 = m/2\pi$ is the density of states of free electrons in the two-dimensional space, and γ is the Born scattering frequency.

The corrections to the density of states found above can be determined from the following heuristic considerations. Let us consider the interaction of an electron in the state ν with energy E with electrons in the states with energy E_F. The relative correction to its wave function in first-order of perturbation theory would be

$$\frac{\delta\varphi_\nu}{\varphi_\nu} \sim \int_0^\infty dt H_{\mathrm{int}}(t), \qquad (4.52)$$

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*In case of cutoff scheme proposed by Lee [136] instead of power-law (4.49) we get the logarithmic divergence of the density of states correction at the mobility edge.

where at $t = 0$ the interaction is "switched" on and $H_{\rm int}(t)$ is used in the interaction representation. The electron "diffuses" within the limits of the volume $(D_{E_F} t)^{d/2}$ during the time t. The matrix element for the short-range repulsive interaction can then be estimated as $\sim v_0 (D_{E_F} t)^{-d/2}$. Thus we find

$$\frac{\delta\varphi_\nu}{\varphi_\nu} \sim v_0 \int_{t_{\min}}^{t_{\max}} dt (D_{E_F} t)^{-d/2} \sim \frac{v_0}{D_{E_F}^{d/2}} \{ t_{\min}^{1-(d/2)} - t_{\max}^{1-(d/2)} \}. \quad (4.53)$$

Here $t_{\min}$ can be determined from the condition of applicability of "diffusion" approximation $(D_{E_F} t_{\min})^{1/2} \sim p_F^{-1}$, i.e., $t_{\min} \sim (D_{E_F} p_F^2)^{-1} \sim \tilde{E}^{-1}$. The time $t_{\max}$ is determined by two factors. First, the interaction matrix element vanishes over the time intervals $t > |E - E_F|^{-1}$ because of the time-dependent oscillations of the wave functions. Secondly, the interacting electrons in the region of localized states cannot move apart to a distance greater than $R_{\rm loc}(E_F)$ from each other and the "diffusion" approximation is valid as long as the time $t \leq R_{\rm loc}^2(E_F)/D_{E_F} \sim [\omega_0^2(E_F)\tau_{E_F}]^{-1}$. Therefore, $t_{\max} \sim \min\{|E - E_F|^{-1};\ [\omega_0^2(E_F)\tau_{E_F}]^{-1}\}$. Setting $\delta N(E)/N_0(E_F) \sim \delta\varphi_\nu/\varphi_\nu$ [compare with (1.4)], we immediately find (4.42). This estimate is, of course, purely explanatory in nature.

These results represents a simple generalization of the analysis of Aronov and Al'tshuler [9, 128, 129] for the insulator side of the Anderson transition. There is no "Coulomb gap" in these approximations [130–133] principally because of the short-range nature of the interaction and possibly because of the crudeness of the model which is based solely on the first-order perturbation theory corrections in terms of the interaction. Although the advantage of the formalism which has been used is its clarity, it obviously cannot be used with any measure of success for a generalization in which higher-order corrections in terms of the interaction could be incorporated, because in this case we have to know the behavior of "higher-order" spectral densities.

4.2. Electron–Electron Interaction in the Self-Consistent Theory of Localization

We will consider below the role of the first-order corrections in the perturbation theory over interaction within the framework of the standard diagram formalism of the self-consistent theory of localization. By using this approach we can improve the accuracy of the results obtained above and make some generalizations. Clearly, the diagram formalism, in principle, is not limited to the first-order perturbation theory in terms of the interaction.

Following the procedure of Al'tshuler and Aronov [128], we will examine a very simple correction to a single-electron Green's function, illustrated graphically in Fig. 15a, where the "triangular" vertex is given by the equation, illustrated graphically in Fig. 15b,

$$\gamma(\mathbf{q}\omega) = 1 + \int \frac{d^d\mathbf{p}'}{(2\pi)^d} \Gamma_{\mathbf{pp}'}(\mathbf{q}\omega) G(\epsilon + \omega\mathbf{p}'_+) G(\epsilon\mathbf{p}_-). \tag{4.54}$$

To avoid confusion we wish to point out that here and elsewhere in the text we have $\epsilon = E - E_F$; i.e., this expression denotes the energy reckoned from the Fermi level. The correction to a single-electron density of states will then be

$$\begin{aligned}\frac{\delta N(\epsilon)}{N_0(E_F)} &= -\frac{1}{\pi N_0(E_F)} \int \frac{d^d\mathbf{p}}{(2\pi)^d} \operatorname{Im} \delta G^R(\epsilon\mathbf{p}) \\ &= -\frac{1}{\pi N_0(E_F)} \operatorname{Im} \int \frac{d^d\mathbf{p}}{(2\pi)^d}\, i[G^R(\epsilon\mathbf{p})]^2 \\ &\quad \times \int \frac{d^d\mathbf{q}}{(2\pi)^d} \int_\epsilon^\infty \frac{d\omega}{2\pi}\, \gamma^2(\mathbf{q}\omega) G^A(\epsilon + \omega\mathbf{p} + \mathbf{q}) \\ &= -\frac{1}{2\gamma^2(E_F)} \operatorname{Im} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \int_\epsilon^\infty \frac{d\omega}{2\pi}\, \gamma^2_{RA}(\mathbf{q}\omega) v(\mathbf{q}). \end{aligned} \tag{4.55}$$

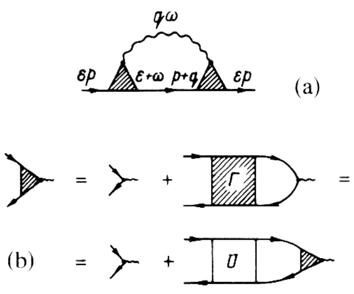

Figure 15. (a) A simplest correction to a single-electron Green's function. (b) An equation for a "triangular" vertex.

The single-electron Green's functions in (4.55) are taken in a simplest approximation (2.34). Clearly, for a "triangular" vertex with external legs we would have (see Fig. 3)

$$\begin{aligned}\mathcal{T}^{RA}_{\mathbf{p}}(\mathbf{q}\omega) &\equiv G^R(\epsilon+\omega\mathbf{p}_+)G^A(\epsilon\mathbf{p}_-)\gamma^{RA}(\mathbf{q}\omega)\\ &= -2\pi i\int\frac{d^d\mathbf{p}'}{(2\pi)^d}\,\phi^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega). \qquad (4.56)\end{aligned}$$

Using (2.50) and (2.54)–(2.57), we then find an expression for the relevant vertex in the self-consistent theory of localization

$$\begin{aligned}\gamma^{RA}(\mathbf{q}\omega) &= \left\{\omega+\frac{1}{m}\,\mathbf{p}\cdot\mathbf{q}+2i\gamma(E_F)\right\}\frac{\omega+M_{E_F}(\mathbf{q}\omega)-\dfrac{1}{m}\,\mathbf{p}\cdot\mathbf{q}}{\omega^2+\omega M_{E_F}(\mathbf{q}\omega)-\dfrac{2E_F}{dm}\,\mathbf{q}^2}\\ &\approx\frac{2\gamma(E_F)}{-i\omega+D_{E_F}(\mathbf{q}\omega)\mathbf{q}^2}, \qquad (4.57)\end{aligned}$$

where the last expression is valid only for small values of ω and $\mathbf{q}$. Accordingly, the vertex $\gamma(\mathbf{q}\omega)$ in the self-consistent theory has the same form as in a "dirty" metal [9, 128, 129], although the classical diffusion coefficient is replaced by a generalized diffusion coefficient which was defined in (2.71). We see that in the localization region ($E_F < E_c$) Eqs. (2.73) and (2.74) imply in the limit of $\omega \to 0$

$$\gamma^{RA}(\mathbf{q}\omega)\underset{\omega\to 0}{\approx}\frac{2i\gamma(E_F)}{\omega+i\delta}\,A_{E_F}(\mathbf{q}). \qquad (4.58)$$

This rather general result may be attributed to the localization criterion formulated in Sec. 1. The use of general relation (2.26) in (2.56) in fact shows that in the self-consistent theory we would have

$$\psi^{\mathbf{q}}_{\mathbf{p}}(E_F)\simeq\frac{\operatorname{Im} G^R(E_F\mathbf{p})}{\pi N_0(E_F)}\,\chi_{\mathbf{q}}(E_F), \qquad (4.59)$$

where $\chi_{\mathbf{q}}(E_F)$ is given by (2.62). We then find from (2.26) and (4.56)

$$\mathcal{T}^{RA}_{\mathbf{p}} = 2\pi i \frac{\psi^{\mathbf{q}}_{\mathbf{p}}(E_F)\chi_{-\mathbf{q}}(E_F)}{\omega + i\delta} + \cdots. \tag{4.60}$$

Using (4.59) and (2.50), this yields (4.58) in the limit of $\omega \to 0$, taking into account the general relation (2.29).

Using (4.57) in (4.55), we find

$$\frac{\delta N(\epsilon)}{N_0(E_F)} = -\frac{1}{\pi}\,\mathrm{Im}\int \frac{d^d\mathbf{q}}{(2\pi)^d} \times \int_{\epsilon}^{\infty} d\omega \frac{1}{[-i\omega + D_{E_F}(\mathbf{q}\omega)\mathbf{q}^2]^2}. \tag{4.61}$$

Using (2.73) ($\mathbf{q} \to 0$), we can write (4.61) in the form

$$\frac{\delta N(\epsilon)}{N_0(E_F)} = -\frac{2}{\pi}\int_{\epsilon}^{\infty} d\omega \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q}) D_{E_F}\mathbf{q}^2 \times \left\{ \frac{\omega}{[\omega^2 + (\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2)^2]^2} + \frac{\omega_0^2(E_F)\tau_{E_F}}{\omega} \frac{\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2}{[\omega^2 + (\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2)^2]^2} \right\}. \tag{4.62}$$

Calculating the integral over ω in (4.62), we find

$$\frac{\delta N(\epsilon)}{N_0(E_F)} = \left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{I}} + \left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{II}},$$

where

$$\left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{I}} = -\frac{1}{\pi}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q}) \times \frac{D_{E_F}\mathbf{q}^2}{\epsilon^2 + [\omega_0^2(E_F)\tau_{E_F} + D_{E_F}\mathbf{q}^2]^2} \tag{4.63}$$

is in agreement with expression (4.41) derived above, and

$$\begin{aligned}\left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{II}} &= \frac{1}{\pi}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q}) D_{E_F}\mathbf{q}^2 \left\{ \frac{1}{\epsilon^2+[\omega_0^2(E_F)\tau_{E_F}+D_{E_F}\mathbf{q}^2]^2} \right.\\ &\quad \times \frac{\omega_0^2(E_F)\tau_{E_F}}{\omega_0^2(E_F)\tau_{E_F}+D_{E_F}\mathbf{q}^2} + \frac{\omega_0^2(E_F)\tau_{E_F}}{[\omega_0^2(E_F)\tau_{E_F}+D_{E_F}\mathbf{q}^2]^3}\\ &\quad \left. \times \ln \frac{\epsilon^2}{\epsilon^2+[\omega_0^2(E_F)\tau_{E_F}+D_{E_F}\mathbf{q}^2]^2}\right\}\\ &\underset{|\epsilon|\to 0}{\approx} \frac{2}{\pi}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, v(\mathbf{q}) \frac{\omega_0^2(E_F)\tau_{E_F}}{[\omega_0^2(E_F)\tau_{E_F}+D_{E_F}\mathbf{q}^2]^3} D_{E_F}\mathbf{q}^2\\ &\quad \left. \times \ln \frac{|\epsilon|}{\omega_0^2(E_F)\tau_{E_F}+D_{E_F}\mathbf{q}^2}\right\}. \end{aligned} \tag{4.64}$$

At $\omega_0^2(E_F)=0$, i.e., in the metallic region, contribution (4.64) vanishes. The fact that it exists shows that, in general, the diagram formalism used here does not match uniquely the formalism of the exact eigenfunctions which was used to obtain the results in the preceding section. Such a contribution can, in principle, be linked with the second term in Eq. (2.18) which was dropped in the derivation of Eq. (2.74). An estimate of the integral in (4.64) on the basis of the point-like interaction model yields

$$\left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{II}} \simeq \frac{v_0}{\pi}\frac{S_d}{4-d} D_{E_F}^{-d/2}[\omega_0^2(E_F)\tau_{E_F}]^{(d-2)/2} \ln \frac{|\epsilon|}{\omega_0^2(E_F)\tau_{E_F}}. \tag{4.65}$$

We see that Eq. (4.65) plays a dominant role only in an exponentially small region near the Fermi level (or the mobility edge), whereas contribution (4.42) accounts for the main correction in a broader energy region and, in this sense, is the principal contribution. However, the logarithmic singularity in (4.65) unequivocally shows that the higher-order corrections in terms of the interaction must be taken into account if we wish to describe correctly the immediate neighborhood of the Fermi level (mobility edge) in the localization region. Using (4.46) and (4.47), we find

$$\left(\frac{\delta N(\epsilon)}{N_0(E_F)}\right)_{\mathrm{II}} \sim v_0 m^{d/2} E_F^{(d/2)-1} \ln\left\{\frac{|\epsilon|}{\lambda E_F}[p_F R_{\mathrm{loc}}(E_F)]^d\right\}$$

$$\sim v_0 N_0(E_F) \ln\left\{\frac{|\epsilon|}{\lambda E_F}\left|\frac{E_F - E_c}{E_c}\right|^{-d\nu}\right\}. \tag{4.66}$$

Using (4.50) for $d = 2$ we find in a similar way, that for $|\epsilon| \ll E_F \exp(-1/\lambda)$ the contribution (4.64) is always dominant

$$\frac{\delta N(\epsilon)}{N_0} \sim v_0 N_0 \frac{1}{\lambda} \ln \frac{|\epsilon|}{E_F}, \tag{4.67}$$

so that the second expression in (4.51) has, in fact, no region of applicability. The results obtained in the formalism of the exact eigenfunctions can thus be substantially modified by using the diagram formalism. This method can also be used to analyze the role of other diagrams, whose contributions are different from that of the simplest "Fock" diagram shown in Fig. 15a. There are, in fact, several other diagrams of the first-order perturbation theory in terms of the interaction. Some examples of these diagrams are shown in Fig. 16. Let us examine first of all the "Hartree" diagram of Fig. 16a. We can estimate its contribution within the context of a self-consistent theory of localization analogously to how it was done in Refs. 9, 129, and 151. In a self-consistent theory the full vertex

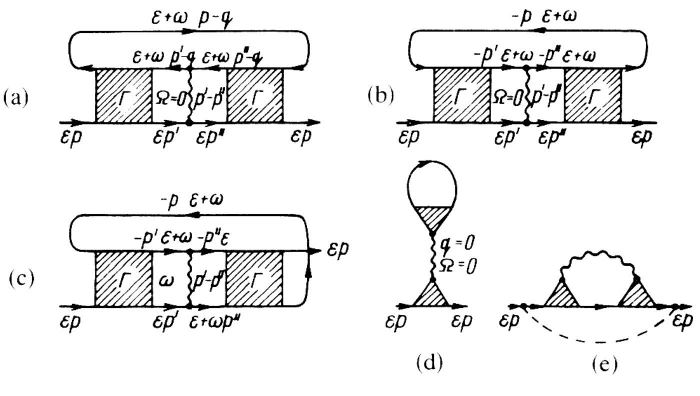

Figure 16. Examples of various first-order diagrams, whose contribution is compared with the contribution of the diagram in Fig. 15a.

$\Gamma_{\mathbf{pp}'}(\mathbf{q}\omega)$ (see Fig. 3) can be written in the following simple form, for small values of $\mathbf{q}$ and ω:

$$\Gamma^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega) \approx \frac{2\gamma(E_F)\rho V^2}{-i\omega + D_{E_F}(\mathbf{q}\omega)\mathbf{q}^2}. \tag{4.68}$$

This expression generalizes the "diffusion" expression (2.86) which is derived by taking the sum over the ladder diagrams in Fig. 5b. The result in (4.68) is basically self-evident. It is obvious, in particular, that its use in Eq. (4.54) yields (4.57). Equation (4.68) can also be derived formally. In this connection, let us examine an auxiliary vertex $\tilde{\Gamma}^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega)$, which is determined graphically in Fig. 17 (compare with Fig. 3). We find

$$\begin{aligned}\tilde{\Gamma}^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega) &= (\rho V^2)^2(-2\pi i)\int \frac{d^d\mathbf{p}''}{(2\pi)^d}\int \frac{d^d\mathbf{p}'''}{(2\pi)^d}\,\phi^{RA}_{\mathbf{p}''\mathbf{p}'''}(\mathbf{q}\omega)\\ &= -2\pi i(\rho V^2)^2\phi^{RA}(\mathbf{q}\omega) \simeq \frac{2\gamma(E_F)\rho V^2}{-i\omega + D_{E_F}(\mathbf{q}\omega)\mathbf{q}^2},\end{aligned} \tag{4.69}$$

where in the last equality we have used (2.69). Let us assume that in the limit of $\mathbf{q}\to 0$ we have $\Gamma^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega) \approx \tilde{\Gamma}^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega)$. We can then reproduce (4.69) by using (4.68) in the second diagram on the right side of Fig. 17, whereas the first diagram is nonsingular in the limits of $\omega\to 0$ and $\mathbf{q}\to 0$. This demonstrates that our assumption is self-consistent. We can now compare directly the contributions to the electron self-energy from the diagrams in Figs. 16a and 15a:

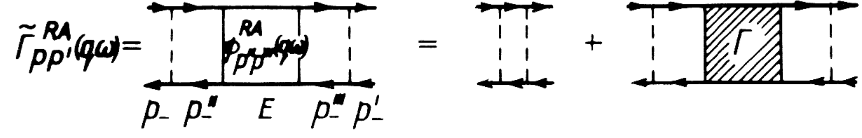

Figure 17. Graphic determination of the auxiliary vertex $\tilde{\Gamma}^{RA}_{\mathbf{pp}'}(\mathbf{q}\omega)$.

$$\Sigma_H^R(\epsilon\mathbf{p}) = \int \frac{d^d\mathbf{p}'}{(2\pi)^d} \int \frac{d^d\mathbf{p}''}{(2\pi)^d} \int \frac{d^q\mathbf{q}}{(2\pi)^d} \int_\epsilon^\infty \frac{d\omega}{2\pi} v(\mathbf{p}'-\mathbf{p}'')$$
$$\times \frac{4(\rho V^2)\gamma^2(E_F)}{[-i\omega + D_{E_F}(\omega)\mathbf{q}^2]^2} G^R(\epsilon\mathbf{p}')G^R(\epsilon\mathbf{p}'')G^A(\mathbf{p}''-\mathbf{q}\epsilon+\omega)$$
$$\times G^A(\mathbf{p}'-\mathbf{q}\epsilon+\omega)G^A(\mathbf{p}-\mathbf{q}\epsilon+\omega) \tag{4.70}$$

$$\Sigma_F^R(\epsilon\mathbf{p}) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \int_\epsilon^\infty \frac{d\omega}{2\pi} v(\mathbf{q}) \frac{4(\rho V^2)^2\gamma^2(E_F)}{[-i\omega + D_{E_F}(\omega)\mathbf{q}^2]^2} G^A(\epsilon+\omega\mathbf{p}-\mathbf{q}). \tag{4.71}$$

An important point here is that in contrast with the "Fock" diagram, the momentum transferred along the interaction line in the "Hartree" diagram is not small, and the frequency transfer is zero. We can then easily estimate the ratio

$$F = \frac{\Sigma_H^R(\epsilon\mathbf{p})}{\Sigma_F^R(\epsilon\mathbf{p})} \approx \left\{\int \frac{d^d\mathbf{q}}{(2\pi)^d} v(0)\right\}^{-1} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \int \frac{d^d\mathbf{p}'}{(2\pi)^d}$$
$$\times \int \frac{d^d\mathbf{p}''}{(2\pi)^d} (\rho V^2)^2 G^R(\epsilon\mathbf{p}')G^A(\epsilon\mathbf{p}')v(\mathbf{p}'-\mathbf{p}'')G^R(\epsilon\mathbf{p}'')G^A(\epsilon\mathbf{p}'')$$
$$\approx \left\{\int d\Omega v(0)\right\}^{-1} \int d\Omega \frac{1}{N_0^2} \int \frac{d^d\mathbf{p}'}{(2\pi)^d} \int \frac{d^d\mathbf{p}''}{(2\pi)^d}$$
$$\times \delta\left(E_F - \frac{\mathbf{p}'^2}{2m}\right) v(\mathbf{p}'-\mathbf{p}'')\delta\left(E_F - \frac{\mathbf{p}''^2}{2m}\right). \tag{4.72}$$

and finally find the same result as that for a metal [9, 129, 151]:

$$F = \frac{\int d\Omega v\left(2p_F \sin\frac{\theta}{2}\right)}{\int d\Omega v(0)}, \tag{4.73}$$

where the integration is over the solid angle at the Fermi surface (θ is the angle between two momenta at the Fermi sphere). We easily see that $F < 1$ if the interaction potential falls off at a distance greater than the inverse Fermi momentum. If, for example, we would use a screened Coulomb interaction as a model potential,* we would find, for $d = 3$ (Ref. 9),

$$F = \frac{1}{2}\int_0^\pi d\theta \sin\theta \frac{1}{1 + \dfrac{4p_F^2}{\kappa_D^2}\sin^2\dfrac{\theta}{2}} = \frac{\kappa_D^2}{4p_F^2}\ln\left\{1 + \frac{4p_F^2}{\kappa_D^2}\right\}, \tag{4.74}$$

where $\kappa_D^2 = 4\pi e^2 N_0(E_F)$ is the inverse square of the screening length. We can infer from this that $F \ll 1$ if $\kappa_D^2 \ll p_F^2$, but $F \to 1$ if $p_F^2 \ll \kappa_D^2$.

Since $\gamma(E_F) \sim \rho V^2 N_0(E_F)$ (2.35), we find $F \sim \kappa_D^2/p_F^2 \sim e^2/m\rho V^2\gamma(E_F)/E_F \sim e^2/m\rho V^2\lambda(E_F)$, so that $F \sim e^2/m\rho V^2$ near the mobility edge ($E_F \sim E_c \sim E_{sc}$). For a point-like interaction we would have $F = 1$ and, after incorporating the spin, the Hartree contribution in Fig. 16a would be twice as large as the Fock contribution. In this case, all the corrections to the density of states which we have considered would simply change sign, as we have mentioned above. The contribution of the diagram in Fig. 16b, where the interaction occurs in the Cooper channel, in the absence of interactions which break down the time-reversal invariance (magnetic field, magnetic impurities, etc.), is equal to the contribution of the diagram in Fig. 16a (Ref. 152), so that it is also small over the parameter F. A similar small parameter is also found for the Fock diagram in Fig. 16c, where the interaction occurs in the Cooper channel. The difference between this diagram and the one in Fig. 15a is again linked to the appreciable momentum transfer along the interaction line. The diagram in Fig. 16d is canceled out because of the total electrical neutrality of the system, and for the contribution of the diagram in Fig. 16a we easily find

$$\operatorname{Im}\Sigma^R(\epsilon\mathbf{p}) = -\pi\rho V^2\delta N(\epsilon), \tag{4.75}$$

where $\delta N(\epsilon)$ is the correction to the density of states due to the diagram in Fig. 15a. Diagrams of this type therefore have an additional smallness over the parameter ρV^2.

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*We will see below that the static (zero frequency-transfer) interaction in a Coulomb system in the localization region is given by a screened Coulomb potential. This example is therefore quite realistic.

The problem of the relevance of higher-order interaction corrections remains essentially unresolved. The solution of this problem, in addition to the analysis of many new diagrams, should also consider how the interaction can be incorporated into the determination of the current-relaxation kernel of the self-consistent theory $M_{E_F}(\mathbf{q}\omega)$. This can be done, for example, by introducing the interaction lines and vertices into the calculation of the irreducible kernel $U_{\mathbf{pp}'}^{E_F}(\mathbf{q}\omega)$ in (2.58). Clearly, such effects could be ignored only in the first-order corrections over interaction, a circumstance which was exploited above. The higher-order corrections in terms of the interaction are clearly important in the immediate vicinity of the mobility edge and the Fermi level, but the first-order corrections are presumably adequate in the remaining energies. The diagram formalism creates the opportunity, at least in principle, to analyze these problems systematically.

4.3. Polarization Operator, Screening, and Coulomb Interaction

Let us consider in general terms and within the self-consistent theory how the localization affects the behavior of the polarization operator; i.e., let us consider, in fact, the screening of the electric field in Fermi glass.

Working again in the representation of exact eigenfunctions of a single-electron problem, we find the following expression for the Fourier transform of the polarization operator of noninteracting electrons:

$$\begin{aligned}\Pi(\mathbf{q}\omega) &= \left\langle \sum_{\mu\nu\mathbf{pp}'} \frac{f_\mu - f_\nu}{\epsilon_\nu - \epsilon_\mu + \omega + i\delta\,\mathrm{sign}\,\omega}\, \varphi_\nu(\mathbf{p}_+)\varphi_\nu^*(\mathbf{p}'_+)\varphi_\mu(\mathbf{p}'_-)\varphi_\mu^*(\mathbf{p}_-)\right\rangle \\ &= \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\Omega\, \frac{f(E) - f(E+\Omega)}{\Omega + \omega + i\delta\,\mathrm{sign}\,\omega}\, N_0(E)\langle\langle \rho_E \rho_{E+\Omega}\rangle\rangle_{\mathbf{q}}^F \qquad (4.76)\end{aligned}$$

in the zero-temperature formalism, or

$$\Pi(\mathbf{q}\omega_m) = \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\Omega\, \frac{f(E) - f(E+\Omega)}{\Omega + i\omega_m}\, N_0(E)\langle\langle \rho_E \rho_{E+\Omega}\rangle\rangle_{\mathbf{q}}^F \qquad (4.77)$$

in the Matsubara technique ($\omega_m = 2\pi mT$). Substituting into this expression the singular part of (4.15), at $T = 0$ we find

$$\Pi_{\text{loc}}(\mathbf{q}\omega) = 0 \tag{4.78}$$

and the nonvanishing contribution arises only from the regular part of (4.15),

$$\Pi(\mathbf{q}\omega) = \Pi_{\text{reg}}(\mathbf{q}\omega) = \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\Omega \, \frac{f(E) - f(E+\Omega)}{\Omega + \omega + i\delta \operatorname{sign} \omega} N_0(E) \rho_E^F(\mathbf{q}\Omega). \tag{4.79}$$

The situation is quite different in the Matsubara technique ($T \neq 0$):

$$\begin{aligned} &\Pi_{\text{loc}}(\mathbf{q}\omega_m \neq 0) = 0 \\ &\Pi_{\text{loc}}(\mathbf{q}\omega_m = 0) = \int_{-\infty}^{\infty} dE \left(-\frac{\partial f(E)}{\partial E}\right) N_0(E) A_E(\mathbf{q}) \approx N_0(E_F) A_{E_F}(\mathbf{q}). \end{aligned} \tag{4.80}$$

We thus find

$$\begin{aligned} \Pi(\mathbf{q}\omega_m) = {} & \frac{1}{T} \delta_{\omega_m 0} \int_{-\infty}^{\infty} dE f(E)[1 - f(E)] N_0(E) A_E(\mathbf{q}) \\ & + \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} d\Omega \, \frac{f(E) - f(E+\Omega)}{\Omega + i\omega_m} N_0(E) \rho_E^F(\mathbf{q}\Omega). \end{aligned} \tag{4.81}$$

Taking into account the explicit form of the "regular" part of the Berezinskiĭ–Gor'kov spectral density which arises in the self-consistent theory of localization [see Eq. (2.73)], we find from (4.79), after some straightforward calculations the following expression:

$$\Pi(\mathbf{q}\omega) = \Pi_{\text{reg}}(\mathbf{q}\omega) = N_0(E_F) \frac{D_{E_F} \mathbf{q}^2}{D_{E_F} \mathbf{q}^2 + \omega_0^2(E_F) \tau_{E_F} - i(\omega + i\delta \operatorname{sign} \omega)}. \tag{4.82}$$

In the metal region we would have $\omega_0^2(E_F) = 0$, and (4.82) reduces to the familiar expression for the polarization operator of a "dirty" metal [9, 128, 135]. In the localization region, after incorporating $\omega_0^2(E_F)\tau_{E_F} = D_{E_F} R_{\rm loc}^{-2}(E_F)$ [compare with (2.74)], we find

$$\Pi_{\rm reg}(\mathbf{q}0) = N_0(E_F)\frac{\mathbf{q}^2}{\mathbf{q}^2 + R_{\rm loc}^{-2}(E_F)}. \tag{4.83}$$

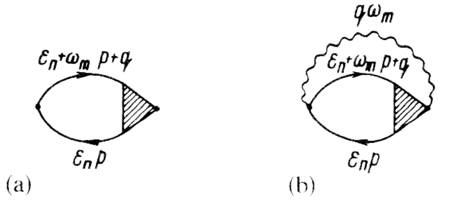
For the Matsubara polarization operator some analogous calculations yield

$$\begin{aligned}\Pi(\mathbf{q}\omega_m) = N_0(E_F)\Bigg\{ & A_{E_F}(\mathbf{q})\delta_{\omega_m 0} + \frac{D_{E_F}(\omega_m)\mathbf{q}^2}{\omega_m + D_{E_F}(\omega_m)\mathbf{q}^2}\theta(\omega_m) \\ & + \frac{D_{E_F}(-\omega_m)\mathbf{q}^2}{-\omega_m + D_{E_F}(-\omega_m)\mathbf{q}^2}\theta(-\omega_m)\Bigg\}; \quad \theta(\omega_m) = \begin{cases} 1 & m \geqslant 0 \\ 0 & m < 0 \end{cases}\end{aligned} \tag{4.84}$$

where $A_{E_F}(\mathbf{q})$ is given by expression (2.74′), and the generalized diffusion coefficient in the Matsubara technique is

$$D_{E_F}(\omega_m) = \frac{2E_F}{dm}\frac{i}{M_{E_F}(\omega_m)}; \qquad M_{E_F}(\omega_m) = \frac{i}{\tau_{E_F}} - \frac{\omega_0^2(E_F)}{i\omega_m}. \tag{4.85}$$

The result in (4.84) can also be obtained by a direct calculation from the diagram in Fig. 18a if we take into account that the triangular vertex of the self-consistent theory in the Matsubara technique is



(a) (b)

Figure 18. (a) A diagram for the polarization operator of noninteracting electrons. (b) "Fock" contribution to the thermodynamic potential.

given by

$$\gamma(\mathbf{q}\omega_m\epsilon_n) = \theta(\epsilon_n)\theta(\epsilon_n + \omega_m) + \theta(-\epsilon_n)\theta(-\epsilon_n - \omega_m)$$
$$+ 2\gamma(E_F)\left\{\frac{\theta(\epsilon_n)\theta(-\epsilon_n - \omega_m)}{-\omega_m + D_{E_F}(-\omega_m)\mathbf{q}^2}\right.$$
$$\left. + \frac{\theta(-\epsilon_n)\theta(\epsilon_n + \omega_m)}{\omega_m + D_{E_F}(\omega_m)\mathbf{q}^2}\right\}; \quad \epsilon_n = (2n+1)\pi T. \tag{4.86}$$

This expression is the well-known Aronov–Al'tshuler result [128], in which the generalized diffusion coefficient (4.85) is substituted for the classical diffusion coefficient.

We see that the "localization" contribution

$$\Pi_{\mathrm{loc}}(\mathbf{q}\omega_m) = \int_{-\infty}^{\infty} dE\left(-\frac{\partial f(E)}{\partial E}\right) N_0(A)A_E(\mathbf{q})\delta_{\omega_m 0}$$
$$= \frac{1}{T}\delta_{\omega_m 0}\int_{-\infty}^{\infty} dEf(E)[1 - f(E)]N_0(E)A_E(\mathbf{q})$$
$$\underset{T\to 0}{\approx} N_0(E_F)A_{E_F}(\mathbf{q})\delta_{\omega_m 0} \tag{4.87}$$

arises in polarization operators (4.81) and (4.84). Let us consider the simplest (Fock) contribution to the thermodynamic potential, shown schematically in Fig. 18b. The localization contribution in (4.87) would then correspond to

$$\langle\delta\Omega_{\mathrm{loc}}\rangle = \frac{1}{2}T\sum_m\int\frac{d^d\mathbf{q}}{(2\pi)^d}v(\mathbf{q})\Pi_{\mathrm{loc}}(\mathbf{q}\omega_m)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} dEf(E)[1 - f(E)]N_0(E)\int\frac{d^d\mathbf{q}}{(2\pi)^d}v(\mathbf{q})A_E(\mathbf{q})$$
$$= -\frac{1}{2}\int_{-\infty}^{\infty} dE[1 - f(E)]N_0(E)\tilde{\Sigma}^F_{E_{\mathrm{loc}}}. \tag{4.88}$$

In the last equality in (4.88) we have used (4.18). We have found the exact analog of the localization contribution in (4.28). The term with unity, as was noted in (4.29), goes into the renormalization of the chemical potential (taking the account of the regular contribution) and the factor $\frac{1}{2}$ is canceled by summing over the spin. Equation (4.88) then reduces to the localization part of (4.30). Consequently, we find the diagram "recipe" for determining the localization contribution to the self-energy $\tilde{\Sigma}_E$ which, as we can see from its definition (4.6), has no rigorous diagrammatic meaning.

The difference in the behavior of the polarization operator at $T = 0$ and in the Matsubara technique, which is seen only at zero frequency (static-field screening), is a consequence of the known difference between the adiabatic (Kubo) static response and the isothermal static response in systems with nonergodic behavior [39] manifested, as was noted in Sec. 1, in a $\delta(\omega)$ contribution to the Berezinskiĭ–Gor'kov spectral density. As we can see from a general analysis carried out by Kwok and Schulz [40], the Matsubara response is sensitive to nonergodic behavior, whereas the response determined by the commutator Green's functions (Kubo) does not "feel" it. At $\omega_m = 0$ the corresponding Matsubara Green's function determines the *isothermal* response, whereas the limit $\omega \to 0$ of the corresponding commutator Green's function (which can be obtained by continuing the Matsubara function analytically $i\omega_m \to \omega + i\delta$) determines the *adiabatic* response. The physical difference between these two responses is that the adiabatic (Kubo) response describes a system which is isolated from the heat bath after the external perturbation is applied, whereas the isothermal response corresponds to a system inside the heat bath during the time this perturbation is active [39]. The polarization operator, as we know, is associated with the electronic compressibility and the dielectric constant. For an isothermal static compressibility we would have

$$\kappa^T(\mathbf{q}0) = \Pi(\mathbf{q}\omega_m = 0), \tag{4.89}$$

whereas the adiabatic compressibility can be determined by means of

$$\kappa^A(\mathbf{q}0) = \Pi_{\rm reg}(\mathbf{q}\omega \to 0). \tag{4.90}$$

From (4.79)–(4.81) we then find

$$\kappa^T(\mathbf{q}0) - \kappa^A(\mathbf{q}0) = N_0(E_F)A_{E_F}(\mathbf{q}) = \frac{N_0(E_F)}{1 + R_{\text{loc}}^2(E_F)\mathbf{q}^2}. \tag{4.91}$$

The last equation in (4.91) is valid in the self-consistent theory of localization. Götze [50] and Prelovšek [63] have pointed out for the first time that the quantity $A_{E_F}(\mathbf{q})$ determines the difference between the isothermal static compressibility and the adiabatic static compressibility.

For an isothermal static polarization operator from (4.80) and (4.83) we find

$$\begin{aligned}\Pi^T(\mathbf{q}0) &= \Pi(\mathbf{q}\omega_m = 0) = \Pi_{\text{loc}}(\mathbf{q}0) + \Pi_{\text{reg}}(\mathbf{q}0)\\ &= N_0(E_F)\left\{\frac{1}{1 + R_{\text{loc}}^2(E_F)\mathbf{q}^2} + \frac{\mathbf{q}^2}{\mathbf{q}^2 + R_{\text{loc}}^{-2}(E_F)}\right\} = N_0(E_F).\end{aligned} \tag{4.92}$$

Correspondingly, for an *adiabatic* static dielectric constant we find

$$\begin{aligned}\epsilon^A(\mathbf{q}\omega \to 0) &= 1 + \frac{4\pi e^2}{\mathbf{q}^2}\Pi_{\text{reg}}(\mathbf{q}\omega \to 0)\\ &= \begin{cases} 1 + \dfrac{\kappa_D^2}{\mathbf{q}^2}; & \mathbf{q} \gtrsim R_{\text{loc}}^{-1}(E_F)\\ 1 + \kappa_D^2 R_{\text{loc}}^2(E_F); & \mathbf{q} \lesssim R_{\text{loc}}^{-1}(E_F),\end{cases}\end{aligned} \tag{4.93}$$

where $\kappa_D^2 = 4\pi e^2 N_0(E_F)$, consistent with (2.100), whereas the *isothermal* static dielectric constant is

$$\epsilon^T(\mathbf{q}0) = 1 + \frac{4\pi e^2}{\mathbf{q}^2}\Pi^T(\mathbf{q}0) = 1 + \frac{\kappa_D^2}{\mathbf{q}^2}, \tag{4.94}$$

consistent with the standard Thomas–Fermi result for a *metal*. This last dielectric constant corresponds to a real experiment on screening of the external static field. Rice [153] and Jäckle [154] have obtained this important result for the first time at the qualitative level, and we have shown here the formal "mechanism" of this behavior. Using the exact eigenfunction formalism, we can demonstrate directly that the first-order interaction corrections to $\Pi(\mathbf{q}0)$ cancel each other. These corrections account for the density of states behavior of the type shown in (4.42). There are no such features in $\Pi(\mathbf{q}0)$, in complete agreement with the important circumstance pointed out by Lee [136] and Finkel'shtein [137]: The screening range is determined by the quantity $dn/d\mu = \Pi(\mathbf{q} \to 00)$, rather than by the density of states. McMillan [134] did not take this fact into account. The physical mechanism of screening in a Fermi glass is connected with the electrons adjusting themselves in such a manner as to ensure total screening at any temperature, however small, as a result of hopping conductivity along the localized states in an Anderson dielectric. The typical time for such an adjustment is determined by the reciprocal frequency $\omega^* \sim D_{\text{hop}}\mathbf{q}^2$, where D_{hop} is the diffusion coefficient determined by the hopping conductivity [153, 154]. In the formalism described above, which ignores the hopping conductivity, the static nature of the field (and of the response) should be understood in the context of the condition $\omega \ll \omega^*$. In a real experiment, we would have $q \sim L^{-1}$, where L is the typical length-scale of the nonuniformity of the external field (which is determined by the size of the sample, etc.). The divergence of the dielectric constant, observed near the insulator-metal transition in certain experiments on Si doped with P [155, 156], is probably attributable to the divergence of the localization length $R_{\text{loc}}(E_F \to E_c)$ in Eqs. (2.100) and (4.93). These experiments were carried out at the external-field frequencies in the range from several hundred MHz to the infrared region of the optical spectrum, i.e., they measured a response of the type in (4.93). It would be clearly worthwhile to measure experimentally the dielectric constant of this system in a static field.

We can now turn from the analysis of the short-range repulsive potential to the case of real Coulomb interaction. In the following, we will carry out all the calculations for the three-dimensional case, using the Matsubara technique. We will consider, as in Ref. 128, only the contribution from the diagram in Fig. 15a, where the wavy line corresponds to the dynamically screened Coulomb interaction

$$V(\mathbf{q}\omega_m) = \frac{4\pi e^2}{\mathbf{q}^2 \epsilon(\mathbf{q}\omega_m)}. \tag{4.95}$$

The dielectric constant in (4.95) is

$$\epsilon(\mathbf{q}\omega_m) = 1 + \frac{4\pi e^2}{\mathbf{q}^2}\Pi(\mathbf{q}\omega_m) \tag{4.96}$$

and the polarization operator is taken from (4.84) (Fig. 18a). The "localization" contribution discussed above is now important, since in the calculation of the diagram in Fig. 15a it reduces to the contribution which vanishes in the limit of $T \to 0$. We can therefore confine ourselves below solely to the regular contribution. From Fig. 15a we find

$$\delta G(\epsilon_n\mathbf{p}) = T\sum_m \int \frac{d^d\mathbf{q}}{(2\pi)^d} V(\mathbf{q}\omega_m)\gamma^2(\mathbf{q}\omega_m)G^2(\epsilon_n\mathbf{p})G(\epsilon_n + \omega_m\mathbf{p} + \mathbf{q}), \tag{4.97}$$

where the screened interaction for small values of $\mathbf{q}$ and ω_m is

$$\begin{aligned} V(\mathbf{q}\omega_m \to 0) \approx {} & \frac{-\omega_m + D_{E_F}(-\omega_m)\mathbf{q}^2}{N_0(E_F)D_{E_F}(-\omega_m)\mathbf{q}^2}\theta(-\omega_m) \\ & + \frac{\omega_m + D_{E_F}(\omega_m)\mathbf{q}^2}{N_0(E_F)D_{E_F}(\omega_m)\mathbf{q}^2}\theta(\omega_m). \end{aligned} \tag{4.98}$$

Introducing, by definition,

$$\delta N(\epsilon_n) = -\frac{1}{\pi}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\delta G(\epsilon_n\mathbf{p}), \tag{4.99}$$

we find

$$\begin{aligned} \delta N(\epsilon_n) = {} & -\frac{1}{\pi}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\, T\sum_m V(\mathbf{q}\omega_m)\gamma^2(\mathbf{q}\omega_m\epsilon_n) \\ & \times \int \frac{d^3\mathbf{p}}{(2\pi)^3} G^2(\epsilon_n\mathbf{p})G(\epsilon_n + \omega_m\mathbf{p} + \mathbf{q}) \\ \approx {} & -iN_0(E_F)T\sum_m \int \frac{d^3\mathbf{q}}{(2\pi)^3} V(\mathbf{q}\omega_m) \\ & \times \left\{ \frac{\theta(\epsilon_n)\theta(-\epsilon_n - \omega_m)}{[-\omega_m + D_{E_F}(-\omega_m)\mathbf{q}^2]^2} - \frac{\theta(-\epsilon_n)\theta(\epsilon_n + \omega_m)}{[\omega_m + D_{E_F}(\omega_m)\mathbf{q}^2]^2} \right\}. \end{aligned} \tag{4.100}$$

Using (4.98), for $\epsilon_n > 0$, we find

$$\begin{aligned}\delta N(\epsilon_n > 0) &\approx -\frac{iT}{2\pi^2}\sum_{\omega_m=-\infty}^{-\epsilon_n} D_{E_F}^{-1}(\omega_m)\int_0^\infty d\mathbf{q}\,\frac{1}{D_{E_F}(-\omega_m)\mathbf{q}^2-\omega_m}\\ &= -\frac{iT}{2\pi^2 D_{E_F}^{3/2}}\sum_{\omega_m=-\infty}^{-\epsilon_n}\omega_m^{-2}\{\omega_m-\omega_0^2(E_F)\tau_{E_F}\}^2\\ &\quad\times\int_{-\infty}^{\infty}\frac{dx}{x^2-\omega_m+\omega_0^2(E_F)\tau_{E_F}}.\end{aligned} \tag{4.101}$$

The finite sum over the Bose frequencies appearing here can be calculated from the relation

$$\begin{aligned}T\sum_{m=-\infty}^{-n}\Phi(i\omega_m) &= \int_{-i\epsilon_n+\infty}^{-i\epsilon_n-\infty}\frac{dz}{2\pi i}f_B(z)\Phi(z)\\ &= \int_{-\infty}^{\infty}\frac{dz}{2\pi i}f_B(z-i\epsilon_n)\Phi(z-i\epsilon_n)\\ &= \int_{-\infty}^{\infty}\frac{dz}{2\pi i}f(z)\Phi(z-i\epsilon_n),\end{aligned} \tag{4.102}$$

where $f_B(z)=\{\exp(z/T\}-1\}^{-1}$ is the Bose function, and $f(z)=\{\exp(z/T)+1\}^{-1}$ is the Fermi function. In our case, we would have

$$\Phi(z)=\{iz+\omega_0^2(E_F)\tau_{E_F}\}\frac{1}{2z^2}\int_0^\infty dx\{x^2+iz+\omega_0^2(E_F)\tau_{E_F}\}^{-1}. \tag{4.103}$$

Carrying out in (4.102) and (4.103) an analytic continuation $i\epsilon_n \to E+i\delta$ and calculating the imaginary part, taking the account of $\operatorname{Im}\Phi(z)=-\operatorname{Im}\Phi(-z)$, we find the correction to the density of states corresponding to (4.101),

$$\delta N(\epsilon)=\frac{1}{4\pi^3 D_{E_F}^{3/2}}\int_0^{\tilde{E}} dz\{f(z+\epsilon)+f(z-\epsilon)-1\}\operatorname{Im}\Phi(z). \tag{4.104}$$

Expression (4.102), strictly speaking, is valid for $\Phi(z)$ which falls off rapidly enough in the limit $|z|\to\infty$. Since the function $\Phi(z)$ in

(4.103) generally does not satisfy this condition, we have introduced a cutoff at the upper limit of $\tilde{E}$, where $\tilde{E}$ has the same meaning as in Eq. (4.43), in order to take into account the "constant" that arises in (4.104). Using the asymptotic expression

$$\operatorname{Im} \Phi(z) \approx \begin{cases} \dfrac{\pi}{4\sqrt{2z}}; & z \gg \omega_0^2(E_F)\tau_{E_F} \\ \dfrac{3\pi}{4\sqrt{2}}\dfrac{1}{z}[\omega_0^2(E_F)\tau_{E_F}]^{1/2}; & z \ll \omega_0^2(E_F)\tau_{E_F}, \end{cases} \tag{4.105}$$

we find

$$\delta N(\epsilon) \approx \begin{cases} \dfrac{1}{2^{5/2}\pi^2 D_{E_F}^{3/2}}\left\{T^{1/2}\varphi\left(\dfrac{|\epsilon|}{2T}\right) - \tilde{E}^{1/2}\right\}; & |\epsilon|, T \gg \omega_0^2(E_F)\tau_{E_F} \\ \dfrac{1}{2^{5/2}\pi^2 D_{E_F}^{3/2}}\{(\omega_0^2(E_F)\tau_{E_F})^{1/2} - \tilde{E}^{1/2}\} & \\ + \dfrac{3(\omega_0^2(E_F)\tau_{E_F})^{1/2}}{2^{5/2}\pi^2 D_{E_F}^{3/2}} \ln \dfrac{\max\{|\epsilon|, T\}}{\omega_0^2(E_F)\tau_{E_F}}; & |\epsilon|, T \ll \omega_0^2(E_F)\tau_{E_F}, \end{cases} \tag{4.106}$$

where (Ref. 128)

$$\varphi(x) = \frac{1}{\sqrt{2}}\int_0^{\infty} dy\sqrt{y}\left\{\frac{1}{ch^2(x-y)} + \frac{1}{ch^2(x+y)}\right\} \approx \begin{cases} 1.07; & x \ll 1 \\ \sqrt{2x}; & x \gg 1. \end{cases} \tag{4.107}$$

The results which we have obtained extend directly the corresponding expressions, derived by Aronov and Al'tshuler [128], to the insulator side of the Anderson transition. It is easy to see that our correction to the density of states, due to dynamically screened Coulomb interaction, is the same within a constant as the results given in (4.42) and (4.65). Consequently, there is no "Coulomb gap" even in the Coulomb-interaction model, which probably may be attributed to the crudeness of our model. We should emphasize, however, that a standard analysis [130–133] of the Coulomb gap is carried out without regard for the screening and quantum effects. The problems which arise here can in our opinion be solved only through a systematic analysis of the higher-order corrections in terms of the electron–electron interaction.

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SUPERCONDUCTIVITY AND LOCALIZATION

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Contents

Abstract

We present a review of theoretical and experimental works on the problem of mutual interplay of Anderson localization and superconductivity in strongly disordered systems. Superconductivity occurs close to the metal–insulator transition in some disordered systems such as amorphous metals, superconducting compounds disordered by fast neutron irradiation, etc. High-temperature superconductors are especially interesting from this point of view. Only bulk systems are considered in this review. The superconductor–insulator transition in purely two-dimensional disordered systems is not discussed.

We start with a brief discussion of the modern aspects of localization theory including the basic concept of scaling, self-consistent theory and interaction effects. After that we analyze disorder effects on Cooper pairing and superconducting transition temperature as well as the Ginzburg–Landau equations for superconductors which are close to those for the Anderson transition. A necessary generalization of the usual theory of "dirty" superconductors is formulated which

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allows to analyze anomalies of the main superconducting properties close to the disorder-induced metal–insulator transition. Under very rigid conditions superconductivity may persist even in the localized phase (Anderson insulator).

Strong disordering leads to considerable reduction of superconducting transition temperature T_c and to important anomalies in the behavior of the upper critical field H_{c2}. Fluctuation effects are also discussed. In the vicinity of the Anderson transition, inhomogeneous superconductivity appears due to statistical fluctuations of the local density of states.

We briefly discuss a number of experiments demonstrating superconductivity close to the Anderson transition both in traditional and high-T_c superconductors. In traditional systems superconductivity is in most cases destroyed before the metal–insulator transition. In the case of high-T_c superconductors a number of anomalies show that superconductivity is apparently conserved in the localized phase before it is suppressed by a strong enough disorder.

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Keywords: Superconductivity; Localization phenomena

1. Introduction

The concept of electron localization [1] is basic for the understanding of electron properties of disordered systems [2,3]. In recent years a number of review papers have appeared, extensively discussing this problem [4–7]. According to this concept introduction of sufficiently strong disorder into a metallic system leads to spatial localization of electronic states near the Fermi level and thus to a transition to dielectric state (Anderson transition). After this transition DC conductivity (at zero temperature, $T = 0$) vanishes, despite the finite value of electronic density of states at the Fermi level (at least in one-electron approximation).

At the same time it is well known that even the smallest attraction of electrons close to the Fermi level leads to formation of Cooper pairs and the system becomes superconducting at sufficiently low temperatures [8,9]. It is known that the introduction of disorder which does not break the time-reversal invariance (normal, nonmagnetic impurities, etc.) does not seriously influence the superconducting transition temperature T_c and superconductivity in general (Anderson theorem) [10–13].

Thus a problem appears of the mutual interplay of these two possible electronic transitions in a disordered system which leads to quite different (even opposite) ground states (insulator or superconductor). This problem is very important both from theoretical and experimental points of view. Actually, superconducting properties of many compounds depend strongly on structural disorder. In this respect we can mention amorphous systems (metallic glasses) and superconductors disordered by different forms of irradiation by high-energy particles (fast neutrons, electrons, heavy-ions, etc.). It appears that in many of these systems superconductivity is realized when the system in normal state is quite close to the metal–insulator transition induced by disorder. In this case many anomalies of superconducting properties appear which cannot be satisfactorily explained within the standard theory of "dirty" superconductors [9–13]. These include rather strong dependence of T_c on disorder in apparent contradiction with Anderson's theorem, as well as some unusual behavior of the upper critical field H_{c2}.

The discovery of high-temperature superconductivity in metallic oxides [14,15] has lead to the entirely new opportunities in the studies of strong disorder effects in superconductors. Very soon it had been established that high-T_c superconductors are quite sensitive to structural disordering which leads to a rather fast destruction of superconductivity and metal–insulator transition. However, the high values of initial T_c, as well as a small size of Cooper pairs and the quasi-two-dimensional nature of electronic states in these systems are very appropriate for studies of the mutual interplay of localization and superconductivity [16]. It may be stated with some confidence that in these systems superconductivity can be observed even in the region of localization (Anderson insulator).

This review is mainly concerned with theoretical aspects of localization and superconductivity close to the Anderson transition. However, we shall pay some attention to experiments demonstrating the importance of localization phenomena for the correct analysis of superconductivity in strongly disordered systems. Special emphasis will be on the experiments with high-T_c superconductors. We shall limit our discussion to three-dimensional and quasi-two-dimensional (in case of HTSC) systems, practically excluding any discussion of purely two-dimensional systems, which are quite special both in respect to localization and superconductivity. In this case we refer the reader to recent reviews [17–19] which are specifically concerned with the two-dimensional case.

I must stress that the material presented in this review is concerned mainly with the personal interests of its author and I apologize to those people whose important contributions in this field would not be discussed in detail or even would be missed because of the lack of space.

The usual theory of "dirty superconductors" [9–13] is a cornerstone in our understanding of superconducting properties of disordered metals. It is based on the following main statements:

1. As impurity concentration (disorder) grows, a transition takes place from the "pure" limit, when the electron mean-free path l is much larger than the superconducting coherence length ξ_0: $l \gg \xi_0 = \hbar v_F/\pi\Delta_0$ to a "dirty" superconductor with $\xi_0 \gg l \gg \hbar/p_F$ (Here v_F, p_F are the Fermi velocity and momentum, Δ_0 is the zero temperature energy gap). Transition temperature T_c changes only slightly, mainly due to small changes of Debye frequency ω_D and of the pairing constant λ_p, which are due to relatively small changes in the electronic density of states under disordering. Transition from the free electron motion to diffusive one does not change T_c at all (Anderson's theorem). These statements ignore any disorder dependence of microscopic pairing interaction, which is assumed to be some constant as in the simplest BCS model.

2. Superconducting coherence length ξ (at $T = 0$) determining the spatial scale of the superconducting order-parameter (the size of a Cooper pair) diminishes with l so that $\xi \approx \sqrt{\xi_0 l}$ in the limit of $\hbar/p_F \ll l \ll \xi_0$.

3. As ξ diminishes the critical region near T_c where thermodynamic fluctuations are important widens and is of the order of $\tau_G T_c$, where $\tau_G \sim [T_c N(E_F)\xi^3]^{-2}$ is the so-called Ginzburg's parameter ($N(E_F)$ is the electronic density of states at the Fermi level E_F). For "pure" superconductors $\tau_G \sim (T_c/E_F)^4 \ll 1$ and as l drops τ_G grows and ξ drops. However, in the limit of $l \gg \hbar/p_F$ the value of τ_G still remains very small.

Theory of "dirty" superconductors is the basis of our understanding of superconducting properties of many disordered alloys. However, the main results of this theory must be modified for the mean-free path values l of the order of inverse Fermi momentum $\hbar/p_F$ (i.e. of the order of interatomic distance). In three-dimensional systems the growth of disorder leads to destruction of diffusive motion of electrons and transition from extended to localized states at critical disorder determined by $l_c \approx \hbar/p_F$, i.e. to transition to Anderson insulator. This metal–insulator transition is reflected in a continuous drop to zero of the static metallic conductivity (at $T = 0$) as $l \to l_c$. For $l \gg l_c$ conductivity is determined by the usual Drude formula $\sigma_0 \sim l$, while for $l \to l_c$ it drops as $\sigma \sim (l - l_c)^\nu$, where ν is some critical exponent. Transition from diffusion to localization is realized at the conductivity scale of the order of the so-called "minimal metallic conductivity" $\sigma_c \approx (e^2 p_F/\pi^3\hbar^2) \approx (2\text{–}5)\times 10^2\ \Omega^{-1}\,\mathrm{cm}^{-1}$. The usual theory of "dirty" superconductors does not consider localization effects and is valid for conductivities in the interval $(E_F/T_c)\sigma_c \gg \sigma \gg \sigma_c$.

At present the following results are well established for superconductors close to localization transition (i.e. $\sigma \leq \sigma_c$):

1. Assuming independence of the density of states at the Fermi level $N(E_F)$ and of the pairing constant λ_p from the value of the mean-free path l (disorder) we can show that T_c drops as disorder grows due to respective growth of Coulomb pseudopotential μ^*. This effect is due to the growth of retardation effects of Coulomb interaction within the Cooper pair as diffusion coefficient drops close to the Anderson transition [20]. T_c degradation starts even for $\sigma \gg \sigma_c$ and becomes fast for $\sigma < \sigma_c$ [21,22]. The growth of spin fluctuations and changes in the density of states due to interaction effects may also lead to the drop of T_c, though these mechanisms were not analyzed in detail up to now.

2. Close to the Anderson transition the usual expression for superconducting coherence length for a "dirty" limit $\xi = \sqrt{\xi_0 l}$ should be replaced by $\xi \approx (\xi_0 l^2)^{1/3}$ and it remains finite even below the Anderson transition (i.e. in insulating phase) [21–25], signalling the possibility of superconductivity in Anderson insulator. Obviously, these results are valid only in the case of finite T_c close to Anderson transition, which is possible only if very rigid conditions are satisfied.

3. The growth of disorder as the system moves to the Anderson transition leads to the growth of different kinds of fluctuations of the superconducting order-parameter both of thermodynamic nature and due to fluctuations of electronic characteristics of the system.

In our review we shall present an extensive discussion of these and some of the other problems concerning the interplay of superconductivity and localization. However, first of all we shall briefly describe the main principles of the modern theory of electron localization and physics of metal–insulator transition in disordered systems, which will be necessary for a clear understanding of the main problem under discussion. After that we shall give rather detailed presentation of the theoretical problem of superconductivity close to the Anderson transition. Finally, we shall describe the present experimental situation. We shall briefly describe some of the experiments with traditional superconductors, but our main emphasis will be on high-T_c oxides. We shall concentrate on the experiments with high-temperature-superconductors disordered by fast neutron irradiation. We consider this as one of the best methods to introduce disorder in a controlled fashion without any chemical (composition) changes. In this sense our review of experiments is also far from being complete, but we hope that it is full enough to claim that high-T_c systems are especially good for testing some of the main theoretical ideas, expressed throughout this review. Also we believe that better understanding of their properties under disordering may be important for the development of the general theory of high-temperature superconductivity. The preliminary version of this review has been published in Ref. [26].

2. Anderson localization and metal–insulator transition in disordered systems

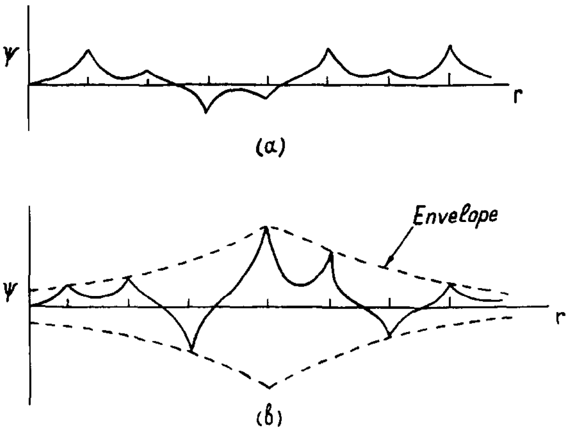
2.1. Basic concepts of localization

In recent years a number of review papers have appeared dealing with basic aspects of Anderson localization [4–7, 27–29]. Here we shall recall the main points of this theory and introduce the accepted terminology.

In 1958 Anderson [1] showed for the first time that the wave function of a quantum particle in a random potential can qualitatively change its nature if randomness becomes large enough. Usually, when disorder is small, the particle (e.g. electron) is scattered randomly and the wave function changes at the scale of the order of mean free path l. However, the wave function remains as extended plane-wave-like (Bloch wave-like) throughout the system. In the case of a strong enough disorder, the wave function becomes localized so that its amplitude (envelope) drops exponentially with distance from the center of localization $\mathbf{r}_0$:

$$|\psi(\mathbf{r})| \sim \exp(|\mathbf{r} - \mathbf{r}_0|/R_{\mathrm{loc}}) , \tag{2.1}$$

where R_{loc} is the localization length. This situation is shown qualitatively in Fig. 1. The physical meaning of Anderson localization is relatively simple: coherent tunneling of electrons is possible



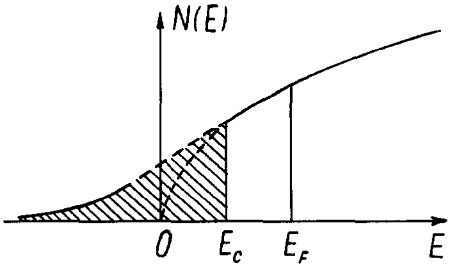


Fig. 1. Electron wave function in a disordered system: (a) extended state, (b) localized state.

Fig. 2. Electron density of states near the band edge in a disordered system. Dashed line is the region of localized states and E_c the mobility edge.

only between energy levels with the same energy (e.g. between equivalent sites in crystalline lattice). However, in case of strong randomness the states with the same energy are too far apart in space for tunneling to be effective.

At small disorder, DC conductivity of a metal at $T = 0$ is determined by the Drude expression:

$$\sigma_0 = (ne^2/m)\tau = (ne^2/p_F)l \,, \tag{2.2}$$

where τ is the mean free time, n the electron density and e its charge. The usual kinetic theory can be applied if

$$p_F l/\hbar \gg 1 \quad \text{or} \quad E_F\tau/\hbar \gg 1 \tag{2.3}$$

which is a condition of weak scattering (disorder). From Eqs. (2.2) and (2.3), taking into account $n = p_F^3/(3\pi^2\hbar^3)$, we can estimate the lower limit of conductivity for which the Drude approximation is still valid:

$$\sigma_0 = \frac{e^2 p_F}{3\pi^2\hbar^2}\left(\frac{p_F l}{\hbar}\right) \gg \frac{e^2 p_F}{3\pi^2\hbar^2} \,. \tag{2.4}$$

The conductivity value:

$$\sigma_c \approx e^2 p_F/3\pi^2\hbar^2 \tag{2.5}$$

is usually called the "minimal metallic conductivity" [2, 3]. As disorder grows the mean free path diminishes and becomes of the order of lattice spacing a, so that we reach $p_F l/\hbar \sim 1$, and the usual kinetic theory based upon Boltzmann equation becomes inapplicable. This was first noted by Ioffe and Rege [30], who observed that at such disorder the qualitative form of the wave function must change, transforming from extended to localized accompanied by a metal–insulator transition.

From Eq. (2.5) it is clear that this transition takes place at the conductivity scale of the order of $\sigma_c \sim (2\text{–}5)\times 10^2\,\Omega^{-1}\,\mathrm{cm}^{-1}$ for typical $\hbar/p_F \sim a \sim (2\text{–}3)\times 10^{-8}$ cm.

The qualitative form of the energy spectrum near the band-edge of a disordered system is shown in Fig. 2. When the Fermi level lies in the high-energy region electronic states close to it are slightly distorted plane waves. As Fermi energy moves towards the band edge (or with the growth of disorder) the critical energy E_c (mobility edge) separating extended and localized states crosses the Fermi level. If E_F belongs to the region of localized states the system becomes insulating, conductivity is possible only for $T > 0$ or by exciting the carriers by an alternating electric field. The appearance of these hopping mechanisms of conductivity signals the Anderson transition [2,3].

One of the main problems is the qualitative behavior of conductivity when the Fermi level E_F crosses the mobility edge E_c (at $T = 0$). While Mott assumed the discontinuous drop of conductivity from σ_c to zero [2,3] modern approach [4–6,29] based mainly on the scaling theory to localization [31] demonstrates a *continuous* transition. Experiments at low temperatures clearly confirm this type of behavior [6], and σ_c acts as a characteristic conductivity scale close to transition. Static conductivity of a metal at $T = 0$ close to Anderson transition within this approach is written as

$$\sigma = A\frac{e^2}{\hbar\xi_{\mathrm{loc}}} \approx \sigma_c\left|\frac{E_F - E_c}{E_c}\right|^{(d-2)\nu}, \tag{2.6}$$

where A is a numerical constant, d the space dimension, and $\sigma_c \approx Ae^2/(\hbar a^{d-2})$. Here we introduced the correlation length of scaling theory diverging at the transition

$$\xi_{\mathrm{loc}} \approx \frac{\hbar}{p_F}\left|\frac{E_F - E_c}{E_c}\right|^{-\nu}. \tag{2.7}$$

Critical exponent ν determines this divergence. In the one-electron approximation and in the absence of magnetic scattering, $\nu \sim 1$ [6,7,29,32]. In the region of localized states (i.e. for $E_F < E_c$) ξ_{loc} coincides with the localization length of electrons R_{loc}. In the metallic region, ξ_{loc} determines the effective size of a sample at which "ohmic" behavior appears, i.e. conductivity becomes independent of the sample size [6,33]. "Minimal metallic conductivity" σ_c determines, as we noted, the conductivity scale close to a transition.

In the vicinity of Anderson transition, conductivity acquires an important frequency dependence [51,34]. For $E_F = E_c$, i.e. at the transition we have

$$\sigma(\omega) \approx \sigma_c(\mathrm{i}\omega\tau)^{(d-2)/d} \tag{2.8}$$

which is also valid close to the transition (from either side) for frequencies $\omega \gg \omega_c \sim [N(E_F)\xi_{\mathrm{loc}}^d]^{-1}$. For $d = 3$ this is sometimes referred to as Gotze's [35] law $\omega^{1/3}$, although this particular derivation was later acknowledged to be wrong [36].

The spatial dimension $d = 2$ is the so-called "lower critical dimensionality" [4–7]. For $d = 2$ all electronic states are localized for infinitesimal disorder [31], and there is no Anderson transition.

Quasi-two-dimensional systems are especially interesting, mainly because most of the high-T_c oxides demonstrate strongly anisotropic electronic properties. Here we shall make the simplest estimates for such systems on the line of Ioffe–Regel approach. Consider a system made of highly-conducting "planes" where the current carriers are "nearly free", while the interplane tunneling is possible only due to some small transfer integral $w \ll E_F$ (E_F is the Fermi energy of the two-dimensional gas within the plane). Conductivity within the plane is determined for small disorder as

$$\sigma_{\parallel} = e^2 D_{\parallel} N(E_F) , \tag{2.9}$$

where $D_{\parallel} = v_F^2 \tau/2$, $N(E_F) = m/(\pi a_{\perp} \hbar^2)$, $a_{\perp}$ is the interplane spacing, which is noticeably larger than the interatomic distance within the plane. Interplane conductivity is given by

$$\sigma_{\perp} = e^2 D_{\perp} N(E_F) , \tag{2.10}$$

where $D_{\perp} = (wa_{\perp})^2 \tau/\hbar^2$. The appropriate mean free paths are $l_{\parallel} = v_F \tau$, $l_{\perp} = wa_{\perp}\tau/\hbar$, Ioffe–Regel criterion for a quasi-two-dimensional system can be written as

$$l_{\perp} = wa_{\perp}\tau/\hbar \sim a_{\perp} \tag{2.11}$$

which is equivalent to $w\tau/\hbar \sim 1$ – the condition of breaking of coherent tunneling between the planes. Elementary estimate shows that this corresponds to

$$\sqrt{\sigma_{\parallel}\sigma_{\perp}} \sim e^2/\sqrt{2}\pi\hbar a \sim \sigma_c , \tag{2.12}$$

where a is the interatomic distance *within the planes.* In the isotropic case this reduces to Eq. (2.5). For a strongly anisotropic system, when $\sigma_{\parallel} \gg \sigma_{\perp}$ it is clear that Eq. (2.12) can be satisfied even for $\sigma_{\parallel} \gg \sigma_c$, because of small values of $\sigma_{\perp}$. Formally, for $\sigma_{\perp} \to 0$, the critical value of $\sigma_{\parallel}$ diverges, which reflects, at this elementary level, the tendency towards complete localization in the purely two-dimensional case.

The important property of the energy spectrum in the region of localized states is its local discreteness. As we noted above, the physical meaning of localization itself leads to a picture of close energy levels being far apart in space, despite the continuous nature of the average density of states. Due to exponential decay of the localized wave functions it leads to the absence of tunneling [1]. The energy spacing between levels of electrons localized within a sphere of radius of the order of $R_{loc}(E)$ can be estimated [2, 3] as

$$\delta_{E_F} \approx [N(E_F) R_{loc}^d]^{-1} . \tag{2.13}$$

As the metallic system moves toward the Anderson transition, i.e. as the mean free path drops to interatomic distances and conductivity becomes less than $\sim 10^3\,\Omega^{-1}\,\mathrm{cm}^{-1}$, there appear the well-known anomalies like the negative temperature coefficient of resistivity [30, 37]. These anomalies are apparently closely connected with the localization phenomena [6].

So far we discussed the Anderson transition, neglecting electron interactions. Its importance in the problem of metal–insulator transitions in disordered systems was known for a long time [2]. In recent years, there has been considerable progress in the general approach to the theory of "dirty" metals, based on the analysis of interference of impurity scattering and Coulomb interactions [38–40]. Later, we shall review its implications for the general picture of Anderson transition. Apparently, the continuous nature of metal–insulator transition is not changed though interaction leads to a number of specific effects, e.g. in the behavior of the density of states at the Fermi level, as well as to the growth of magnetic (spin) fluctuations. Here we shall briefly describe the concept of "soft" Coulomb gap appearing below the transition in the region of localized states [41–44]. Coulomb interaction between localized electrons can be estimated as $e^2/\varepsilon R_{\mathrm{loc}}$, and it is obviously important if lthis energy is comparable with the local level spacing $[N(E_{\mathrm{F}})R_{\mathrm{loc}}^3]^{-1}$ (for three-dimensional system). As a result, a Coulomb pseudogap appears at the Fermi level with the width:

$$\Delta_{\mathrm{C}} \approx (e^3/\varepsilon^{3/2})[N(E_{\mathrm{F}})]^{1/2}\,, \tag{2.11}$$

where ε is the dielectric constant. We shall see later that close to the Anderson transition $\varepsilon \approx 4\pi e^2 N(E_{\mathrm{F}})R_{\mathrm{loc}}^2$ and accordingly,

$$\Delta_{\mathrm{C}} \approx [N(E_{\mathrm{F}})R_{\mathrm{loc}}^3]^{-1} \approx \delta_{E_{\mathrm{F}}} \tag{2.15}$$

so that in this case Coulomb effects are comparable with the effects of the discreteness of energy spectrum in the localized phase. At the moment there is no complete theory connecting the localization region with the metallic phase within the general approaches of interaction theory.

2.2. *Elementary scaling theory of localization*

The behavior of the electronic system close to the Anderson transition can be described by a scaling theory similar to that used in the theory of critical phenomena [45–47]. The main physical idea of this approach is based upon a series of scale transformations from smaller to larger "cells" in coordinate space with the appropriate description of a system by transformed parameters of the initial Hamiltonian. These transformations usually constitute the renormalization group. In the theory of critical phenomena, this approach is usually motivated by the growth of correlation length of the order-parameter fluctuations near the critical point [45]. This is analogous to the growth of the localization length on approaching the mobility edge from the Anderson insulator.

The accepted scaling approach to the localization problem was proposed by Abrahams et al. [31]. In this theory localization is described in terms of *conductance* g as a function of the sample size L. For a small disorder ($p_{\mathrm{F}}l/\hbar \gg 1$) the system is in a metallic state and conductivity σ is determined by Eq. (2.2) and is independent of the sample size if this size is much larger than the mean free path, $L \gg l$. Conductance is determined in this case just by Ohm's law and for a d-dimensional hypercube we have

$$g(L) = \sigma L^{d-2}\,. \tag{2.16}$$

If electronic states near the Fermi level are localized, conductivity of an infinite system at $T = 0$ is zero and matrix elements for transitions between different electronic states drop exponentially on distances of the order of R_{loc}. Then it can be expected that for $L \gg R_{\mathrm{loc}}$, the effective conductance becomes exponentially small:

$$g(L) \sim \exp(-L/R_{\mathrm{loc}})\,. \tag{2.17}$$

Elementary scaling theory of localization assumes that in the general case the conductance of a hypercube of size L satisfies the simplest differential equation of a renormalization group:

$$\mathrm{d}\ln g(L)/\mathrm{d}\ln L = \beta_d(g(L))\,. \tag{2.18}$$

The most important assumption here is the dependence of $\beta_d(g)$ only on one variable g (one parameter scaling). Then the qualitative behavior of β_d can be analyzed in the simplest possible way by interpolating between limiting forms given by Eqs. (2.16) and (2.17). For the metallic phase (large g), we get from Eqs. (2.16) and Eq. (2.18)

$$\lim_{g\to\infty} \beta_d(g) \to d-2\,. \tag{2.19}$$

For insulator ($g \to 0$), it follows from Eqs. (2.18) and (2.17) that

$$\lim_{g\to 0} \beta_d(g) \to \ln(g/g_c)\,. \tag{2.20}$$

Assuming the existence of two perturbation expansions over the "charge" g in the limits of weak and strong "couplings" we can write the corrections to Eqs. (2.20) and (2.19) in the following form:

$$\beta_d(g\to 0) = \ln(g/g_c)(1 + bg + \cdots)\,, \tag{2.21}$$

$$\beta_d(g\to\infty) = d-2-(\alpha/g) + \cdots, \quad \alpha > 0\,. \tag{2.22}$$

Following these *assumptions* and supposing now a monotonous and continuous form of $\beta_d(g)$, it is easy to plot it qualitatively for all g, as shown in Fig. 3. All the previous equations are written for dimensionless conductance, which is measured in natural units of $e^2/\hbar \approx 2.5\times 10^{-4}\,\Omega^{-1}\,\mathrm{cm}^{-1}$. We see that $\beta_d(g)$ definitely has no zeros for $d < 2$. If expansion Eq. (2.22) is valid there is no zero for $d = 2$ also. For $d > 2$, function β_d must have a zero: $\beta_d(g_c) = 0$. It is clear that $g_c \sim 1$ and no form of perturbation theory is valid near that zero. The existence of a zero of $\beta_d(g)$ corresponds to existence of an unstable fixed point of Eq. (2.18). The state of a system is supposedly determined by disorder at microscopic distances of the order of interatomic spacing a, i.e. by $g_0 = g(L = a)$. Using g_0 as an initial value and integrating Eq. (2.18) it is easy to find that for $g_0 > g_c$ conductivity $\sigma_L = g(L)L^{2-d}$ tends for $L \to \infty$ to a constant (metallic) value. For $g < g_c$ in the limit of $L \to \infty$ we get insulating

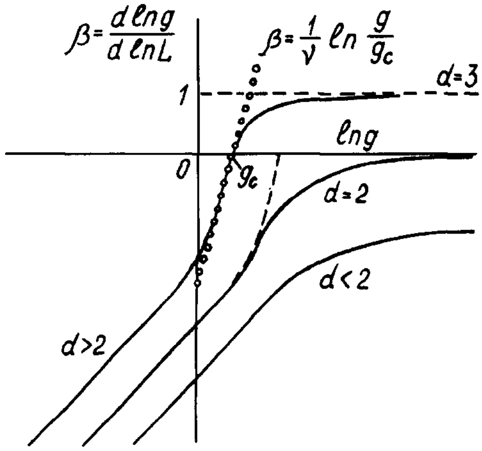


Fig. 3. Qualitative form of $\beta_d(g)$ for different d. Dashed line shows the behavior necessary to get discontinuous drop of conductivity at the mobility edge for $d = 2$.

behavior. Using for $g \sim g_c$ an approximation (shown with circles in Fig. 3)

$$\beta_d(g) \approx (1/\nu)\ln(g/g_c) \approx (1/\nu)(g-g_c)/g_c\,, \tag{2.23}$$

we obtain from Eq. (2.18) for $g_0 > g_c$ the following behavior of conductivity for $L \to \infty$:

$$\sigma \approx A\frac{e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(\ln\frac{g_0}{g_c}\right)^{(d-2)\nu} \approx A\frac{e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(\frac{g_0-g_c}{g_c}\right)^{(d-2)\nu}, \tag{2.24}$$

where $A = \text{const.}$ and we have explicitly introduced the conductivity scale of the order of σ_c (cf. Eq. (2.5)). We see that the existence of a fixed point leads to the existence of a mobility edge, and behavior of $\beta_d(g)$ close to its zero determines the critical behavior at the Anderson transition. Under these assumptions conductivity continuously goes to zero for $g_0 \to g_c$, and the value of $\sigma_c \approx e^2/(\hbar a^{d-2})$ is the characteristic scale of conductivity at the metal–insulator transition. To get a discontinuous drop of conductivity at the mobility edge, $\beta_d(g)$ must be nonmonotonic as shown by the dashed line for $d = 2$ in Fig. 3. This behavior seems more or less unphysical.

Integrating Eq. (2.18) with $\beta_d(g)$ from Eq. (2.23) with initial $g_0 < g_c$ gives

$$g(L) \approx g_c \exp\left\{-A|\ln(g_0/g_c)|^{\nu}\frac{L}{a}\right\}. \tag{2.25}$$

From this it is clear (cf. Eq. (2.7)) that

$$R_{\text{loc}} \sim a|(g_0-g_c)/g_c|^{-\nu} \tag{2.26}$$

and ν is the critical exponent of the localization length. For $d = 2$ we have $\beta_d(g) < 0$ in the whole interval of g, so that $\sigma_{L\to\infty} \to 0$ for any initial value of g and there is no mobility edge and all states are localized.

For $d > 2$, limiting ourselves by those terms of the perturbation expansion in g^{-1} shown in Eq. (2.22) we can solve $\beta_d(g_c) = 0$ to find

$$g_c = \alpha/(d-2)\,. \tag{2.27}$$

We can see that for $d \to 2$ the mobility edge goes to infinity which corresponds to complete localization in the two-dimensional case. Now we have

$$\beta_d(g \sim g_c) \approx (d-2)(g_0-g_c)/g_c \tag{2.28}$$

and for the critical exponent of localization length we get (cf. Eq. (2.23))

$$\nu = 1/(d-2)\,. \tag{2.29}$$

which may be considered as the first term of the ε-expansion near $d = 2$ (where $\varepsilon = d-2$), i.e. near the "lower critical dimension" for localization [31, 82, 48]. Note that the expansion of Eq. (2.22) can be reproduced in the framework of standard perturbation theory over impurity scattering [49, 50]. For $d = 3$ this gives $\alpha = \pi^{-3}$ (cf. Ref. [6]).

Let us now define the correlation length of a localization transition as

$$\xi_{\text{loc}} \sim a|(g_0-g_c)/g_c|^{-\nu}\,. \tag{2.30}$$

For $g_0 < g_c$ this length coincides with the localization length R_{loc}. It is easy to see that Eq. (2.24) can be written as [51]

$$\sigma \approx A g_c (e^2/\hbar \xi_{loc}^{d-2}) . \tag{2.31}$$

It follows that for $g > g_c$ correlation length ξ_{loc} determines the behavior of conductivity close to the mobility edge, when this length becomes much larger than the interatomic distance and mean free path.

Let us consider the three-dimensional case in more details. Integrating Eq. (2.18) with $\beta_3(g) = 1 - g_c/g$ where $g_c = \alpha$ gives $g(L) = (\hbar/e^2)\sigma_L L = (\hbar/e^2)\sigma + g_c$ so that for a finite sample close to the mobility edge ($\xi_{loc} \gg l$) we obtain

$$\sigma_L = \sigma + (e^2 g_c/\hbar L) , \tag{2.32}$$

where in correspondence with Eq. (2.31)

$$\sigma \approx A g_c \frac{e^2}{\hbar \xi_{loc}} . \tag{2.33}$$

It follows that for $L \gg \xi_{loc} \gg l$ conductivity $\sigma_L \to \sigma$ while for $l \ll L \ll \xi_{loc}$ conductivity σ_L and the appropriate diffusion coefficient, determined by Einstein relation $\sigma = e^2 D N(E_F)$ are given by

$$\sigma_L \approx e^2 g_c/\hbar L , \tag{2.34}$$

$$D_L \approx [g_c/N(E_F)](1/\hbar L) , \tag{2.35}$$

respectively, where $N(E_F)$ is the electron density of states at the Fermi level. Thus, in this latest case, conductivity is not Ohmic, i.e. diffusion of electrons is "non-classical" [20, 6]. From this discussion it is clear that the characteristic length ξ_{loc} in the metallic region determines the scale at which conductivity becomes independent of the sample size. Close to the mobility edge when $\xi_{loc} \to \infty$ only the samples with growing sizes $L \gg \xi_{loc}$ can be considered as macroscopic. These considerations allow us to understand the physical meaning of the diverging length ξ_{loc} of scaling theory in the metallic region [33]. Close to mobility, ξ_{loc} is considered as the only relevant length in the problem (with an exception of a sample size L) and the scaling hypothesis is equivalent to the assumption

$$g(L) = f(L/\xi_{loc}) , \tag{2.36}$$

where $f(x)$ is some universal (for a given dimensionality d) function. In the metallic region, for $L \gg \xi_{loc} \gg l$ it is obvious that $f(x) \sim x^{d-2}$ which reproduces Eq. (2.31).

For finite frequencies ω of an external electric field a new length appears in the system [34]:

$$L_\omega = [D(\omega)/\omega]^{1/2} , \tag{2.37}$$

where $D(\omega)$ is the frequency dependent diffusion coefficient. L_ω is a length of electron diffusion during one cycle of an external field. Close to the mobility edge ξ_{loc} is large and for $L_\omega < \xi_{loc}$, L and L_ω become the relevant length scale. In general, for finite ω localization transition is smeared, a sharp transition is realized only for $L^{-1} = L_\omega^{-1} = 0$. Thus, for the finite frequency case the scaling hypothesis of Eq. (2.36) can be generalized as [34]

$$g(L,\omega) = f(L/\xi_{loc}, L_\omega/\xi_{loc}) , \tag{2.38}$$

where g denotes a real part of conductance. In the metallic phase, for $L \gg \xi_{\mathrm{loc}}$ we have $g \sim L^{d-2}$ so that

$$\sigma(\omega) = \frac{e^2}{\hbar} L^{2-d} f\left(\frac{L}{\xi_{\mathrm{loc}}}, \frac{L_\omega}{\xi_{\mathrm{loc}}}\right) \to \frac{e^2}{\hbar} \xi_{\mathrm{loc}}^{2-d} f\left(\infty, \frac{L_\omega}{\xi_{\mathrm{loc}}}\right) \equiv \frac{e^2}{\hbar \xi_{\mathrm{loc}}^{d-2}} F\left(\frac{\xi_{\mathrm{loc}}}{L_\omega}\right). \tag{2.39}$$

For small frequencies, when $L_\omega \gg \xi_{\mathrm{loc}}$, we can write down the universal function $F(x)$ as $F(x) \approx Ag_{\mathrm{c}} + Bx^{d-2}$ which reproduces Eq. (2.31) and the small frequency corrections found earlier in [49]. For $L_\omega \ll \xi_{\mathrm{loc}}$ i.e. for high frequencies or close to the mobility edge, the relevant length is L_ω and the frequency dependent part of conductivity is dominating. In particular, at the mobility edge itself the length ξ_{loc} drops out and must cancel in Eq. (2.38) which leads to

$$\sigma(\omega, E_{\mathrm{F}} = E_{\mathrm{c}}) \sim L_\omega^{2-d} \sim [\omega/D(\omega)]^{(d-2)/2}\,. \tag{2.40}$$

On the other hand, according to Einstein relation we must have $\sigma(\omega) \sim D(\omega)$. Accordingly, from $[\omega/D(\omega)]^{(d-2)/2} \sim D(\omega)$ we get at the mobility edge

$$\sigma(\omega, E_{\mathrm{F}} = E_{\mathrm{c}}) \sim D(\omega) \sim \omega^{(d-2)/d}\,. \tag{2.41}$$

For $d = 3$ this leads [51,35] to $\sigma(\omega) \sim D(\omega) \sim \omega^{1/3}$. The crossover between different types of frequency dependence occurs for $L_\omega \sim \xi_{\mathrm{loc}}$ which determines the characteristic frequency [34]:

$$\omega_{\mathrm{c}} \sim \frac{1}{\hbar \xi_{\mathrm{loc}}^d N(E_{\mathrm{F}})}\,. \tag{2.42}$$

The $\omega^{(d-2)/d}$ behavior is realized for $\omega \gg \omega_{\mathrm{c}}$, while for $\omega \ll \omega_{\mathrm{c}}$ we get small corrections of the order of $\sim\omega^{(d-2)/2}$ to Eq. (2.31).

Finally, we must stress that for finite temperatures there appear *inelastic* scattering processes which destroy the phase correlations of wave functions at distances greater than a characteristic length of the order of $L_\varphi = \sqrt{D\tau_\varphi}$, where D is the diffusion coefficient due to *elastic* scattering processes considered above and τ_φ is the "dephasing" time due to inelastic processes [39]. For $T > 0$ this length L_φ effectively replaces the sample size L in all expressions of scaling theory when $L \gg L_\varphi$, because on distances larger than L_φ all information on the nature of wave functions (e.g. whether they are localized or extended) is smeared out. Taking into account the usual low-temperature dependence like $\tau_\varphi \sim T^{-p}$ (where p is some integer, depending on the mechanism of inelastic scattering) this can lead to a nontrivial temperature dependence of conductivity, in particular, to the possibility of a negative temperature coefficient of resistivity of "dirty" metals [33] which are close to the localization transition. It is important to stress that similar expressions determine the temperature dependence of conductivity also for the localized phase until $L_\varphi < R_{\mathrm{loc}}$. Only for $L_\varphi > R_{\mathrm{loc}}$ the localized nature of the wave functions reveals itself in the temperature dependence of conductivity and the transition to exponentially activated hopping behavior takes place, which becomes complete for $T < [N(E_{\mathrm{F}})R_{\mathrm{loc}}^d]^{-1}$.

2.3. *Self-consistent theory of localization*

2.3.1. *Isotropic systems*

It is obvious that the qualitative scaling picture of Anderson transition described in the previous section requires microscopic justification. At the same time, we need a practical method of explicit calculations for any physical characteristic of the electronic system close to the mobility edge. Here we shall briefly describe the main principles of the so-called self-consistent theory of localization which while leaving aside some important points, leads to an effective scheme for analysis of the relevant physical characteristics important for us. This approach, first formulated by Gotze [52,35] was later further developed by Vollhardt and Wolfle and others [53–56,32,7].

Complete information concerning the Anderson transition and transport in a disordered system is contained in the two-particle Green's function

$$\Phi^{\mathrm{RA}}_{pp'}(E\omega \boldsymbol{q}) = -\frac{1}{2\pi \mathrm{i}}\langle G^{\mathrm{R}}(\boldsymbol{p}_+\boldsymbol{p}'_- E+\omega)G^{\mathrm{A}}(\boldsymbol{p}'_-\boldsymbol{p}_- E)\rangle\ , \tag{2.43}$$

where $\boldsymbol{p}_{+-} = \boldsymbol{p} \pm \frac{1}{2}\boldsymbol{q}$, in most cases below E just coincides with the Fermi energy E_{F}. Angular brackets denote averaging over disorder. Graphically this Green's function is shown in Fig. 4. It is well known that this Green's function is determined by the Bethe–Salpeter equation also shown graphically in Fig. 4 [57,58,53]:

$$\Phi^{\mathrm{RA}}_{pp'}(E\boldsymbol{q}\omega) = G^{\mathrm{R}}(E+\omega \boldsymbol{p}_+)G^{\mathrm{A}}(E\boldsymbol{p}_-)\left\{-\frac{1}{2\pi \mathrm{i}}\delta(\boldsymbol{p}-\boldsymbol{p}') + \sum_{p''} U^{\mathrm{E}}_{pp''}(\boldsymbol{q}\omega)\Phi^{\mathrm{RA}}_{p''p'}(E\boldsymbol{q}\omega)\right\}\ , \tag{2.44}$$

where $G^{\mathrm{R,A}}(E\boldsymbol{p})$ is the averaged retarded (advanced) one-electron Green's function, while the irreducible vertex part $U^{\mathrm{E}}_{pp'}(\boldsymbol{q}\omega)$ is determined by the sum of all diagrams which cannot be cut over two electron lines (cf. Fig. 4).

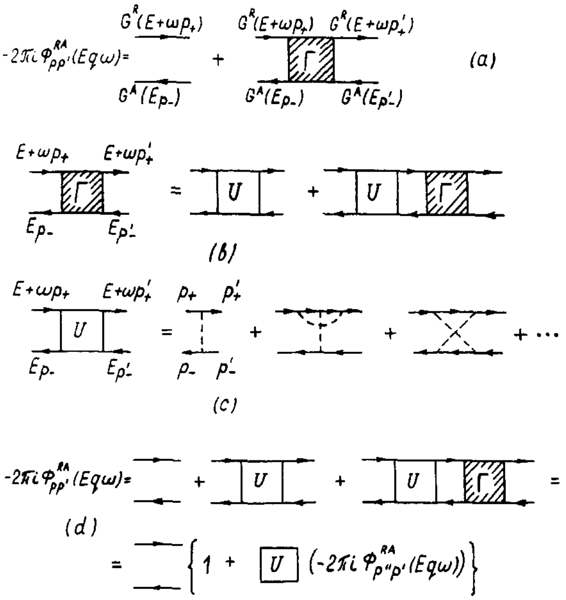


Fig. 4. Graphical representation of (a) a two-electron Green's function $\Phi^{\mathrm{RA}}_{pp'}(E\boldsymbol{q}\omega)$, (b) the equation for the full vertex part $\Gamma^E_{pp'}(\boldsymbol{q}\omega)$, (c) typical diagrams for the irreducible vertex $U^E_{pp'}(\boldsymbol{q};\omega)$; (d) the Bethe–Salpeter equation. Dashed line denotes interaction $U_0(\boldsymbol{p}-\boldsymbol{p}') = \rho|V(\boldsymbol{p}-\boldsymbol{p}')|^2$, where ρ is the density of scatterers and $V(\boldsymbol{p}-\boldsymbol{p}')$ the Fourier transform of a single scatterer potential.

In fact, the two-particle Green's function of Eq. (2.43) contains some extra information and for the complete description of Anderson transition it is sufficient to know the two-particle Green's function summed over $\boldsymbol{pp'}$ [53]:

$$\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) = -\frac{1}{2\pi \mathrm{i}}\sum_{\boldsymbol{pp'}} \langle G^{\mathrm{R}}(\boldsymbol{p}_+\boldsymbol{p}'_+ E+\omega)G^{\mathrm{A}}(\boldsymbol{p}'_-\boldsymbol{p}_- E)\rangle\ . \tag{2.45}$$

Using the Bethe–Salpeter equation, Eq. (2.44), and exact Ward identities we can obtain a closed equation for $\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega)$ [53, 32, 7], and for small ω and $\boldsymbol{q}$ the solution of this equation has a typical diffusion-pole form

$$\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) = -N(E)\frac{1}{\omega + \mathrm{i}D_E(\boldsymbol{q}\omega)q^2}\ , \tag{2.46}$$

where $N(E)$ is the electron density of states at energy E and the *generalized* diffusion coefficient $D_E(\boldsymbol{q}\omega)$ is expressed through the so-called relaxation kernel $M_E(\boldsymbol{q}\omega)$:

$$D_E(\boldsymbol{q}\omega) = \mathrm{i}\frac{2E}{dm}\frac{1}{M_E(\boldsymbol{q}\omega)} = \frac{v_{\mathrm{F}}^2}{d}\frac{\mathrm{i}}{M_E(\boldsymbol{q}\omega)}\ , \tag{2.47}$$

where v_{F} is the Fermi velocity of an electron. The retarded density–density response function at small ω and $\boldsymbol{q}$ is given by

$$\chi^{\mathrm{R}}(\boldsymbol{q}\omega) = \omega\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) + N(E) + \mathrm{O}(\omega, q^2) \tag{2.48}$$

or from Eq. (2.46)

$$\chi^{\mathrm{R}}(\boldsymbol{q}\omega) = N(E)\frac{\mathrm{i}D_E(\boldsymbol{q}\omega)q^2}{\omega + \mathrm{i}D_E(\boldsymbol{q}\omega)q^2}\ . \tag{2.49}$$

For the relaxation kernel $M_E(\boldsymbol{q}\omega)$ (or for the generalized diffusion coefficient) a self-consistency equation can be derived, which is actually the main equation of the theory [53, 29, 32]. The central point in this derivation is some approximation for the irreducible vertex part $U_{\boldsymbol{pp'}}^E(\boldsymbol{q}\omega)$ in the Bethe–Salpeter equation. The approximation of Vollhardt and Wolfle is based upon the use for $U_{\boldsymbol{pp'}}^E(\boldsymbol{q}\omega)$ of the sum of "maximally-crossed" graphs shown in Fig. 5. This series is easily summed and we get the so-called "Cooperon" [49, 53]:

$$U_{\boldsymbol{pp'}}^{EC}(\boldsymbol{q}\omega) = \frac{2\gamma\rho V^2}{D_0(\boldsymbol{p}+\boldsymbol{p}')^2 + \mathrm{i}\omega}\ , \tag{2.50}$$

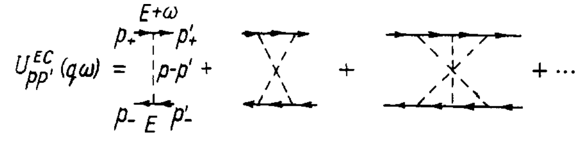


Fig. 5. "Maximally-crossed" diagrams for the irreducible vertex part of Bethe–Salpeter equation ("Cooperon").

where

$$D_0 = E/md\gamma = v_F^2\tau/d \tag{2.51}$$

is the classical (bare) diffusion coefficient determining the Drude conductivity (Eq. (2.2)). For point scatterers randomly distributed with spatial density ρ (V is the scattering amplitude) we have

$$\gamma = 1/2\tau = \pi\rho V^2 N(E_F) . \tag{2.52}$$

These "maximally crossed" diagrams lead to the following quantum correction to the diffusion coefficient:

$$\frac{\delta D(\omega)}{D_0} = -\frac{1}{\pi N(E)}\sum_{|\boldsymbol{k}|<k_0}\frac{1}{-\mathrm{i}\omega + D_0 k^2} . \tag{2.53}$$

Appropriate correction to the relaxation kernel can be expressed via the correction to diffusion coefficient as

$$\delta M_E(\omega) = -\mathrm{i}\frac{2E_F}{dm}\frac{\delta D(\omega)}{D(\omega)^2} = -\frac{M_E(\omega)}{D(\omega)}\delta D(\omega) . \tag{2.54}$$

Considering the usual Drude metal as the zeroth approximation we get

$$\delta M_E(\omega) = -(M_0/D_0)\delta D(\omega) . \tag{2.55}$$

The central point of the self-consistent theory of localization [52] reduces to the replacement of the Drude diffusion coefficient D_0 in the diffusion pole of Eq. (2.53) by the generalized one $D(\omega)$. Using this relation in Eq. (2.55) we obtain the main equation of the self-consistent theory of localization determining the relaxation kernel $M(0\omega)$ (for $\boldsymbol{q} = 0$) [53, 32]:

$$M_E(\omega) = 2\mathrm{i}\gamma\left\{1 + \frac{1}{\pi N(E)}\sum_{|\boldsymbol{k}|<k_0}\frac{\mathrm{i}}{\omega + (2E/dm)(k^2/M_E(\omega))}\right\} \tag{2.56}$$

or the equivalent equation for the generalized diffusion coefficient itself:

$$\frac{D_0}{D_E(\omega)} = 1 + \frac{1}{\pi N(E)}\sum_{|\boldsymbol{k}|<k_0}\frac{1}{-\mathrm{i}\omega + D_E(\omega)k^2} . \tag{2.57}$$

Cut-off in momentum space in Eqs. (2.53), (2.56), (2.57) is determined by the limit of applicability of diffusion-pole approximation of Eq. (2.46) or Eq. (2.50) [7]:

$$k_0 \approx \mathrm{Min}\{p_F, l^{-1}\} . \tag{2.58}$$

Close to the mobility edge, $p_F \sim l^{-1}$. Note that from here on we are generally using natural units with Planck constant $\hbar = 1$; however, in some of the final expressions we shall write $\hbar$ explicitly.

Conductivity can be expressed as [53, 32]

$$\sigma(\omega) = (ne^2/m)[\mathrm{i}/(\omega + M_E(\omega))] \to e^2 D_E(\omega)N(E) \quad \text{for } \omega \to 0 , \tag{2.59}$$

where we have used $n/N(E) = 2E/d$. It is clear that for the metallic phase $M_E(\omega \to 0) = \mathrm{i}/\tau_E$, where τ_E is the generalized mean free time. Far from the Anderson transition (for weak disorder) $\tau_E \approx \tau$ from Eqs. (2.52) and (2.59) reduces to the standard Drude expression.

If the frequency behavior of relaxation kernel leads to the existence of a limit $\lim_{\omega\to 0}\omega M_E(\boldsymbol{q}\omega)$ a singular contribution appears in Eq. (2.46) for $\omega \to 0$ [35, 7]:

$$\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) \approx -\frac{N(E)}{\omega}\frac{1}{1-(2E/md)(q^2/\omega M_E(\boldsymbol{q}\omega))} \approx -\frac{N(E)}{\omega}\frac{1}{1+R_{\mathrm{loc}}^2 q^2}, \tag{2.60}$$

where we have defined

$$R_{\mathrm{loc}}^2(E) = -(2E/md)\lim_{\omega\to 0}(1/\omega M_E(\omega)). \tag{2.61}$$

According to the general criterion of localization [59, 7] (cf. Appendix A) this behavior corresponds to the region of localized states. Using Eq. (A.16) we immediately obtain from Eq. (2.60) the singular contribution to Gorkov–Berezinskii spectral density (cf. Eqs. (A.8), (A.9)):

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_{\boldsymbol{q}}^F = (1/\pi N(E))\,\mathrm{Im}\,\Phi_E^{\mathrm{RA}}(\boldsymbol{q}\omega) = A_E(\boldsymbol{q})\delta(\omega), \tag{2.62}$$

where

$$A_E(\boldsymbol{q}) = 1/(1+R_{\mathrm{loc}}^2(E)q^2) \to 1 - R_{\mathrm{loc}}^2(E)q^2 \quad \text{for } q\to 0. \tag{2.63}$$

From the above expression and from Eq. (A.11) we can see that $R_{\mathrm{loc}}(E)$ as defined in Eq. (2.61) is actually the localization length. It is useful to define a characteristic frequency [53]:

$$\omega_0^2(E) = -\lim_{\omega\to 0}\omega M_E(\omega) > 0 \tag{2.64}$$

so that

$$R_{\mathrm{loc}}(E) = \sqrt{(2E/md)}(1/\omega_0(E)). \tag{2.65}$$

Thus, the localization transition is signalled by the divergence of the relaxation kernel for $\omega \to 0$ [53], so that two characteristic types of its behavior for $\boldsymbol{q} = 0$ and $\omega \to 0$ appear:

$$M_E(0\omega) \approx \begin{cases} \dfrac{\mathrm{i}}{\tau_E} & \text{for } E \geq E_c, \\[2ex] \dfrac{\mathrm{i}}{\tau_E} - \dfrac{\omega_0^2(E)}{\omega} & \text{for } E \leq E_c. \end{cases} \tag{2.66}$$

The frequency $\omega_0(E)$ is in some crude sense analogous to the order parameter in the usual theory of phase transitions. It appears in the localized phase signalling about the Anderson transition.

From Eq. (A.16), for $\omega \to 0$ and $\boldsymbol{q} = 0$, neglecting the nonsingular contribution from $\mathrm{Im}\,\Phi_E^{\mathrm{RR}}(\boldsymbol{q}\omega)$ we can get an explicit expression for the Berezinskii–Gorkov spectral density which is valid for small ω and $\boldsymbol{q}$ [60, 7]:

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_{\boldsymbol{q}}^F = \begin{cases} \dfrac{1}{\pi}\dfrac{D_E q^2}{\omega^2 + (D_E q^2)^2} & \text{(Metal)}, \\[2ex] A_E(\boldsymbol{q})\delta(\omega) + \dfrac{1}{\pi}\dfrac{D_E q^2}{\omega^2 + [\omega_0^2(E)\tau_E + D_E q^2]^2} & \text{(Insulator)}, \end{cases} \tag{2.67}$$

where we have introduced the renormalized diffusion coefficient, determined by relaxation time τ_E:

$$D_E = \frac{2E}{dm}\tau_E = \frac{1}{d}v_F^2\tau_E . \tag{2.68}$$

Substituting Eq. (2.66) into the equation (Eq. (2.56)) for self-consistency we can obtain equations for τ_E and $\omega_0(E)$ [54, 55, 7] and thus determine all the relevant characteristics of the system. For $d > 2$, Eqs. (2.56) and (2.57) do really describe a metal–insulator transition [54, 55, 7, 29]. For $d = 2$ all the electronic states are localized [53].

Below we present some of the results of this analysis which will be important for the following. For $2 < d < 4$ a correlation length similar to that of Eqs. (2.7) and (2.30) appears:

$$\xi_{\mathrm{loc}}(E) \sim \frac{1}{p_F}\left|\frac{E - E_c}{E_c}\right|^{-\nu} \quad \text{for } E \sim E_c , \tag{2.69}$$

where $\nu = 1/(d-2)$. The position of the mobility edge is determined by the condition:

$$\left.\frac{E}{\gamma}\right|_{E=E_c} = \frac{d}{\pi(d-2)} \tag{2.70}$$

which follows if we assume the cut-off $k_0 = p_F$ in Eqs. (2.56) and (2.57). Static conductivity in the metallic phase ($E > E_c$) is given by (cf. Eq. (2.31))

$$\sigma = \frac{\sigma_0}{[p_F\xi_{\mathrm{loc}}(E)]^{d-2}} , \tag{2.71}$$

where $\sigma_0 = (ne^2/m)\tau$ is the usual Drude conductivity. In particular, for $d = 3$

$$\left.\frac{E}{\gamma}\right|_{E=E_c} = p_F l|_{E=E_c} = \frac{3}{\pi} \tag{2.72}$$

in complete accordance with Ioffe–Regel criterion, and

$$\sigma = \frac{\sigma_0}{p_F\xi_{\mathrm{loc}}(E)} . \tag{2.73}$$

Critical exponent $\nu = 1$. The mean free path which follows from Eq. (2.72) corresponds to Drude conductivity:

$$\sigma_c = \left.\frac{ne^2}{m}\tau\right|_{E=E_c} = \left.\frac{e^2p_F}{3\pi^2\hbar^2}\left(\frac{p_F l}{\hbar}\right)\right|_{E=E_c} = \frac{e^2p_F}{\pi^3\hbar^2} \tag{2.74}$$

which is equivalent to the elementary estimate of Eq. (2.5).

Eq. (2.73) can also be rewritten as [22]

$$\sigma = \sigma_0\left\{1 - \frac{\sigma_c}{\sigma_0}\right\} = \sigma_0 - \sigma_c , \tag{2.75}$$

where the Drude conductivity σ_0 is now the measure of disorder. It is obvious that for small disorder (large mean free path) $\sigma_0 \gg \sigma_c$, Eq. (2.75) reduces to $\sigma \approx \sigma_0$. As disorder grows (mean free path drops) conductivity $\sigma \to 0$ for $\sigma_0 \to \sigma_c$.

In the dielectric phase ($E < E_c$) we have $\xi_{loc}(E) = R_{loc}(E)$ and finite $\omega_0^2(E)$ from Eq. (2.64) which tends to zero as $E \to E_c$ from below. This frequency determines the dielectric function of the insulating phase [7]:

$$\varepsilon(\omega \to 0) = 1 + \frac{\omega_p^2}{\omega_0^2(E)} = 1 + \kappa_D^2 R_{loc}^2(E) \sim \left| \frac{E - E_c}{E_c} \right|^{-2\nu}, \tag{2.76}$$

where $\omega_p^2 = 4\pi ne^2/m$ is the square of the plasma frequency, $\kappa_D^2 = 4\pi e^2 N(E)$ is the square of the inverse screening length of a metal.

Thus the main results of the self-consistent theory of localization coincide with the main predictions of elementary scaling theory of localization. Vollhardt and Wolfle have shown [54, 32] that equations of this theory and especially the main differential equation of renormalization group Eq. (2.18) for conductance may be explicitly derived from self-consistency equations (Eqs. (2.56) and (2.57)) reformulated for a finite system by introduction of low-momentum cut-off at $k \sim 1/L$, where L is the system size.

The results considered up to now are valid for $\omega \to 0$. Self-consistent theory of localization allows to study the frequency dependence of conductivity (generalized diffusion coefficient) [32]. At finite frequency the main Eq. (2.57) for the generalized diffusion coefficient for $d = 3$ can be rewritten as [36, 32]

$$\frac{D_E(\omega)}{D_0} = 1 - \left(\frac{E_c}{E}\right)^{1/2} + \frac{\pi}{2}\left(\frac{E_c}{E}\right)^{1/2} \left\{ -\frac{i\omega}{2\gamma} \frac{D_0}{D_E(\omega)} \right\}^{1/2} \tag{2.77}$$

which can be solved explicitly. For the level of accuracy we are aiming, this solution may be written as

$$D_E(\omega) \approx \begin{cases} D_E\,, & \omega \ll \omega_c, \quad E \geq E_c \quad \text{(Metal)}\,, \\ D_0 \left(-\dfrac{i\omega}{2\gamma} \right)^{1/3}, & \omega \gg \omega_c \qquad\qquad\quad \text{(Metal and insulator)}\,, \\ D_E \dfrac{-i\omega}{-i\omega + (3D_E/v_F^2)\omega_0^2(E)}\,, & \omega \ll \omega_c, \quad E < E_c \quad \text{(Insulator)}\,, \end{cases} \tag{2.78}$$

where (cf. Eq. (2.42)):

$$\omega_c \sim 2\gamma [p_F \xi_{loc}]^{-d} \sim \frac{1}{N(E)\xi_{loc}^d}\,. \tag{2.79}$$

Here the renormalized diffusion coefficient

$$D_E = \frac{D_0}{p_F \xi_{loc}(E)}\,. \tag{2.80}$$

At the mobility edge itself $\xi_{\mathrm{loc}}(E=E_c)=\infty$, so that $\omega_c=0$ and we get the $\omega^{1/3}$-behavior (cf. Eq. (2.41)):

$$D_E(\omega)=D_0(-\mathrm{i}\omega/2\gamma)^{1/3}\,. \tag{2.81}$$

Note that ω_c is in fact determined by $D_E(\omega_c)\sim D_E\sim D_0(\omega_c/2\gamma)^{1/3}$. The meaning of the limit $\omega\to 0$ used above (cf. e.g. Eq. (2.66)) is just that $\omega\ll\omega_c$. In particular, the expression Eq. (2.67) for the Gorkov–Berezinskii spectral density is valid only for $\omega\ll\omega_c$. For $\omega_c\le\omega\le 2\gamma$, using Eq. (2.81) in Eq. (2.46) we get from Eq. (A.16):

$$\langle\!\langle\rho_E\rho_{E+\omega}\rangle\!\rangle_q^F=\frac{\sqrt{3}}{2\pi}\frac{\alpha^{2/3}\omega^{1/3}q^2}{\omega^2+\alpha^{2/3}\omega^{4/3}q^2+\alpha^{4/3}\omega^{2/3}q^4}\,, \tag{2.82}$$

where $\alpha=D_0v_F/2\gamma=D_0l\sim[N(E)]^{-1}$, where the last estimate is for $l\sim p_F^{-1}$. Eq. (2.82) is valid also at the mobility edge itself where $\omega_c=0$. Obviously, the correct estimate can be obtained from Eq. (2.67) by a simple replacement $D_E\to D_0(\omega/\gamma)^{1/3}$. It should be noted that the self-consistent theory approach to the frequency dependence of conductivity is clearly approximate. For example, it is unable to reproduce the correct $\mathrm{Re}\,\sigma(\omega)\sim\omega^2\ln^4\omega$ dependence for $\omega\to 0$ in the insulating state [3]. This is apparently related to its inability to take the correct account of the locally discrete nature of energy levels in Anderson insulators (cf. below). However, this is unimportant for our purposes while the general nature of frequency dependence at the mobility edge is apparently correctly reproduced.

In the following analysis we will also need a correlator of *local* densities of states defined in Eq. (A.3). This correlator can be expressed via the two-particle Green's function as in Eq. (A.15). For small ω and q and far from the Anderson transition (weak disorder), neglecting the nonsingular contribution from the second term of Eq. (A.15) we can estimate the most important contribution to that correlator from the diagram shown in Fig. 6 [62]. The same contribution comes from the diagram which differs from that in Fig. 6 by the direction of electron lines in one of the loops. Direct calculation gives

$$\begin{aligned}\langle\!\langle\rho_E\rho_{E+\omega}\rangle\!\rangle_q^H&\sim\frac{N(E)}{\gamma^2}(\rho V^2)^2\,\mathrm{Re}\int\mathrm{d}^d\boldsymbol{Q}\frac{1}{-\mathrm{i}\omega+D_0\boldsymbol{Q}^2}\frac{1}{-\mathrm{i}\omega+D_0(\boldsymbol{Q}+\boldsymbol{q})^2}\\&\sim\frac{1}{N(E)}\mathrm{Re}\frac{1}{D_0^{d/2}}\frac{1}{(-\mathrm{i}\omega+D_0q^2)^{2-d/2}}\,.\end{aligned} \tag{2.83}$$

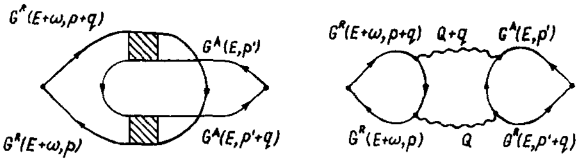


Fig. 6. Two equivalent forms of the diagram for the correlator of local density of states. Wavy lines denote the diffusion propagator, i.e. the sum of ladder diagrams.

A similar result for this correlator for some special model was first obtained by Oppermann and Wegner [63]. From Eq. (2.83) for $d = 3$ we find

$$\langle\langle \rho_E \rho_{E+\omega} \rangle\rangle_q^H \sim \frac{1}{N(E) D_0^{3/2}} \left\{ \frac{D_0 q^2}{\omega^2 + (D_0 q^2)^2} + [\omega^2 + (D_0 q^2)^2]^{-1/2} \right\}^{1/2} . \tag{2.84}$$

It is obvious that for the estimates close to the mobility edge we can in the spirit of self-consistent theory of localization replace D_0 in Eqs. (2.83) and (2.84) by the generalized diffusion coefficient $D(\omega)$. In particular, for a system at the mobility edge $(\omega_c = 0) D_0 \to D_0(\omega/\gamma)^{1/3}$ in Eq. (2.84).

Surely, the self-consistent theory of localization is not free from some difficulties. Apparently the main problem is an uncontrollable nature of the self-consistency procedure itself. These are discussed in Refs. [7, 29] in more detail. Here we shall concentrate only on some problems relevant for the future discussion. From the definition of the generalized diffusion coefficient in Eq. (2.47) it is clear that it may be a function of both ω and $\boldsymbol{q}$, i.e. it can also possess spatial dispersion. Self-consistent theory of localization deals only with the limit of $D_E(\boldsymbol{q} \to 0\omega)$. At present it is not clear whether we can in any way introduce spatial dispersion into the equations of self-consistent theory. Using scaling considerations the q dependence of $D_E(\boldsymbol{q}\omega \to 0)$ can be estimated as follows [6, 101]. We have seen above that for the system of finite size of $L \ll \xi_{\mathrm{loc}}$ elementary scaling theory of localization predicts the L-dependent diffusion coefficient $D_E \approx (g_c/N(E))/L^{d-2}$ (cf. Eq. (2.35) for $d = 3$). From simple dimensional considerations we can try the replacement $L \to q^{-1}$ and get

$$D_E(\omega \to 0\boldsymbol{q}) \approx \begin{cases} D_E & \text{for } q\xi_{\mathrm{loc}} \ll 1 , \\ \alpha q^{d-2} & \text{for } q\xi_{\mathrm{loc}} \gg 1 , \end{cases} \tag{2.85}$$

where $\alpha \sim g_c/N(E) \sim D_0 l$ and $E \sim E_c$, $l^{-1} \sim p_F$. Obviously, an attempt to incorporate such q-dependence into the equations of self-consistent theory of localization (like Eqs. (2.56) and (2.57)) will radically change its structure. At the same time the L-dependence like $D_E \sim \alpha/L^{d-2}$ (for $L \ll \xi_{\mathrm{loc}}$) can be directly derived from Eq. (2.57) as equations of elementary scaling theory are derived from it [54, 32, 29]. Thus the foundations for simple replacements like $L \to q^{-1}$ in Eq. (2.85) are not completely clear. More detailed analysis of the wave number dependence of the diffusion coefficient leading to Eq. (2.85) was given by Abrahams and Lee [65] within the scaling approach. However, the complete solution of this problem is apparently still absent. In a recent paper [66] it was shown that Eq. (2.85) actually contradicts the general localization criterion of Berezinskii and Gorkov, from which it follows directly that at the localization transition the static diffusion coefficient $D(\omega = 0, \boldsymbol{q})$ vanishes for all q simultaneously. The detailed analysis performed in Ref. [66] demonstrates the absence of any significant spatial dispersion of the diffusion coefficient on the scale of $q \sim \xi^{-1}$, while its presence on the scale of $q \sim p_F$ is irrelevant for the critical behavior of the system close to the Anderson transition. In fact, in Ref. [66] it is claimed that the *exact* critical behavior at the mobility edge coincides with that predicted by the self-consistent theory of localization.

Finally, we would like to stress that the self-consistent theory of localization cannot be applied "deep" inside the localization region. Its derivation is based on a kind of extrapolation of "metallic" expressions and it does not take into account local discreteness of the energy spectrum in the region of localized states as discussed in the previous section. This is reflected in the form of the one-particle Green's function used in the self-consistent theory [53, 32, 29, 7]. It does not describe the effects of local level repulsion, though it does not contradict it [67]. Thus the self-consistent theory of localization can be applied within the localized region only until local energy spacing given by Eq. (2.13) is much smaller than the other relevant energies of the problem under consideration. In fact, this always leads to a condition of sufficiently large localization length R_{loc}, i.e. the system must be in some sense close to the mobility edge.

2.3.2. Quasi-two-dimensional systems

Self-consistent theory of localization for quasi-two-dimensional systems was first analyzed by Prigodin and Firsov [68]. The electronic spectrum of a quasi-two-dimensional system can be modeled by nearly-free electrons within highly conducting planes and tight binding approximation for interplane electron transfer:

$$E(\boldsymbol{p}) - E_{\mathrm{F}} = v_{\mathrm{F}}(|\boldsymbol{p}_{\parallel}| - p_{\mathrm{F}}) - w\varphi(p_{\perp}) . \tag{2.86}$$

Here w is the interplane transfer integral and $\varphi(p_{\perp}) = \cos p_{\perp} a_{\perp}$, where $-\pi/a_{\perp} \le p_{\perp} \le \pi/a_{\perp}$. Then the equations of self-consistent theory of localization for the anisotropic generalized diffusion coefficient take the following form [68]:

$$D_j(\omega) = D_j^0 - \frac{1}{\pi N(E_{\mathrm{F}})} \int \frac{\mathrm{d}^3 \boldsymbol{q}}{(2\pi)^3} \frac{D_j(\omega)}{-\mathrm{i}\omega + D_{\parallel}(\omega) q_{\parallel}^2 + D_{\perp}(\omega)(1 - \varphi(q_{\perp}))} , \tag{2.87}$$

where $j = \parallel, \perp$, and $D_{\parallel}^0 = v_{\mathrm{F}}^2 \tau/2$, $D_{\perp}^0 = (w a_{\perp})^2 \tau$ are inplane and interplane bare Drude diffusion coefficients, τ is the mean free time due to elastic scattering (disorder). This approach is in complete correspondence with the analysis of Wolfle and Bhatt [69] who has shown that the effects of anisotropy can be completely absorbed into the anisotropic diffusion coefficient. It can be seen that the initial anisotropy of the diffusion coefficient does not change as disorder grows up to the Anderson transition and in fact we have only to find one unknown ratio

$$d(\omega) = D_j(\omega)/D_j^0 = \sigma_j(\omega)/\sigma_j^0 \tag{2.88}$$

which is determined by the algebraic equation following from Eq. (2.87):

$$d(\omega) = 1 - \frac{1}{2\pi E_{\mathrm{F}} \tau} \ln \frac{2}{[-\mathrm{i}\omega\tau/d(\omega)] + (w\tau)^2 + [(-\mathrm{i}\omega\tau/d(\omega))(-\mathrm{i}\omega/d(\omega) + 2w^2\tau^2)]^{1/2}} . \tag{2.89}$$

Due to a quasi-two-dimensional nature of the system there is no complete localization for any degree of disorder which is typical for a purely two-dimensional system. However, the tendency for a system to become localized at lower disorder than in the isotropic case is clearly seen. All states at the Fermi level become localized only for $w < w_{\mathrm{c}}$, where

$$w_{\mathrm{c}} = \sqrt{2}\tau^{-1} \exp(-\pi E_{\mathrm{F}} \tau) . \tag{2.90}$$

Thus the condition for localization is actually more stringent than given by the simplest Ioffe–Regel type estimate as in Eq. (2.11). For fixed w the mobility edge appears at

$$E_{\mathrm{F}} = E_{\mathrm{c}} = (1/\pi\tau) \ln(\sqrt{2}/w\tau) . \tag{2.91}$$

Thus in the case of strong anisotropy when $w\tau \ll 1$ localization can in principle take place even in the case of $E_F \gg \tau^{-1}$, i.e. at relatively weak disorder. These estimates are in qualitative accordance with Eq. (2.11), which is valid in the case of relatively strong disorder $E_F\tau \sim 1$.

In the metallic phase close to the Anderson transition,

$$\sigma_j = \sigma_j^0 (E_F - E_c)/E_c \,. \tag{2.92}$$

For $w \to 0$ we have $E_c \to \infty$ which reflects complete localization in two dimensions. We can also define in-plane Drude conductivity at $E_F = E_c$ as a kind of a "minimal metallic conductivity" in this case as a characteristic conductivity scale at the transition:

$$\sigma_\parallel^c = e^2 N(E_F) D_\parallel^0(E_F = E_c) = \frac{1}{\pi^2}\frac{e^2}{\hbar a_\perp}\ln\left(\frac{\sqrt{2}\hbar}{w\tau}\right) \approx \frac{1}{\pi^2}\frac{e^2}{\hbar a_\perp}\ln\left(\frac{E_F}{w}\right), \tag{2.93}$$

where we have used $N(E_F) = m/(\pi a_\perp \hbar^2)$, m is the in-plane effective mass, and the last equality is valid for $E_F\tau/\hbar \sim 1$, i.e. for a case of sufficiently strong disorder. For the time being we again use $\hbar$ explicitly. From these estimates it is clear that in-plane "minimal conductivity" is logarithmically enhanced in comparison with the usual estimates (cf. Eq. (2.5)). This logarithmic enhancement grows as the interplane overlap of the electronic wave functions diminishes. Accordingly, in case of small overlap ($w\tau/\hbar \ll 1$) this conductivity scale may be significantly larger than $(3–5) \times 10^2\,\Omega^{-1}\,\mathrm{cm}^{-1}$ which is characteristic for isotropic systems. Thus, in the quasi-two-dimensional case, Anderson transition may take place at relatively high values of inplane conductivity. For a typical estimate in a high-T_c system we can take something like $E_F/w > 10$ so that the value of $\sigma_\parallel^c$ may exceed $10^3\,\Omega^{-1}\,\mathrm{cm}^{-1}$. Obviously, these estimates are in qualitative accordance with elementary estimates based upon the Ioffe–Regel criterion of Eqs. (2.11) and (2.12). Similar conclusions can be deduced from the analysis presented in Ref. [70] where it was shown by a different method that in the case of the anisotropic Anderson model the growth of anisotropy leads to a significant drop of a critical disorder necessary to localize all the states in a conduction band.

Now let us quote some results for the frequency dependence of the generalized diffusion coefficient in the quasi-two-dimensional case which follow from the solution of Eq. (2.89) [68]. We shall limit ourselves only to the results valid close to the mobility edge in metallic phase:

$$d(\omega) \approx \begin{cases} \dfrac{E_F - E_c}{E_c}\,, & \omega \ll \omega_c\,, \\ (2\pi E_F w\tau^2)^{-2/3}(-\mathrm{i}\omega\tau)^{1/3}\,, & \omega_c \ll \omega \ll \omega^2\tau\,, \\ 1 - \dfrac{1}{2\pi E_F\tau}\ln(1/-\mathrm{i}\omega\tau)\,, & w^2\tau \ll \omega \ll \tau^{-1}\,, \end{cases} \tag{2.94}$$

where

$$\omega_c \approx [2\pi E_F w\tau^2]^2 (1/\tau)\,|(E_F - E_c)/E_c|^3 \,. \tag{2.95}$$

From these expressions we can see the crossover from the $\omega^{1/3}$-behavior typical for isotropic three-dimensional systems to a logarithmic dependence on frequency which is characteristic of two-dimensional systems.

2.3.3. Self-consistent theory of localization in a magnetic field

An early version of the self-consistent theory of localization as proposed by Vollhardt and Wolfle was essentially based upon the time-reversal invariance [53, 32]. This property is obviously absent in the presence of an external magnetic field. In this case in addition to Eq. (2.45) we have to consider the two-particle Green's function in the particle–particle (Cooper) channel:

$$\Psi_E^{\mathrm{RA}}(\boldsymbol{q},\omega) = -\frac{1}{2\pi\mathrm{i}}\sum_{\boldsymbol{p}_+\boldsymbol{p}'_-}\langle G^{\mathrm{R}}(\boldsymbol{p}_+,\boldsymbol{p}'_+,E+\omega)G^{\mathrm{A}}(-\boldsymbol{p}'_-,-\boldsymbol{p}_-,E\rangle \tag{2.96}$$

which for small ω and $\boldsymbol{q}$ again has a diffusion-pole form like that of Eq. (2.46), but with a *different* diffusion coefficient. Appropriate generalization of the self-consistent theory of localization was proposed by Yoshioka et al. [71]. This theory is based on the following system of coupled equations for relaxation kernels $M_j(\boldsymbol{q},\omega)$, corresponding to the diffusion coefficients in the particle–hole and particle–particle channels:

$$M_1 = 2\mathrm{i}\gamma\left\{1 - \frac{1}{\pi N(E)}\sum_{n=0}^{N_0}\frac{2}{\pi L_H}\int\limits_0^{\sqrt{q_0^2-4m\omega_H(n+1/2)}}\frac{\mathrm{d}q_z}{2\pi}\frac{1}{\omega-(D_0/\tau M_2)[q_z^2+4m\omega_H(n+1/2)]}\right\}, \tag{2.97}$$

$$M_2 = 2\mathrm{i}\gamma\left\{1 - \frac{1}{\pi N(E)}\sum_{|\boldsymbol{q}|<q_0}\frac{1}{\omega - D_0q^2/(\tau M_1)}\right\}. \tag{2.98}$$

Here $\omega_H = eH/mc$ is the cyclotron frequency, $L_H = (c/eH)^{1/2}$ is the magnetic length and $N_0 = q_0^2/4m\omega_H$. These equations form the basis of the self-consistent theory of localization in the absence of time-reversal invariance and were extensively studied in Refs. [71–75]. Alternative formulations of self-consistent theory in a magnetic field were given in Refs. [76–80]. All these approaches lead to qualitatively similar results. Here we shall concentrate on formulations given in Ref. [75].

Let us introduce the dimensionless parameter $\lambda = \gamma/\pi E$ as a measure of disorder and the generalized diffusion coefficients in diffusion and Cooper channels D_1 and D_2 defined as in Eq. (2.47) with M replaced by M_1 and M_2, respectively. We shall use dimensionless $d_j = D_j/D_0$ ($j = 1, 2$) in the following.

We are mainly interested in the diffusion coefficient in the Cooper channel, which as we shall see defines the upper critical field of a superconductor. Both this coefficient as well as the usual one are determined by the following equations which follow from Eqs. (2.97) and (2.98) after the use of Poisson summation over Landau levels in the first equation which allows one to separate the usual diffusion coefficient independent of magnetic field and the field-dependent part:

$$d_1 = (1 + (3\lambda - \delta_2 - \Delta_2)/d_2)^{-1}, \qquad d_2 = (1 + (3\lambda - \delta_1)/d_1)^{-1}, \tag{2.99}$$

where

$$\delta_j = (3/2\pi\lambda)^{3/2}(-\mathrm{i}\omega/E)^{1/2}d_j^{-1/2} \tag{2.100}$$

and

$$\Delta_2 = -3\lambda\sum_{p=1}^{\infty}(-1)^p\int_0^1\mathrm{d}x\,2\int_0^{\sqrt{1-x}}\mathrm{d}y\,\frac{\cos(2\pi p x_0^2/c^2)}{y^2 + x + 3/2\pi\lambda(-\mathrm{i}\omega/E)/(d_2x_0^2)}, \tag{2.101}$$

where $c = (2\omega_H/E)^{1/2}$. In the following we have to solve Eqs. (2.99) for the case of small δ_j and Δ_2. Limiting ourselves to terms linear in δ_1, δ_2 and Δ_2 we obtain

$$d_1/d_2 = 1 + \Delta_2/(1 + 3\lambda) . \tag{2.102}$$

Using Eq. (2.102) in Eqs. (2.99) we get an equation for the diffusion coefficient in the Cooper channel:

$$d_2 = 1 - 3\lambda + \delta_2 + [3\lambda/(1 + 3\lambda)]\Delta_2 . \tag{2.103}$$

Introducing Δ_1 which differs from Δ_2 by the replacement of d_2 by d_1 we can also write down the approximate equation for the usual diffusion coefficient:

$$d_1 = 1 - 3\lambda + \delta_1 + [1/(1 + 3\lambda)]\Delta_1 . \tag{2.104}$$

In the absence of the magnetic field ($\Delta_1 = \Delta_2 = 0$) Eqs. (2.103) and (2.104) are the same and lead to standard results of self-consistent theory quoted above. Eq. (2.103) can be written as

$$2mD_2 = \pm\left(\frac{\omega_c}{E}\right)^{1/3} + \left(-\frac{i\omega}{E}\right)^{1/2}(2mD_2)^{-1/2} + \frac{3\lambda}{1 + 3\lambda}\Delta_2 , \tag{2.105}$$

where + corresponds to metallic, and − to insulating phases, while the characteristic frequency

$$\omega_c = (|1 - 3\lambda|/(\tfrac{3}{2}\pi\lambda))^3 E \tag{2.106}$$

can be considered as a measure of disorder and separate regions with different frequency dependence of the diffusion coefficient.

Neglecting in Eq. (2.101) terms oscillating with the magnetic field (these oscillations are connected with the sharp cut-off in the momentum space used above and disappear for smooth cut-off) we get

$$\Delta_2 = -(2\omega_H/E)^{1/2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p^{1/2}} f(2\pi p\kappa) , \tag{2.107}$$

where

$$f(y) = \sqrt{2/\pi}\int_0^{\infty} \frac{\cos(t)\,dt}{\sqrt{t + y}} ; \qquad \kappa = \frac{-i\omega/E}{2\omega_H/E}\frac{1}{2mD_2} . \tag{2.108}$$

This gives

$$\Delta_2 = \begin{cases} W(2\omega_H/E)^{1/2} , & |\kappa| \ll 1 , \\ \dfrac{1}{48}\left((-i\omega/E)\dfrac{1}{2mD_2}\right)^{-3/2}(2\omega_H/E)^2 , & |\kappa| \gg 1 , \end{cases} \tag{2.109}$$

where $W = -\sum_{p=1}^{\infty}(-1)^p/p^{1/2} \approx 0.603$.

Solutions of Eq. (2.105) for different limiting cases can be found in Ref. [75]. Comparison of Eq. (2.104) and Eq. (2.103) shows that the usual diffusion coefficient D_1 is given by the same expressions as D_2 with the replacement of the coefficient $3\lambda/(1 + 3\lambda)$ before the field-dependent correction by

$1/(1+3\lambda)$. Here we only quote the results for D_2 in case of $\omega_c/E \ll (\omega_H/E)^{3/2}$, valid close to the transition in the absence of magnetic field:

$$D_2 = \frac{1}{2m}\left\{\pm(\omega_c/E)^{1/3} + \left[\frac{3\lambda}{1+3\lambda}\right] W(2\omega_H/E)^{1/2}\right\} \approx \frac{1}{4m} W(2\omega_H/E)^{1/2}\,, \quad \omega \ll \omega_c^*\,, \tag{2.110}$$

$$D_2 = \frac{1}{2m}\left\{\left(\frac{-\mathrm{i}\omega}{E}\right)^{1/3} + \frac{2}{3}\left[\frac{3\lambda}{1+3\lambda}\right]\frac{1}{48}\frac{(2\omega_H/E)^2}{(-\mathrm{i}\omega/E)}\right\}, \quad \omega \gg \omega_c^*\,, \tag{2.111}$$

where $\omega_c^* = (W/2)^3(2\omega_H/E)^{3/2}E$.

Note that for high frequencies larger than ω_c^* the correction term becomes quadratic in field which differs from the usual square root behavior at low frequencies.

It is easy to see that in the absence of the external magnetic field these equations reduce to the usual self-consistency equation as derived by Vollhardt and Wolfle with a single relaxation kernel.

Let us finally quote some results for the purely two-dimensional case [81]. Self-consistent equations for the diffusion coefficients take now the following form:

$$\begin{aligned}\frac{D_0}{D_2} &= 1 + \frac{1}{\pi N(E)}\sum_{|\mathbf{q}|<q_0}\frac{1}{\omega + D_1 q^2}\,,\\ \frac{D_0}{D_1} &= 1 + \frac{1}{\pi N(E)}\sum_{|\mathbf{k}|<q_0}\frac{1}{\omega + D_2 k^2}\,,\end{aligned} \tag{2.112}$$

where $k^2 = 4m\omega_H(n+\frac{1}{2})$, and we assume that ω here is the imaginary (Matsubara) frequency, which simplifies the analysis. Actually, only the dependence on the Matsubara's frequencies are important for further applications to superconductivity.

Introduce again the dimensionless diffusion coefficients $d_1 = D_1/D_0$, $d_2 = D_2/D_0$, so that Eqs. (2.112) are rewritten as

$$\begin{aligned}\frac{1}{d_2} &= 1 + \frac{\lambda}{d_1}\ln\left(1 + d_1\frac{1}{2\omega\tau}\right),\\ \frac{1}{d_1} &= 1 + \frac{\lambda}{d_2}\sum_{n=0}^{N_0}\frac{1}{n+\frac{1}{2}+(\omega/4m\omega_H D_0)(1/d_2)}\,,\end{aligned} \tag{2.113}$$

where $N_0 = 1/8m\omega_H D_0\tau$ is the number of Landau levels below the cut-off. We assume that the magnetic field is low enough, so that $N_0 \gg 1$, i.e.

$$H \ll \Phi_0/D_0\tau\,. \tag{2.114}$$

With accuracy sufficient for further use we can write down the following solution for the diffusion coefficient in the Cooper channel:

For weak magnetic field $\omega_H \ll \lambda \mathrm{e}^{-1/\lambda}/\tau$

$$d_2 = \begin{cases} 1 & \text{for } \omega \gg \mathrm{e}^{-1/\lambda}/2\tau\,,\\ 2\omega\tau\mathrm{e}^{1/\lambda} & \text{for } \omega \ll \mathrm{e}^{-1/\lambda}/2\tau\,,\end{cases} \tag{2.115}$$

and we can neglect the magnetic field influence upon diffusion.

For larger fields $\omega_H \gg \lambda e^{-1/\lambda}/\tau$

$$d_2 = \begin{cases} 1 & \text{for } \omega \gg e^{-1/\lambda}/2\tau\,, \\ 1/\lambda \ln(1/2\omega\tau) & \text{for } e^{-1/\lambda^2 \ln Q}/2\tau \ll \omega \ll e^{-1/\lambda}/2\tau\,, \\ 2\omega\tau\lambda \ln Q\, e^{1/\lambda^2 \ln Q} & \text{for } \omega \ll e^{-1/\lambda^2 \ln Q}/2\tau\,, \end{cases} \tag{2.116}$$

where $Q = \pi\gamma\lambda/\tau\omega_H$, $\gamma \approx 1.781$.

Here we neglect the magnetic field corrections small in comparison to the d_2 value in the absence of the magnetic field given by Eq. (2.115).

2.4. Phase transition analogy and scaling for correlators

Scaling description of a system close to the Anderson transition can be developed also on the basis of some analogies with usual phase transitions [4, 7, 6]. Most successful in this respect is an approach initially proposed by Wegner [82–84].

Let us consider Eqs. (2.67) and (2.83) which define basic electronic correlators (spectral densities) in a disordered system. For the metallic region we can write:

$$K_F(\boldsymbol{q}\omega) \equiv N(E)\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_{\boldsymbol{q}}^F \sim \mathrm{Re}\,\frac{N(E)}{-\mathrm{i}\omega + D_E q^2}\,, \tag{2.117}$$

$$K_H(\boldsymbol{q}\omega) \equiv N(E)\langle\!\langle \rho_E \rho_{E+\omega} \rangle\!\rangle_{\boldsymbol{q}}^H \sim \mathrm{Re}\,\frac{D_E^{d/2}}{(-\mathrm{i}\omega + D_E q^2)^{2-d/2}}\,. \tag{2.118}$$

Wegner has noted [83, 63] that these expressions are in some sense similar to analogous expressions for the transverse and longitudinal susceptibilities of a ferromagnet [47]:

$$\chi_\perp(\boldsymbol{q}) = M/(H + \rho_s q^2)\,, \tag{2.119}$$

$$\chi_\parallel(\boldsymbol{q}) \sim \frac{1}{(H + \rho_s q^2)^{2-d/2}}\,, \tag{2.120}$$

where M is the magnetization, H the external magnetic field and ρ_s is the spin-stiffness coefficient. Comparing Eqs. (2.117) with Eq. (2.119) and Eq. (2.118) with Eq. (2.120) we can write down a correspondence between electron diffusion in a random system and a ferromagnet as given in Table 1.

Now we can use the main ideas of the scaling approach in the theory of critical phenomena [45–47, 85] and formulate similar expressions for the electronic system close to the Anderson transition. As was noted above, scaling theory is based upon an assumption that a singular behavior of the physical parameters of a system close to a phase transition appears due to large scale (long wave-length) fluctuations of the order-parameter (e.g. magnetization) close to the critical temperature T_c. Scaling hypothesis claims that singular dependence on $T - T_c$ reflect the divergence of the correlation length of these fluctuations ξ and this length is the only relevant length-scale in the critical region. Scaling approach is based upon an idea of scale transformations

Table 1
Anderson transition and ferromagnet close to Curie point T_c

Localization	Ferromagnet
$E - E_c$	$T - T_c$
K_F	$\chi_{\perp}$
K_H	$\chi_{|}$
$-i\omega$	H
$N(E)$	M
D_E	ρ_s
ξ_{loc}	ξ

and dimensional analysis. Under the scale transformation, the spatial interval Δx changes to $\Delta x'$, according to

$$\Delta x \to \Delta x' = s^{-1}\Delta x \,. \tag{2.121}$$

Accordingly, for the wave vector:

$$\boldsymbol{q} \to \boldsymbol{q}' = s\boldsymbol{q} \,. \tag{2.122}$$

Scaling dimension [85] of a physical quantity A is equal to λ if under scale transformations defined by Eqs. (2.121) and (2.122) we get

$$A \to A' = As^{\lambda} \,. \tag{2.123}$$

Scaling dimensions for the main characteristics of a ferromagnet are given in terms of standard critical exponents [85] in Table 2.

Correlation length of the theory of critical phenomena behaves like

$$\xi \sim |T - T_c|^{-\nu} \,. \tag{2.124}$$

The knowledge of scaling dimension of a given physical quantity allows to determine its dependence on ξ, i.e. on $T - T_c$. For example, magnetization M behaves according to Table 2 as

$$M \sim \xi^{-1/2(d-2+\eta)} \sim |T - T_c|^{\beta} \,, \tag{2.125}$$

where the critical exponent of magnetization equals

$$\beta = \tfrac{1}{2}\nu(d - 2 + \eta) \,. \tag{2.126}$$

Magnetic susceptibility is given by

$$\chi(\boldsymbol{q}, T - T_c) = \xi^{2-\eta} g(q\xi) \,, \tag{2.127}$$

where $g(x)$ is some universal function, such that $g(0) \sim$ const., $g(x \to \infty) \sim x^{-(2-\eta)}$. From Eq. (2.127) we get the standard results:

$$\chi(0, T - T_c) \approx \xi^{2-\eta} g(0) \sim |T - T_c|^{-\gamma} \,, \tag{2.128}$$

Table 2
Scaling dimensions in the theory of critical phenomena

ξ	q	M	H
-1	$+1$	$1/2(d-2+\eta)$	$1/2(d+2-\eta)$

where $\gamma = (2-\eta)\nu$ is the susceptibility exponent. Analogously,

$$\chi(\boldsymbol{q}, T = T_c) \sim q^{-2+\eta}\,. \tag{2.129}$$

Here η is sometimes called Fisher's exponent.

It is easy to see that Eq. (2.127) is equivalent to the scaling relation (H-dependence is taken from Table 2)

$$\chi(s\boldsymbol{q}, s^{-1}\xi, s^{1/2(d+2-\eta)}H) = s^{-(2-\eta)}\chi(\boldsymbol{q}, \xi, H)\,. \tag{2.130}$$

It is convenient to make the transformation $|T-T_c| \to b|T-T_c|$ so that $\xi \to b^{-\nu}\xi$ which is equivalent to the choice of $s = b^{\nu}$. Then Eq. (2.130) transforms to

$$\chi(b^{\nu}\boldsymbol{q}, b^{-\nu}\xi, b^{\nu(d+2-\eta)/2}H) = b^{-\gamma}\chi(\boldsymbol{q}, \xi, H)\,. \tag{2.131}$$

Finally, note that close to Curie point the spin-stiffness coefficient ρ_s satisfies the so-called Josephson relation [47]:

$$\rho_s \sim |T-T_c|^{(d-2)\nu} \tag{2.132}$$

and tends to zero as $T \to T_c$ from within the condensed phase.

Consider now the analogy formulated in Table 1. Density of states $N(E)$ is nonsingular at the mobility edge [27, 7]. Then considering $N(E)$ as an analog of magnetization M we have to assume $\beta = 0$, i.e. at the localization transition

$$\eta = 2 - d \tag{2.133}$$

and the "order-parameter" $N(E)$ is nonsingular at the transition $E = E_c$. Accordingly, we have $\gamma = d\nu$. Josephson relation Eq. (2.132) now takes the form

$$D_E \sim |E-E_c|^{(d-2)\nu}\,, \tag{2.134}$$

i.e. it is in fact is equivalent to Wegner's relation for conductivity given by Eq. (2.31). Correlation length exponent ν remains unknown.

For electronic correlators of Eqs. (2.117) and (2.118) we obtain from Eq. (2.131) scaling relations [83, 84]

$$K_{F,H}(b^{\nu}\boldsymbol{q}, b^{d\nu}\omega, b(E-E_c)) = b^{-d\nu}K_{F,H}(\boldsymbol{q}, \omega, E-E_c)\,. \tag{2.135}$$

Taking $\nu = 1/(d-2)$ from Eq. (2.29) for $d = 3$ and $E = E_c$ (i.e. at the mobility edge itself) we get from Eq. (2.135):

$$K_{F,H}(b\boldsymbol{q}, b^3\omega) = b^{-3}K_{F,H}(\boldsymbol{q}\omega) \tag{2.136}$$

which is equivalent to

$$K_{F,H}(\boldsymbol{q}\omega) = L_\omega^3 F_{F,H}(qL_\omega) \tag{2.137}$$

where $F_{F,H}(x)$ is some universal function and we introduced the characteristic length

$$L_\omega = [\omega N(E)]^{-1/3}\ . \tag{2.138}$$

Note that the same scaling dependence follows, e.g. for $K_H(\boldsymbol{q}\omega)$ from Eq. (2.83) or Eq. (2.84) after a simple replacement of D_0 by a diffusion coefficient given by

$$D_{E=E_c}(\boldsymbol{q}\omega) = L_\omega^{-1} f(qL_\omega)\ , \tag{2.139}$$

where $f(x \to 0) \to 1$ and $f(x \to \infty) \to x$. In particular, in the limit of $qL_\omega \to 0$ we get $F(x) = (1 + x^4)^{-1/4}$ and the replacement $D_0 \to D_0(\omega/\gamma)^{1/3}$ mentioned in connection with Eq. (2.84) is valid. On the other hand, from Eq. (2.133) it follows that at $\omega = 0$ we get from Eq. (2.129)

$$K(\boldsymbol{q}, \omega = 0, E = E_c) \sim q^{-d} \tag{2.140}$$

which is equivalent to Eq. (2.67) if we take $D_{E=E_c}(\omega = 0, \boldsymbol{q}) = \alpha q^{d-2}$ (cf. Eq. (2.85)).

Microscopic justification for this scaling hypothesis can be provided with one or other variant of the field-theory approach based upon nonlinear σ-model [82–84]. There exist several alternative schemes of "mapping" of the problem of an electron in a random field onto field-theoretic formalism of nonlinear σ-models [86,88–92]. The main physical justification of this approach is to represent an effective Hamiltonian of an electronic system in a form similar to the analogous Hamiltonian of the Heisenberg ferromagnet below Curie point:

$$\mathscr{H} = \tfrac{1}{2}(\partial \boldsymbol{M}/\partial x_\alpha)^2 - \boldsymbol{H}\boldsymbol{M}\ ; \quad \boldsymbol{M}^2 = \text{const}\ . \tag{2.141}$$

As a result, an effective Hamiltonian for an electron in a random field in terms of interacting modes responsible for the critical behavior close to mobility edge appears. Following Ref. [88] we can introduce an "order-parameter" as a $2n \times 2n$ matrix $\hat{Q}$ (n-integer). Every matrix element of $\hat{Q}$ can be represented as

$$Q_{ij} = \begin{pmatrix} D_{ij} & \Delta_{ij} \\ -\Delta_{ij}^* & D_{ij}^* \end{pmatrix}, \tag{2.142}$$

where $D_{ij} = D_{ji}^*$ and $\Delta_{ij} = -\Delta_{ji}^*$, i.e. they are elements of Hermitian and antisymmetric matrices, respectively. Analogously, $\boldsymbol{M}^2$ = const. in a ferromagnet. $\hat{Q}$-matrix must satisfy the condition:

$$\hat{Q}^2 = 1\ ; \qquad \operatorname{Tr}\hat{Q} = 0\ . \tag{2.143}$$

The effective Hamiltonian for diffusion modes takes the following form [82,83]:

$$\mathscr{H} = D_0 \operatorname{Tr}(-\mathrm{i}\nabla\hat{Q})^2 - \mathrm{i}\omega \operatorname{Tr}\hat{\Lambda}\hat{Q}\ . \tag{2.144}$$

Here $\hat{\Lambda}$ is the diagonal matrix with the first n elements equal to 1 and the remaining n are -1. Correlation function of D-elements corresponds to diffusion, while that of Δ-elements to Cooperon. Parameter n should be put equal to zero at the end of calculations in the spirit of the famous "replica trick" in the theory of disordered systems [85,4].

This formalism is useful also for the analysis of different kinds of external perturbations, such as external magnetic field, magnetic impurities, spin–orbital scattering etc. [88]. Standard methods of renormalization group using perturbation theory over $(p_F l)^{-1} \ll 1$ reproduces all the main results of elementary scaling theory of localization, including the qualitative form of the β-function as in Fig. 3. However, the formalism of the σ-model approach is quite complicated and practically does not allow to get explicit expressions for the physical characteristics of the system, especially in the localized phase.

Many problems of fundamental nature still remain unresolved. Most important are questions concerning the role of nonperturbative contributions close to the mobility edge [4, 7, 92–94]. Note, especially, the strong criticism about one-parameter scaling in Refs. [93–94]. Among the several results obtained within the σ-model approach we wish to mention an important paper by Lerner [95], where a distribution function for the local density of states in a system close to the Anderson transition was determined and shown to be essentially non-Gaussian.

For our future analysis it is important to stress that in most cases the results of the σ-model approach practically coincide with the predictions of the self-consistent theory of localization which also neglects all nonperturbative effects, except those determined by some infinite resummation of diagrams. It must be stressed that self-consistent theory is based upon some uncontrollable ad hoc assumptions and in this respect it is not as well justified as the σ-model approach. However, this simple theory as we have seen above allows practical calculation of any interesting characteristic of an electronic system close to the mobility edge including the localized phase.

2.5. *Interaction effects and Anderson transition*

The main unresolved problem of the theory of metal–insulator transition in disordered systems is the role of electron–electron interactions. The importance of interactions for this problem is known for a long time [2]. In recent years the decisive importance of interactions was revealed in the theory of "dirty metals" [38–40], as well as in the concept of the Coulomb gap at the Fermi level of strongly localized electrons [41–44]. We have already briefly discussed the Coulomb gap. It appears for strongly localized states. In case of "dirty metals", the diffusive nature of electronic transport leads to special interference effects between the Coulomb interaction and disorder scattering [38, 40]. Most important is an appearance of some kind of a precursor to the Coulomb gap already in the metallic state. It is connected with simple exchange correction to the electron self-energy (cf. Fig. 7) which leads to the following cusp-like correction to one-particle density of states in case of the screened Coulomb interaction in three-dimensional systems [38]:

$$\delta N(E) = \frac{|E - E_F|^{1/2}}{2\sqrt{2}\pi^2 D_0^{3/2}}, \tag{2.145}$$

where D_0 is the usual Drude diffusion coefficient. In the two-dimensional case this correction is logarithmic [96, 40]. General belief is that this cusp somehow transforms into the Coulomb gap as the system moves from metal to insulator. However, up to now there is no complete solution for this problem.

An early attempt to describe electron–electron interactions in Anderson insulators in a Fermi-liquid-like scheme is discussed in Ref. [97]. Simple generalization of the theory of "dirty metals"

[38–40] along the lines of self-consistent theory of localization was proposed in Refs. [98, 60, 7]. However, the most general approach to this problem was introduced by McMillan [99] who proposed to describe the metal–insulator transition in a disordered system by a scaling scheme similar in spirit to the elementary scaling theory of localization of noninteracting electrons discussed above. He formulated a simple system of coupled differential equations of the renormalization group for two effective "charges": dimensionless conductance g and single-particle density of states $N(E)$. Later it was realized that this simple scheme cannot be correct because it assumed for conductivity a relation like Eq. (2.59) with the density of states while the correct Einstein relation for the interacting system contains electron compressibility $dn/d\zeta$ (ζ is chemical potential) [100–102], which is not renormalized close to the metal–insulator transition as opposed to density of states. The most comprehensive approach to a scaling description of the metal–insulator transition in disordered systems was formulated by Finkelstein [102–105]. Unfortunately, more or less explicit solutions were only obtained neglecting the scattering and interaction processes in the Cooper channel which are mainly responsible, as we have seen above, for localization itself. Some attempts in this direction were undertaken only in Ref. [104]. This approach is still under very active discussion [107–115] and demonstrates the fundamental importance of interactions. However, the problem is still unresolved and most of these works consider only the metallic side of transition with no serious attempts to analyze the insulating state.

Below we consider only some qualitative results of this approach, following mainly Refs. [108, 109]. Fermi liquid theory survives the introduction of disorder [119], although with some important corrections [38, 40], and is actually valid up to the metal–insulator transition [102, 103, 108, 109].

In the absence of translation invariance there is no momentum conservation and we have to use some unknown exact eigenstate $\phi_\nu(\boldsymbol{r})$ representation for electrons in a random field to characterize quasi-particles with energies ε_ν (cf. Ref. [120]). The free energy as a functional of the quasi-particle distribution function $n_s(\varepsilon_\nu, \boldsymbol{r})$ (s-spin variable) is written as is usual in the Fermi liquid theory:

$$F\{n_s(\varepsilon_\nu,\boldsymbol{r})\} = \sum_{s,\nu}\int d\boldsymbol{r}\, n_s(\varepsilon_\nu \boldsymbol{r})(\varepsilon_\nu - \zeta) + \frac{1}{2}\sum_{ss'}\int d^d\boldsymbol{r}\,\delta N_s(\boldsymbol{r})\delta N_{s'}(\boldsymbol{r}) f_{ss'} , \tag{2.146}$$

where $N_s = \sum_\nu n_s(\varepsilon_\nu \boldsymbol{r})$ is the total density per spin and $f_{ss'} = f^s + ss'f^a$ is the quasi-particle interaction function. The angular dependence of the f-function in the dirty case can be neglected, because $n_s(\varepsilon_\nu \boldsymbol{r})$ is assumed to describe electrons on distances larger than the mean free path where only s-wave scattering is important and the Fermi-liquid interaction becomes point-like. In an external spin dependent field V_s the quasi-particle distribution function obeys a kinetic equation:

$$\frac{\partial}{\partial t} n_s - D\nabla^2 n_s + (\partial n_s/\partial\varepsilon)(-D\nabla^2)\left[V_s + \sum_{s'} f_{ss'} N_{s'}\right] = 0 , \tag{2.147}$$

where D is the quasi-particle diffusion coefficient. Eq. (2.147) is obtained from the usual Fermi-liquid kinetic equation [120] by replacing $v_F\partial/\partial\boldsymbol{r}$ by $-D\nabla^2$ which reflects a crossover from ballistic to diffusive transport in disordered systems. Solving Eq. (2.147) for density–density and spin–spin response functions one gets [102, 103, 107]

$$\chi_\rho(\boldsymbol{q}\omega) = (dn/d\zeta) D_\rho q^2/(D_\rho q^2 - i\omega) , \tag{2.148}$$

$$\chi_s(\boldsymbol{q}\omega) = \chi D_s q^2/(D_s q^2 - i\omega) , \tag{2.149}$$

where $\mathrm{d}n/\mathrm{d}\zeta = N(E_F)/(1 + F_0^s)$, $\chi = N(E_F)\mu_B^2/(1 + F_0^a)$ (μ_B is Bohr's magneton) and

$$D_\rho = D(1 + F_0^s)\,, \tag{2.150}$$

$$D_s = D(1 + F_0^a)\,. \tag{2.151}$$

Landau parameters $F_0^{s,a}$ are defined by

$$N(E_F)f^s = F_0^s\,, \quad N(E_F)f^a = F_0^a\,, \tag{2.152}$$

where $N(E_F)$ is the quasi-particle density of states at the Fermi level (for both spin directions). If we neglect Fermi-liquid renormalization effects Eq. (2.148) reduces to Eq. (2.49). Conductivity is given now by $\sigma = e^2D(\mathrm{d}n/\mathrm{d}\zeta)$.

As the system moves towards the metal–insulator transition Hubbard-like interaction of electrons close to a given impurity site becomes more and more important. It is known for a long time [2, 7] that this interaction leads to the appearance of a band of single-occupied states just below the Fermi level of a system on the dielectric side of the Anderson transition. These states actually simulate paramagnetic centers and lead to Curie-like contribution (diverging as temperature $T_c \to 0$) [2, 7]. Thus on the metallic side of transition static magnetic susceptibility χ is expected to diverge since it is infinite (at $T = 0$) on the insulating side. At the same time, $\mathrm{d}n/\mathrm{d}\zeta$ remains finite. Therefore, $D_s/D_\rho = (\mathrm{d}n/\mathrm{d}\zeta)/\chi$ goes to zero, i.e. *spin diffusion is much slower than charge diffusion* close to the metal–insulator transition. This fact was first noted in Ref. [104] where it was assumed that it leads to a possibility of local magnetic effects appearing in the metallic phase before a transition. It is interesting to note that the slowing down of spin diffusion due to interactions was actually discovered long before [106] it appeared in the context of the interaction picture of the metal–insulator transition. This idea was further elaborated in Refs. [112–114], where extensive discussion of this magnetic transition was given. There is an interesting problem why these localized moments are not quenched by the Kondo effect. This can apparently be explained by the local fluctuations of Kondo temperature due to fluctuations of local density of states induced by disorder [116]. The resulting distribution of Kondo temperatures is shown to be singular enough to induce diverging magnetic susceptibility as $T \to 0$.

The idea of paramagnetic moments appearing already in the metallic phase apparently can much simplify the analysis of the metal–insulator transition and allow its description by equations of elementary scaling theory of localization [117, 118, 40]. In the general case, electron interactions in the diffusion channel can be classified by the total spin of an electron and hole j [40]. It can be shown that all interaction corrections with $j = 0$ do not depend on the electron–electron coupling constant (charge) and are universal [40]. If paramagnetic scattering is operating in the system it dumps scattering processes in the Cooper (localization) channel [121] as well as interaction processes in the diffusion channel with $j = 1$ [40]. In this case, only interaction processes with $j = 0$ determine corrections to the classical (Drude) conductivity. Due to the universal nature of these corrections (independence of electronic charge) their structure actually coincides with that of localization corrections (Cooperon) [117, 118]. This means that renormalization group has only one effective "charge" – dimensionless conductance g. In this case, the differential equation for the conductance of a finite system is again given by Eq. (2.18) with the same asymptotic forms of $\beta_d(g)$. This approach is valid for systems with linear size $L < L_T = \sqrt{\hbar D/T}$. This length L_T replaces in the theory of interacting electrons the characteristic length of phase coherence L_φ of the noninteracting theory. The appearance of this new length is due to the fact that the characteristic time of the interaction processes [40] is $\sim\hbar/T$. We must stress that these arguments are probably oversimplified as Refs. [102–104,107] have demonstrated the relevance of interaction in the sense of appearance of additional coupling constants ("charges"). Also it is in no way clear that local moments appearing within this approach are acting just as the usual paramagnetic scatterers. However, the simple scheme following from Refs. [117,118] seems to be too attractive on physical grounds just to be neglected.

As in the noninteracting case for $d = 3$, Eq. (2.18) again possess an unstable fixed point responsible for the existence of the mobility edge and absence of minimal metallic conductivity at the metal–insulator transition. However, in this case there are no special reasons to believe that the critical exponent ν of the localization correlation length $\xi_{\rm loc}$ will coincide with its value from the noninteracting theory. At finite temperatures, as in the usual scaling picture, conductivity for $d = 3$ is given by [117,118,40]

$$\sigma \approx (e^2/\hbar\xi_{\rm loc})f(\xi_{\rm loc}/L_T)\,. \tag{2.153}$$

As the system approaches the insulating phase, $\xi_{\rm loc} \to \infty$. For $\xi_{\rm loc} \ll L_T$ we have $f(\xi_{\rm loc}/L_T) = A + B(\xi_{\rm loc}/L_T)$, where A and B are some numerical constants. Thus, in this region, conductivity corrections are proportional to $\sqrt{T}$ [38]. In case of $\xi_{\rm loc} \gg L_T$, i.e. very close to transition:

$$\sigma \approx C(e^2/\hbar L_T) = C(e^2/\hbar)\sqrt{T/D\hbar}\,, \tag{2.154}$$

where again $C \sim 1$. Using the Einstein relation [100] $\sigma = e^2 D(\mathrm{d}n/\mathrm{d}\zeta)$ we immediately obtain

$$D = (C^{2/3}/\hbar)T^{1/3}(\mathrm{d}n/\mathrm{d}\zeta)^{-2/3} \tag{2.155}$$

and

$$\sigma = C^{2/3}(e^2/\hbar)(T\,\mathrm{d}n/\mathrm{d}\zeta)^{1/3} \tag{2.156}$$

which is valid for $L_T < \xi_{\rm loc}$, where $L_T = [C/(T\,\mathrm{d}n/\mathrm{d}\zeta)]^{1/3}$.

In the case of a system in an alternating electric field with frequency $\omega \gg T/\hbar$ the relevant length becomes $L_\omega = [D/\omega]^{1/2}$ as in Eq. (2.37). Accordingly, for $L_\omega \ll \xi_{\rm loc}$ instead of Eq. (2.156) we get

$$\sigma(\omega) \approx (e^2/\hbar)(\omega\,\mathrm{d}n/\mathrm{d}\zeta)^{1/3} \tag{2.157}$$

which is analogous to Eqs. (2.41) and (2.81). However, we must note that this result cannot be considered very reliable since the dynamical critical exponent in the general case is an independent one [103,104].

The metal–insulator transition can be viewed as a gradual breakdown of the Fermi liquid state [109]. As we approach the transition, different Fermi-liquid parameters, such as D, $N(E_{\rm F})$, χ, etc. change continuously and at a critical point some of these may either diverge or go to zero. This behavior is related to the divergence of the correlation length $\xi_{\rm loc}$ characterized by a critical exponent ν. On the insulating side of the transition this length can also be interpreted as the scale inside which a Fermi liquid description of the system still holds.

At present, we are in need of some kind of new approach to the theory of interacting electrons in disordered systems which probably may be formulated along the lines of the self-consistent theory of localization. The attempt is to provide an effective formalism to calculate the basic physical properties of the system in an interpolating scheme from metallic to insulating state. Below, we briefly describe an attempt to construct such a self-consistent approach [122].

The basic idea in equal footing (additive) is the treatment of both localization and interaction corrections to the current relaxation kernel defining the generalized diffusion coefficient in Eq. (2.47). As a zeroth approximation we take the Drude metal and consider the simplest localization and interaction corrections, so that the relaxation kernel takes the following form:

$$M(\omega) = M_0 + \delta M(\omega)\,, \tag{2.158}$$

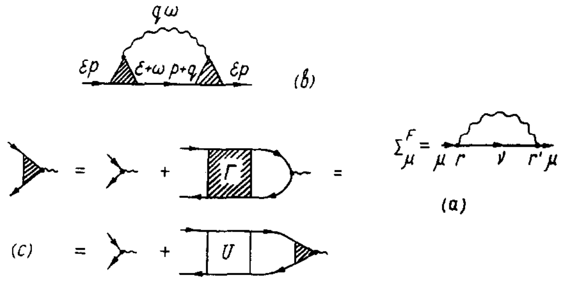
where $\delta M(\omega) = \delta M_1(\omega) + \delta M_c(\omega) = -(M_0/D_0)(\delta D_1(\omega) + \delta D_c(\omega))$. Here the localization correction to the diffusion coefficient $D_1(\omega)$ is defined by the usual sum of "maximally crossed" diagrams which yields

$$\frac{\delta D(\omega)}{D_0} = -\frac{1}{\pi N_0(E_F)} \sum_{|q|<k_0} \frac{1}{-i\omega + D_0 q^2}\,, \tag{2.159}$$

while the Coulomb correction $D_c(\omega)$ is given by

$$\frac{\delta D_c(\omega)}{D_0} = \frac{\delta\sigma(\omega)}{2e^2 N_0(E_F) D_0}$$
$$= \frac{8i}{\pi d}\mu D_0 \frac{1}{\pi N_0(E_F)} \int_\omega^\infty d\Omega \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(-i(\Omega+\omega) + D(\Omega+\omega)q^2)(-i\Omega + D(\Omega)q^2)^2}\,, \tag{2.160}$$

where $\mu = N_0(E_F)v_0$ is the dimensionless point-like interaction with $N_0(E_F)$ now denoting the single-spin density of states at the Fermi level for the noninteracting case. The lowest order interaction corrections are shown in Fig. 7. Conductivity correction $\delta\sigma$ due to interactions was



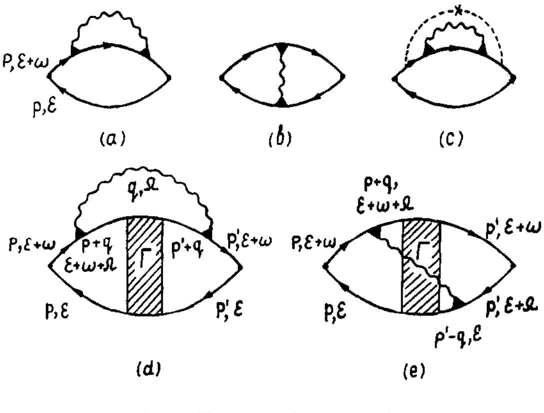


Fig. 7. Lowest order interaction corrections: (a) Simplest Fock correction for self-energy in exact eigenstate representation. (b) Equivalent diagram in momentum representation. (c) "Triangular" vertex defining diffusion renormalization. U – irreducible impurity scattering vertex, Γ – full impurity scattering vertex, and wavy line denotes interelectron interaction.

Fig. 8. Lowest order interaction corrections to conductivity.

defined by the lowest-order diagrams shown in Fig. 8 which were for the first time analyzed in Ref. [123], neglecting localization corrections. It was shown in Ref. [123] that the total contribution of diagrams (a)–(c) is actually zero and conductivity correction reduces to that determined by diagrams (d) and (e). Here we neglect also the so-called Hartree corrections to conductivity [40, 123], which is valid in the limit of $2k_F/\kappa_D \gg 1$, where κ_D is the inverse screening length. This inequality, strictly speaking, is valid for systems with low electronic density, which are most interesting for experimental studies of disorder induced metal–insulator transitions. Also, if we remember the divergence of the screening length at the metal–insulator transition, we can guess that this approximation becomes better as we approach the transition. The point-like interaction model used above has to be understood only in this sense.

Self-consistency procedure is reduced to the replacement of D_0 by the generalized diffusion coefficient in the denominators of all the diffusion forms. As a result, we obtain the following *integral* equation for the generalized diffusion equation:

$$\frac{D_0}{D(\omega)} = 1 + \frac{1}{\pi N_0(E_F)} \int \frac{d^dq}{(2\pi)^d} \frac{1}{-i\omega + D(\omega)q^2}$$
$$- \frac{8i}{\pi d} \mu D_0 \frac{1}{\pi N_0(E_F)} \int_\omega^\infty d\Omega \int \frac{d^dq}{(2\pi)^d} \frac{q^2}{(-i(\Omega+\omega) + D(\Omega+\omega)q^2)(-i\Omega + D(\Omega)q^2)^2} \,. \tag{2.161}$$

This equation forms the basis of the proposed self-consistent approach. In the absence of interactions ($\mu = 0$) it obviously reduces to the usual self-consistent theory of localization. Let us transform it to dimensionless imaginary Matsubara frequencies which is the only case we need for further applications to the superconducting state: $-i\omega/D_0k_0^2 \to \omega$, $-i\Omega/D_0k_0^2 \to \Omega$, and also introduce the dimensionless diffusion coefficient $d(\omega) = D(\omega)/D_0$. In these notations integral equation (2.161) takes the following form:

$$\frac{1}{d(\omega)} = 1 + \frac{1}{d(\omega)} d\lambda x_0^{d-2} \int_0^1 \frac{dy\, y^{d-1}}{y^2 + (\omega/d(\omega))}$$
$$+ \frac{8}{\pi} \mu\lambda x_0^{d-2} \int_\omega^\infty \frac{d\Omega}{d(\omega+\Omega)d^2(\Omega)} \int_0^1 \frac{dy\, y^{d+1}}{(y^2 + (\omega+\Omega)/d(\omega+\Omega))(y^2 + \Omega/d(\Omega))^2} \,, \tag{2.162}$$

where $\lambda = \gamma/\pi E_F = 1/2\pi E_F\tau$ is the usual disorder parameter. In the following we shall limit ourselves only to the case of spatial dimension $d = 3$. The diffusion coefficient of the usual self-consistent theory of localization (2.78) in these notations reduces to

$$d(\omega) = \begin{cases} \alpha = 1 - 3\lambda x_0 \approx \dfrac{E_F - E_c}{E_c} \,, & \omega \ll \omega_c \,,\ \alpha > 0 \quad \text{(Metal)} \,, \\ \left(\dfrac{\pi}{2} 3\lambda x_0\right)^{2/3} \omega^{1/3} \,, & \omega \gg \omega_c \quad \text{(Metal and insulator)} \,, \\ \dfrac{((\pi/2)3\lambda x_0)^2}{\alpha^2} \omega = (\xi_{loc}k_0)^2 \omega \,, & \omega \ll \omega_c \,,\ \alpha < 0 \quad \text{(Insulator)} \,, \end{cases} \tag{2.163}$$

where $\omega_c = |\alpha|^3/((\pi/2)3\lambda x_0)^2$ and $\xi_{\rm loc}$ is the localization length and x_0 the dimensionless cutoff. Let us introduce $K(\omega) = \omega/d(\omega)$ and analyze Eq. (2.162) assuming that $K(\omega)$, $K(\Omega)$ and $K(\omega + \Omega) \ll 1$. Expanding the right-hand side of Eq. (2.162) over these small parameters we obtain

$$\frac{\alpha}{d(\omega)} = 1 - \frac{\pi}{2}\frac{3\lambda x_0}{d(\omega)}K^{1/2}(\omega) + 2\mu\lambda x_0 \int_{\omega}^{\infty} \frac{d\Omega}{d(\omega + \Omega)d^2(\Omega)} \frac{K^{1/2}(\Omega) + 2K^{1/2}(\omega + \Omega)}{(K^{1/2}(\Omega) + K^{1/2}(\omega + \Omega))^2}\,. \tag{2.164}$$

Consider the metallic phase and look for the solution for the diffusion coefficient $d(\omega)$ in the following form:

$$d(\omega) = \begin{cases} d\,, & \omega \ll \omega_c\,, \\ d(\omega/\omega_c)^{1/3}\,, & \omega \gg \omega_c\,. \end{cases} \tag{2.165}$$

Substituting (2.165) into Eq. (2.164) we find d and ω_c and for the diffusion coefficient we obtain

$$d(\omega) = \begin{cases} \alpha - \alpha^*\,, & \omega \ll \omega_c\,, \\ (\frac{1}{2}\pi\,3\lambda x_0)^{2/3}\omega^{1/3}\,, & \omega \gg \omega_c\,, \end{cases} \tag{2.166}$$

where $\omega_c = |\alpha - \alpha^*|^3/(\frac{1}{2}\pi\,3\lambda x_0)^2$, $\alpha^* = c\mu$, $c \approx 0.89$.

Thus for the metallic phase we come to a very simple qualitative conclusion: Anderson transition persists and the conductivity exponent remains as $\nu = 1$. The transition itself has shifted to the region of weaker disorder $\alpha = \alpha^* = c\mu$ – interaction facilitates transition to the insulating state. The frequency behavior of the diffusion coefficient in metallic phase is qualitatively similar to that in the usual self-consistent theory of localization (2.163). In the region of high frequencies $\omega \gg \omega_c$ the behavior of diffusion coefficient remains unchanged after the introduction of interelectron interactions.

Consider now the insulating phase. In the region of high frequencies $\omega \gg \omega_c$ the diffusion coefficient obviously possesses the frequency dependence like $d(\omega) \sim \omega^{1/3}$. Assume that for small frequencies it is also some power of the frequency:

$$d(\omega) = \begin{cases} d\left(\dfrac{\omega}{\omega_c}\right)^{\delta}\,, & \omega \ll \omega_c\,, \\ d\left(\dfrac{\omega}{\omega_c}\right)^{1/3}\,, & \omega \gg \omega_c\,, \end{cases} \tag{2.167}$$

where δ is some exponent to be determined.

Substituting (2.167) into (2.164) and considering the case of $\alpha < 0$ (insulating phase of the usual self-consistent theory of localization) and $|\alpha| \gg \alpha^*$, we get

$$d(\omega) = \begin{cases} \dfrac{(\frac{1}{2}\pi\,3\lambda x_0)^2}{\alpha^2}\,\omega = (\xi_{\rm loc}k_0)^2\omega\,, & \omega^* \ll \omega \ll \omega_c\,, \\ (\frac{1}{2}\pi\,3\lambda x_0)^{2/3}\omega^{1/3}\,, & \omega \gg \omega_c\,, \end{cases} \tag{2.168}$$

where $\omega_c = |\alpha|^3/(\frac{1}{2}\pi\,3\lambda x_0)^2$, while $\omega^* \approx 0.1\mu\,\alpha^2/(\frac{1}{2}\pi\,3\lambda x_0)^2 = 0.1\,\mu/(\xi_{\mathrm{loc}}k_0)^2$ is some new characteristic frequency defined by the interactions. Note that $\omega^* \to 0$ as we approach the transition point when $\xi_{\mathrm{loc}} \to \infty$.

Thus, sufficiently deep inside the insulating phase when $\alpha < 0$ and $|\alpha| \gg \alpha^*$ and for the frequencies $\omega \gg \omega^*$, the diffusion coefficient remains the same as in the self-consistent theory of localization, i.e. at small frequencies it is linear over frequency, while for the higher frequencies it is $\sim\omega^{1/3}$.

The analysis of Eq. (2.164) shows that for the frequencies $\omega \ll \omega^*$ it is impossible to find the power-like dependence for $d(\omega)$, i.e. the diffusion coefficient in the insulating phase apparently cannot be represented in the form of $d(\omega) = d\frac{\omega^*}{\omega_c}(\frac{\omega}{\omega^*})^\delta$, where δ is some unknown exponent. Because of this we were unable to find any analytical treatment of Eq. (2.164) in the region of $\omega \ll \omega^*$ within the insulating phase.

Consider now the behavior of the system not very deep inside the insulating phase when $\alpha - \alpha^* < 0$ while $\alpha > 0$, that is when the system without interaction would be within the metallic phase. Let us assume that the frequency behavior of the diffusion coefficient for $\omega \ll \omega_c$ has the power-like form, i.e. the diffusion coefficient is defined by the expression (2.167). Substituting (2.167) into (2.164) we get $\delta = \frac{1}{3}$. As a result, for the diffusion coefficient we get

$$d(\omega) = \begin{cases} \left(4{,}2\dfrac{\mu\lambda x_0}{\alpha}\right)^{2/3}\omega^{1/3}\,, & \omega \ll \omega_c\,, \\ (\frac{1}{2}\pi\,3\lambda x_0)^{2/3}\omega^{1/3}\,, & \omega \gg \omega_c\,, \end{cases} \tag{2.169}$$

where $\omega_c = |\alpha - \alpha^*|^3/(\frac{1}{2}\pi\,3\lambda x_0)^2$. Naturally, the exact solution for the diffusion coefficient should show a continuous change of frequency around $\omega \sim \omega_c$.

Thus, within the insulating phase close to transition point, where the system without interactions should have been metallic, the diffusion coefficient behaves as $\sim\omega^{1/3}$ everywhere, though for the low frequency region the coefficient of $\omega^{1/3}$ differs from that of the usual self-consistent theory of localization and explicitly depends upon the interaction constant.

We have also performed the numerical analysis of the integral equation (2.162) for the wide region of frequencies, both for metallic (Fig. 9) and insulating phases (Fig. 10). Solution was achieved by a simple iteration procedure using the results of the usual self-consistent theory of localization as an initial approximation. Numerical data are in good correspondence with our analytical estimates. In the region of high frequencies, both for metallic and insulating phases, the frequency behavior of the diffusion coefficient is very close to that defined by the usual self-consistent theory of localization. In the region of small frequencies within the metallic phase diffusion coefficient $d(\omega)$ diminishes as interaction grows. Dependence of the static generalized diffusion coefficient on disorder for $\mu = 0.24$ is shown as an inset in Fig. 9, and is practically linear. Metal–insulator transition in this case is observed at $\alpha = \alpha^* = c\mu$, where $c \approx 0.5$, which is also in good correspondence with our qualitative analysis. Within the insulating phase for the region of small frequencies ($\omega \ll \omega^*$) we observe significant deviations from predictions of the usual self-consistent theory of localization. Diffusion coefficient is apparently nonanalytic in the frequency here and we clearly see the tendency to the formation of some kind of an effective gap for the frequencies $\omega \ll \omega^*$, with this “gap” closing as interactions are turned off.

Our numerical analysis was performed in the Matsubara frequency region, which was used in writing down Eq. (2.162). Analytical continuation of our numerical data to the real frequencies was

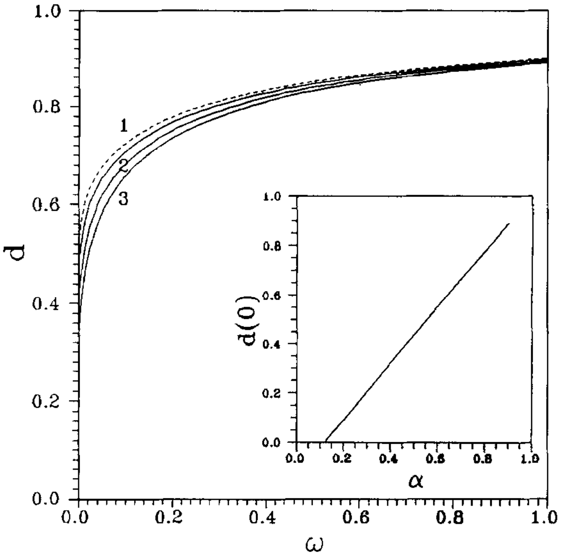


Fig. 9. Dependence of the dimensionless generalized diffusion coefficient on dimensionless Matsubara frequency in metallic phase ($\alpha = 0.5$), obtained by numerical solution for different values of μ: (1) 0.24; (2) 0.6; (3) 0.95; dashed line – the usual self-consistent theory of localization, $\mu = 0$. Inset: dependence of the static diffusion coefficient ($d = D(0)/D_0$) on disorder for $\mu = 0.24$.

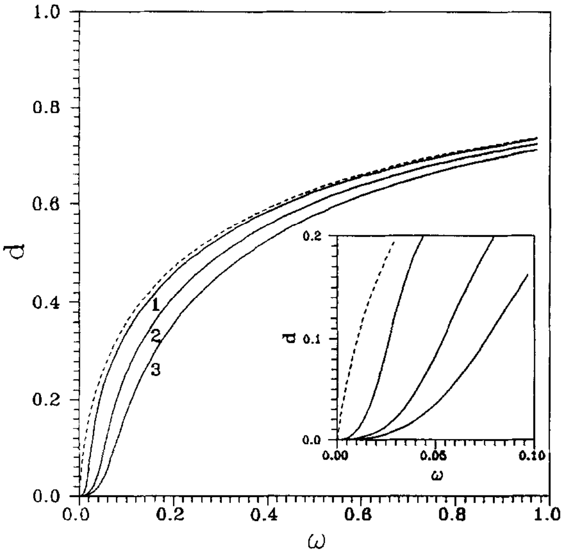


Fig. 10. Dependence of dimensionless generalized diffusion coefficient on dimensionless Matsubara frequency in the dielectric phase ($\alpha = -0.5$), obtained by numerical solution for different values of μ: (1) 0.12; (2) 0.6; (3) 1.2; dashed line – the usual self-consistent theory of localization, $\mu = 0$.

not attempted, but as we stressed above the Matsubara frequency behavior is sufficient for our studies of the superconducting state discussed below.

In Ref. [122] we were also able to study the gradual evolution of the tunneling density of states from metallic to insulating region, demonstrating the continuous transformation of a cusp singularity of Eq. (2.145) in a metal into a kind of interaction induced pseudogap at the Fermi level in an insulator, which is in some respects similar to the Coulomb gap of Refs. [41–44].

For high-T_c superconductors, problems of interplay of localization and interactions become especially important because of the unusual nature of the normal state of these systems. In the absence of an accepted theory of this normal state we shall limit ourselves only to a few remarks on one specific model. The so-called "marginal" Fermi-liquid theory [124, 125] is a promising semi-phenomenological description of both normal and superconducting properties of these systems. We shall see that localization effects are apparently greatly enhanced in this case [126, 127].

Basically, the idea of "marginal" Fermi-liquid is expressed by the following form of one-particle Green's function [124].

$$G(Ep) = Z_p/(\varepsilon - \xi_p - i\gamma_p) + G_{\text{incoh}}\,, \tag{2.170}$$

where ξ_p is the renormalized quasi-particle energy, $\gamma_p \sim \text{Max}[\varepsilon, T]$ is the anomalous (linear) decay-rate for these quasiparticles which is quite different from quadratic in ε or T decay-rates of the usual Fermi-liquid theory [120]. The concept of "marginality" arises due to the peculiar behavior of the quasi-particle residue:

$$Z_p^{-1} \approx \ln(\tilde{\omega}_c/|\xi_p|) \approx \ln(\tilde{\omega}_c/|\varepsilon|)\,, \qquad (2.171)$$

where $\tilde{\omega}_c$ is the characteristic frequency scale of some kind of electronic excitations, which is the phenomenological parameter of the theory. From Eq. (2.171) it is clear that quasi-particle contribution to Green's function Eq. (2.170) vanishes precisely at the Fermi level, while it exists close to it though with logarithmically reduced weight. Note that in the case of the usual Fermi-liquid $Z_p \approx 1$ [120].

For the disordered system we can estimate the impurity contribution to the scattering rate of quasi-particles as [126]

$$\gamma = 2\rho V^2 Z_p \operatorname{Im} \sum_p \Lambda^2(\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}) G(\boldsymbol{p}+\boldsymbol{q}\varepsilon) \approx 2\pi\rho V^2 Z^2 \Lambda^2(\boldsymbol{q}\to 0) N(E_F) \approx Z\Lambda^2\gamma_0\,, \qquad (2.172)$$

where Λ is the appropriate vertex-part renormalized by Fermi-liquid effects, ρ again is the impurity concentration, V the impurity potential and $N(E_F) = Z^{-1}N_0(E_F)$ is the renormalized density of states in the Fermi-liquid. Here $N_0(E_F)$ is the density of states for noninteracting electrons at the Fermi level, γ_0 is the scattering rate for the noninteracting case. To get the last relation in Eq. (2.172) a weak dependence of vertices and self-energy on momentum was assumed. Now we can use the Ward identity for $\Lambda(\boldsymbol{q}\to 0\omega = 0)$ vertex of the disordered Fermi-liquid theory [119, 108, 109]:

$$\Lambda(\boldsymbol{q}\to 0\omega = 0) = (1 + F_0^s)^{-1} Z^{-1}\,, \qquad (2.173)$$

where F_0^s is Landau parameter introduced above. As a result, we can easily get a simple relation between the mean free paths of interacting and noninteracting quasi-particles [126, 127]:

$$l = (p_F/m^*)\gamma^{-1} = (p_F/m)\gamma_0^{-1}/\Lambda^2(\boldsymbol{q}\to 0) = l_0(1 + F_0^s)^2 Z^2\,. \qquad (2.174)$$

Here $m^* = Z^{-1}m$ is the effective mass of the quasi-particle. Assuming $F_0^s \approx \text{const.} < 1$ and using Eq. (2.171) we get at $T = 0$

$$l = l_0/[\ln(\tilde{\omega}_c/|\varepsilon|)]^2\,. \qquad (2.175)$$

Then from the usual Ioffe–Regel criterion for localization $p_F l \approx 1$ we obtain that all the quasi-particle states within a region of the order of

$$|\varepsilon_c| \approx \tilde{\omega}_c \exp(-\sqrt{p_F l}) \qquad (2.176)$$

around the Fermi-level in high-T_c oxides are localized even for the case of weak impurity scattering $p_F l \gg 1$. For realistic estimates of $\tilde{\omega}_c \approx 0.1$–$0.2$ eV [124] and $p_F l < 5$ the width of this localized band may easily be of the order of hundreds of degrees Kelvin, while for $p_F l \approx 10$ and $\tilde{\omega}_c \approx 1000$ K we get $|\varepsilon_c| \approx 40$ K. Obviously, this band grows with disorder as the mean free path l_0 drops. We can safely neglect this localization for $T \gg |\varepsilon_c|$, but for low enough temperatures localization effects become important and all states are localized in the ground state.

Of course, the formal divergence of the mean free path denominator in Eq. (2.175) is unphysical. Single-impurity scattering cannot overcome the so-called unitarity limit [126], so that we must always have

$$l \geq p_F^2/4\pi\rho \,. \tag{2.177}$$

In a typical metal with $p_F \sim a^{-1}$ this leads to $l \geq 1/4\pi\rho a^2$ and Ioffe–Regel criterion $l \leq a$ can be easily satisfied for large impurity concentrations $\rho \sim a^{-3}$. Thus the singularity in Eq. (2.175) does not mean that localization can appear for arbitrarily low concentration of impurities. We can safely speak only about the significant enhancement of localization effects in marginal Fermi liquids. These ideas are still at this elementary level and we may quote only one paper attempting to put them on a more sound basis of scaling theory of the metal–insulator transition of interacting electrons [128].

3. Superconductivity and localization: statistical mean-field approach

3.1. BCS model and Anderson theorem

We shall start our analysis of superconductivity in strongly disordered systems within the framework of the simple BCS-model [8, 9] which assumes the existence of some kind of effective electron–electron attraction within the energy region of the order of $2\langle\omega\rangle$ around the Fermi level. In the usual superconductors, $\langle\omega\rangle \sim \omega_D$, where ω_D is the Debye frequency, because pairing is determined by electron–phonon mechanism; however, we shall use some effective $\langle\omega\rangle$ as an average frequency of some kind of Bose-like excitations responsible for pairing, e.g. in high-T_c superconductors. At the moment, we shall not discuss the microscopic nature of this attraction which in general case is determined by the balance of attraction due to Boson-exchange and Coulomb repulsion. Here we just assume (as always is done in simple BCS-approach) that this effective attraction is described by some interaction constant g, which is considered just as a parameter. More detailed microscopic approach will be given in later sections.

Nontrivial results concerning superconductivity in disordered systems were obtained very soon after the formulation of the BCS-theory [10–13]. The concept of a "dirty" superconductor described the experimentally very important case of the mean free path l short in comparison with the superconducting coherence length $\xi_0 \sim \hbar v_F/T_c$, i.e. the case when

$$\xi_0 \gg l \gg \hbar/p_F \,. \tag{3.1}$$

Already in this case of the not so strongly disordered (in the sense of closeness to metal–insulator transition) system, Cooper pairing takes place not between electrons with opposite momenta and spins as in regular case, but between time-reversed exact eigenstates of electrons in a disordered system [13, 9].

$$(\boldsymbol{p}_\uparrow, -\boldsymbol{p}_\downarrow) \Rightarrow (\phi_\nu(\boldsymbol{r})_\uparrow, \phi_\nu^*(\boldsymbol{r})_\downarrow) \,. \tag{3.2}$$

In the following we consider only singlet isotropic (s-wave) pairing. Some aspects of anisotropic pairing are analyzed in Appendix C. The underlying physics is simple: in disordered systems such as e.g. an alloy the electron momentum is poorly determined due to the lack of translational invariance. However, in a random potential field we can always define exact eigenstates $\phi_\nu(\boldsymbol{r})$, which are just solutions of the Schroedinger equation in this random field (for a given configuration of this field). We do not need to know the explicit form of these eigenstates at all, the pairing partner of $\phi_\nu(\boldsymbol{r})$ is being given by time-reversed $\phi_\nu^*(\boldsymbol{r})$. This leads to a relative stability of a superconducting state with respect to disordering in the absence of scattering mechanisms which break the time-reversal invariance such as e.g. of magnetic impurities.

Within the standard Green's function approach, the superconducting system is described by Gorkov equations [58, 129] which in the coordinate representation take the form:

$$\mathscr{G}_\uparrow(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = G_\uparrow(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) - \int \mathrm{d}\boldsymbol{r}''\, G_\uparrow(\boldsymbol{r}\boldsymbol{r}''\varepsilon_n)\Delta(\boldsymbol{r}'')\mathscr{F}(\boldsymbol{r}''\boldsymbol{r}'\varepsilon_n)\,, \tag{3.3}$$

$$\mathscr{F}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \int \mathrm{d}\boldsymbol{r}''\, G_\downarrow^*(\boldsymbol{r}\boldsymbol{r}''\varepsilon_n)\Delta^*(\boldsymbol{r}'')\mathscr{G}_\uparrow(\boldsymbol{r}''\boldsymbol{r}'\varepsilon_n)\,, \tag{3.4}$$

where $G(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ is an exact one-electron Matsubara Green's function of the normal state and the superconducting order-parameter (gap) $\Delta(\boldsymbol{r})$ is determined by the self-consistent gap equation:

$$\Delta(\boldsymbol{r}) = gT\sum_{\varepsilon_n}\mathscr{F}^*(\boldsymbol{r}\boldsymbol{r}\varepsilon_n)\,, \tag{3.5}$$

where $\mathscr{F}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ is (antisymmetric over spin variables) anomalous Gorkov Green's function, $\varepsilon_n = (2n+1)\pi T$.

If we consider temperatures close to the superconducting transition temperature T_c, when $\Delta(\boldsymbol{r})$ is small, $\mathscr{F}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ can be obtained from the linearized equation:

$$\mathscr{F}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \int \mathrm{d}\boldsymbol{r}''\, G_\downarrow^*(\boldsymbol{r}\boldsymbol{r}''\varepsilon_n)\Delta^*(\boldsymbol{r}'')G_\uparrow(\boldsymbol{r}''\boldsymbol{r}'\varepsilon_n)\,. \tag{3.6}$$

Then the linearized gap equation determining T_c takes the form

$$\Delta(\boldsymbol{r}) = gT\int \mathrm{d}\boldsymbol{r}'\sum_{\varepsilon_n} K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)\Delta(\boldsymbol{r}')\,, \tag{3.7}$$

where the kernel

$$K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = G_\uparrow(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)G_\downarrow^*(\boldsymbol{r}'\boldsymbol{r}\varepsilon_n) \tag{3.8}$$

is formed by exact one-electron Green's functions of a normal metal. Now we can use an exact eigenstate representation for an electron in a random field of a disordered system to write (cf. Eq. (A.13))

$$G_\uparrow(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \sum_\nu \frac{\phi_{\nu\uparrow}(\boldsymbol{r})\phi_{\nu\uparrow}^*(\boldsymbol{r}')}{\mathrm{i}\varepsilon_n - \varepsilon_\nu}\,, \tag{3.9}$$

where ε_ν are exact energy levels of an electron in the disordered system. Then

$$K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = Tg\sum_{\mu\nu}\frac{\phi_{\nu\uparrow}(\boldsymbol{r})\phi_{\nu\uparrow}^*(\boldsymbol{r}')\phi_{\mu\downarrow}^*(\boldsymbol{r}')\phi_{\mu\downarrow}(\boldsymbol{r})}{(\mathrm{i}\varepsilon_n - \varepsilon_\nu)(-\mathrm{i}\varepsilon_n + \varepsilon_\mu)}\,. \tag{3.10}$$

In the following for brevity we shall drop spin variables always assuming singlet pairing. In the case of a system with time-reversal invariance (i.e. in the absence of an external magnetic field, magnetic impurities, etc.) Eq. (3.10) can be rewritten as

$$K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = G(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)G(\boldsymbol{r}'\boldsymbol{r}-\varepsilon_n) = \sum_{\mu\nu}\frac{\phi_\nu(\boldsymbol{r})\phi_\nu^*(\boldsymbol{r}')\phi_\mu(\boldsymbol{r}')\phi_\mu^*(\boldsymbol{r})}{(\mathrm{i}\varepsilon_n-\varepsilon_\nu)(-\mathrm{i}\varepsilon_n-\varepsilon_\mu)}\,. \tag{3.11}$$

Averaging over disorder we get

$$\langle\Delta(\boldsymbol{r})\rangle = gT\int \mathrm{d}\boldsymbol{r}'\sum_{\varepsilon_n}\langle K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)\Delta(\boldsymbol{r}')\rangle\,. \tag{3.12}$$

Practically, in all the papers on superconductivity in disordered systems it is assumed that we can make the simplest decoupling in Eq. (3.12) to get the following linearized equation for the average order-parameter:

$$\langle\Delta(\boldsymbol{r})\rangle = gT\int \mathrm{d}\boldsymbol{r}'\sum_{\varepsilon_n}K(\boldsymbol{r}-\boldsymbol{r}'\varepsilon_n)\langle\Delta(\boldsymbol{r}')\rangle\,, \tag{3.13}$$

where the averaged kernel in the case of time-invariance is given by

$$\begin{aligned}K(\boldsymbol{r}-\boldsymbol{r}'\varepsilon_n) &= K^*(\boldsymbol{r}-\boldsymbol{r}'\varepsilon_n) = \langle K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)\rangle\\ &= \left\langle\sum_{\mu\nu}\frac{\phi_\nu(\boldsymbol{r})\phi_\mu^*(\boldsymbol{r})\phi_\mu(\boldsymbol{r}')\phi_\nu^*(\boldsymbol{r}')}{(\mathrm{i}\varepsilon_n-\varepsilon_\nu)(-\mathrm{i}\varepsilon_n-\varepsilon_\mu)}\right\rangle\\ &= \int_{-\infty}^{\infty}\mathrm{d}E\,N(E)\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\langle\langle\rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^{\mathrm{F}}}{(\mathrm{i}\varepsilon_n+E)(E+\omega-\mathrm{i}\varepsilon_n)}\,,\end{aligned} \tag{3.14}$$

where we have introduced Gorkov–Berezinskii spectral density [59] (cf. Eq. (A.2)):

$$\langle\langle\rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^{\mathrm{F}} = \frac{1}{N(E)}\left\langle\sum_{\mu\nu}\phi_\nu^*(\boldsymbol{r})\phi_\mu(\boldsymbol{r})\phi_\mu^*(\boldsymbol{r}')\phi_\nu(\boldsymbol{r}')\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'})\right\rangle\,. \tag{3.15}$$

Here $N(E)$ is an exact electron density of states per *spin direction* as it always appears in superconductivity theory (above, while discussing localization we always used density of states for both spin directions).

Usually, the decoupling procedure used in Eq. (3.12) to reduce it to Eq. (3.13) is justified by the assumption that the averaging of $\Delta(\boldsymbol{r})$ and of Green's functions in Eq. (3.12) forming the kernel can be performed independently because of the essentially different spatial scales [12]. $\Delta(\boldsymbol{r})$ changes at a scale of the order of coherence length (Cooper pair size) ξ, while $G(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ are oscillating on the scale of interatomic distance $a \sim \hbar/p_{\mathrm{F}}$, and we always have $\xi \gg a$. Actually, it is clear that this decoupling is valid only if the order-parameter is *self-averaging* (i.e. in fact nonrandom) quantity: $\Delta(\boldsymbol{r}) = \langle\Delta(\boldsymbol{r})\rangle$, $\langle\Delta^2(\boldsymbol{r})\rangle = \langle\Delta(\boldsymbol{r})\rangle^2$. Below we shall see that for a system close to the mobility edge the property of self-averageness of $\Delta(\boldsymbol{r})$ is absent and situation is actually highly nontrivial. In this case, the so-called *statistical* fluctuations [62] leading to the inequality of $\langle\Delta^2(\boldsymbol{r})\rangle$ and $\langle\Delta(\boldsymbol{r})\rangle^2$ become quite important. However, we shall start with what we call the statistical mean-field approach which completely neglects these fluctuations and allows the simple analysis using

Eq. (3.13), as a necessary first step to understand superconductivity in strongly disordered systems, which will allow us to find most of the important deviations from the usual theory of "dirty" superconductors. The role of statistical fluctuations will be analyzed later.

If we look for the solution of Eq. (3.13) $\Delta(\boldsymbol{r}) = \text{const.}$ (homogeneous gap), we immediately obtain the following equation for the superconducting transition temperature T_c:

$$\begin{aligned} 1 &= gT_c \int \mathrm{d}\boldsymbol{r} \sum_{\varepsilon_n} K(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n) \\ &= gT_c \int \mathrm{d}\boldsymbol{r} \sum_{\varepsilon_n} \int_{-\infty}^{\infty} \mathrm{d}E\, N(E) \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^F}{(E + \mathrm{i}\varepsilon_n)(E + \omega - \mathrm{i}\varepsilon_n)} \end{aligned} \tag{3.16}$$

Using the general sum-rule given in Eq. (A.5) [59]:

$$\int \mathrm{d}\boldsymbol{r} \langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^F = \delta(\omega) , \tag{3.17}$$

we immediately reduce Eq. (3.16) to a standard BCS form

$$1 = gT_c \int_{-\infty}^{\infty} \mathrm{d}E\, N(E) \sum_{\varepsilon_n} \frac{1}{E^2 + \varepsilon_n^2} = g \int_0^{\langle\omega\rangle} \mathrm{d}E\, N(E) \frac{1}{E} \tanh(E/2T_c) , \tag{3.18}$$

where we introduced the usual cut off at $\varepsilon_n \sim 2\langle\omega\rangle$. Note that $N(E)$ here is an exact one-particle density of states (per spin direction) in a normal state of a disordered system. From Eq. (3.18) we get the usual result:

$$T_c = \frac{2\gamma}{\pi} \langle\omega\rangle \exp(-1/\lambda_p) , \tag{3.19}$$

where $\lambda_p = gN(E_F)$ is the dimensionless pairing constant and $\ln\gamma = C = 0.577\ldots$ the Euler constant. This is the notorious Anderson theorem: in the absence of scattering processes breaking time-reversal invariance, disorder influences T_c only through possible changes of the density of states $N(E_F)$ under disordering (which are usually relatively small).

Due to the sum-rule of Eq. (3.17), all the singularities of the Berezinskii–Gorkov spectral density, reflecting a possible localization transition, do not appear in the equation determining T_c: there is no explicit contribution from the $\delta(\omega)$ term of Eq. (A.8) and Eq. (3.18) has the same form in both metallic and localized phases (cf. Ref. [130]).

The only limitation here which appears on physical grounds is connected with the local discreteness of the electronic spectrum in the localized phase discussed above. It is clear that Cooper pairing is possible in the localized phase only between electrons with centers of localization within a distance of the order of $\sim R_{\text{loc}}(E)$, because only in that case their wave functions overlap [21,22]. However, these states are split in the energy by δ_E defined in Eq. (2.13). Obviously, we have to demand that the superconducting gap Δ (at $T = 0$, $\Delta \sim T_c$) be much larger than this δ_E:

$$\Delta \sim T_c \gg \delta_E \sim 1/N(E)R_{\text{loc}}^3(E) , \tag{3.20}$$

i.e. on the energy interval of the order of $\Delta \sim T_c$ there must be many discrete levels, with centers of localization within a distance $\sim R_{\text{loc}}(E)$ from each other. In this case, the problem of Cooper pairs formation within $\sim R_{\mathrm{loc}}(E)$ is qualitatively the same as in the metallic state, e.g. we can replace the summation over discrete levels ε_ν by integration. An analogous problem was considered previously in the case of Cooper pairing of nucleons in finite nuclei [120] and also of Cooper pairing of electrons in small metallic particles (granular metals) [131, 132]. For strongly anisotropic high-T_c systems we must similarly have [16]

$$\Delta \sim T_c \gg [N(E) R^{a}_{\mathrm{loc}} R^{b}_{\mathrm{loc}} R^{c}_{\mathrm{loc}}]^{-1} , \tag{3.21}$$

where we have introduced the appropriate values of localization lengths along the axes of an orthorhombic lattice.

Obviously, Eq. (3.20) is equivalent to a condition of large enough localization length:

$$R_{\mathrm{loc}}(E) \gg [N(E)\Delta]^{-1/3} \sim (\xi_0/p_F^2)^{1/3} \sim (\xi_0 l^2)^{1/3} , \tag{3.22}$$

i.e. the system must be close enough to the mobility edge or just slightly localized. Here we used the usual estimate of mean free path close to the Anderson transition $l \sim p_F^{-1}$. Below we shall see that Eq. (3.22) is just a condition that Cooper pairs must be much smaller than the localization length, only in that case Cooper pairing is possible in the localized phase [21, 22].

3.2. T_c degradation

In the usual BCS model discussed above pairing interaction g is assumed to be a given constant in the vicinity of the Fermi level. In a more realistic approach this interaction is determined by the balance of interelectron attraction, due e.g. to electron–phonon coupling (as in traditional superconductors) or some other Boson-exchange mechanism (as is apparently the case in high-T_c superconductors), and Coulomb repulsion. It is clear that in a strongly disordered system all these interactions can, in principle, be strongly renormalized in comparison with the "pure" case. The aim of this section is to discuss these effects in the context of the metal–insulator transition induced by disorder.

Usually the Coulomb repulsion within a Cooper pair is strongly reduced in comparison with the electron–phonon attraction due to the retarded nature of the electron–phonon coupling [9]. The characteristic time of electron–phonon interaction is of the order of ω_D^{-1}, while for the Coulomb interaction in "pure" metal it is determined by $\sim \hbar/E_F$ – the time during which electrons "pass" each other in the pair. Due to metallic screening both interactions are more or less point-like. However, in a disordered metal, ballistic transport changes to diffusion and as disorder grows electron motion becomes slower effectively leading to the growth of Coulomb repulsion within the Cooper pair and the appropriate drop of T_c as was first claimed by Anderson et al. [20]. Actually, electron–phonon interaction can also change under disordering but a common belief is that these changes are less significant than in the case of Coulomb interaction [134, 135]. This problem is still under active discussion and some alternative points of view have been expressed [136–138]. However, the general agreement is that some kind of diffusion renormalization of the effective interaction of the electrons within a Cooper pair provides an effective mechanism of T_c degradation under disordering. Below we shall mainly use the approach of Ref. [22], with the main aim of finding the possibility of superconductivity surviving up to the Anderson transition.

Later in this section we shall also consider the possible mechanisms of T_c degradation under disordering due to magnetic fluctuations (or local moments) which appear close to the metal–insulator transition. A possible relation of these mechanisms to enhanced Coulomb effects will also be discussed.

The general problem of T_c degradation under disordering becomes much more complicated in the case of high-temperature superconductors because of the unknown nature of pairing in these systems. However, we believe that the mechanism based upon the growth of Coulomb repulsion within the Cooper pair is also operational here, while of course it is difficult to say anything about disorder effects upon attractive interactions leading to Cooper pair formation in these systems.

If we assume a spin-independent Boson-exchange (phonons, excitons, etc.) model of the pairing interaction, T_c can be obtained from the generalized Eliashberg equations and thus be given by the famous Allen–Dynes expression [139]:

$$T_c = \frac{f_1 f_2}{1.20}\,\omega_{\log}\exp\left\{-\frac{1.04(1+\lambda)}{\lambda-\mu^*(1+0.62\lambda)}\right\}, \tag{3.23}$$

where

$$f_1 = [1+(\lambda/\lambda_1)^{3/2}]^{1/3}\,; \qquad f_2 = 1+\frac{[\langle\omega^2\rangle^{1/2}/\omega_{\log}-1]\lambda^2}{\lambda^2+\lambda_2^2}\,,$$

$$\lambda_1 = 2.46(1+3.8\mu^*)\,; \qquad \lambda_2 = 1.82(1+6.3\mu^*)\frac{\langle\omega^2\rangle^{1/2}}{\omega_{\log}}\,. \tag{3.24}$$

Here $\omega_{\log}$ is the mean logarithmic frequency and $\langle\omega\rangle^2$ is the mean square frequency of Bosons responsible for pairing (the averaging is over the spectrum of these Bosons), μ^* is the Coulomb pseudopotential, λ is the dimensionless pairing constant due to Boson-exchange. Strictly speaking, the Allen–Dynes formula has been derived for the electron–phonon model, with certain assumptions about the phonon spectrum. Its use for the general Boson-exchange model here serves only for illustrative purposes. At relatively weak coupling $\lambda \leq 1.5$, Allen–Dynes expression effectively reduces to McMillan formula [140]:

$$T_c = \frac{\omega_{\log}}{1.20}\exp\left\{-\frac{1.04(1+\lambda)}{\lambda-\mu^*(1+0.62\lambda)}\right\} \tag{3.25}$$

which in the weak coupling limit gives the usual BCS result $T_c \sim \langle\omega\rangle\exp(-1/\lambda-\mu^*)$. For a very strong pairing interaction, Eq. (3.23) gives the asymptotic behavior $T_c \approx 0.18\sqrt{\lambda\langle\omega^2\rangle}$. In most parts of this review we shall limit ourselves to the weak coupling approximation. The Coulomb pseudopotential μ^* in the "pure" system is given by

$$\mu^* = \frac{\mu}{1+\mu\ln(E_F/\langle\omega\rangle)}, \tag{3.26}$$

where μ is the dimensionless Coulomb constant. The mechanism of T_c degradation under disordering due to the growth of Coulomb repulsion is reflected in the appropriate growth of μ^* [20,22].

The singlet gap function with a simple s-wave symmetry which we have discussed above has a nonzero amplitude at zero separation of the two electrons in the pair. Thus it must pay the energy price for the short-range repulsion due to a finite μ. In recent years, a number of new mechanisms of superconducting pairing were proposed which try to eliminate the effect of repulsion assuming a pair wave function which vanishes at zero separation. This is equivalent to the requirement that the sum over all momentum of the BCS gap function Δ must vanish [141]:

$$\Delta(\boldsymbol{r}) = \langle \psi\uparrow(\boldsymbol{r})\psi\downarrow(\boldsymbol{r})\rangle = \sum_{\boldsymbol{k}} \Delta(\boldsymbol{k}) = 0 . \tag{3.27}$$

A number of rather exotic schemes for this were proposed [141], but probably the simplest way of satisfying this requirement is by means of higher angular momentum pairing, e.g. d-wave which became rather popular as a possible explanation of high-T_c superconductivity within the spin-fluctuation exchange mechanism [142–145]. The sum in Eq. (3.27) is then zero because the gap changes sign as $\boldsymbol{k}$ goes around the Fermi surface. This leads, to a large extent, to the cancellation of Coulomb pseudopotential effects. However, this type of pairing is extremely sensitive to any kind of disordering (cf. Appendix C) and superconductivity is destroyed long before the localization transition. For these reasons we shall not discuss the disorder effects in such superconductors in this review. The same applies to more exotic pairing schemes such as the odd-gap pairing [146], where the usual scattering suppression of T_c is also very strong.

Among mechanisms discussed for high-T_c superconductors we should also mention different types of the so-called van-Hove scenarios [147–150], which are based upon the idea of T_c-enhancement due to some kind of the density of states singularity close to the Fermi level. For all such mechanisms, a rather strong T_c suppression may be due to the potential scattering smoothing out these singularities. Again we shall not discuss these mechanisms in our review as having nothing to do with localization effects.

3.2.1. Coulomb kernel

Let us use again the exact eigenstate $\phi_\nu(\boldsymbol{r})$ representation for an electron in a random system, with exact energy levels ε_ν. These functions and energies may correspond either to extended or to localized states. Consider the one-electron Green's function in this representation and take its diagonal element $G_{\nu\nu}(\varepsilon)$. The influence of interaction is described by the appropriate irreducible self-energy $\Sigma_\nu(\varepsilon)$ [120, 151]:

$$G_{\nu\nu}(\varepsilon) = 1/[\varepsilon - \varepsilon_\nu - \Sigma_\nu(\varepsilon)] . \tag{3.28}$$

Here energy zero is at the Fermi level. Let us introduce a "self-energy" $\Sigma_E(\varepsilon)$ averaged over some surface of constant energy $E = \varepsilon_\nu$ and over random field configurations [151]:

$$\Sigma_E(\varepsilon) = \frac{1}{N(E)} \left\langle \sum_\nu \delta(E - \varepsilon_\nu)\Sigma_\nu(\varepsilon) \right\rangle . \tag{3.29}$$

Consider a model with short-range static interelectron interaction $v(\boldsymbol{r} - \boldsymbol{r}')$. Then for the simplest Fock diagram shown in Fig. 7 we find

$$\Sigma_\mu^{\mathrm{F}} = -\int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}' v(\boldsymbol{r} - \boldsymbol{r}') \sum_\nu f_\nu \phi_\mu^*(\boldsymbol{r}')\phi_\nu^*(\boldsymbol{r})\phi_\mu(\boldsymbol{r})\phi_\nu(\boldsymbol{r}') , \tag{3.30}$$

where $f_\nu = f(\varepsilon_\nu)$ is the Fermi distribution function. Accordingly, from Eq. (3.29) we get [60]

$$\Sigma_E^F = -\int_{-\infty}^{\infty} d\omega\, f(E+\omega) \int d\boldsymbol{r} \int d\boldsymbol{r}'\, v(\boldsymbol{r}-\boldsymbol{r}') \langle\langle \rho_E(\boldsymbol{r}) \rho_{E+\omega}(\boldsymbol{r}') \rangle\rangle^F , \tag{3.31}$$

where we again introduced Berezinskii–Gorkov spectral density defined in Eqs. (A.2) and (3.15).

Let us define the Coulomb kernel by the following functional derivative:

$$K_C(E-E') = -\delta\Sigma_E^F/\delta f(E') \tag{3.32}$$

which characterizes the change of electron energy due to a variation of its distribution function. It is easy to see that

$$\begin{aligned} K_C(\omega) &= \frac{1}{N(E)} \left\langle \sum_{\mu\nu} \langle \mu\nu | v(\boldsymbol{r}-\boldsymbol{r}') | \nu\mu \rangle \delta(E-\varepsilon_\nu) \delta(E+\omega-\varepsilon_\mu) \right\rangle \\ &= \int d\boldsymbol{r} \int d\boldsymbol{r}'\, v(\boldsymbol{r}-\boldsymbol{r}') \langle\langle \rho_E(\boldsymbol{r}) \rho_{E+\omega}(\boldsymbol{r}') \rangle\rangle^F \end{aligned} \tag{3.33}$$

is actually a Fock-type matrix element of interaction averaged over two surfaces of constant energy E and $E' = E + \omega$ and over disorder. We can use $K_C(\omega)$ as a kernel in the linearized gap equation (cf. Appendix B) determining T_c which is a reasonable generalization of a Coulomb kernel used in the theory of ordered superconductors [152]. In the momentum representation,

$$K_C(\omega) = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} v(\boldsymbol{q}) \langle\langle \rho_E \rho_{E+\omega} \rangle\rangle_{\boldsymbol{q}}^F . \tag{3.34}$$

In the weak coupling approximation over pairing interaction it is the only relevant Coulomb contribution in the gap equation (cf. Appendix B), in case of strong coupling there are additional contributions, e.g. connected with diffusional renormalization of the density of states, Eq. (2.145) [136–138, 153, 154]. We refer to these papers for a detailed analysis of the density of states effects upon T_c.

In the following we assume point-like interaction: $v(\boldsymbol{q}) = v_0$. During our discussion of localization we have discovered that for small $\omega \ll \gamma$ and $q \ll l^{-1}$, the Gorkov–Berezinskii spectral density acquires a diffusional contribution:

$$\langle\langle \rho_E \rho_{E+\omega} \rangle\rangle_{\boldsymbol{q}}^{F_{\mathrm{diff}}} = \frac{1}{\pi N(E)} \operatorname{Im} \Phi_E^{RA}(\boldsymbol{q}\omega) , \tag{3.35}$$

where

$$\Phi_E^{RA}(\boldsymbol{q}\omega) = -N(E)/(\omega + iD_E(\omega)q^2) \tag{3.36}$$

and the generalized diffusion coefficient in the metallic phase is given by

$$D_E(\omega) \approx \begin{cases} D_E , & |\omega| \ll \omega_c \approx 2\gamma(\sigma/\sigma_c)^3 , \\ D_0\left(-\dfrac{i\omega}{2\gamma}\right)^{1/3} , & |\omega| \gg \omega_c . \end{cases} \tag{3.37}$$

In the absence of disorder this diffusional contribution disappears and the kernel $K_c(\omega)$ for $|\omega| < E_F$ reduces to the usual Coulomb potential $\mu = N(E)v_0$ [9,152]. Accordingly, we can use the following approximation [22]:

$$K_c(\omega) \approx \mu\theta(E_F - |\omega|) + K_c^{\text{diff}}(\omega)\,, \tag{3.38}$$

where

$$K_c^{\text{diff}}(\omega) = \int \frac{d^3 q}{(2\pi)^3} v_0 \langle\langle \rho_E \rho_{E+\omega} \rangle\rangle_q^{F_{\text{diff}}}\,. \tag{3.39}$$

This form of the Coulomb kernel gives correct interpolation between the strong disorder limit and the "pure" case. Note that in the case of a disordered system, besides the diffusional contribution which contains singularities associated with the Anderson transition there also appear "regular" contributions to $K_c(\omega)$ which may be modelled by μ, making it different from its value in the "pure" system. The diffusional term in $K_c(\omega)$ is connected with diffusion renormalization of the electron–electron interaction vertex [38,39,151,101,98,60]. Fig. 7 shows diagrams of standard perturbation theory responsible for this renormalization. In the approach based upon the self-consistent theory of localization "triangular" vertex defined by Fig. 7(c) is given by [98,155]

$$\gamma^{RA}(\boldsymbol{q}\omega) \approx \frac{2\gamma}{-i\omega + D_E(\omega)q^2}\,, \quad \omega \ll \gamma,\ q \ll l^{-1}\,. \tag{3.40}$$

Singularity of Eq. (3.40) for small ω and q leads to a significant growth of the interaction in the disordered system. Actually, this expression is the same as in a "dirty" metal [38] but with the replacement of the Drude diffusion coefficient by the generalized one.

3.2.2. Electron–phonon interaction

The case of electron–phonon interaction is different. Diffusion renormalization of the electron–phonon vertex is unimportant because the relevant corrections compensate each other if we take into account the impurity vibrations [133–135]. Surely, the value of the electron–phonon contribution to the pairing interaction does change in a disordered system in comparison with the "pure" case [134]. However, these changes are relatively insignificant in the sense of absence of drastic changes at the Anderson transition. We shall demonstrate the absence of diffusion renormalization of the electron–phonon vertex using the lowest order diagrams of perturbation theory following the approach of Ref. [135].

Let us limit our analysis to a homogeneous continuous medium. The appearance of deformation $\boldsymbol{u}$ leads to the variation of density of the medium given by $\delta\rho = -\rho\,\mathrm{div}\,\boldsymbol{u}$. Accordingly, taking into account the electroneutrality condition we get the variation of electron density as $\delta n = -n\,\mathrm{div}\,\boldsymbol{u}$. This leads to the following change of the free electron Green's function:

$$\begin{aligned}\delta G^{-1}(E\boldsymbol{p}) &= -n\,\mathrm{div}\,\boldsymbol{u}\frac{d}{dn}[E - v_F(|\boldsymbol{p}| - p_F)] \\ &= -nv_F\,\mathrm{div}\,\boldsymbol{u}\frac{dp_F}{dn} = -\frac{1}{3}v_F p_F\,\mathrm{div}\,\boldsymbol{u}\,,\end{aligned} \tag{3.41}$$

where we have used $n = p_F^3/(3\pi^2)$. Let us define the electron–phonon vertex Λ by

$$\frac{\delta G}{\delta \boldsymbol{u}} = G\Lambda G = -G(\delta G^{-1}/\delta \boldsymbol{u})G\,; \qquad \Lambda = -\delta G^{-1}/\delta \boldsymbol{u}\,. \tag{3.42}$$

For $\boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{u}\exp(\mathrm{i}\boldsymbol{qr} - \mathrm{i}\omega t)$ we get from Eq. (3.41)

$$\delta G^{-1}(Ep) = -\tfrac{1}{3}\mathrm{i}\boldsymbol{qu}\, v_F p_F \tag{3.43}$$

so that the "bare" electron–phonon vertex (i is vector index)

$$\Lambda_{1i}^{(0)} = \tfrac{1}{3}\mathrm{i}q_i v_F p_F\,. \tag{3.44}$$

Consider the system with impurities randomly placed at points $\boldsymbol{R}_n$ which create the potential

$$U(\boldsymbol{r}) = \sum_n V(\boldsymbol{r} - \boldsymbol{R}_n)\,. \tag{3.45}$$

Vibrations of the medium lead to vibrations of impurity atoms, so that $\boldsymbol{R}_n \to \boldsymbol{R}_{0n} + \boldsymbol{u}_n(t)$ with $\boldsymbol{u}_n(t) = \boldsymbol{u}\exp(\mathrm{i}\boldsymbol{qR}_{0n} - \mathrm{i}\omega t)$. The random field of static impurities leads to a simplest self-energy correction given by Fig. 11(a) [57, 58]. Impurity vibrations can be accounted for by the additional interaction term:

$$\delta V(\boldsymbol{r} - \boldsymbol{R}_n) = \frac{\partial V(\boldsymbol{r} - \boldsymbol{R}_{n0})}{\partial \boldsymbol{R}_{n0}}\,\boldsymbol{u}\exp(\mathrm{i}\boldsymbol{qR}_{n0} - \mathrm{i}\omega t)$$

so that

$$\begin{aligned}\Lambda_{2i}u_i = \frac{\delta\Sigma}{\delta u_i}u_i = \Bigg\langle \sum_n \Bigg\{ &\frac{\partial V(\boldsymbol{r} - \boldsymbol{R}_{n0})}{\partial R_{n0}^i} G(\boldsymbol{r}t, \boldsymbol{r}'t') V(\boldsymbol{r}' - \boldsymbol{R}_{n0}) u_{in} \\ &+ V(\boldsymbol{r} - \boldsymbol{R}_{n0}) G(\boldsymbol{r}t, \boldsymbol{r}'t') \frac{\partial V(\boldsymbol{r}' - \boldsymbol{R}_{n0})}{\partial R_{n0}^i} u_{in} \Bigg\} \Bigg\rangle\,,\end{aligned} \tag{3.46}$$

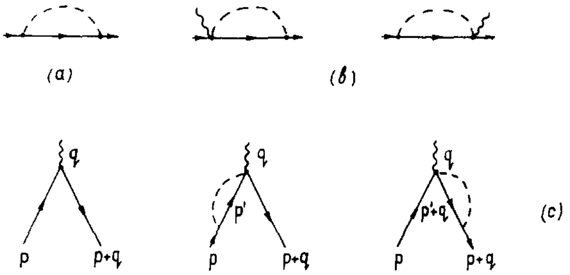
where the angular brackets define as usual the averaging over random impurity positions. In the momentum representation and for point-like impurities we get in the lowest order over ω/E_F and q/p_F

$$\begin{aligned}\Lambda_{2i}(\boldsymbol{p},\boldsymbol{q}) &= \rho V^2 \int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}\left[-\mathrm{i}(p_i - p_i')G(E\boldsymbol{p}') + \mathrm{i}(p_i' - p_i)G(E\boldsymbol{p}')\right] \\ &= 2\rho V^2 \int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}\left[-\mathrm{i}(p_i - p_i')G(E\boldsymbol{p}')\right] = 2\pi\rho V^2 N(E) p_i = 2\gamma p_i\,.\end{aligned} \tag{3.47}$$

The relevant diagrams are shown in Fig. 11(b) [156]. A "bare" electron–phonon vertex is thus given by the sum of three diagrams shown in Fig. 11(b) and reduces to

$$\Lambda_i^{(0)} = \Lambda_{1i}^{(0)} + \Lambda_{2i}^{(0)} = \tfrac{1}{3}\mathrm{i}q_i v_F p_F + 2\gamma p_i\,. \tag{3.48}$$

Diffusion renormalization of the electron–phonon vertex can appear due to impurity scattering ladder corrections as shown in Fig. 12(a). Similar diagrams shown in Fig. 7(c) lead to diffusion renormalization of the Coulomb vertex. However, in the case of electron–phonon interaction we



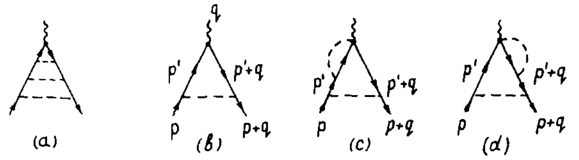


Fig. 11. Electron–phonon interaction and impurity scattering: (a) Self-energy due to impurity scattering, (b) Diagrams representing changes of (a) due to impurity vibrations, (c) diagrams for the "bare" electron–phonon vertex in the case of vibrating impurities.

Fig. 12. Electron–phonon vertex renormalization: (a) Impurity "ladder" (diffusion) renormalization. (b)–(d) Simplest corrections due to impurity vibrations.

have to make the same renormalization of the three diagrams of Fig. 7(c). Let us consider the simplest corrections shown in Fig. 12(b)–(d). For the contribution of graph of Fig. 12(b) we have

$$\begin{aligned}\Lambda^{(1)}_{1i} &= \tfrac{1}{3}\rho V^2 \mathrm{i} q_i v_{\mathrm{F}} p_{\mathrm{F}} \int \frac{\mathrm{d}^3 \boldsymbol{p}'}{(2\pi)^3} G(E\boldsymbol{p}')G(E+\omega\boldsymbol{p}'+\boldsymbol{q}) \\ &\approx \tfrac{1}{3}\mathrm{i} q_i v_{\mathrm{F}} p_{\mathrm{F}}[1+\mathrm{i}\omega/2\gamma - D_0 q^2/2\gamma] \approx \tfrac{1}{3}\mathrm{i} q_i v_{\mathrm{F}} p_{\mathrm{F}}\,, \quad \omega, q \to 0 \end{aligned} \tag{3.49}$$

and for the sum of graphs of Fig. 12(c), (d):

$$\begin{aligned}\Lambda^{(1)}_{2i} &= 2\rho V^2 \gamma \int \frac{\mathrm{d}^3 \boldsymbol{p}'}{(2\pi)^3} G(E\boldsymbol{p}')G(E+\omega\boldsymbol{p}'+\boldsymbol{q})p'_i \\ &\approx 2\rho V^2 \gamma q_i \int \frac{\mathrm{d}^3 \boldsymbol{p}'}{(2\pi)^3} p'_i G(E\boldsymbol{p}') \frac{\partial}{\partial p_i} G(E+\omega\boldsymbol{p}') \\ &\approx 2\gamma\rho V^2 q_i p_{\mathrm{F}} \int \frac{\mathrm{d}^3 \boldsymbol{p}'}{(2\pi)^3} \frac{v_{\mathrm{F}}}{3} G(E\boldsymbol{p}')G^2(E\boldsymbol{p}') = -\tfrac{1}{3}\mathrm{i} q_i v_{\mathrm{F}} p_{\mathrm{F}}\,. \end{aligned} \tag{3.50}$$

Thus for $\omega \to 0$, $q \to 0$ we obtain

$$\Lambda^{(1)}_{1i} + \Lambda^{(1)}_{2i} = 0 \tag{3.51}$$

and we have total cancellation of initial diagrams contributing to the diffusion ladder. Apparently there is no diffusion renormalization of the electron–phonon vertex (for $\omega, q \to 0$): this cancellation is valid for any graph obtained from the simplest corrections by adding further impurity lines to the ladder. Similar cancellation takes place when corrections due to maximally crossed impurity lines (Cooper channel) are added to diagrams of Fig. 12(b)–(d). Thus there is no significant change of the electron–phonon vertex due to Cooperon and the only relevant contribution to the electron–phonon vertex in an impure system is defined by the sum of diagrams of Fig. 11(b) leading to Eq. (3.48) which does not contain diffusion type renormalization. Localization appears via a generalized diffusion coefficient which replaces the Drude one. Thus localization singularities does not appear in the electron–phonon vertex, though surely this interaction is really changed by disorder scattering in comparison with the "pure" case. Of course, the question of whether localization effects contribute to a renormalization of the electron–phonon coupling is still open to discussion [136]. Probably, a more important aspect of this problem is reflected by the fact that superconductivity is actually determined not by the electron–phonon vertex itself, but by the famous integral expression over the phonon spectrum of the Eliashberg function $\alpha^2(\omega)F(\omega)$ which defines the pairing constant λ [139]. This integration will apparently smooth out all possible singularities.

In the following we shall model pairing interaction due to phonon exchange by some constant λ as in the BCS model. Of course, we must stress that this constant is different from that in a regular metal. It is constant in the sense that it does not contain singularities due to the metal–insulator transition. Electron–phonon kernel in the linearized gap equation (cf. Appendix B) can be taken in the simplest form

$$K_{\mathrm{ph}}(E,E') = \begin{cases} -\lambda & |E|,|E'| < \omega_{\mathrm{D}} , \\ 0 & |E|,|E'| > \omega_{\mathrm{D}} , \end{cases} \tag{3.52}$$

and consider λ as relatively weakly dependent on disordering. More detailed discussion of electron–phonon pairing in disordered systems can be found in Refs. [134, 136, 137].

As we mentioned above it is quite difficult to speculate on the disorder dependence of the pairing interaction in high-temperature superconductors. In the "marginal" Fermi-liquid approach [124, 125] pairing interaction can be modelled as in Eq. (3.52) with the replacement of Debye frequency ω_{D} by some phenomenological *electronic* frequency $\tilde{\omega}_{\mathrm{c}}$ which we briefly mentioned before while discussing localization in a "marginal" Fermi-liquid. In the following we shall just assume that this pairing interaction is weakly dependent on disorder as in the case of the phonon mechanism of pairing.

3.2.3. Metallic region

In the metallic region we can use Eqs. (3.34)–(3.36) and (3.38) and find the diffusional contribution to the Coulomb kernel:

$$\begin{aligned} K_{\mathrm{c}}^{\mathrm{diff}}(\omega) &= -\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} v_0 \,\mathrm{Im}\, \frac{1}{\omega + \mathrm{i}D_E q^2} \approx \frac{v_0}{2\pi^3}\left[\frac{1}{|D_E(\omega)l|} - \frac{|\omega|^{1/2}}{|D_E^{3/2}(\omega)|}\right] \\ &\approx \frac{v_0}{2\pi^3}\begin{cases} \dfrac{1}{D_E l} - \dfrac{|\omega|^{1/2}}{D_E^{3/2}} , & |\omega| \ll \omega_{\mathrm{c}} , \\ \dfrac{1}{D_0 l}\left(\dfrac{\omega}{2\gamma}\right)^{-1/3} , & |\omega| \gg \omega_{\mathrm{c}} . \end{cases} \end{aligned} \tag{3.53}$$

Accordingly, for the Coulomb kernel defined by Eq. (3.38) we get [22]

$$K_{\mathrm{c}}(\omega) = \mu\theta(E_{\mathrm{F}} - |\omega|) + \frac{\mu}{p_{\mathrm{F}} l}\begin{cases} \sigma_{\mathrm{c}}/\sigma & |\omega| < \omega_{\mathrm{c}} \\ (1/p_{\mathrm{F}} l)(\omega/2\gamma)^{-1/3} & \omega_{\mathrm{c}} < \omega < \gamma \sim E_{\mathrm{F}} \end{cases} . \tag{3.54}$$

Upper limit cut-off in the integral in Eq. (3.53) was taken $\sim l^{-1}$. Rough estimate of the contribution of higher momenta can be achieved introducing cut-off $\sim p_{\mathrm{F}}$ (cf. Ref. [153]). This will cancel $(p_{\mathrm{F}} l)^{-1}$ in Eq. (3.54). Close to the Anderson transition $l^{-1} \sim p_{\mathrm{F}}$ and this correction is irrelevant. We shall assume that far from transition these higher momenta corrections can be included in the definition of μ. From Eq. (3.54) we can see that diffusion renormalization of the Coulomb kernel leads to substantial growth of the Coulomb repulsion close to Anderson transition (i.e. when conductivity drops below σ_c – "minimal metallic conductivity").

Superconducting transition temperature T_c is determined by the linearized gap equation [152] which in the weak coupling approximation can be written as (cf. Appendix B) [157, 158]

$$\Delta(\omega) = \lambda\theta(\langle\omega\rangle - \omega)\int_0^{\langle\omega\rangle}\frac{d\omega'}{\omega'}\Delta(\omega')\tanh(\omega'/2T_c)$$

$$-\theta(E_F - \omega)\int_0^{E_F}\frac{d\omega'}{\omega'}K_c(\omega-\omega')\Delta(\omega')\tanh(\omega'/2T_c)\,. \tag{3.55}$$

Consider the metallic region and take $\omega_c \gg \langle\omega\rangle$ which in accordance with the ω_c estimate given in Eq. (3.37) roughly corresponds to $\sigma \geq \sigma_c$ for typical $E_F/\langle\omega\rangle \sim 10^2$, so that the system is not very close to the Anderson transition. The change of T_c due to the diffusion contribution in the Coulomb kernel (Eq. (3.54)) can be determined by perturbation theory over $K_c^{\text{diff}}(\omega)$ in the gap equation. First iteration of Eq. (3.55) gives

$$\frac{\delta T_c}{T_{co}} \approx \frac{\int_0^\infty (d\omega/\omega)\int_0^\infty (d\omega'/\omega')\Delta_0(\omega)\tanh(\omega/2T_{co})K_c^{\text{diff}}(\omega-\omega')\Delta_0(\omega')\tanh(\omega'/2T_{co})}{(1/2T_{co})\int_0^\infty d\omega\,[\Delta_0(\omega)]^2[\cosh(\omega/2T_{co})]^{-2}}\,, \tag{3.56}$$

where $\Delta_0(\omega)$ is the usual "two-step" solution of Eq. (3.55) [9, 152] which is valid for the standard form of the Coulomb kernel $K_c(\omega) = \mu\theta(E_F - |\omega|)$,

$$T_{co} = 1.13\langle\omega\rangle\exp(-1/(\lambda - \mu_0^*)) \tag{3.57}$$

is a critical temperature in the regular superconductor when the Coulomb pseudopotential is given by

$$\mu_0^* = \frac{\mu}{1 + \mu\ln(E_F/\langle\omega\rangle)}\,. \tag{3.58}$$

Using the first relation in Eq. (3.54) we get from Eq. (3.56)

$$\delta T_c/T_{co} \approx -\frac{\mu}{(\lambda - \mu_0^*)^2}\frac{1}{p_F l}\frac{\sigma_c}{\sigma}\,. \tag{3.59}$$

This change of T_c is equivalent to the following change of Coulomb pseudopotential [22]

$$\delta\mu^* \approx \mu\sigma_c^2/[\sigma(\sigma + \sigma_c)]\,, \tag{3.60}$$

where we have used Eq. (2.75) and $p_F l \approx \sigma_0/\sigma_c = (\sigma + \sigma_c)/\sigma_c$ to replace $p_F l$ in Eq. (3.59). As we noted above this later factor disappears from Eq. (3.59) if we use cut-off at $q \sim p_F$ in Eq. (3.53). According to Eq. (3.60) Coulomb pseudopotential μ^* grows as σ drops and this dependence is stronger than a similar one obtained in Ref. [20], which is connected with our use of the results of the self-consistent theory of localization. The method of Ref. [20] is based upon the use of the

q-dependence of the diffusion coefficient as given by Eq. (2.85). Our expression for $\delta\mu^*$ leads to a significant growth of μ^* for conductivities $\sigma \leq 10^3\ \Omega^{-1}\,\mathrm{cm}^{-1}$. This growth can easily explain the typical T_c degradation in "very dirty" superconductors as their conductivity in normal state drops upon approaching the Ioffe–Regel limit [30]. At the same time expressions for μ^* proposed in Ref. [20] can explain experimental data only under the assumption that a characteristic conductivity scale determining μ^* is an order of magnitude larger than the Ioffe–Regel limit, for which we see no serious grounds. A more extensive discussion can be found in Ref. [136].

Let us consider now the situation at the mobility edge itself, when $\sigma = 0$ and $\omega_c = 0$ so that $K_c(\omega)$ is determined by the second expression in Eq. (3.54) for all the frequencies below $\gamma \sim E_F$. In this case we can show [22] that the influence of the Coulomb repulsion on T_c is again described by the effective pseudopotential μ^* which can be estimated as

$$\mu^* \sim \alpha\mu(\langle\omega\rangle/2\gamma)^{-1/3}\,, \quad \alpha \sim 1\,. \tag{3.61}$$

In this case T_c may remain finite at the mobility edge only under very strict conditions: both $E_F \sim \gamma$ and μ must be very small, while λ must be at least close to unity. As a crude estimate we can demand something like $\lambda \sim 1$, $\mu \leq 0.2$ and $E_F \leq 10^3\,T_{c0}$. Obviously, only some narrow band superconductors like Chevrel phases can satisfy these conditions among traditional systems. High-T_c superconductors are especially promising. The experimental situation will be discussed later.

Using Eq. (3.60) and Eq. (3.61) we can write down a simple interpolation formula for the conductivity dependence of μ^* [22]:

$$\mu^* \approx \mu_0^* + \frac{\alpha\mu(\langle\omega\rangle/2\gamma)^{-1/3} - \mu_0^*}{1 + (\langle\omega\rangle/2\gamma)^{-1/3}\sigma(\sigma + \sigma_c)/\sigma_c^2}\,. \tag{3.62}$$

To get an expression via observable parameters, take into account $\langle\omega\rangle/\gamma \approx (\langle\omega\rangle/E_F)(1 + \sigma/\sigma_c)$. These expressions describe continuous crossover from the region of weak localization corrections to the vicinity of Anderson transition where its influence upon T_c becomes very strong. This crossover takes place at $\omega_c \sim \langle\omega\rangle$.

3.2.4. Localization region

Let us now consider an Anderson insulator. According to Eqs. (3.34) and (A.9) the Coulomb kernel acquires in this case a $\delta(\omega)$-contribution:

$$K_c^{\mathrm{loc}}(\omega) = v_0 A_E \delta(\omega) = v_0 \frac{1}{N(E)} \left\langle \sum_\nu \delta(E - \varepsilon_\nu)|\phi(\boldsymbol{r})|^2|\phi(\boldsymbol{r})|^2 \right\rangle, \tag{3.63}$$

$$A_E = A_E(\boldsymbol{r} - \boldsymbol{r}')|_{\boldsymbol{r} = \boldsymbol{r}'} \sim R_{\mathrm{loc}}^{-3} \tag{3.64}$$

which is actually connected with the "Hubbard-like" repulsion of electrons in a single quantum state becoming nonzero in the localization region [159, 98, 7]. This mechanism contributes in addition to the diffusion contributions in the Coulomb pseudopotential μ^* considered above, which are due to the "regular" part of the Gorkov–Berezinskii spectral density. Using Eq. (3.64) as a full Coulomb vertex in the linearized gap equation (3.55) we can solve it exactly [22] and find

$$\Delta(\omega) = \frac{\theta(\langle\omega\rangle - |\omega|)\Delta_1}{1 + (\mu A_E/2N(E))(1/\omega)\tanh(\omega/2T_c)}\,, \tag{3.65}$$

where

$$\Delta_1 = \lambda \int_0^{\langle\omega\rangle} \mathrm{d}\omega\, \Delta(\omega)(1/\omega)\tanh(\omega/2T_c) \tag{3.66}$$

and the equation for T_c takes the form

$$1 = \lambda \int_0^{\langle\omega\rangle} \mathrm{d}\omega \frac{\tanh(\omega/2T_c)}{\omega + (\mu A_E/2N(E))\tanh(\omega/2T_c)}\,. \tag{3.67}$$

To account for the "regular" diffusion contributions to μ^* we can just replace here $\lambda \to \lambda^* = \lambda - \mu^*$, where μ^* is given by Eq. (3.61). Then our equation for T_c can be approximately represented by [22]

$$\ln\frac{T^*}{T_c} \approx \psi\left(\frac{1}{2} + \frac{\mu A_E}{4T_c N(E)}\right) - \psi\left(\frac{1}{2}\right), \tag{3.68}$$

where $\psi(x)$ is digamma function, and T^* is taken to be equal to T_c of the system at the mobility edge which is given by Eq. (3.57) with μ_0^* replaced by μ^* from Eq. (3.61). Here we slightly overestimate the role of the Coulomb repulsion in the localization region. We can see that this additional "Hubbard-like" repulsion acts upon T_c as magnetic impurities [9, 152] with an effective spin-flip scattering rate:

$$\frac{1}{\tau_{\mathrm{sf}}} = \pi\mu A_E/N(E) \sim \mu/(N(E)R_{\mathrm{loc}}^3(E))\,. \tag{3.69}$$

Obviously, this result is connected with the appearance of singly occupied electron states of width [159, 98, 7, 160] $v_0 R_{\mathrm{loc}}^{-3}$ below the mobility edge of the "band". Superconductivity persists until $\tau_{\mathrm{sf}}^{-1} < 0.57T_c^*$, i.e. until

$$R_{\mathrm{loc}}(E) > [\mu/N(E)T_c^*]^{1/3} \sim (\xi_0 p_F^{-2})^{1/3} \sim (\xi_0 l^2)^{1/3}\,, \tag{3.70}$$

where the last estimates are valid for typical values of parameters and correspond to the simple estimate of Eq. (3.20). Thus the Coulomb repulsion in a single (localized) quantum state leads to a sharp reduction of T_c below the mobility edge even if superconductivity survived up to the Anderson transition. Another interpretation of this effect is the influence of "free" spins of Mott's band of singly occupied states below the Fermi level of an Anderson insulator.

Coulomb gap [41–44] effects can be neglected here [22] because according to the estimates given in Eqs. (2.14) and (2.15) the Coulomb gap width

$$\Delta_c \sim [N(E)R_{\mathrm{loc}}^3(E)]^{-1} \ll T_c \sim \Delta\,, \tag{3.71}$$

i.e. is small in comparison to the superconducting gap Δ (or T_c) under the conditions given by Eq. (3.20) which is necessary for the observation of superconductivity in the localization region.

3.2.5. *Spin fluctuations*

As we mentioned during our discussion of interaction effects upon the Anderson transition the role of magnetic fluctuations (spin effects) in general becomes stronger as we approach the metal–insulator transition. A band of single-occupied states is being formed below the Fermi level of the Anderson insulator, which is equivalent to the appearance of localized moments [159, 7, 160]. These effects actually may become important even before the metal–insulator transition [104, 112–114, 117, 118, 40] and lead to the additional mechanism of T_c degradation. Unfortunately, there is no complete theoretical understanding of these effects and accordingly only few estimates can be done concerning superconductivity. Here we shall mention only some of these crude estimates following Refs. [161–163].

In the framework of the Hubbard model with weak disorder it can be shown [161] that the spin susceptibility is represented by

$$\chi_s = \frac{\chi_0}{1 - UN(E) + \gamma_0 - \gamma'} = \frac{\chi_0}{\eta_0 - \gamma'} \equiv \frac{\chi_0}{\eta}, \tag{3.72}$$

where χ_0 is the spin susceptibility of free electrons, $\eta_0 = 1 - UN(E) + \gamma_0$ is the enhancement factor for the ordered case (U is the Hubbard interaction, γ_0 the correlation correction to the RPA approximation), γ' is the correction due to the interference of Hubbard interaction and disorder scattering:

$$\gamma' = B\lambda^2\,, \quad B = 6\sqrt{3}\pi^2[N(E)U]^2\{1 - \tfrac{1}{2}UN(E)\}\,. \tag{3.73}$$

Here $\lambda = 1/(2\pi E\tau) = 1/(p_F l)$ is the usual perturbation theory parameter for disorder scattering. As $\gamma' > 0$ we can see from Eq. (3.72) that disordering leads to a diminishing denominator $\eta = \eta_0 - \gamma'$. If we reach a critical disorder defined by

$$\lambda_c = \sqrt{\frac{\eta_0}{B}}\,, \quad p_F l|_c = 2\sqrt{\frac{B}{\eta_0}}\,, \tag{3.74}$$

we get $\chi_s \to \infty$. It should be stressed that this divergence of χ_s in a disordered system must not be identified with any kind of ferromagnetic instability but may signify something like the appearance of a spin-glass state or just of localized moments. In any case, it means the growth of spin dependent effects under disordering.

If the initial enhancement of spin susceptibility is strong enough (e.g. due to a large U), i.e. $\eta_0 \ll 1$, the critical disorder defined by Eq. (3.74) may be lower than the critical disorder for Anderson localization, appearing at $p_F l \sim 1$. Then these spin dependent effects may become important well before Anderson transition. In the opposite case, these effects will appear only very close to the metal–insulator transition. In the general case, the relation between these two transitions depends on parameters.

If spin fluctuations are strong enough ($\eta \ll 1$) a strong mechanism for T_c degradation in the superconducting state appears [162] analogous to the similar effect due to magnetic impurities [9, 152]

$$\ln\frac{T_{c0}}{T_c} = \psi\left(\frac{1}{2} + \rho\right) - \psi\left(\frac{1}{2}\right), \tag{3.75}$$

where [162]

$$\rho = \frac{9\sqrt{3}\pi}{2}\lambda^2\frac{UN(E)}{\eta} = \frac{9\sqrt{3}}{2}\left[\frac{UN(E)}{B}\right]\frac{\lambda^2}{\lambda_c^2 - \lambda^2}\,. \tag{3.76}$$

As ρ from Eq. (3.76) diverges as $(\lambda_c - \lambda)^{-1}$ for $\lambda \to \lambda_c$ superconducting transition, temperature T_c drops to zero.

If $\lambda_c \ll 1$, which is possible for $\eta_0 \ll 1$, superconductivity will be destroyed long before the metal–insulator transition. In the opposite case this mechanism may lead to its destruction on either side of the metal–insulator transition depending on the parameters of the system, such as U. In general, we need a more accurate analysis which must include the mutual interplay of magnetic fluctuations and disorder scattering leading to a metal–insulator transition. In any case, magnetic mechanisms of T_c degradation close to the metal–insulator transition may be as important as the Coulomb effects considered above.

3.3. *Ginzburg–Landau theory and Anderson transition*

3.3.1. *General analysis*

The main result of the previous analysis may be formulated as follows. Despite many mechanisms leading to T_c degradation and destruction of superconductivity in strongly disordered systems there seems to be no general rule prohibiting the possibility of a superconducting state in the Anderson insulator. Of course we must meet very rigid conditions if we hope to observe this rather exotic state. There is almost no chance to observe it in traditional superconductors but high-T_c systems seem promising. The following analysis will be based on the general assumption that T_c survives in a strongly disordered system or even in an Anderson insulator, i.e. that these strict conditions are met. Our aim is to study superconducting properties of such a strongly disordered system to determine specific characteristics which will make this case different from the usual case of "dirty" superconductors. We shall see that even before the Anderson transition there are significant deviations from the predictions of standard theory which make a strongly disordered system different. So, on the practical side, our aim is simply to generalize the usual theory of "dirty" superconductors for the case of a strong disorder in the sense of the mean free path of the order of the interatomic spacing or $l \sim p_F^{-1}$.

To claim that superconductivity is possible close to a disorder-induced metal–insulator transition it is not sufficient just to demonstrate the finite values of T_c. Even more important is to show the existence of the superconducting response to an external electromagnetic potential $\mathbf{A}$. In the general case, the analysis of response functions of a superconductor with strong disorder seems to be a difficult task. However, close to T_c, significant simplifications take place and actually we only have to show that the free-energy density of the system can be expressed in the standard Ginzburg–Landau form [164, 12, 9]:

$$F = F_n + A|\Delta|^2 + \tfrac{1}{2}B|\Delta|^4 + C|(\nabla - 2ie\mathbf{A}/\hbar c)\Delta|^2 , \tag{3.77}$$

where F_n is the free energy density of the normal state. Our problem is thus reduced to a microscopic deviation of expressions for the coefficients A, B, and C of the Ginzburg–Landau expansion Eq. (3.77) taking into account the possibility of electron localization. This will be the generalization of the famous Gorkov's derivation [12] of similar expressions for the case of "dirty" superconductors. Such an analysis was first done by Bulaevskii and Sadovskii [21, 22] and later by Kotliar and Kapitulnik [23, 24]. Recently, the same results were obtained by Kravtsov [166].

Within the BCS model, coefficients A and B which determine the transition temperature and the equilibrium value of the order-parameter Δ do not change in comparison with their values found in the theory of "dirty" superconductors, even if the system is close to the Anderson transition. This corresponds to the main statement of the Anderson theorem. Less trivial is the behavior of the coefficient C, which in fact defines the superconducting response. In the usual theory of "dirty" superconductors [12] this coefficient is proportional to the diffusion coefficient of electrons, i.e. to conductivity (at $T = 0$). As the Fermi level approaches the mobility edge, conductivity drops to zero. However, we shall see that the coefficient C remains finite in the vicinity of the Anderson transition, even in the region of localized states.

To derive Ginzburg–Landau coefficients we must know the two-electron Green's function in the normal state [12]. Let us introduce the following two-particle Matsubara Green's functions in momentum representation [22]:

$$\Psi_E(\boldsymbol{q}, \omega_m, \varepsilon_n) = -\frac{1}{2\pi\mathrm{i}} \sum_{\boldsymbol{p}_+\boldsymbol{p}'_-} \langle G(\boldsymbol{p}_+, \boldsymbol{p}'_+, -\varepsilon_n + \omega_m) G(-\boldsymbol{p}'_-, -\boldsymbol{p}_-, -\varepsilon_n)\rangle \tag{3.78}$$

$$\Phi_E(\boldsymbol{q}, \omega_m, \varepsilon_n) = -\frac{1}{2\pi\mathrm{i}} \sum_{\boldsymbol{p}_+\boldsymbol{p}'_-} \langle G(\boldsymbol{p}_+, \boldsymbol{p}'_+, -\varepsilon_n + \omega_m) G(\boldsymbol{p}'_-, \boldsymbol{p}_-, -\varepsilon_n)\rangle \tag{3.79}$$

where $\boldsymbol{p}_{+-} = \frac{1}{2}(\boldsymbol{p} \pm \boldsymbol{q})$ and $\omega_m = 2\pi m T$. Graphically, these functions are represented in Fig. 13. Then Ginzburg–Landau coefficients are defined by [12, 165]

$$A = (1/g) + 2\pi\mathrm{i}T \sum_{\varepsilon_n} \Psi_E(\boldsymbol{q} = 0\omega_m = 2\varepsilon_n) \tag{3.80}$$

$$C = \mathrm{i}\pi T \sum_{\varepsilon_n} \frac{\partial^2}{\partial q^2} \Psi_E(\boldsymbol{q}\omega_m = 2\varepsilon_n)|_{q=0} \,. \tag{3.81}$$

Thus the superconducting properties are determined by the Green's function Ψ_E describing the propagation of the electron (Cooper) pair. At the same time we have seen that the Green's function Φ_E determines transport properties of a normal metal and Anderson transition. In the case of time-invariance (i.e. in the absence of the external magnetic field or magnetic

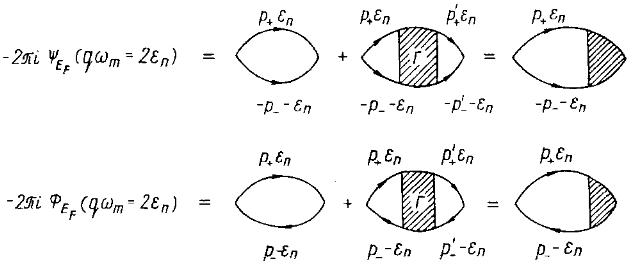


Fig. 13. Graphical representation of the two-particle Green's functions $\Psi_E(\boldsymbol{q}\omega_m)$ and $\Phi_E(\boldsymbol{q}\omega_m)$ (for $\omega_m = 2\varepsilon_n$). There is no summation over ε_n in the loops.

impurities) we have [71]

$$\Psi_E(\boldsymbol{q}\omega_m\varepsilon_n) = \Phi_E(\boldsymbol{q}\omega_m\varepsilon_n) \qquad (3.82)$$

and it is sufficient to know only $\Phi_E(\boldsymbol{q}\omega_m = 2\varepsilon_n)$ to determine the Ginzburg–Landau coefficients.

As a one-electron model of Anderson transition we can take the self-consistent theory of localization which will allow us to perform all calculations explicitly. We only have to formulate the main equations of this theory in the Matsubara formalism (finite T) [22]. For small q and ω_m, analogous to Eq. (2.46), we have

$$\Phi_E(\boldsymbol{q}\omega_m) = -\frac{N(E)}{\mathrm{i}|\omega_m| + D_E(|\omega_m|)q^2}\,, \quad \omega_m = 2\pi mT\,, \qquad (3.83)$$

where the generalized diffusion coefficient $D_E(\omega_m)$ is determined by the self-consistency equation analogous to Eq. (2.57):

$$\frac{D_0}{D_E(\omega_m)} = 1 - \frac{\mathrm{i}}{\pi N^2(E)} \sum_{|q| < k_0} \Phi_E(\boldsymbol{q}\omega_m)\,. \qquad (3.84)$$

In the three-dimensional case, Eq. (3.84) reduces to (cf. Eq. (2.77))

$$\frac{D_E(\omega_m)}{D_0} = 1 - \frac{\lambda}{\lambda_c} + \frac{\pi}{2}\frac{\lambda}{\lambda_c}\left[\frac{D_0}{D_E(\omega_m)}\frac{\omega_m}{2\gamma}\right]^{1/2}, \qquad (3.85)$$

where we have used the same notations as in our discussion of the self-consistent theory of localization. Analogous to Eq. (2.78) and with accuracy sufficient for our purpose we can write down the solution of Eq. (3.85) as

$$D_E(\omega_m) \approx \mathrm{Max}\left\{D_E\frac{\omega_m}{\omega_m + 3D_E\omega_0^2(E)/v_F^2}\,;\ D_0\left(\frac{\omega_m}{2\gamma}\right)^{1/3}\right\}, \qquad (3.86)$$

where D_E is the renormalized diffusion coefficient defined in Eq. (2.80) and ω_0 is the fundamental frequency defined by Eq. (2.64), which signals a transition to insulator.

As we have already noted Ginzburg–Landau coefficients A and B are given by the usual expressions valid also for "dirty" superconductors [12, 22]:

$$A = N(E_F)\ln\frac{T}{T_c} \approx N(E_F)\frac{T - T_c}{T_c}\,, \qquad (3.87)$$

where T_c is given by the usual BCS relation of Eq. (3.19), and

$$B = (7\zeta(3)/8\pi^2 T_c^2)N(E_F)\,, \qquad (3.88)$$

where $\zeta(x)$ is the Riemann zeta-function ($\zeta(3) = 1.202\ldots$). These coefficients depend on disorder only through the appropriate disorder dependence of $N(E_F)$ and are valid even in the localized phase. This is equivalent to the main statement of Anderson theorem.

Significant changes appear in the gradient term coefficient C. Using Eqs. (3.81)–(3.83) with Eq. (3.86) we can find that in different limiting cases this coefficient can be expressed as [21,22]:

$$C \equiv N(E_F)\xi^2 \approx N(E_F)\begin{cases} \dfrac{\pi}{8T_c} D_{E_F}\,, & \xi_{\mathrm{loc}}(E_F) < (\xi_0 l^2)^{1/3}\,;\; E_F > E_c\,, \\ \left(\dfrac{D_0 l}{T_c}\right)^{2/3} \approx (\xi_0 l^2)^{2/3}\,, & \xi_{\mathrm{loc}}(E_F) > (\xi_0 l^2)^{1/3}\,;\; E_F \sim E_c\,, \\ R^2_{\mathrm{loc}}(E_F) \ln \dfrac{1.78 D_{E_F}}{\pi T_c R^2_{\mathrm{loc}}(E_F)}\,, & R_{\mathrm{loc}}(E_F) < (\xi_0 l^2)^{1/3}\,;\; E_F < E_c\,, \end{cases} \tag{3.89}$$

where we have defined the coherence length ξ, and $\xi_0 = 0.18 v_F/T_c$ is the BCS coherence length, l as usual is the mean free path. Practically, the same results were obtained in Refs. [23,24] using the approach based upon elementary scaling theory of localization, which is as we already noted is equivalent to our use of the self-consistent theory of localization. In Ref. [166] the same results were confirmed using the σ-model approach to localization.

In the metallic state, as the Fermi level E_F moves towards the mobility edge E_c localization correlation length ξ_{loc} grows and the coefficient C initially drops as the generalized diffusion coefficient D_{E_F}, i.e. as conductivity of a system in the normal state. However, in the vicinity of Anderson transition, while $\sigma \to 0$ the drop in C saturates and it remains finite even for $E_F < E_c$, i.e. in Anderson insulator. With further lowering of E_F into the localization region (or E_c growth with disorder) the C coefficient is determined by the localization radius R_{loc} which diminishes as E_F moves deep into the insulating state. However, remembering Eqs. (3.20) and (3.22) we recognize that our analysis is valid only for large enough values of localization length, which satisfy Eq. (3.22). In this sense the last expression in Eq. (3.89) is actually outside these limits of applicability.

The finite value of the coefficient C in the Ginzburg–Landau expansion in the vicinity of Anderson transition signifies the existence of a superconducting (Meissner) response to an external magnetic field. Accordingly, for $T < T_c$ the system can undergo a transition from Anderson insulator to superconductor. The physical meaning of this result can be understood from the following qualitative picture (cf. Ref. [131] where similar estimates were used for the granular metal). In an Anderson insulator all electrons with energies E close to the Fermi level are localized in spatial regions of the size of $\sim R_{\mathrm{loc}}(E)$. Nearby regions are connected by a tunneling amplitude $\mathscr{V}$ which determines the probability of electron transition between such regions as

$$P_T \approx 2\pi |\mathscr{V}|^2 N(E) R^3_{\mathrm{loc}}(E)\,. \tag{3.90}$$

However, Anderson localization means that

$$|\mathscr{V}| < 1/(N(E) R^3_{\mathrm{loc}}(E)) \tag{3.91}$$

and coherent tunneling between states localized in these regions is impossible, and we have $P_T < 2\pi N^{-1}(E) R^{-3}_{\mathrm{loc}}$. At the same time if conditions given by Eqs. (3.20) or (3.22) are satisfied inside each region $\sim R_{\mathrm{loc}}$, Cooper pairs may form and a superconducting gap Δ appears in the spectrum. Then a kind of “Josephson” coupling appears between regions of localized states which determines the possibility of *pairs* tunneling:

$$E_J \approx \pi^2 [N(E) R^3_{\mathrm{loc}}(E)]^2 |\mathscr{V}|^2 \Delta\,. \tag{3.92}$$

It is easy to see that for

$$\Delta > (2/\pi)\, 1/(N(E) R^3_{\mathrm{loc}}(E)) \tag{3.93}$$

we have $E_J > P_T$, so that if Eq. (3.20) is satisfied we can get $E_J \gg N^{-1}(E) R^{-3}(E)$ irrespective of Eq. (3.91) and tunneling of pairs between nearby regions of localized states is possible, even in the absence of single-particle tunneling.

It is convenient to rewrite Eq. (3.89) using the relation between a generalized diffusion coefficient and conductivity similar to Eq. (2.59) as well as Eqs. (2.73), (2.75). Then using the Ginzburg–Landau expansion and the expressions for its coefficients we can easily find the temperature dependent coherence length $\xi(T)$ [9,21,22]:

$$\xi^2(T) = \frac{T_c}{T_c - T} \begin{cases} \xi_0 l \dfrac{\sigma}{\sigma + \sigma_c}\,, & \sigma > \sigma^* \quad (E_F > E_c)\,, \\ (\xi_0 l^2)^{2/3}\,, & \sigma < \sigma^* \quad (E_F \sim E_c)\,, \end{cases} \tag{3.94}$$

where $\sigma_c = e^2 p_F/(\pi^3 \hbar^2)$ and characteristic conductivity scale σ^* is given by

$$\sigma^* \approx \sigma_c (p_F \xi_0)^{-1/3} \approx \sigma_c (T_c/E_F)^{1/3}\,. \tag{3.95}$$

Thus in the region of very small conductivities $\sigma < \sigma^*$ the scale of $\xi(T)$ is defined not by $\xi \sim \sqrt{\xi_0 l}$ as in the usual theory of "dirty" superconductors [12,9] but by the new length $\xi \sim (\xi_0 l^2)^{1/3} \sim (\xi_0/p_F^2)^{1/3}$, which is now the characteristic size of Cooper pair close to Anderson transition.

If the $\omega^{1/3}$-law for a diffusion coefficient at the mobility edge is invalid and we have ω^{δ}-behavior, with some unknown critical exponent δ (which is possible because the modern theory actually cannot guarantee precise values of critical exponents at the Anderson transition [51,7]) we can easily show in a similar way that for conductivities $\sigma < \sigma^* \approx \sigma_c (p_F \xi_0)^{-\delta}$ the coherence length is defined by $\xi \sim \xi_0^{(1-\delta)/2} l^{(1+\delta)/2}$. Qualitatively, this leads to the same type of behavior as above.

From Eq. (3.94) we can see that $\xi^2(T)$ initially diminishes as we approach the metal–insulator transition proportionally to σ as in the case of a "dirty" superconductor. However, as in the metallic region for $\sigma < \sigma^*$ it diminishes more slowly remaining finite both at the transition itself and below.

The superconducting electron density n_s can be defined as [9]

$$n_s(T) = 8mC\Delta^2(T) = 8mC(-A)/B\,. \tag{3.96}$$

Close to the Anderson transition we can estimate

$$n_s \sim mN(E_F)\xi^2\Delta^2 \sim mp_F(\xi_0/p_F^2)^{2/3}\Delta^2 \sim n(T_c^{1/2}/E_F^2)^{2/3}(T_c - T)\,, \tag{3.97}$$

where $n \sim p_F^3$ is the total electron density. If we take here $T \sim 0.5T_c$, i.e. more or less low temperatures, we get a simpler estimate:

$$n_s \sim n(T_c/E_F)^{4/3} \tag{3.98}$$

which is actually valid up to $T = 0$, as we shall see below. From these estimates we can see that only a small fraction of electrons are superconducting in a strongly disordered case. However, this confirms the possibility of a superconducting response of the Anderson insulator.

Characteristic conductivity σ^* defined in Eq. (3.95) gives an important conductivity scale at which significant influence of localization effects upon superconducting properties appear [22]. While σ_c is of the order of Mott's "minimal metallic conductivity" [2, 3], σ^* is in general even lower. However, for small enough Cooper pairs (i.e. small ξ_0 which is characteristic of strong coupling and high-T_c superconductors) it is more or less of the order of σ_c. Experimentally, it can be defined as a conductivity scale at which significant deviations from predictions of the standard theory of "dirty" superconductors appear under disordering.

We must stress that these results show the possibility of Cooper pairs being delocalized in an Anderson insulator, while single-particle excitations of such a superconductor are apparently localized, which may lead to some peculiar transport properties of "normal" electrons for $T < T_c$. First attempts to explore this peculiar situation were undertaken in Refs. [167–170].

These results are easily generalized for the case of strongly anisotropic quasi-two-dimensional systems such as high-T_c superconducting oxides. Using the analysis of such systems within the self-consistent theory of localization [68] we can write down the following Matsubara generalization of Eq. (2.94):

$$\frac{D_j(\omega_m)}{D_j^0} \approx \begin{cases} \operatorname{Max}\left[\dfrac{E_F - E_c}{E_c};\ (2\pi E_F w\tau^2)^{-2/3}(\omega_m\tau)^{1/3}\right], & \omega_m \ll w^2\tau\ , \\ 1 - \dfrac{1}{2\pi E_F\tau}\ln\left(\dfrac{1}{\omega_m\tau}\right), & \omega_m \gg w^2\tau\ , \end{cases} \tag{3.99}$$

where $j = \parallel, \perp$. Now carrying out calculations similar to that of Ref. [22] we obtain for the coefficients of gradient terms in the Ginzburg–Landau expansion [16, 171]:

$$C_{\parallel,\perp} = N(E_F)\xi^2_{\parallel,\perp}\ , \tag{3.100}$$

where for the coherence lengths $\xi_{\parallel,\perp}$ we obtain a number of different expressions, depending on the value of the ratio $w^2\tau/2\pi T_c\hbar$ which determines as we shall see the "degree of two-dimensionality" of the problem under study. For the case of $w^2\tau/2\pi T_c\hbar \gg 1$, corresponding to an anisotropic but three-dimensional system, we have

$$\xi^2_{\parallel,\perp} = \frac{\pi}{8T_c} D^0_{\parallel,\perp}\left(\frac{E_F - E_c}{E_c}\right) \approx \xi^0_{\parallel,\perp} l_{\parallel,\perp}\left(\frac{E_F - E_c}{E_c}\right), \tag{3.101}$$

where $\xi^0_\parallel \sim \hbar v_F/T_c$, $\xi^0_\perp \sim wa_\perp/T_c$, $l_\parallel = v_F\tau$ and $l_\perp = wa_\perp\tau/\hbar$ are the longitudinal and transverse BCS coherence lengths and mean free paths. The above expressions are valid in the conductivity region $\sigma_\parallel > \sigma^*$, where

$$\sigma^* \sim \sigma^c_\parallel(\xi^0_\parallel/l_\parallel)(T_c^2/E_F w)^{2/3}\ . \tag{3.102}$$

where $\sigma^c_\parallel$ is defined as in Eq. (2.93). The condition of $w^2\tau/2\pi T_c\hbar \gg 1$ is equivalent to the requirement:

$$\xi_\perp \sim \sqrt{\xi^0_\perp l_\perp} \gg a_\perp \tag{3.103}$$

which clarifies its physical meaning: the transverse size of a Cooper pair must be much greater than the interplane lattice spacing. In this case we have just anisotropic three-dimensional superconductivity.

In the immediate vicinity of the Anderson transition, for $\sigma_\parallel < \sigma^*$ we have

$$\xi^2_{\parallel,\perp} \approx (1 - 2^{-5/3})(16\pi^4)^{-1/3}\zeta(5/3)\frac{D_{\parallel,\perp}}{(E_F T_c w)^{2/3}\tau} \approx (\xi^0_{\parallel,\perp})^2\left(\frac{T_c^2}{E_F w}\right)^{2/3} . \tag{3.104}$$

It is easy to see that for $w \sim E_F$ all these expressions naturally go over to those derived above for the three-dimensional case.

For $w^2\tau/2\pi T_c\hbar < 1$ which corresponds to the “almost two-dimensional” case of

$$\xi_\perp \sim \sqrt{\xi^0_\perp l_\perp} \leq a_\perp , \tag{3.105}$$

i.e. of transverse size of Cooper pairs smaller than the interplane spacing, we have

$$\xi^2_{\parallel,\perp} \approx \left\{\begin{matrix} \dfrac{D^0_{\parallel,\perp}}{\pi T_c}\dfrac{E_F - E_c}{E_c} , & (\sigma_\parallel > \sigma^*) \\ \dfrac{D^0_{\parallel,\perp}}{(4\pi^2 E_F T_c w)^{2/3}\tau} , & (\sigma_\parallel < \sigma^*) \end{matrix}\right\} + (\pi^2/8 - 1)\frac{D^0_{\parallel,\perp}}{\pi T_c}\left(1 - \frac{1}{2\pi E_F\tau}\ln\frac{1}{2\pi T_c\tau}\right) . \tag{3.106}$$

Essential difference from just the anisotropic case of Eqs. (3.101) and (3.104) is the appearance here of a second term of “two-dimensional” type. In a purely two-dimensional problem ($w = 0$) we have [165]:

$$\xi^2_\parallel = \frac{\pi D^0_\parallel}{8T_c}\left(1 - \frac{1}{2\pi E_F\tau}\ln\frac{1}{2\pi T_c\tau}\right) . \tag{3.107}$$

For high-T_c oxides it is reasonable to estimate $\xi^0_\parallel \sim l_\parallel$, $T_c \sim w$, $T_c \sim 0.1E_F$, so that $\sigma^* \sim \sigma^c_\parallel$, i.e. these systems are always more or less close to the Anderson transition. For $T_c \sim w$ and $\hbar/\tau \sim E_F$ which is characteristic of the rather strongly disordered case, we have $w^2\tau/2\pi T_c\hbar < 1$, so that for these systems we can realize almost two-dimensional behavior, though in general high-T_c oxides are apparently an intermediate case between the strongly anisotropic three-dimensional and nearly two-dimensional superconductors.

The significant change of Ginzburg–Landau coefficients and the new scale of coherence length close to the Anderson transition lead to an increased width of the critical region of thermodynamic fluctuations near T_c [23, 24]. These are well known to be important for any second-order phase transition. The width of the critical region is defined by the so-called Ginzburg criterion [45, 47] which may be expressed via the coefficients of Landau expansion. Mean-field approximation for the order parameter in Landau theory is valid (for $d = 3$) for [45, 47]

$$1 \gg \left|\frac{T - T_c}{T_c}\right| \gg \frac{B^2 T_c^2}{\alpha C^3} \equiv \tau_G , \tag{3.108}$$

where α is defined by $A = \alpha(T - T_c)/T_c$. In the case of a superconducting transition we have: $\alpha = N(E_F)$, $B \sim N(E_F)/T_c^2$ and $C = N(E_F)\xi^2$. Accordingly, from Eq. (3.108) we get the following estimate for the critical region:

$$\tau_G \sim \frac{1}{N^2(E_F)\xi^6 T_c^2} \sim \left(\frac{E_F}{T_c}\right)^2 \frac{1}{\xi^6 p_F^6} . \tag{3.109}$$

In the "pure" limit $\xi = \xi_0 \sim v_F/T_c$ and we get $\tau_G \sim (T_c/E_F)^4$, so that the critical region is practically unobservable. In a "dirty" superconductor $\xi \sim \sqrt{\xi_0 l}$ and

$$\tau_G \sim (T_c/E_F)[1/(p_F l)^3] \tag{3.110}$$

and again we have $\tau_G \ll 1$. However, for a superconductor close to the mobility edge $\xi \sim (\xi_0/p_F^2)^{1/3}$ and from Eq. (3.109) we get [23,24]

$$\tau_G \sim 1 \;. \tag{3.111}$$

Note that in fact τ_G may still be small because of numerical constants which we have dropped in our estimates. Anyhow, the critical region in this case becomes unusually wide and the superconducting transition becomes similar in this respect to the superfluid transition in Helium. Fluctuation effects may thus become observable even in a bulk three-dimensional superconductor. Note that in localized phase $\xi \sim R_{loc}$ and $\tau_G \sim [N^2(E_F)R_{loc}^6 T_c^2]^{-1} > 1$ if the condition given by Eq. (3.20) is violated.

Finally we should like to mention that thermodynamic fluctuations lead [23,24] to an additional mechanism of T_c degradation for a system which is close to Anderson transition. This follows from the general result on the reduction of mean-field transition temperature due to critical fluctuations. If these fluctuations are small (and we can use the so-called one-loop approximation) for a three-dimensional system it can be shown that [23,24]:

$$T_c = T_{c0} - \frac{7\zeta(3)}{16\pi^4\xi^3 N(E_F)}\,, \tag{3.112}$$

where T_{c0} is the mean-field transition temperature. If we use here our expressions for ξ valid close to the metal–insulator transition we easily find for $\sigma > \sigma^*$ [24]:

$$T_c \approx T_{c0}[1 - 0.5(\sigma_c/\sigma)^{3/2}(T_{c0}/E_F)^{1/2}] \;. \tag{3.113}$$

For $\sigma < \sigma^*$ this fluctuation correction saturates as the further drop of coherence length stops there. Obviously, higher-order corrections are important here, but unfortunately little is known on the importance of this mechanism of T_c degradation outside the limits of one loop approximation.

3.3.2. Upper critical field

Direct information on the value of $\xi^2(T)$ can be obtained from the measurements of the upper critical field H_{c2} [9]:

$$H_{c2} = \phi_0/2\pi\xi^2(T)\,, \tag{3.114}$$

where $\phi_0 = \pi c\hbar/e$ is the superconducting magnetic flux quantum. Using Eq. (3.94) we obtain the following relation between normal state conductivity σ, the slope of the upper critical field at $T = T_c$ given by $(\mathrm{d}H_{c2}/\mathrm{d}T)_{T_c}$ and the value of the electronic density of states at the Fermi level (per one spin direction) $N(E_F)$ [21,22]:

$$-\frac{\sigma}{N(E_F)}\left(\frac{\mathrm{d}H_{c2}}{\mathrm{d}T}\right)_{T_c} \approx \begin{cases} \dfrac{8e^2}{\pi^2\hbar}\phi_0\,, & \sigma > \sigma^*\,, \\ \phi_0\dfrac{\sigma}{N(E_F)(\xi_0 l^2)^{2/3}T_c} \approx \phi_0\dfrac{\sigma}{[N(E_F)T_c]^{1/3}}, & \sigma < \sigma^*\,. \end{cases} \tag{3.115}$$

For $\sigma > \sigma^*$ the r.h.s. of Eq. (3.115) contains only the fundamental constants. This so-called Gorkov's relation [12] is often used to interpret experimental data in "dirty" superconductors. Using it we may find $N(E_F)$ for different degrees of disorder from measurements of $(dH_{c2}/dT)_{T_c}$ and conductivity σ. On the other hand, $N(E_F)$ can in principle be determined from independent measurements e.g. of the electronic contribution to specific heat. However, our expression for $\sigma < \sigma^*$ which is valid close to the metal–insulator transition shows that in this region Gorkov's relation becomes invalid and its use can "simulate" the drop of $N(E_F)$ with the growth of resistivity (disorder). Roughly speaking, Eq. (3.115) shows that under the assumption of relatively smooth change of $N(E_F)$ and T_c with disorder the usual growth of $(dH_{c2}/dT)_{T_c}$ with disorder saturates in the conductivity region of $\sigma < \sigma^*$ close to the Anderson transition and the slope of the upper critical field becomes independent of resistivity. This stresses the importance of independent measurements of $N(E_F)$.

Note that the qualitative behavior given by Eq. (3.115) is retained also when the dependence of the diffusion coefficient at the mobility edge is ω^δ (with some arbitrary critical exponent δ); only the expression for σ^* is changed as noted above. Thus this behavior is not related to any specific approximations of the self-consistent theory of localization, except the general concept of continuous transition.

For an anisotropic (quasi-two-dimensional) system we have similar relations:

$$\left(\frac{dH_{c2}^{\perp}}{dT}\right)_{T_c} = -\frac{\phi_0}{2\pi\xi_{\parallel}^2 T_c}, \tag{3.116}$$

$$\left(\frac{dH_{c2}^{\parallel}}{dT}\right)_{T_c} = -\frac{\phi_0}{2\pi\xi_{\parallel}\xi_{\perp} T_c} \tag{3.117}$$

with $\xi_{\parallel,\perp}$ given in our discussion above after Eq. (3.100). This leads to relations and qualitative behavior similar to Eq. (3.115). However, we should like to note an especially interesting relation for the anisotropy of the slopes of the upper critical field [16]:

$$\frac{(dH_{c2}^{\parallel}/dT)_{T_c}}{(dH_{c2}^{\perp}/dT)_{T_c}} = \frac{\xi_{\parallel}}{\xi_{\perp}} = \frac{v_F}{wa/\hbar}. \tag{3.118}$$

We see that the anisotropy of $(dH_{c2}/dT)_{T_c}$ is actually determined by the anisotropy of the Fermi velocity irrespective of the superconductivity regime: from the "pure" limit, through the usual "dirty" case, up to the vicinity of the Anderson transition.

The above derivation of C coefficient of Ginzburg-Landau expansion explicitly used the time-reversal invariance expressed by Eq. (3.82). This is valid in the absence of the external magnetic field and magnetic impurities. Accordingly the previous results for the upper critical field are formally valid in the limit of infinitesimal external field and this is sufficient for the demonstration of superconducting (Meissner) response and for the determination of $(dH_{c2}/dT)_{T_c}$, because $H_{c2} \to 0$ as $T \to T_c$. In a finite external field we must take into account its influence upon localization. The appropriate analysis was performed in Refs. [75, 172] and with a slightly different method in Ref. [80]. The results are essentially similar and below we shall follow Ref. [75]. The standard scheme for the analysis of the superconducting transition in an external magnetic field

[12,9,173,174] gives the following equation determining the temperature dependence of $H_{c2}(T)$:

$$\ln\frac{T}{T_c} = 2\pi T\sum_{\varepsilon_n}\left\{\frac{1}{2|\varepsilon_n| + 2\pi D_2(2|\varepsilon_n|)H/\phi_0} - \frac{1}{2|\varepsilon_n|}\right\}, \tag{3.119}$$

where $D_2(2|\varepsilon_n|)$ is the generalized diffusion coefficient in the Cooper channel as defined after Eqs. (2.97) and (2.98). Eq. (3.119) is valid [9] for

$$R_H = mcv_F/eH \gg \xi , \tag{3.120}$$

where R_H is the Larmor radius of an electron in a magnetic field and ξ the coherence length. Note that Eq. (3.119) describes only the orbital motion contribution to H_{c2}. In fact H_{c2} is also limited by the paramagnetic limit [9,174]:

$$\tfrac{1}{2}g_0\mu_B H < \Delta , \tag{3.121}$$

where g_0 is the usual g-factor of an electron and μ_B the Bohr magneton.

Standard approach of the theory of "dirty" superconductors is based upon the replacement of $D_2(2|\varepsilon_n|)$ in Eq. (3.119) by the Drude diffusion coefficient D_0 which is valid for a metal with $l \gg p_F^{-1}$. For a system which is close to the Anderson transition we must take into account both the frequency dependence of the diffusion coefficient and the fact that in a magnetic field D_2 is not equal to the usual diffusion coefficient determining electronic transport, D_1. Actually, we shall see that the external magnetic field's influence upon localization leads to rather small corrections to $H_{c2}(T)$ practically everywhere except in the region of localized states [75]. Thus we may really neglect this influence as a first approximation as was done in Refs. [21,22] and start with the replacement of D_2 in Eq. (3.119) by $D_1 = D_E$, where D_E is the frequency dependent generalized diffusion coefficient in the absence of magnetic field. Detailed analysis of Eq. (3.119) can be found in Ref. [75].

Summation over Matsubara frequencies in Eq. (3.119) must be cutoff at some frequency of the order of $\langle\omega\rangle$ the characteristic frequency of Bose excitations responsible for the pairing interaction. It is convenient here to measure the distance from the Anderson transition (degree of disorder) via frequency ω_c defined in Eqs. (2.42), (2.79) or Eq. (2.106). If a system is far from Anderson transition, so that $\omega_c \gg \langle\omega\rangle$ we can completely neglect the frequency dependence of the diffusion coefficient and find the usual results of the theory of "dirty" superconductors:

$$H_{c2}(T) = \frac{4}{\pi^2}\frac{\phi_0 T_c}{D_0}\ln\frac{T_c}{T}, \quad T \sim T_c , \tag{3.122}$$

$$H_{c2} = \frac{1}{2\gamma}\frac{\phi_0 T_c}{D_0}\left[1 - \frac{1}{24}\left(\frac{4\gamma T}{T_c}\right)^2\right], \quad T \ll T_c , \tag{3.123}$$

where $\gamma = 1.781]\ldots$. For the H_{c2} derivative at $T = T_c$ we find from here the first relation of Eq. (3.115), and $H_{c2}(T = 0)$ is conveniently expressed as [12,173]

$$-H_{c2}(0)/[T_c(\mathrm{d}H_{c2}/\mathrm{d}T)_{T_c}] = \pi^2/8\gamma \approx 0.69 . \tag{3.124}$$

In this case, the $H_{c2}(T)$ curve is convex at all temperatures below T_c [12,173,9,174]. Very close to the Anderson transition, when $\omega_c \ll 2\pi T$, only $\omega^{1/3}$ behavior of the diffusion coefficient is important in Eq. (3.119) and it takes the following form [75]:

$$\ln\frac{T}{T_c}=\sum_{n=0}^{\infty}\{[(n+\tfrac{1}{2})+(n+\tfrac{1}{2})^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)]^{-1}-[n+\tfrac{1}{2}]^{-1}\}\,, \tag{3.125}$$

where $\omega_H = eH/mc$. From the above we get

$$H_{c2}(T)=m(\phi_0/\pi)[(4\pi)^{2/3}c_1]\,T^{2/3}E^{1/3}\ln(T_c/T)\,,\quad T\sim T_c\,, \tag{3.126}$$

$$H_{c2}(T)=m(\phi_0/\pi)(\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}[1-\tfrac{2}{3}c_2(4\gamma T/T_c)^{2/3}]\,,\quad T\ll T_c\,, \tag{3.127}$$

where $c_1=\sum_{n=0}^{\infty}(n+\frac{1}{2})^{-5/3}\approx 4.615$ and $c_2\approx 0.259$. From these expressions we get

$$-\frac{1}{N(E)}\left(\frac{\mathrm{d}H_{c2}}{\mathrm{d}T}\right)_{T_c}=\frac{(4\pi)^{2/3}}{\pi c_1}m\phi_0(E/T_c)^{1/3}=\frac{2\pi}{c_1}\frac{\phi_0}{[N(E)T_c]^{1/3}} \tag{3.128}$$

which makes precise the second relation in Eq. (3.115), while for $H_{c2}(T=0)$ we obtain

$$-H_{c2}(0)/(T_c(\mathrm{d}H_{c2}/\mathrm{d}T)_{T_c})=c_1/(4\gamma)^{2/3}\approx 1.24\,. \tag{3.129}$$

As was first noted in Refs. [21,22] this ratio for the system at the mobility edge is significantly larger than its classical value 0.69. In this case $H_{c2}(T)$ curve is concave for all temperatures below T_c [22]. Detailed expressions for the intermediate disorder when $2\pi T \ll \omega_c \ll \langle\omega\rangle$ can be found in Ref. [75].

In Fig. 14 we present the results of the numerical solution of Eq. (3.119) for different values of the characteristic frequency ω_c, i.e. for different disorders. A smooth crossover from the classical behavior of the theory of "dirty" superconductors [173, 9, 174] to anomalous temperature dependence close to the Anderson transition [22] is clearly seen.

Below the mobility edge (i.e. in the Anderson insulator) and for $\omega_c = 1/(2\pi^2 N(E)R_{\mathrm{loc}}^3) \ll 2\pi T$, i.e. very close to the mobility edge we can again use the $\omega^{1/3}$ behavior of the diffusion coefficient and find the same temperature dependence of H_{c2} as at the mobility edge itself or just above it. For $2\pi T \ll \omega_c \ll 2\pi T_c$ Eq. (3.119) takes the form [75]

$$\begin{aligned}\ln\frac{T}{T_c}=&\sum_{n=0}^{n_0-1}\{(n+\tfrac{1}{2})[1+(E/\omega_c)^{2/3}(\omega_H/E)]\}^{-1}\\&+\sum_{n=n_0}^{\infty}\{(n+\tfrac{1}{2})+(n+\tfrac{1}{2})^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)\}^{-1}-\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{-1}\,,\end{aligned} \tag{3.130}$$

where $n_0=\omega_c/4\pi T_c$ corresponds to a change of frequency behavior of the diffusion coefficient. Defining $x=\omega_H/\omega_c^{2/3}E^{1/3}$ we can reduce Eq. (3.130) to

$$\ln(T/T_c)=x\ln(\gamma\omega_c/\pi T_c)+\tfrac{3}{2}(1+x)\ln(1+x) \tag{3.131}$$

which implicitly defines $H_{c2}(T)$ and shows [75] that now $H_{c2}(T)\to\infty$ for $T\to 0$ (logarithmic divergence). Numerical solution of Eq. (3.130) is shown in the inset in Fig. 14. Below we shall see, however, that this divergence of H_{c2} is lifted by the inverse influence of magnetic field upon diffusion.

Let us now turn to the problem of the influence of magnetic field upon diffusion and its consequences for H_{c2} temperature behavior. If we are far from the Anderson transition, magnetic

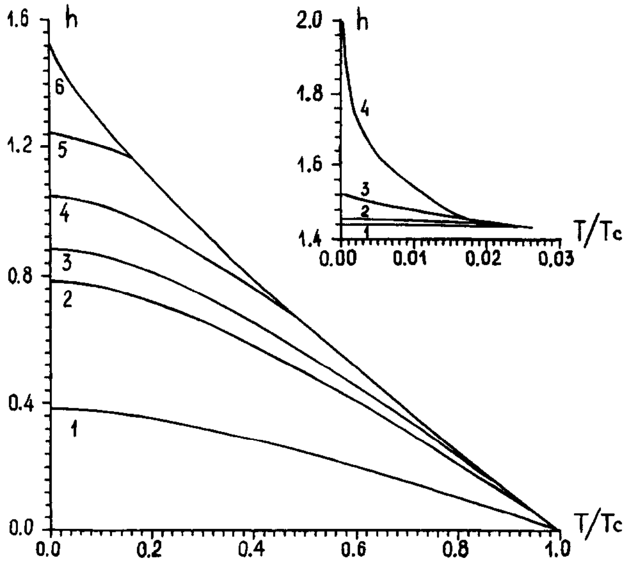


Fig. 14. Temperature dependence of the upper critical field H_{c2}. Numerical solution for the dependence of $h = \omega_H/T_c^{2/3}E_{1/3}$ on T/T_c for different values of $\theta = \omega_c/T_c$: (1) $\theta = 100$; (2) $\theta = 10$; (3) $\theta = 2\pi$; (4) $\theta = 3$; (5) $\theta = 1$; (6) $\theta = 0$ (mobility edge). Metallic state, no magnetic field influence on diffusion. At the inset: Low temperature dependence of h on T/T_c close to the Anderson transition; mobility edge ($\theta = 0$) with magnetic field influence on diffusion; metallic phase ($\theta = 0.1$), no magnetic field influence; mobility edge ($\theta = 0$), no magnetic field influence; and insulating phase ($\theta = 0.1$), no magnetic field influence. Numerical cut-off was taken at $\langle\omega\rangle = 100T_c$.

field influence is small on parameter $\sim\sqrt{\omega_H/E}$ and its influence upon H_{c2} is insignificant. Close to the transition, magnetic field correction may overcome the value of $D(H = 0)$ and we have to consider its influence in detail [75]. Accordingly, we shall limit ourselves to the case of $\omega_c/E \ll (\omega_H/E)^{2/3}$ for which we have already discussed the magnetic field behavior of the generalized diffusion coefficient in the Cooper channel. This was given in Eqs. (2.110) and (2.111). In this case, we have seen that the characteristic frequency ω_c is replaced by

$$\omega_c' = (\varphi\omega_H/E)^{3/2}E\ , \tag{3.132}$$

where $\varphi = \frac{1}{2}W^2 \approx 0.18$. ($W$ was defined during our discussion of localization in a magnetic field.) For $T \sim T_c$ there is no change in the slope of H_{c2} given by Eq. (3.128) as was noted already in Ref. [22]. Here we shall consider the case of $T \ll T_c$.

For $2\pi T > \omega_c'$ in all sums over Matsubara frequencies we can take $D(\omega) \sim \omega^{1/3}$ and actually we can neglect the magnetic field's influence upon diffusion. In this case, $H_{c2}(T)$ behaves as in Eq. (3.127) i.e. as at the mobility edge in the absence of magnetic field effects. For $2\pi T < \omega_c'$ the equation for $H_{c2}(T)$ takes the form [75]

$$\ln\frac{T}{T_c} = \sum_{n=0}^{n_0-1}\left[(n+\tfrac{1}{2}) + (\omega_c/E)^{1/3}(\omega_H/4\pi T)\right]^{-1} + \sum_{n=n_0}^{\infty}\left\{(n+\tfrac{1}{2}) + (n+\tfrac{1}{2})^{1/3}(E/4\pi T)^{2/3}(\omega_H/E)\right\}^{-1} - \sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{-1} \tag{3.133}$$

where $n_0 = \omega_c'/4\pi T$. In this case we find

$$H_{c2}(T) = m\frac{\phi_0}{\pi}(1+\varphi)^{-1/3}(\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}\left[1-\frac{4\gamma}{3\varphi^{1/3}(1+\varphi)}\frac{T}{T_c}\right]. \tag{3.134}$$

Accordingly, we have

$$-\frac{H_{c2}(0)}{T_c(\mathrm{d}H_{c2}/\mathrm{d}T)_{T_c}} = (1+\varphi)^{-1/3}\frac{c_1}{(4\gamma)^{2/3}} \approx 1.18 \tag{3.135}$$

and the change when compared with Eq. (3.129) is actually small. However, for $2\pi T < \omega_c'$ the $H_{c2}(T)$ curve becomes convex. The inflexion point can be estimated as $T^* = \omega_c'/2\pi \approx 0.02T_c$. This behavior is shown in the inset of Fig. 14.

Consider now the insulating region. We shall see that the magnetic field effects on diffusion lead to the effective cut-off of the weak divergence of H_{c2} as $T \to 0$ noted above. Generalized diffusion coefficient D_2 in the insulating phase and at low enough frequencies is determined by the following equation [75]:

$$2mD_2 = -(\omega_c/E)^{1/3} + (-\mathrm{i}\omega/E)^{1/2}(2mD_2)^{-1/2} + \tfrac{1}{2}W(2\omega_H/E)^{1/2}. \tag{3.136}$$

Now we can see that the external field defined by

$$\frac{W}{2}\sqrt{2\omega_H/E} > (\omega_c/E)^{1/3} \tag{3.137}$$

transfers the system from the insulating to the metallic state. If the system remains close to the mobility edge we can estimate the upper critical field as above by $\omega_H \approx (\pi/\gamma)^{2/3}T_c^{2/3}E^{1/3}$ and Eq. (3.137) reduces to

$$\omega_c \approx \frac{1}{2\pi^2 N(E)R_{\mathrm{loc}}^3} < \frac{\pi}{\gamma}(W/\sqrt{2})^3 T_c \approx 0.14T_c \tag{3.138}$$

and practically in the entire interval of localization lengths, where according to our main criterion of Eq. (3.20) we can have superconductivity in an Anderson insulator, *the upper critical field in fact destroys localization* and the system becomes metallic. Accordingly there is no way to observe the divergence of the upper critical field as $T \to 0$ and the $H_{c2}(T)$ curves in the "insulating" phase all belong to the region between the curves of $H_{c2}(T)$ at the mobility edge defined in the absence of magnetic field (curve 3 in the inset on Fig. 14) and at the mobility edge defined in the magnetic field (curve 1 in the inset). This result actually shows that it may be difficult to confirm the insulating ground state of a strongly disordered superconducting system just by applying a strong enough magnetic field to destroy superconductivity and perform the usual transport measurements at low temperatures.

Note that another mechanism for the change of $H_{c2}(T)$ at low temperatures was proposed by Coffey et al. [175]. They have found the enhancement of H_{c2} at low temperatures due to the magnetic field dependence of the Coulomb pseudopotential μ^* which appears via the magnetic field dependence of the diffusion coefficient. Magnetic field suppression of localization effects leads to the reduction of Coulomb pseudopotential enhancement due to these effects [20]. Accordingly, we get the enhancement of H_{c2} at low temperatures. Unfortunately, the apparently more important effects of the frequency dependence of the generalized diffusion coefficient were dropped.

Returning to the general criteria for the validity of Eq. (3.119) we note that the condition of $R_H \gg \xi$ is reduced to $\omega_H \ll T_c^{1/3} E_F^{2/3}$ which is obviously satisfied in any practical case. Note, however, that our estimates for H_{c2} at low temperatures lead to $\omega_H = \Delta_0 (E_F/\Delta_0)^{1/3} > \Delta_0$ which can easily overcome the paramagnetic limit. In this case, the experimentally observed H_{c2} of course will be determined by the paramagnetic limit and the anomalous behavior due to localization will be unobservable at low temperatures. At the same time in case of H_{c2} being determined by the paramagnetic limit it may become possible to obtain the insulating ground state of the system by applying a strong enough magnetic field. Note that the effective masses entering to cyclotron frequency and the paramagnetic splitting may actually be very different and there may be realistic cases when the orbital critical field may dominate at low T. For $T \sim T_c$, H_{c2} is always determined by the orbital contribution.

Similar analysis can be performed for the two-dimensional and quasi-two-dimensional cases [81], which are important mainly due to the quasi-two-dimensional nature of high-temperature superconductors. We shall limit ourselves only to the case of a magnetic field perpendicular to the highly conducting planes, when the temperature dependence of $H_{c2}(T)$ is again determined by Eq. (3.119) with $D_2(\omega)$ having the meaning of a diffusion coefficient in the Cooper channel along the plane.

If we neglect the magnetic field influence upon diffusion the frequency dependence of the diffusion coefficient in the purely two-dimensional case is determined by Eq. (2.115). It is easy to see that the possible anomalies in the temperature behavior of the upper critical field due to the frequency dependence of the diffusion coefficient will appear only at temperatures $T \ll \mathrm{e}^{-1/\lambda}/\tau$. At higher temperatures we obtain the usual dependence of the "dirty" limit. Accordingly, from Eq. (3.119) we obtain two different types of behavior of $H_{c2}(T)$:

1. For $T_c \gg \mathrm{e}^{-1/\lambda}/\tau$

$$H_{c2}(T) = \frac{4}{\pi^2} \frac{\phi_0}{D_0} T \ln\left(\frac{T_c}{T}\right) \quad \text{for } T \sim T_c\,, \tag{3.139}$$

$$H_{c2}(T) = \frac{1}{2\gamma} \frac{\phi_0 T_c}{D_0} \left(1 - 2.12\left(\frac{T}{T_c}\right)^2\right) \quad \text{for } \frac{\mathrm{e}^{-1/\lambda}}{\tau} \ll T \ll T_c\,. \tag{3.140}$$

For $T \ll \mathrm{e}^{-1/\lambda}/\tau$ the upper critical field is defined by the equation:

$$\ln\left(\frac{\gamma}{2\pi} \frac{\mathrm{e}^{-1/\lambda}}{\tau T}\right) = \left(1 + 4\pi \frac{D_0}{\phi_0} \frac{\tau H_{c2}}{\mathrm{e}^{-1/\lambda}}\right) \ln\left(\frac{\gamma}{2\pi} \frac{\mathrm{e}^{-1/\lambda}}{\tau T_c} \left(1 + 4\pi \frac{D_0}{\phi_0} \frac{\tau H_{c2}}{\mathrm{e}^{-1/\lambda}}\right)\right) \tag{3.141}$$

from which we can explicitly obtain the dependence of $T(H_{c2})$.
Thus, up to very low temperatures of the order of $\sim \mathrm{e}^{-1/\lambda}/\tau$, the upper critical field is determined by the Drude diffusion coefficient and we obtain the standard $H_{c2}(T)$ dependence of a "dirty" superconductor. The ratio $-H_{c2}(T)/[T_c(\mathrm{d}H_{c2}/\mathrm{d}T)]_{T_c}$ for $\mathrm{e}^{-1/\lambda}/\tau \ll T \ll T_c$ is equal to the usual value of 0.69. For low temperatures $T \ll \mathrm{e}^{-1/\lambda}/\tau$ we obtain significant deviations from the predictions of the usual theory of "dirty" superconductors. $H_{c2}(T)$ dependence acquires positive

curvature and the upper critical field diverges as $T \to 0$. The behavior of the upper critical field for the case of $T_c \gg e^{-1/\lambda}/\tau$ is shown in Fig. 15, curve 1.

2. For $T_c \ll e^{-1/\lambda}/\tau$ the upper critical field behavior for any temperature is defined by Eq. (3.141). $H_{c2}(T)$-dependence acquires positive curvature and H_{c2} diverges for $T \to 0$. For small fields $H_{c2} \ll (\phi_0/D_0)\, e^{-1/\lambda}/\tau$, i.e. for $T \sim T_c$, Eq. (3.141), gives the explicit expression for H_{c2}:

$$H_{c2} = \frac{1}{4\pi}\frac{\phi_0}{D_0}\frac{e^{-1/\lambda}}{\tau}\frac{\ln\left(\dfrac{T_c}{T}\right)}{\ln\left(\dfrac{\gamma}{2\pi}\dfrac{e^{-1/\lambda}}{\tau T}\right)}. \tag{3.142}$$

The slope of $H_{c2}(T)$ at the superconducting transition is determined by

$$-\frac{\sigma_0}{N(E)}\left(\frac{dH_{c2}}{dT}\right)_{T_c} = \frac{e^2}{2\pi}\phi_0\frac{e^{-1/\lambda}}{\tau T_c \ln\left(\dfrac{\gamma}{2\pi}\dfrac{e^{-1/\lambda}}{\tau T_c}\right)}. \tag{3.143}$$

The behavior of the upper critical field for the case of $T_c \ll e^{-1/\lambda}/\tau$ is shown in Fig. 16, curve 1.

It is clearly seen from Eqs. (2.115) and (2.116) that the magnetic field's influence upon diffusion becomes relevant only for high enough magnetic fields $H_{c2} \gg (\phi_0/D_0)(e^{-1/\lambda}/\tau)$, i.e. for very low

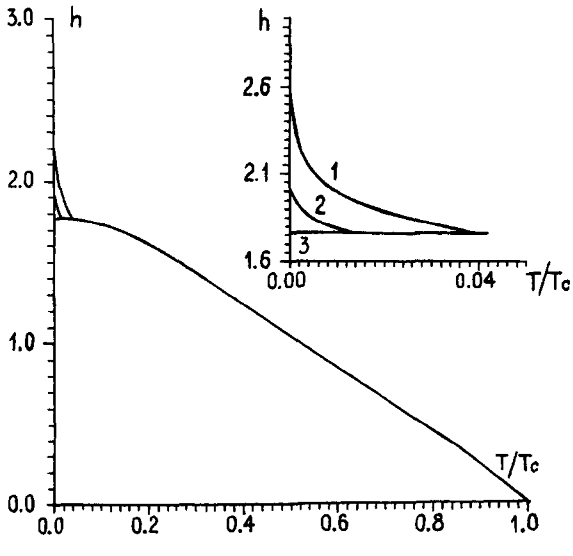


Fig. 15. Temperature dependence of the upper critical field for the two-dimensional superconductor $(e^{-1/\lambda}/\tau T_c) = 0.4$, $\lambda = 0.1$, $h = \omega_H/\pi\lambda T_c$): (1) no magnetic field influence upon diffusion, (2) with magnetic field influence upon diffusion and (3) standard theory of "dirty" superconductors.

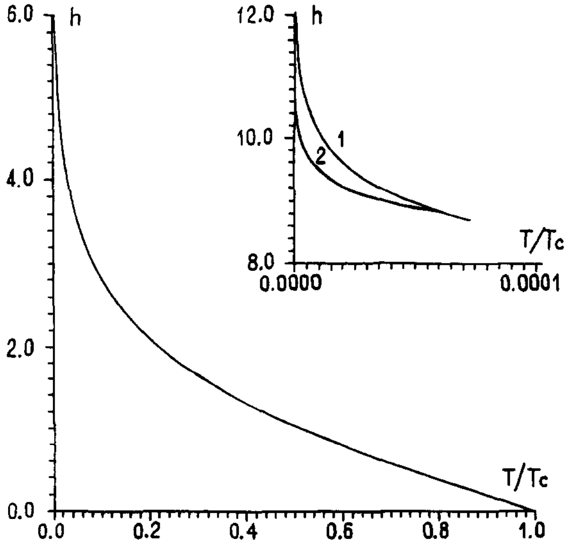


Fig. 16. Temperature dependence of the upper critical field for the two-dimensional superconductor $(e^{-1/\lambda}/\tau T_c = 4$, $\lambda = 0.126$, $h = \omega_H/\pi\lambda T_c)$: (1) no magnetic field influence upon diffusion and (2) with magnetic field influence upon diffusion.

temperatures $T \ll T_c$. If we use Eq. (2.116) in the main equation (3.119), we obtain the following results:

1. The case of $e^{-1/\lambda}/\tau \ll T_c \ll 1/\tau$: For high enough temperatures $T \gg e^{-1/\lambda}/\tau$ the diffusion coefficient entering Eq. (3.119) coincides with Drude's D_0 and the upper critical field is determined by Eqs. (3.139) and (3.140).

For $\exp[-1/\lambda^2 \ln(\gamma^2/\pi)(1/\tau T_c)]/\tau \ll T \ll e^{-1/\lambda}/\tau$ we obtain:

$$H_{c2}(T) = (1/2\gamma)(\phi_0 T_c/D_0)(1 - 3.56(T/T_c)) . \tag{3.144}$$

Eq. (3.144) differs from Eq. (3.140) only by the temperature dependent corrections and we can say that the magnetic field's influence upon diffusion in this case leads to the widening of the temperature region where we can formally apply the usual theory of "dirty" superconductors.

For $T \ll \exp[-1/\lambda^2 \ln(\gamma^2/\pi)(1/\tau T_c)]/\tau \ll T \ll e^{-1/\lambda}/\tau$ the upper critical field is defined by

$$\ln\left(\frac{\gamma}{2\pi e}\frac{e^{-1/\lambda^2 \ln Q}}{\tau T}\right) = \frac{2\gamma}{Q}\frac{\lambda \ln Q}{e^{-1/\lambda^2 \ln Q}} \ln\left(\frac{\gamma^2}{\pi Q}\frac{1}{\tau T_c}\right) , \tag{3.145}$$

where $Q = (\gamma/2\pi)(\phi_0/D_0 H_{c2})(1/\tau)$. From Eq. (3.145) we can obtain the explicit dependence $T(H_{c2})$. The upper critical field in this case is slightly concave as in Eq. (3.141) where we have neglected the magnetic field influence upon diffusion. However, now we have no divergence of H_{c2} for $T \to 0$ and

$$H_{c2}(T = 0) = (\gamma/2\pi)(\phi_0/D_0)(1/\tau) . \tag{3.146}$$

In fact, the value of $H_{c2}(T = 0)$ will be even smaller, because for these values of the field the number of Landau levels below the cutoff will be of the order of unity and we are now outside the limits of applicability of Eqs. (2.113). However, the order of magnitude of $H_{c2}(T = 0)$ given by Eq. (3.146) is correct. $H_{c2}(T)$ behavior with the influence of magnetic field upon diffusion is shown in Fig. 15, curve 2.

2. The case of $T_c \ll e^{-1/\lambda}/\tau$: For small fields $H_{c2} \ll (\phi_0/D_0)(e^{-1/\lambda}/\tau)$, i.e. for $T \sim T_c$, the magnetic field's influence upon diffusion is irrelevant and the upper critical field is determined by Eq. (3.142). For low temperatures, $H_{c2}(T)$ is determined by Eq. (3.145), i.e. the magnetic field's influence upon diffusion liquidates the divergence of the upper critical field as $T \to 0$. The behavior of $H_{c2}(T)$ for $T_c \ll e^{-1/\lambda}/\tau$ is shown in Fig. 16, curve 2.

It should be noted that the case of $T_c \ll e^{-1/\lambda}/\tau$ is possible only for a sufficiently strong disorder. For typical $T_c \sim 10^{-4}E_F$, this case can occur only for $\lambda > 0.2$. Superconducting pairing can exist only when a condition similar to Eq. (3.20) is satisfied. In the two-dimensional case, this condition leads to the inequality $T_c \gg \lambda e^{-1/\lambda}/\tau$ which makes the region under discussion rather narrow.

The quasi-two-dimensional case was extensively discussed in Ref. [81]. The situation here is in many respects similar to that of two-dimensions, e.g. the anomalies in the upper critical field behavior due to the frequency dependence of the diffusion coefficient appear only for temperatures $T \ll e^{-1/\lambda}/\tau$, while at higher temperatures $H_{c2}(T)$ is well described by the usual theory of "dirty" superconductors. As the interplane transfer integral w grows, the smooth transition from the purely two-dimensional behavior to that of a three-dimensional isotropic system can be demonstrated. When $T_c \gg e^{-1/\lambda}/\tau$, deviations from the usual temperature behavior of H_{c2} is observed only for very low temperatures $T \ll e^{-1/\lambda}/\tau$, while close to T_c there are no significant changes from the standard dependence of $H_{c2}(T)$. For $T_c \ll e^{-1/\lambda}/\tau$, as the interplane transfer integral w grows, the temperature dependence of $H_{c2}(T)$ changes from a purely two-dimensional concave behavior for all temperatures to a convex three-dimensional like dependence. In Fig. 17 we show the typical transformations of the $H_{c2}(T)$ behavior, as the transfer integral w changes, driving the system through the metal–insulator transition [81]. This clearly demonstrates the sharp anomalies in the H_{c2} behavior which can appear due to localization effects.

3.4. Fluctuation conductivity near the Anderson transition

Fluctuation conductivity of Cooper pairs (above T_c) is especially interesting in strongly disordered systems because the usual single-particle contribution to conductivity drops to zero as the system moves towards the Anderson transition. We shall use the standard approach [176] which takes into account fluctuational Cooper pairs formation above T_c. We assume that it is possible to neglect the so-called Maki–Thompson correction which describes the increased one-particle contribution to conductivity due to superconducting fluctuations [177]. We expect that these estimates [178] will enable us to find a correct scale of fluctuation conductivity close to the mobility edge.

Consider first the averaged fluctuation propagator:

$$L^{-1}(\boldsymbol{q},\Omega_k) = \lambda^{-1} - \Pi(\boldsymbol{q},\Omega_k)\,, \tag{3.147}$$

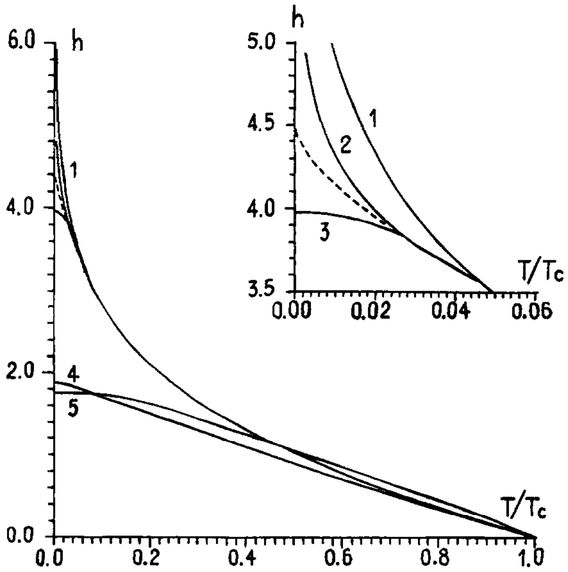


Fig. 17. Temperature dependence of the upper critical field for a quasi-two-dimensional superconductor ($e^{-1/\lambda}/\tau T_c = 4$, $\lambda = 0.126$, $h = \omega_H/\pi\lambda T_c$) for different values of the interplane transfer integral around the critical value of w_c corresponding to Anderson transition at a given disorder: (1) purely two-dimensional behavior ($w = 0$), (2) dielectric side close to Anderson transition ($L = |2\ln(w/w_c)| = 0.7$), (3) metallic side close to Anderson transition ($L = 2\ln(w/w_c) = 0.7$), and (4) metallic state far from the Anderson transition ($L = 3$). Dashed line represents the behavior at the Anderson transition ($L = 0$).

where the polarization operator

$$\Pi(\boldsymbol{q},\Omega_k) = \sum_{\varepsilon_n}\sum_{pp'} \langle G(\boldsymbol{p}_+\boldsymbol{p}'_+\,\varepsilon_n + \Omega_k)G(\boldsymbol{p}_-\boldsymbol{p}'_-\, - \varepsilon_n)\rangle$$
$$= -2\mathrm{i}\pi T\sum_{\varepsilon_n}\Phi_E(\boldsymbol{q},\omega_m = -2\varepsilon_n + \Omega_k)\,, \quad \omega_m = 2\pi m T\,. \tag{3.148}$$

During our analysis of Ginzburg–Landau coefficients we were interested in $\omega_m = 2\varepsilon_n$, so that one of the Green's functions in Φ_E was automatically retarded, while the other was advanced. Now we need a more general expression of Eq. (3.148) with $\omega_m = 2\varepsilon_n + \Omega_k$. Accordingly, instead of Eq. (3.83) we must use the following expression with an additional θ-function:

$$\Phi_E(\boldsymbol{q},\omega_m = 2\varepsilon_n + \Omega_k) = -\frac{N(E)\theta[\varepsilon_n(\varepsilon_n+\Omega_k)]}{\mathrm{i}|2\varepsilon_n + \Omega_k| + \mathrm{i}D_E(|2\varepsilon_n+\Omega_k|)q^2}\,, \tag{3.149}$$

where the generalized diffusion coefficient is again determined by Eqs. (3.85) and (3.86). From Eqs. (3.147)–(3.149), performing summation over ε_n we get the following form of the fluctuation propagator for small $\boldsymbol{q}$ ($D_E q^2 < T$):

$$L^{-1}(\boldsymbol{q},\Omega_k) = -N(E)\left\{\ln\frac{T}{T_\mathrm{c}} + \psi\left(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}\right) - \psi\left(\frac{1}{2}\right) + \eta(|\Omega_k|)q^2\right\}, \tag{3.150}$$

where

$$\eta(|\Omega_k|) = 4\pi T\sum_{n=0}^{\infty}\frac{D_E(2\varepsilon_n + |\Omega_k|)}{(2\varepsilon_n + |\Omega_k|)^2}$$
$$= \begin{cases} \dfrac{D_E}{4\pi T}\psi'\left(\dfrac{1}{2}+\dfrac{|\Omega_k|}{4\pi T}\right), & \xi_\mathrm{loc} < (\xi_0 l^2)^{1/3}\,,\ E > E_\mathrm{c}\,, \\ \dfrac{D_0}{(4\pi T)^{2/3}(2\gamma)^{1/3}}\zeta\left(\dfrac{5}{3};\dfrac{1}{2}+\dfrac{|\Omega_k|}{4\pi T}\right), & \xi_\mathrm{loc} > (\xi_0 l^2)^{1/3}\,. \end{cases} \tag{3.151}$$

It is also useful to know the form of the fluctuation propagator for $|\Omega_k| \gg T$. In this case, close to the Anderson transition, we may replace the sum over ε_n in Eq. (3.148) by an integral, while far from the transition it can be calculated exactly. As a result, we get

$$L^{-1}(\boldsymbol{q},\Omega_k) =$$
$$-N(E)\begin{cases} \ln\dfrac{T}{T_\mathrm{c}} + \psi\left(\dfrac{1}{2}+\dfrac{|\Omega_k|}{4\pi T}+\dfrac{D_E q^2}{4\pi T}\right) - \psi\left(\dfrac{1}{2}\right), & \xi_\mathrm{loc} < (\xi_0 l^2)^{1/3}\,,\ E > E_\mathrm{c}\,, \\ \ln\dfrac{T}{T_\mathrm{c}} + \dfrac{3}{2}\ln\left[\left(\dfrac{|\Omega|}{4\pi T}\right)^{2/3} + \dfrac{D_0 q^2}{(4\pi T)^{2/3}(2\gamma)^{1/3}}\right], & \xi_\mathrm{loc} > (\xi_0 l^2)^{1/3}\,. \end{cases} \tag{3.152}$$

Diagrams determining fluctuation conductivity are shown in Fig. 18. Contributions of graphs, Fig. 18(a) and (b), are nonsingular close to T_c because at least one of the fluctuation propagators transfers a large momentum of the order of p_F. Thus we have to consider independent contributions $\boldsymbol{B}$ formed by three Green's functions. We can calculate these contributions using the usual approximations of the self-consistent theory of localization taking into account the renormalization of triangular vertices by maximally-crossed graphs [98,155] (cf. Eq. (3.40)) as in Fig. 18(c).

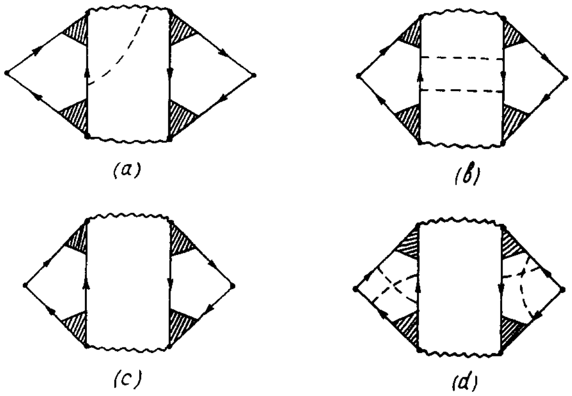


Fig. 18. Diagrams for fluctuation conductivity. Wavy lines denote the fluctuation propagator, and dashed lines represent disorder scattering.

We shall neglect graphs like the ones in Fig. 18(d) where the topology of the disorder scattering lines is not reduced to the renormalization of triangular vertices. We assume that these approximations are sufficient at least for a qualitative inclusion of localization effects. Note that it is sufficient to calculate the contribution of three Green's functions $\boldsymbol{B}(\boldsymbol{q},\Omega_k,\omega_m)$ for small $\boldsymbol{q}$ and zero external frequency $\omega_m = 0$. It can be easily found by differentiating the polarization operator of Eq. (3.148)

$$\boldsymbol{B}(\boldsymbol{q},\Omega_k,0) = \boldsymbol{q}C = -\frac{\partial}{\partial \boldsymbol{q}}\Pi(\boldsymbol{q},\Omega_k)\,. \tag{3.153}$$

The contribution of diagram of Fig. 18(c) to the operator of electromagnetic response [58] is determined by the following expression:

$$Q_{\alpha\beta} = -\frac{4e^2T}{m^2}\sum_{\Omega_k}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}(Cq_\alpha)(Cq_\beta)L(\boldsymbol{q},\Omega_k)L(\boldsymbol{q},\Omega_k+\omega_m)\,. \tag{3.154}$$

Close to T_c we can also neglect the dependence of C on Ω_k. Then C reduces to Eq. (3.89) and we have $C = N(E)\xi^2$. Fluctuation propagator analytically continued to the upper halfplane of complex ω takes the usual form

$$L(\boldsymbol{q},\omega) = -\frac{1}{N(E)[(T-T_c)/T_c] - (\mathrm{i}\pi\omega/8T_c) + \xi^2q^2}\,. \tag{3.155}$$

Further calculations can be performed in a standard way and for fluctuation conductivity for $(T-T_c)/T_c \ll 1$ we get the usual result [176]:

$$\sigma_{\mathrm{AL}} = (e^2/32\xi\hbar)(T_c/(T-T_c))^{1/2} \tag{3.156}$$

but with the coherence length ξ being defined as (cf. Eq. (3.89)):

$$\xi = \begin{cases} \left(\dfrac{\xi_0 l}{p_{\mathrm{F}}\xi_{\mathrm{loc}}}\right)^{1/2}, & \xi_{\mathrm{loc}} < (\xi_0 l^2)^{1/3}\,, \quad E > E_c\,, \\ (\xi_0 l^2)^{1/3} \sim (\xi_0/p_{\mathrm{F}}^2)^{1/3}\,, & \xi_{\mathrm{loc}} > (\xi_0 l^2)^{1/3}\,, \quad E \sim E_c\,. \end{cases} \tag{3.157}$$

From these estimates we can see that as the system approaches the Anderson transition a temperature interval, where the fluctuation contribution to conductivity is important, widens. Fluctuation Cooper pair conductivity becomes comparable to that for a single particle for $\sigma < \sigma^* \approx \sigma_c(p_F\xi_0)^{-1/3} \approx \sigma_c(T_c/E_F)^{1/3}$, i.e. close enough to the mobility edge. In fact this confirms the above picture of Cooper pairs remaining delocalized while single-particle excitations localize as the system undergoes the metal–insulator transition.

It is not difficult to find also the fluctuation contribution to diamagnetic susceptibility [178]. Close to T_c it is determined by a standard expression:

$$\chi_{fl} = \frac{e^2 T_c}{6\pi c^2}\xi\left(\frac{T_c}{T - T_c}\right)^{1/2}, \tag{3.158}$$

where the coherence length is again defined as in Eq. (3.157).

Thus our expressions for fluctuation effects follow more or less obviously from our general picture of the Ginzburg–Landau expansion: for systems close to the Anderson transition we have to only replace the usual coherence length $\sqrt{\xi_0 l}$ of a "dirty" superconductor by $\xi \sim (\xi_0 l^2)^{1/3} \sim (\xi_0/p_F)^{1/3}$.

3.5. Superconductivity in an Anderson insulator at $T = 0$

We have already considered the superconducting response of a system which is close to the Anderson transition within the Ginzburg–Landau approximation, i.e. for temperatures $T \sim T_c$. In fact it is not difficult to obtain similar results also for $T = 0$ [25].

Superconducting current density at $T = 0$ is given by [9]:

$$j_s = -\frac{n_s e^2}{mc} A, \tag{3.159}$$

where n_s is density of the superconducting electrons and A is the vector potential of an external magnetic field. On the other hand, using exact eigenstates representation DeGennes has obtained the following beautiful relation between superconducting response at $T = 0$ and the conductivity of a system in the normal state [9,25]:

$$j_s = \left\{\frac{1}{2\pi c}\int d\xi \int d\xi' L(\xi,\xi')\,\mathrm{Re}\,\sigma(\xi - \xi') - \frac{ne^2}{mc}\right\} A. \tag{3.160}$$

All characteristics of a superconducting state are contained here in the kernel:

$$L(\xi,\xi') = \frac{1}{2}\frac{EE' - \xi\xi' - \Delta_0^2}{EE'(E + E')}, \tag{3.161}$$

where $E = \sqrt{\xi^2 + \Delta_0^2}$ and Δ_0 is the superconducting gap at $T = 0$. Note that in the normal state $j_s = 0$ and we can rewrite Eq. (3.160) as

$$j_s = \frac{1}{2\pi c}\int d\xi \int d\xi' [L(\xi,\xi')|_{\Delta=\Delta_0} - L(\xi,\xi')|_{\Delta=0}]\,\mathrm{Re}\,\sigma(\xi - \xi') A. \tag{3.162}$$

Taking into account that for large $|\xi - \xi'|$ $L(\xi,\xi')|_{\Delta=\Delta_0} - L(\xi,\xi')|_{\Delta=0}$ drops as $|\xi-\xi'|^{-3}$ it is sufficient to know only the low-frequency response of a system in normal state. In particular, for a "pure" system (with no scattering) we have $\mathrm{Re}\,\sigma(\omega) = (ne^2/m)\pi^{-1}\delta(\omega)$ and comparing Eq. (3.159) with Eq. (3.162) it is immediately clear that at $T=0$ we have $n_s = n$, i.e. in an ideal system all electrons are superconducting.

Close to the Anderson transition we can use the results of elementary scaling theory of localization, e.g. Eqs. (2.31) and (2.33) to write

$$\sigma(\omega) \approx \begin{cases} A\dfrac{g_c}{\xi_{\mathrm{loc}}}\,, & \omega < \omega_c\,, \\ A\dfrac{g_c}{\xi_{\mathrm{loc}}}\left(\dfrac{\omega}{\omega_c}\right)^{1/3}, & \omega > \omega_c\,, \end{cases} \tag{3.163}$$

where $\omega_c \sim [N(E)\xi^3_{\mathrm{loc}}]^-$ is defined in Eq. (2.42), g_c is the critical conductance of scaling theory ($g_c \sim 1$) $A \sim 1$. From Eqs. (3.161) and (3.162) it is clear that the main contribution to the integral in Eq. (3.162) comes from $|\xi-\xi'| \sim \Delta_0$, so that the value of n_s depends on the relation between Δ_0 and ω_c. For $\Delta_0 < \omega_c$ we have $\sigma(\Delta_0) = Ag_c/\xi_{\mathrm{loc}}$ and

$$n_s = A(m/e^2)\Delta_0(g_c/\xi_{\mathrm{loc}})\,. \tag{3.164}$$

For $\Delta_0 > \omega_c$ we have $\sigma(\Delta_0) = Ag_c[N(E)\Delta_0]^{1/3}$ and it becomes independent on further growth of ξ_{loc} in the region of $\xi_{\mathrm{loc}} > [N(E)\Delta_0\,]^{1/3}$. Accordingly, n_s does not vanish at the mobility edge but saturates at

$$n_s = A(m/e^2)g_c[N(E)\Delta_0]^{1/3}\,. \tag{3.165}$$

In the localization region we can write instead of Eq. (3.163)

$$\sigma(\omega) \approx \begin{cases} 0\,, & \omega < \omega_c \\ Ag_c[N(E)\omega]^{1/3}\,, & \omega > \omega_c \end{cases} \tag{3.166}$$

which again leads to $\sigma(\Delta_0) \approx Ag_c[N(E)\Delta_0]^{-1/3}$ and Eq. (3.165) remains valid until $R_{\mathrm{loc}} > [N(E)\Delta_0]^{-1/3}$. Thus the density of superconducting electrons n_s remains finite close to the Anderson transition both in the metallic and insulating states.

However, from Eq. (3.165) it is easy to see that close to the Anderson transition

$$n_s/n \sim (\Delta_0/E_F)^{4/3}\,. \tag{3.167}$$

This coincides with an estimate of Eq. (3.98) based upon the Ginzburg–Landau expansion. For typical Δ_0 and E_F only a small part ($\sim 10^{-4}$ in traditional superconductors) of conduction electrons form Cooper pairs. The condition of $R_{\mathrm{loc}} > [N(E)\Delta_0]^{-1/3} \sim a(E_F/\Delta_0)^{1/3}$ as discussed above defines the size of the possible superconducting region in an Anderson insulator. This region is of course quite small, e.g. if metal–insulator transition takes place with a change of some external parameter x (impurity concentration, pressure, fluence of fast neutrons, etc.), so that $R_{\mathrm{loc}} \sim a|(x-x_c)/x_c|^{-\nu}$, then for $\nu \approx 1$ and typical $E_F/\Delta_0 \sim 10^4$ we get $|x - x_c| < 0.1x_c$.

These estimates are in complete accordance with the results of our discussion of Ginzburg–Landau approximation [21, 22] and we can obtain the qualitative picture of superconductivity in an Anderson insulator both for $T \sim T_c$ and $T \to 0$, i.e. in the ground state.

4. Statistical fluctuations of the superconducting order parameter

The previous discussion of superconductivity in a strongly disordered system is based upon the important assumption of the existence of a self-averaging superconducting order-parameter Δ. This assumption was first used in the theory of "dirty" superconductors [10, 11, 13, 9] and also in all early papers on the interplay of localization and superconductivity. It was expected that spatial fluctuations of this order parameter $\Delta(\mathbf{r})$ are actually small and we can always use some disorder averaged parameter $\langle\Delta(\mathbf{r})\rangle$. It seems natural for $\sigma \gg \sigma_e$ and it really can be justified in this region as we shall see below. However, close to the mobility edge there are no special reasons to believe in the correctness of this assumption. In this case electronic characteristics of the system become strongly fluctuating and we shall see that these lead to the strong spatial (*statistical*) fluctuations of the superconducting order parameter, or even to the regime of inhomogeneous superconductivity. At the same time, we must stress that these fluctuations are in some sense similar to the usual thermodynamic critical fluctuations of the order parameter and become important in some new critical region (we call it the statistical critical region) close to T_c. In this sense, all the previous analysis is just a kind of statistical mean-field approximation and of course it is a necessary step for further studies taking into account the statistical fluctuations. The importance of these fluctuations is stressed by the fact that the statistical critical region widens (similarly to the usual critical region) as the system goes to the Anderson transition and apparently the role of fluctuations becomes decisive for the physics of the interplay of localization and superconductivity.

4.1. Statistical critical region

Here we shall start by a demonstration of the appearance of the new type of fluctuations which are at least of the same importance as the usual critical fluctuations of the superconducting order-parameter. We call them statistical fluctuations [62] and their nature is closely connected to the problem of self-averaging properties of this order parameter (i.e. with a possibility of decoupling which transforms Eq. (3.12) into Eq. (3.13)). We shall more or less follow Ref. [62], equivalent results were recently obtained in Ref. [179].

Let us return to the Eq. (3.7) and analyze the situation in more details. We shall use a simple iteration procedure assuming that fluctuations of the kernel $K(\mathbf{rr}')$ due to disorder are small. Similar approach was first used in Ref. [180]. In this case we can represent $K(\mathbf{rr}')$ and $\Delta(\mathbf{r})$ as

$$K(\mathbf{rr}') = K_0(\mathbf{r}-\mathbf{r}') + K_1(\mathbf{rr}'); \quad K_0(\mathbf{r}-\mathbf{r}') = \langle K(\mathbf{rr}')\rangle\,, \quad \Delta(\mathbf{r}) = \langle\Delta\rangle + \Delta_1(\mathbf{r})\,, \tag{4.1}$$

where $\langle\Delta\rangle$ is the solution of the linearized gap equation with averaged kernel $K_0(\mathbf{r}-\mathbf{r}')$ while $\Delta_1(\mathbf{r})$ is the first order correction for the perturbation defined by $K_1(\mathbf{rr}')$. We have seen that the linearized gap equation (Eq. (3.13)) with the averaged kernel $K_0(\mathbf{r}-\mathbf{r}')$ determines the standard transition temperature of BCS theory given by Eq. (3.19) which we shall now denote as T_{c0}. In the first order over K_1 there is no correction to T_{c0}: $\langle K_1\rangle = 0$. In the second order of this perturbation theory we obtain the following change of transition temperature, defined as the temperature of appearance of the homogeneous order-parameter:

$$\frac{T_c - T_{c0}}{T_{c0}} = \frac{1}{\lambda_p}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{K_1(\mathbf{q}_0)K_1(0\mathbf{q})}{1 - K_0(\mathbf{q}, T_c)}\,, \qquad K_0 = \int d\mathbf{r}\, e^{i\mathbf{qr}} K(\mathbf{r}, T_c)\,, \tag{4.2}$$

where

$$K_1(0\boldsymbol{q}) = K_1(-\boldsymbol{q}0) = \int d\boldsymbol{r}\int d\boldsymbol{r}'\, e^{i\boldsymbol{q}\boldsymbol{r}}[K(\boldsymbol{r}\boldsymbol{r}') - K_0(\boldsymbol{r}-\boldsymbol{r}')]$$

$$= \lambda_p \int_0^{\langle\omega\rangle} \frac{dE}{E}\tanh\left(\frac{E}{2T_c}\right)\int d\boldsymbol{r}\, c^{i\boldsymbol{q}\boldsymbol{r}}\left[\frac{1}{N(E)}\sum_\mu |\phi_\mu(\boldsymbol{r})|^2\delta(E-\varepsilon_\mu) - 1\right]. \tag{4.3}$$

Here $\lambda_p = gN(E_F)$ and we have used the completeness and orthonormality of the exact eigenfunctions $\phi_\mu(\boldsymbol{r})$. It is obvious that correction to T_{c0} given by Eq. (4.2) is always positive. After averaging Eq. (4.2) over disorder we get the relative change of the transition temperature due to fluctuations as

$$\frac{\delta T_c}{T_{c0}} = \left\langle \frac{T_c - T_{c0}}{T_{c0}} \right\rangle = \lambda_p \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{\varphi(\boldsymbol{q})}{1-K_0(\boldsymbol{q},T_c)}, \quad \varphi(\boldsymbol{q}) = \int d\boldsymbol{r}\, e^{i\boldsymbol{q}\boldsymbol{r}}\varphi(\boldsymbol{r}), \tag{4.4}$$

where

$$\varphi(\boldsymbol{r}) = \int_0^{\langle\omega\rangle}\frac{dE}{E}\tanh\left(\frac{E}{2T_c}\right)\int_0^{\langle\omega\rangle}\frac{dE'}{E'}\tanh\left(\frac{E'}{2T_c}\right)\left\{\frac{1}{N(E)}\langle\langle\rho_E(\boldsymbol{r})\rho_{E'}(0)\rangle\rangle^H - 1\right\} \tag{4.5}$$

and we have introduced the spectral density of Eq. (A.3):

$$\langle\langle\rho_E(\boldsymbol{r})\rho_{E'}(\boldsymbol{r}')\rangle\rangle^H = \frac{1}{N(E)}\left\langle\sum_{\mu\nu}|\phi_\mu(\boldsymbol{r})|^2|\phi_\nu(\boldsymbol{r}')|^2\delta(E-\varepsilon_\mu)\delta(E'-\varepsilon_\nu)\right\rangle \tag{4.6}$$

which is actually a correlation function of the local densities of states.

Remember now that in a "dirty" system [174]:

$$1 - K_0(\boldsymbol{q},T) = 1 - 2\pi T\lambda_p\sum_{\varepsilon_n}\frac{1}{2|\varepsilon_n| + D_E(2|\varepsilon_n|)q^2}$$

$$\approx \lambda_p\left[\frac{T-T_{c0}}{T_{c0}} + \xi^2q^2\right], \quad \varepsilon_n = (2n+1)\pi T, \tag{4.7}$$

where ξ is the coherence length defined previously, e.g. in Eq. (3.89). The approximate equality here is valid for $|T - T_{c0}//T_c \ll 1$, $\xi^2q^2 \ll 1$. From Eqs. (4.4) and (4.7) we get the change of transition temperature in the following form:

$$\frac{\delta T_c}{T_{c0}} = \int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{\varphi(\boldsymbol{q})}{\xi^2q^2}. \tag{4.8}$$

Here we must cut off integration at $q \sim \xi^{-1}$ in accordance with the limits of applicability of the last expression in Eq. (4.7). However, the contribution of short-wave fluctuations here may be also important.

The Ginzburg–Landau functional expressed via the non-averaged order parameter $\Delta(\boldsymbol{r})$ has the following form [9]:

$$F\{\Delta\} = \int d\boldsymbol{r}\left\{\frac{N(E_F)}{\lambda_p}|\Delta(\boldsymbol{r})|^2 - N(E_F)\int d\boldsymbol{r}'\, K(\boldsymbol{r}\boldsymbol{r}')\Delta(\boldsymbol{r}')\Delta(\boldsymbol{r}) + \tfrac{1}{2}B|\Delta(\boldsymbol{r})|^4\right\}, \tag{4.9}$$

where we have neglected the fluctuations of the pairing interaction λ_p and of the coefficient B, which is defined by the standard expression given in Eq. (3.88). Using Eqs. (4.1)–(4.3) we can find the Ginzburg–Landau equations which describe the slow changes of $\Delta(\boldsymbol{r})$:

$$\left\{N(E_F)\frac{T_{c0}-T}{T_{c0}}+\delta A(\boldsymbol{r})-B|\Delta(\boldsymbol{r})|^2+C\frac{\partial^2}{\partial r^2}\right\}\Delta(\boldsymbol{r})=0\,, \tag{4.10}$$

where

$$\delta A(\boldsymbol{r})=N(E_F)\int_0^{\langle\omega\rangle}\frac{dE}{E}\tanh\left(\frac{E}{2T_{c0}}\right)\left\{\frac{1}{N(E_F)}\sum_{\nu}|\phi_\nu(\boldsymbol{r})|^2\delta(E-\varepsilon_\nu)-1\right\} \tag{4.11}$$

describes the fluctuations of the coefficient A of the Ginzburg–Landau expansion and we have neglected the fluctuations of the C coefficient.

Ginzburg–Landau equations with fluctuating coefficients were analyzed for the first time by Larkin and Ovchinnikov [181]. It was shown that $\delta A(\boldsymbol{r})$-fluctuations lead to a shift of transition temperature given by Eq. (4.8) and the solution of Eq. (4.10) for the order parameter in the first order over fluctuations has the form of Eq. (4.1) with

$$\Delta_1(\boldsymbol{r})=\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\Delta_1(\boldsymbol{q})e^{i\boldsymbol{q}\boldsymbol{r}}\,,\qquad \Delta_1(\boldsymbol{q})=-\frac{\langle\Delta\rangle}{N(E_F)}\frac{\delta A(\boldsymbol{q})}{\xi^2q^2+2\tau}\,, \tag{4.12}$$

where $\tau=(T_c-T)/T_c$ is the temperature measured relative to the new transition temperature. The mean-square fluctuation of the order-parameter itself is determined from Eq. (4.12) by

$$\frac{\langle\Delta^2\rangle}{\langle\Delta\rangle^2}-1=\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{\varphi(\boldsymbol{q})}{[\xi^2q^2+2\tau]^2}\,, \tag{4.13}$$

where $\varphi(\boldsymbol{q})$ was introduced in Eqs. (4.4), (4.5). It is important to note that fluctuations of $\Delta(\boldsymbol{r})$ as opposed to the T_c-shift are determined by the small $\boldsymbol{q}$ behavior of $\varphi(\boldsymbol{q})$.

We can see now that all the physics of statistical fluctuations is described by the correlation function of local densities of states (or spectral density of Eq. (4.6)). This function was determined above in Eqs. (2.83), (2.84) within the self-consistent theory of localization or by Eqs. (2.137), (2.139) which follow from the scaling approach close to the mobility edge.

Using Eq. (2.84) for the metallic state not very close to the mobility edge we can get from Eq. (4.5)

$$\varphi(\boldsymbol{q}=0)\sim\frac{\xi}{N^2(E_F)D_0^2}\,, \tag{4.14}$$

where $\xi=\sqrt{\xi_0 l}$ and D_0 is the Drude diffusion coefficient. Estimating the T_c-shift from Eq. (4.8) we get

$$\frac{\delta T_c}{T_{c0}}\sim\frac{1}{N^2(E_F)D_0^2\xi^2}\sim\frac{T_c}{E_F}\frac{1}{(p_Fl)^3}\sim\tau_G\,, \tag{4.15}$$

where τ_G is the size of the Ginzburg critical region defined by Eq. (3.108). We have seen that in the usual “dirty” superconductor $\tau_G\ll 1$. For the order-parameter fluctuations from Eq. (4.13) we obtain

$$\frac{\langle\Delta^2\rangle}{\langle\Delta\rangle^2}-1\approx\frac{1}{8\pi}\frac{\varphi(\boldsymbol{q}=0)}{\xi^3\sqrt{2|\tau|}}\approx\left(\frac{\tau_D}{|\tau|}\right)^{1/2}. \tag{4.16}$$

From here we can see that the width of the temperature region where statistical fluctuations are important is given by

$$\tau_{\mathrm{D}} \sim \frac{\varphi^2(0)}{\xi^6} \sim \frac{1}{N^4(E_{\mathrm{F}})D_0^4\xi^4} \sim \left(\frac{T_{\mathrm{c}}}{E_{\mathrm{F}}}\right)^2 \frac{1}{(p_{\mathrm{F}}l)^6} \sim \tau_{\mathrm{G}}^2 . \tag{4.17}$$

It is obvious that in a "dirty" superconductor we have $\tau_{\mathrm{D}} \ll \tau_{\mathrm{G}} \ll 1$ and statistical fluctuations are absolutely unimportant.

Situation changes for a system which is close to the mobility edge. Using Eq. (2.84) with D_0 replaced by $D_0(\omega/\gamma)^{1/3}$ or Eqs. (2.137)–(2.139) we obtain

$$\varphi(\boldsymbol{q}) \approx \frac{\gamma^{1/2}}{N^2(E_{\mathrm{F}})D_0^3T_{\mathrm{c}}} \int_0^{T_{\mathrm{c}}} \frac{\mathrm{d}\omega}{\omega^{1/2}} [\omega^2 + D_0\gamma^{-2/3}\omega^{2/3}q^4]^{-1/4} \sim \xi^2 \ln\frac{1}{\xi_q} , \tag{4.18}$$

where $\xi \sim (\xi_0 p_{\mathrm{F}}^{-2})^{1/3}$. Similarly, we get

$$\frac{\langle\Delta^2\rangle}{\langle\Delta\rangle^2} - 1 \approx \int_0^{\xi^{-1}} \frac{\xi^3q^2\,\mathrm{d}q}{(\xi^2q^2 + 2\tau)^2} \ln\frac{1}{q\xi} \sim \frac{1}{\sqrt{|\tau|}} \ln\frac{1}{|\tau|} . \tag{4.19}$$

From Eq. (4.19) it follows that close to the mobility edge statistical fluctuations become important and even overcome thermodynamic fluctuations due to the logarithmic factor in $\varphi(\boldsymbol{q})$. Thus in this region we have $\tau_{\mathrm{D}} > \tau_{\mathrm{G}} \sim 1$.

The crossover from the regime of weak statistical fluctuations ($\tau_{\mathrm{D}} \ll \tau_{\mathrm{G}}$) to the strong fluctuation regime occurs at the conductivity scale $\sigma \sim \sigma^* \approx \sigma_{\mathrm{c}}(p_{\mathrm{F}}\xi_0)^{-1/3}$ which was extensively discussed above. Thus close to the mobility edge the superconducting order-parameter is no more a self-averaging quantity. Here the mean-field theory approach becomes formally invalid due to thermodynamic and also because of statistical fluctuations. Below we shall analyze this situation in more details.

Finally, we shall briefly discuss the region of localized states. The appearance here of a singular $\delta(\omega)$-contribution to the correlator of local densities of states given by Eqs. (A.8)–(A.10) leads to the additional contribution to $\varphi(\boldsymbol{q})$:

$$\begin{aligned} \varphi(\boldsymbol{q}) &= \int_0^{\langle\omega\rangle} \frac{\mathrm{d}E}{E^2} (\tanh(E/2T_{\mathrm{c}})) \frac{A_E(\boldsymbol{q})}{N(E_{\mathrm{F}})} + \cdots \sim \frac{A_{E_{\mathrm{F}}}}{N(E_{\mathrm{F}})T_{\mathrm{c}0}} + \cdots \\ &= \frac{1}{N(E_{\mathrm{F}})T_{\mathrm{c}}(1 + R_{\mathrm{loc}}^2q^2)} + \cdots . \end{aligned} \tag{4.20}$$

Accordingly, a new contribution to $\Delta(\boldsymbol{r})$ fluctuations is given by

$$\frac{\langle\Delta^2\rangle}{\langle\Delta\rangle^2} - 1 \approx \frac{1}{N(E_{\mathrm{F}})T_{\mathrm{c}}} \int_0^{R_{\mathrm{loc}}^{-1}} \frac{q^2\,\mathrm{d}q}{(\xi^2q^2 + 2|\tau|)^2(1 + R_{\mathrm{loc}}^2q^2)} \sim \frac{1}{N(E_{\mathrm{F}})T_{\mathrm{c}}R_{\mathrm{loc}}^3\tau^2} \tag{4.21}$$

and it grows fast as the localization length R_{loc} diminishes. Using our main criterion of superconductivity in localized phase given by Eq. (3.20) we can see that in all regions of possible superconductivity statistical fluctuations of $\Delta(\boldsymbol{r})$ remain of the order of unity and are important in a rather wide temperature interval around T_c.

4.2. *Superconducting transition at strong disorder*

We consider now superconductivity in systems with strong statistical fluctuations of the "local transition temperature" $T_c(\boldsymbol{r})$ as described by Eqs. (4.10) and (4.11). In this analysis we shall follow Refs. [182, 183]. For simplicity we assume a Gaussian nature for these fluctuations. Note, however, that close to the mobility edge the fluctuations of local density of states become strongly non-Gaussian [95] and this can complicate the situation. Unfortunately, the importance of this non-Gaussian behavior for superconductivity has not been studied up to now. We shall see that in our model, depending on the degree of disorder, which we shall measure by the ratio τ_D/τ_G, two types of superconducting transitions are possible. For τ_D smaller than some critical value τ_D^* the superconducting transition is the usual second-order phase transition at $T = T_c$. The superconducting order-parameter is in this case equal to zero for $T > T_c$ and is spatially homogeneous over scales exceeding the correlation length $\xi(T)$ below T_c. Statistical fluctuations lead only to a change of critical exponents at the transition [184, 185].

At $\tau_D > \tau_D^*$ the superconducting state appears in an inhomogeneous fashion even if the correlation length of disorder induced fluctuations of $T_c(\boldsymbol{r})$ is small compared with the superconducting correlation length ξ (microscopic disorder). This case was first analyzed by Ioffe and Larkin [189]. Investigating the case of extremely strong disorder they have shown that as the temperature is lowered the normal phase acquires localized superconducting regions (drops) with characteristic size determined by $\xi(T)$. Far from T_c their density is low, but with further cooling the density and dimensions of the drops increase and they begin to overlap leading to a kind of percolative superconducting transition.

According to our previous estimates, if we take into account only the fluctuations of local density of states, the parameter τ_D/τ_G increases from very small values to a value greater than unity as the system moves towards the mobility edge. The onset of an inhomogeneous superconducting regime is therefore to be expected as the localization transition is approached.

Our treatment of superconductors with large statistical fluctuations will be based on the Ginzburg–Landau functional:

$$F\{A(\boldsymbol{r}),\Delta(\boldsymbol{r})\} = \int \mathrm{d}\boldsymbol{r}\left\{\frac{\boldsymbol{B}^2(\boldsymbol{r})}{8\pi} + N(E_F)\left[(\tau + t(\boldsymbol{r}))|\Delta(\boldsymbol{r})|^2 + \xi^2\left|\left(\nabla - \frac{2ie}{\hbar c}A(\boldsymbol{r})\right)\Delta(\boldsymbol{r})\right|^2 + \frac{1}{2}\lambda|\Delta(\boldsymbol{r})|^4\right]\right\}, \tag{4.22}$$

where $\boldsymbol{B} = \mathrm{rot}\boldsymbol{A}$ is the magnetic field and we have redefined the coefficient of the quartic term as $B = N(E_F)\lambda$. Here $t(\boldsymbol{r})$ is defined by Eq. (4.11) as $\delta A(\boldsymbol{r}) = N(E_F)t(\boldsymbol{r})$ and plays the role of the fluctuation of local "critical temperature", which appears due to fluctuations of the local density of states. In the general case, it can also have contributions from the local fluctuations of the pairing interaction or other types of microscopic inhomogeneities. As noted above, we assume Gaussian statistics of these fluctuations, though real situation close to the mobility edge may be more complicated [95]. Given the distribution of $t(\boldsymbol{r})$, the free energy of the system and the order-parameter correlator are equal to

$$\mathscr{F}\{t(\boldsymbol{r})\} = -T\ln Z\,, \quad Z = \int D\{A,\Delta\}\exp[-F\{A(\boldsymbol{r}),\Delta(\boldsymbol{r})\}/T]\,, \tag{4.23}$$

$$\langle\Delta(\boldsymbol{r})\Delta(\boldsymbol{r}')\rangle = Z^{-1}\int D\{A,\Delta\}\Delta(\boldsymbol{r})\Delta(\boldsymbol{r}')\exp[-F\{A(\boldsymbol{r}),\Delta(\boldsymbol{r})\}/T]\,, \tag{4.24}$$

and must be averaged over the Gaussian distribution of $t(\boldsymbol{r})$. From our definition of $t(\boldsymbol{r})$ and using the approach of the previous section, assuming the short-range of fluctuations of local density of states (on the scale of ξ), it is easy to estimate the correlator of $t(\boldsymbol{r})$ as

$$\langle t(\boldsymbol{r})t(\boldsymbol{r}')\rangle = \gamma\delta(\boldsymbol{r}-\boldsymbol{r}')\,, \quad \gamma \approx \tau_{\mathrm{D}}^{1/2}\xi^3\,. \tag{4.25}$$

Then the probability of a configuration with a given $t(\boldsymbol{r})$ is given by

$$\mathscr{P}\{t(\boldsymbol{r})\} = \exp\left[-\frac{1}{2\gamma}\int \mathrm{d}\boldsymbol{r}\,t^2(\boldsymbol{r})\right]. \tag{4.26}$$

The problem reduces thus to the calculation of the functions $\mathscr{F}\{t(\boldsymbol{r})\}$ and $\langle\Delta(\boldsymbol{r})\Delta(\boldsymbol{r}')\rangle$ and their subsequent averaging over $\mathscr{P}\{t(\boldsymbol{r})\}$.

We shall limit ourselves to the consideration of noninteracting drops and no vortices. Then we can consider the phase of the order-parameter $\Delta(\boldsymbol{r})$ as nonsingular. After the gauge transformation

$$A(\boldsymbol{r}) \to A(\boldsymbol{r}) + (c\hbar/2e)\,\nabla\phi(\boldsymbol{r})\,, \qquad \Delta(\boldsymbol{r}) \to \Delta(\boldsymbol{r})\exp[-\mathrm{i}\phi(\boldsymbol{r})]\,, \tag{4.27}$$

where $\phi(\boldsymbol{r})$ is the *phase of the order parameter*; we can use real $\Delta(\boldsymbol{r})$ and the Ginzburg–Landau functional of Eq. (4.22) becomes

$$F\{A(\boldsymbol{r}),\Delta(\boldsymbol{r})\} = \int \mathrm{d}\boldsymbol{r}\left\{\frac{\boldsymbol{B}^2(\boldsymbol{r})}{8\pi} + N(E_{\mathrm{F}})\left[(\tau + t(\boldsymbol{r}))\Delta^2(\boldsymbol{r}) + \frac{4e^2\xi^2}{c^2\hbar^2}A^2(\boldsymbol{r})\Delta^2(\boldsymbol{r}) + \xi^2(\nabla\Delta(\boldsymbol{r}))^2 + \frac{1}{2}\lambda\Delta^4(\boldsymbol{r})\right]\right\}. \tag{4.28}$$

Integration over phase in Eq. (4.23) gives an inessential constant factor to the partition function which we disregard.

To average the logarithm of the partition function, Eq. (4.23), over $t(\boldsymbol{r})$ we can use the replica trick [186] which permits the averaging to be carried out in explicit form. We express the average free energy, Eq. (4.23), of the system in the form

$$\langle\mathscr{F}\rangle = -T\lim_{n\to 0}\frac{1}{n}[\langle Z^n\rangle - 1]\,. \tag{4.29}$$

To calculate $\langle Z^n\rangle$ in accordance with the idea of the replica method, we first assume n to be an arbitrary integer. Expressing Z^n in terms of an n-fold functional integral over the fields of the replicas A_α, $\Delta_\alpha(\boldsymbol{r})$, $\alpha = 1, \dots, n$ and carrying out exact Gaussian averaging over $t(\boldsymbol{r})$, we get

$$\langle Z^n \rangle = \int D\{A, \Delta\} \exp[-S_n\{A_\alpha, \Delta_\alpha\}] , \tag{4.30}$$

$$S\{A_\alpha, \Delta_\alpha\} = \int \mathrm{d}\boldsymbol{r} \left\{ \sum_\alpha^n \frac{\boldsymbol{B}^2(\boldsymbol{r})}{8\pi T} + \frac{N(E_\mathrm{F})}{T} \sum_\alpha^n \left[\tau \qquad \Delta_\alpha(\boldsymbol{r})^2 + \frac{4e^2\xi^2}{e^2\hbar^2} A_\alpha^2(\boldsymbol{r}) \Delta_\alpha^2(\boldsymbol{r}) \right.\right.$$
$$\left.\left. + \xi^2 (\nabla \Delta_\alpha(\boldsymbol{r}))^2 + \frac{1}{2} \lambda \Delta_\alpha^4(\boldsymbol{r}) \right] - \frac{1}{2} \frac{N(E_\mathrm{F})}{T} \gamma \left[\sum_{\alpha=1}^n \Delta_\alpha^2(\boldsymbol{r}) \right]^2 \right\} .$$

The last expression here represents the "effective action" and $\tilde{\gamma} = \gamma N(E_\mathrm{F})/T_\mathrm{c} \approx \tau_\mathrm{D}^{1/2} N(E_\mathrm{F})/T_\mathrm{c}$ grows with disorder. Note that the random quantities $t(\boldsymbol{r})$ have already dropped out of these expressions, and that the action $S\{A_\alpha, \Delta_\alpha\}$ is translationally invariant. For the correlator of Eq. (4.24) we obtain

$$\langle \Delta(\boldsymbol{r}) \Delta(\boldsymbol{r}') \rangle = \lim_{n \to 0} \frac{1}{n} \int D\{A, \Delta\} \exp[-S_n\{A_\alpha, \Delta_\alpha\}] \sum_{\alpha=1}^n \Delta_\alpha(\boldsymbol{r}) \Delta_\alpha(\boldsymbol{r}') , \tag{4.31}$$

where we have symmetrized over the replica indices.

Far from the region of strong fluctuations of the order parameter $|\tau| \gg \tau_\mathrm{D}$, τ_G the functional integrals in Eqs. (4.30) and (4.24) can be calculated by the saddle-point method. The extrema of the action are determined by classical equations:

$$\left[\tau - \xi^2 \nabla^2 + \lambda \Delta_\alpha^2 - \tilde{\gamma} \sum_{\beta=1}^n \Delta_\beta^2(\boldsymbol{r}) \right] \Delta_\alpha(\boldsymbol{r}) = 0 , \quad A_\alpha = 0 . \tag{4.32}$$

The nontrivial conclusion is that these equations for $\Delta_\alpha(\boldsymbol{r})$, besides having spatially homogeneous solutions do have localized solutions with finite action (*instantons*). These correspond at $\tau > 0$ to the superconducting drops. We shall limit ourselves to a picture of noninteracting drops and consider only instanton solutions above T_c (at $\tau > 0$). We shall be interested only in those solutions that admit analytic continuation as $n \to 0$. We designate them $\Delta_\alpha^{(i)}(\boldsymbol{r})$, where the superscript i labels the type of solution. To find their contribution we must expand the action of Eq. (4.30) up to the terms quadratic in deviations $\varphi_\alpha(\boldsymbol{r}) = \Delta_\alpha(\boldsymbol{r}) - \Delta_\alpha^{(i)}(\boldsymbol{r})$. It can be shown that fluctuations of the fields $A_\alpha(\boldsymbol{r})$ can be neglected if we consider noninteracting drops [182,183].

For $\tau > 0$ and for $\tilde{\gamma} > \lambda$, Eq. (4.32) possesses (besides the trivial solution $\Delta_\alpha = 0$) the following nontrivial solution with finite action (instanton) (cf. Refs. [187,188,7]):

$$\Delta_\alpha^{(i)}(\boldsymbol{r}) = \Delta_0(\boldsymbol{r}) \delta_{\alpha i} , \quad i = 1, \dots, n ,$$
$$\Delta_0(\boldsymbol{r}) = \sqrt{\frac{\tau}{\tilde{\gamma} - \lambda}} \chi\left[\frac{\tau}{\xi(T)} \right] , \quad \xi(T) = \frac{\xi}{\sqrt{\tau}} , \tag{4.33}$$

where the dimensionless function $\chi(x)$ satisfies the condition $\mathrm{d}\chi(x)/\mathrm{d}x|_{x=0} = 0$ and its asymptotic form: $\chi(x) \sim x^{-1} \exp(-x)$ for $x \gg 1$ (for spatial dimension $d = 3$). The qualitative form of this solution is shown in Fig. 19.

From Eq. (4.33) it is seen that instantons are oriented along axes of replica space (there are n types of instanton solutions) which is due to the "cubic anisotropy" term $\lambda \Delta_\alpha^4$ in the effective

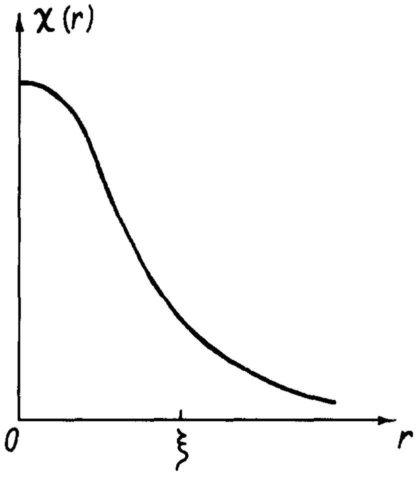


Fig. 19. Qualitative form of instanton solution.

action of Eq. (4.30). Index i characterizes the direction in replica space along which the symmetry breaking takes place. For $\lambda \to 0$ the action becomes O(n)-symmetric and instantons take the form

$$\Delta_\alpha(\mathbf{r}) = \Delta_0(\mathbf{r}) e_\alpha , \quad \sum_{\alpha=1}^{n} e_\alpha^2 = 1 , \tag{4.34}$$

i.e. are oriented along the arbitrary unit vector $\mathbf{e}$ in replica space. Such instantons earlier were studied in the theory of localization [187, 188, 7].

The quadratic expansion of the effective action near the instanton solution takes the form (cf. analogous treatment in Refs. [187, 188, 7]):

$$S\{\Delta_\alpha\} = S\{\Delta_\alpha^{(i)}\} + \frac{1}{2} \int \mathrm{d}\mathbf{r} \sum_{\alpha,\beta} (\varphi_\alpha \hat{M}_{\alpha\beta}^{(i)} \varphi_\beta) , \tag{4.35}$$

where the operator $M_{\alpha\beta}^{(i)}$ on instanton solutions is equal to

$$\hat{M}_{\alpha\beta}^{(i)} = [\hat{M}_{\mathrm{L}} \delta_{\alpha i} + \hat{M}_{\mathrm{T}} (1 - \delta_{\alpha i})] \delta_{\alpha\beta} \tag{4.36}$$

with

$$M_{\mathrm{L,T}} = (2N(E_{\mathrm{F}})/T)[-\xi^2 \nabla^2 + \tau U_{\mathrm{L,T}}(\mathbf{r})] \tag{4.37}$$

where

$$U_{\mathrm{L}}(\mathbf{r}) = 1 - 3\chi^2[r/\xi(T)] , \qquad U_{\mathrm{T}}(\mathbf{r}) = 1 - (1 - \lambda/\tilde{\gamma})^{-1} \chi^2[r/\xi(T)] . \tag{4.38}$$

The value of the Gaussian functional integer is determined by the spectra of eigenstates of operators M_{L} and M_{T}. Detailed analysis can be found in Refs. [182, 183]. The qualitative form of these spectra is shown in Fig. 20. Operator M_{L} always possess an eigenvalue $\varepsilon_1^{\mathrm{L}} = 0$ the so-called translation zero-mode, connected with translation symmetry: instanton center may be placed anywhere in space, the action does not change. However, this is not a lowest eigenvalue of M_{L},

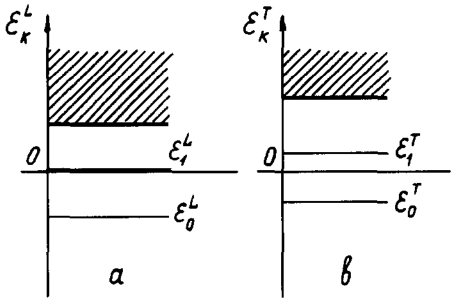


Fig. 20. Qualitative structure of eigenvalues of M_{L} (a) and M_{T} (b) operators. $\varepsilon_1^{\mathrm{L}} = 0$: zero translation mode; $\varepsilon_0^{\mathrm{T}} \to 0$ for $\lambda \to 0$ transforms to zero "rotation" mode. The continuous part of the spectrum is shaded.

there is always a negative eigenvalue $\varepsilon_0^{\mathrm{L}} < \varepsilon_1^{\mathrm{L}} = 0$. It can be shown rigorously that it is the only negative eigenstate of M_{L} [190]. Operator M_{T} possess also a single negative eigenvalue $\varepsilon_0^{\mathrm{T}} < 0$ [182, 183]; however, this eigenvalue tends to zero for $\lambda \to 0$ becoming the "rotation" zero-mode, reflecting the arbitrary "direction" of instanton in replica space in the absence of cubic anisotropy in the action [187, 188, 7]. For $\lambda = \lambda^* = 2/3\tilde{\gamma}$ we have $M_{\mathrm{L}} = M_{\mathrm{T}}$ and the spectra of both operators coincide.

Including the contributions of instantons oriented along all the axes in replica space we obtain the following one-instanton contribution to the partition function entering Eq. (4.29) [182, 183]:

$$\langle Z^n \rangle = n\Omega \left(\frac{J_{\mathrm{L}}}{2\pi}\right)^{d/2} [\mathrm{Det}' M_{\mathrm{L}}]^{-1/2} [\mathrm{Det}\, M_{\mathrm{T}}]^{(1-n)/2} \exp\{-S_0(\tau)\}\,, \tag{4.39}$$

where Ω is the system volume,

$$J_{\mathrm{L}} = \frac{1}{d}\int \mathrm{d}r \left(\frac{\partial \Delta_0}{\partial r}\right)^2 = \frac{T}{2N(E_{\mathrm{F}})} \frac{S_0(\tau)}{\xi^2} \tag{4.40}$$

and the action at the instanton is given by

$$S_0(\tau) = \mathscr{A} \frac{\xi^3 \tau^{1/2}}{\gamma - \lambda T/N(E_{\mathrm{F}})}\,, \tag{4.41}$$

where $\mathscr{A} \approx 37.8$ is a numerical constant [191]. The prime on Det M_{L} means that we must exclude the zero-eigenvalue $\varepsilon_1^{\mathrm{L}} = 0$ from the product of eigenvalues determining this determinant. The condition of applicability of the saddle-point approximation looks like $S_0(\tau) \gg 1$, and in fact all our analyses are valid outside the critical regions both for thermodynamic and statistical fluctuations.

In the limit of $n \to 0$ the total cancellation of imaginary contributions appearing due to negative eigenvalues takes place in Eq. (4.39) and using Eq. (4.29) we get for $\tilde{\gamma} > 3/2\lambda$ the following *real* contribution to the free energy:

$$\mathscr{F} = -\rho_{\mathrm{s}}(\tau) T \Omega \tag{4.42}$$

where the density of superconducting "drops"

$$\rho_{\mathrm{s}}(\tau) = \left[\frac{T}{4\pi N(E_{\mathrm{F}})} S_0(\tau)\right]^{3/2} \xi^{-3} \left[\frac{\mathrm{Det}\, M_{\mathrm{T}}}{\mathrm{Det}' M_{\mathrm{L}}}\right]^{1/2} \exp\{-S_0(\tau)\}\,. \tag{4.43}$$

Thus for $\tilde{\gamma} > 3/2\lambda$ even for $T > T_c$ the superconducting "drops" (instantons) appear in the system which directly contribute to the equilibrium free energy. This contribution given by Eqs. (4.42) and (4.43) exists along the usual thermodynamic fluctuations. The condition of $\tilde{\gamma} > 3/2\lambda$ defines the critical disorder $\tau_D > \tau_D^* > \tau_G$, and this inhomogeneous picture of superconducting transition appears only for the case of sufficiently strong statistical fluctuations. The knowledge of the qualitative structure of spectra of eigenvalues of M_L and M_T allows to analyze different asymptotics of Eq. (4.42) [182, 183]. For $\tilde{\gamma}S_0(\tau) \ll \lambda \ll \lambda^*$ we get

$$\rho_s(\tau) \approx \xi^{-3}(T)(\lambda/\tilde{\gamma})^{1/2} S_0^{3/2}(\tau)\exp[-S_0(\tau)] . \qquad (4.44)$$

For $\lambda \to \lambda^*$, we obtain

$$\rho_s(\tau) \approx \xi^{-3}(T)((\lambda^*/\lambda) - 1)^{3/2} S_0^{3/2}(\tau)\exp[-S_0(\tau)] . \qquad (4.45)$$

Thus the density of superconducting "drops" $\rho_s(\tau)$ vanishes as $\lambda \to \lambda^*$, i.e. they are destroyed by thermodynamic fluctuations.

For the order-parameter correlator of Eq. (4.24) we get the following result:

$$\langle \Delta(\boldsymbol{r})\Delta(\boldsymbol{r}')\rangle \approx \rho_s(\tau) \int d\boldsymbol{R}_0\, \Delta_0(\boldsymbol{r} + \boldsymbol{R}_0)\Delta_0(\boldsymbol{r}' + \boldsymbol{R}_0) . \qquad (4.46)$$

The integration over instanton center $\boldsymbol{R}_0$ here means in fact averaging over different positions of "drops". Note that over large distances this correlator decreases like $\exp[-|\boldsymbol{r} - \boldsymbol{r}'|/\xi(T)]$ and does not contain the usual Ornstein–Zernike factor $|\boldsymbol{r} - \boldsymbol{r}'|^{-1}$.

We have found the free-energy of the inhomogeneous superconducting state in the temperature region $\tau \gg \tau_D$, where the "drop" concentration is exponentially small and the picture of non-interacting "drops" is valid. They give exponentially small contribution to the specific heat and diamagnetic susceptibility. The characteristic size of "drops" is determined by $\xi(T)$ and as $T \to T_c$ the "drops" grow and begin to overlap leading to a percolative superconducting transition. Thus for $\tau_D > \tau_D^* > \tau_G$ superconductivity first appears in isolated "drops". This is similar to the picture of decay of a metastable state in the case of first-order phase transitions [192]. However, in this latter case instantons give imaginary contribution to the free energy determining the decay rate of a "false" equilibrium state (critical bubble formation). Here instanton contributions lead as was noted above to real free energy and "drops" appear in the true equilibrium state.

It is more or less obvious that between isolated "drops" a kind of Josephson coupling may appear and lead to rather complicated phase diagram of the system in an external magnetic field, e.g. including the "superconducting glass" phase [193, 194]. The existence of the inhomogeneous regime of superconductivity will obviously lead to the rounding of BCS-like singularities of the density of states and superconductivity may become gapless. Note that diffusion-enhanced Coulomb interactions can also lead to the gaplessness of strongly disordered superconductors via Coulomb-induced inelastic scattering [195]. Fluctuation conductivity in a similar inhomogeneous superconducting state was studied in Ref. [196]. Note the closely related problem of strongly disordered superfluids [197, 198]. Some results here may be quite useful for the case of strongly disordered superconductors, though the limitations of this analogy are also important.

A major unsolved problem here is the possible influence of statistical fluctuations of the coefficient of the gradient term in the Ginzburg–Landau expansion which has been neglected above, or the equivalent problem (cf. Eq. (3.96)) for superconducting electron density n_s. This problem was briefly considered for the case of weak disorder in Ref. [199]. It was shown that

$$\langle (\delta n_s/n_s)^2 \rangle \sim (\xi_0 p_F^2 l)^{-1} \sim e^4/g(\xi)\,, \tag{4.47}$$

where $g(\xi) = \sigma\xi$ is the conductance of the metallic sample with the size of the order of the superconducting coherence length $\xi = \sqrt{\xi_0 l}$. Extrapolating this estimate up to the Anderson transition using $\xi = (\xi_0/p_F^2)^{1/3}$ we get

$$\langle (\delta n_s/n_s)^2 \rangle \sim \frac{e^4}{\sigma^2 (\xi_0/p_F^2)^{2/3}}\,. \tag{4.48}$$

Obviously, we get $\langle (\delta n_s/n_s)^2 \rangle \geq 1$ for $\sigma \leq \sigma^*$ so that statistical fluctuations of n_s become important close to the Anderson transition in the same region we have discussed above. This further complicates the picture of the superconducting transition and can also be very important for the possible anomalous behavior of H_{c2} which was recently studied on similar lines in Ref. [200]. Some qualitative conjectures for the case of $\langle (\delta n_s/n_s)^2 \rangle \geq 1$ were formulated in Ref. [201], where it was argued that in this case there will occur regions in the sample with locally *negative* values of superfluid density. This is equivalent to a negative sign of the Josephson coupling between the "drops". In this sense, the disordered superconductor is unlike a Bose liquid. This leads to an important prediction that in a small superconducting ring, if there is a segment with negative n_s, the ground state of the ring will spontaneously break the time-reversal invariance. The ground state will have nonzero supercurrent and magnetic flux (or rather random, trapped fluxes in the ground state) and will be two-fold degenerate. At longer times, the symmetry will be restored due to the thermal activation of the macroscopic quantum tunneling between the two states, but according to Ref. [201] it can be expected that for dirty metal rings with conductance of the order of $e^2/\hbar$ there will be "roughly 50% chance that the ground state will break time-reversal symmetry". By the way, this means that in the presence of disorder there may be no way to distinguish between an anyon superconductor [202] and a conventional superconductor. Of course we must stress that these speculations are entirely based upon a simple extrapolation of Eq. (4.47) to the vicinity of metal–insulator transition and there is no complete theory of statistical fluctuations of the gradient term in this region at the moment.

5. Superconductivity in strongly disordered metals: experiment

Our review of experiments on strongly disordered superconductors will be in no sense exhaustive. This is mainly a theoretical review and the author is in no way an expert on experiments. However, we shall try to illustrate the situation with the interplay of Anderson localization and superconductivity in *bulk* (three-dimensional) superconductors, both traditional and high-temperature. Again we must stress that we exclude any discussion of the numerous data on thin films which are to be described by two-dimensional theories. In this case we just refer to existing reviews [17–19]. Here we shall confine ourselves to a limited number of experiments, which we consider most interesting from the point of view of illustration of some of the ideas expressed above, just to convince the reader, that previous discussion, while purely theoretic, has something to do with the real life. More than anywhere else in this review our choice of material is based on personal interests of the author, or our direct involvement in the discussion of experiments. We shall not deal with the general problem of the influence of disorder upon superconductivity, but shall consider only the systems which remain superconducting close to the disorder-induced metal–insulator transition.

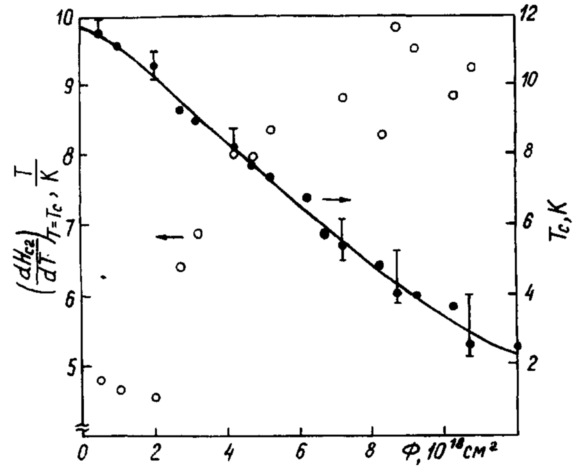
5.1. *Traditional superconductors*

There exists a number of strongly disordered systems which remain superconducting close to the metal–insulator transition induced by disorder.

The drop of T_c with the decrease of conductivity from a value of the order of $10^4\,\Omega^{-1}\,\mathrm{cm}^{-1}$ was observed in amorphous alloys of GeA [203], SiAu [204] and MoRe [205], in Chevrel phase superconductors disordered by fast neutron irradiation, such as $\mathrm{Pb}_{1-x}\mathrm{U}_x\mathrm{Mo}_6\mathrm{S}_8$ [207], $\mathrm{Mo}_6\mathrm{Se}_8$ [208], in amorphous InO_x [209], in $\mathrm{BaPb}_{1-x}\mathrm{Bi}_x\mathrm{O}_3$ in the concentration interval $0.25 < x < 0.30$ [210] and in metallic glass $\mathrm{Zr}_{0.7}\mathrm{Ir}_{0.3}$ [212]. In all these systems, superconducting transition is observed apparently not very far from the metal–insulator transition. For many of these systems, such as $\mathrm{Pb}_{1-x}\mathrm{U}_x\mathrm{Mo}_6\mathrm{S}_8$, $\mathrm{SnMo}_6\mathrm{S}_8$, $\mathrm{Mo}_6\mathrm{Se}_8$, $\mathrm{Zr}_{0.7}\mathrm{Ir}_{0.3}$ and $\mathrm{BaPb}_{0.75}\mathrm{Bi}_{0.25}\mathrm{O}_3$ [210] and some others a characteristic strongly negative temperature resistivity coefficient has been observed. Note, however, that this fact alone in no way indicates that a specimen is on one side or the other of the metal–insulator transition. The drop of T_c close to the mobility edge apparently was also observed in $\mathrm{As}_2\mathrm{Te}_3$ [213]. However, in all of these systems T_c apparently vanishes before the metal–insulator transition. Below we present some of the data on these and other similar systems.

In Fig. 21 we show the dependence of T_c and $|\mathrm{d}H_{c2}/\mathrm{d}T|_{T_c}$ in $\mathrm{SnMo}_5\mathrm{S}_6$ (Chevrel phase superconductor) on the fluence of fast neutron irradiation (the number of neutrons which passed through a cross-section of a sample during irradiation) [207]. In the region of large fluences (large disorder), when the system becomes amorphous, characteristic values of conductivity in the normal state are of the order of $\sim 10^3\,\Omega^{-1}\,\mathrm{cm}^{-1}$, which is not far from the values of "minimal metallic conductivity" $\sigma_c \sim 5\times 10^2\,\Omega^{-1}\,\mathrm{cm}^{-1}$, which define the conductivity scale of a disorder induced metal–insulator transition. A negative temperature coefficient of resistivity was observed in this conductivity range. The experimental data on T_c-decrease with the growth of resistivity in this system were rather well fitted in Ref. [22] using the μ^* dependence on resistivity given by Eq. (3.62). A clear tendency for $|\mathrm{d}H_{c2}/\mathrm{d}T|_{T_c}$ saturation with disorder is also observed. Analogous dependence of T_c and $|\mathrm{d}H_{c2}/\mathrm{d}T|_{T_c}$ on the resistivity in the normal state for $\mathrm{Mo}_6\mathrm{Se}_8$ disordered by fast neutrons is shown in Fig. 22 [208]. Here superconductivity exists up to conductivities $\sigma \sim 250\,\Omega^{-1}\,\mathrm{cm}^{-1}$. Further disordering (irradiation) leads to the destruction of the superconducting state and metal–insulator transition (an unlimited growth of resistivity with decrease of T, with variable-range hopping conduction [2,3] is observed). The slope of the upper critical field $|\mathrm{d}H_{c2}/\mathrm{d}T|_{T_c}$ also has a tendency to saturate with the growth of resistivity. Standard interpretation of such behavior of $|\mathrm{d}H_{c2}/\mathrm{d}T|_{T_c}$ was based upon the use of Gorkov's relation (cf. first relation in Eq. (3.115)) and lead to the conclusion that $N(E_F)$ decreases under disordering. In fact, we have seen that no such conclusion can be reached for systems with conductivities $\sigma < 10^3\,\Omega^{-1}\,\mathrm{cm}^{-1}$, because such saturation behavior may be a natural manifestation of the approaching metal–insulator transition. Similar dependences were observed in other Chevrel phase superconductors [206,214,215].

In Fig. 23 we show the dependence of conductivity and T_c on the parameter $p_F l/\hbar$ in amorphous InO_x alloy [209]. In Fig. 24, the data on the temperature dependence of H_{c2} in amorphous $\mathrm{In}/\mathrm{InO}_x$



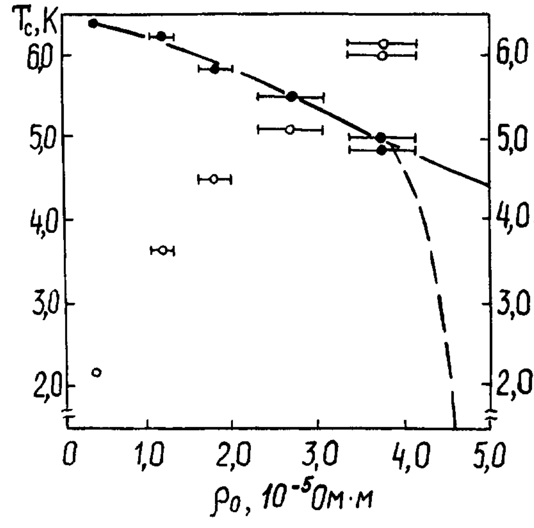
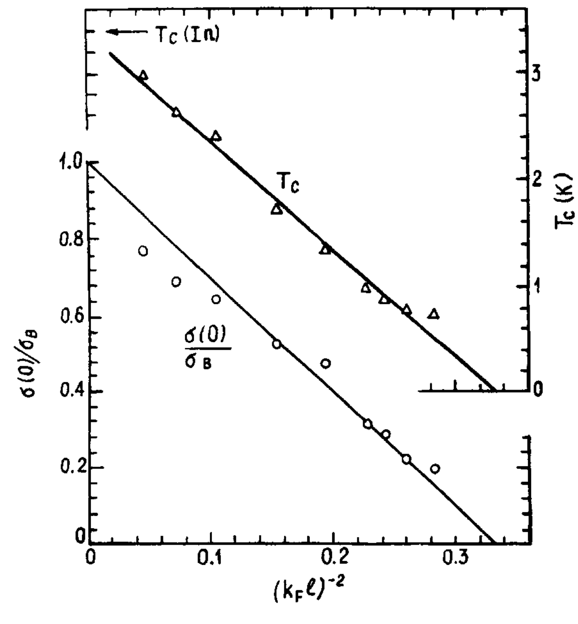


Fig. 21. Fluence dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in $SnMo_5S_6$.

Fig. 22. Resistivity dependence of T_c and $|dH_{c2}/dT|_{T_c}$ in Mo_6Se_8.



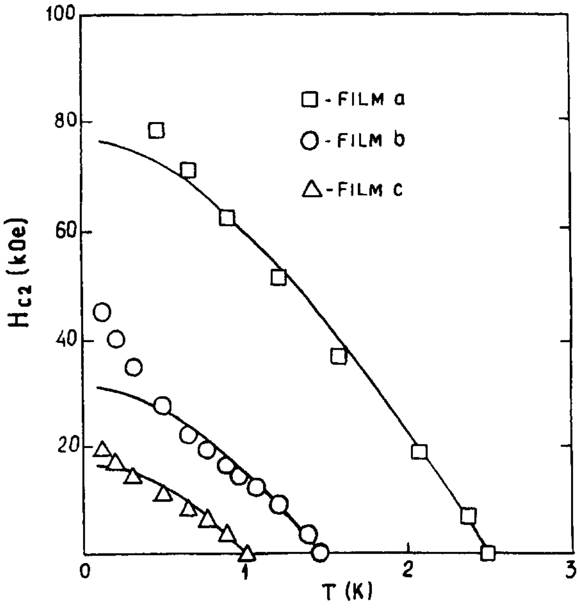
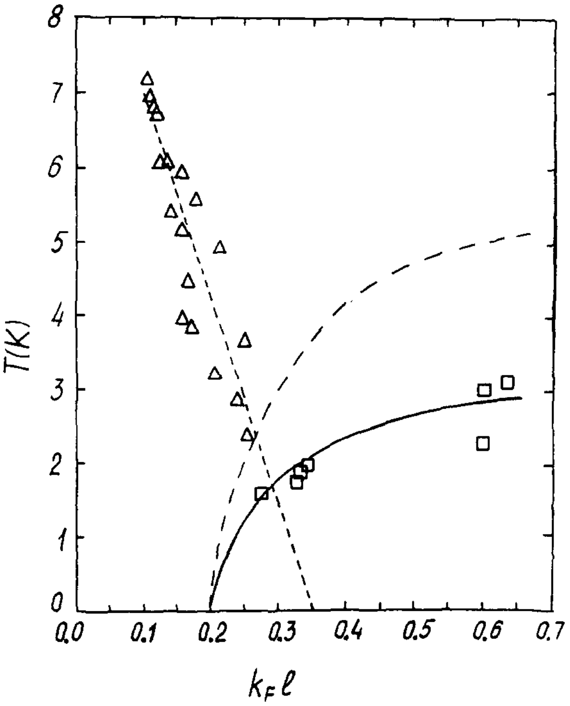


Fig. 23. Conductivity σ and T_c dependence on the parameter $p_F l/\hbar$ in amorphous InO_x; σ_B is the estimated Drude conductivity.

Fig. 24. $H_{c2}(T)$ in amorphous films of In/InO_x. Lines show standard theoretical dependence.

(bulk) films from Ref. [216] are presented for different degrees of disorder. We can see that in the low temperature region $H_{c2}(T)$ deviates from the standard temperature dependence, but apparently confirms the qualitative form predicted above for systems which are close to the Anderson transition. The same system was also studied in Ref. [217]. In Fig. 25 we show the dependence of two characteristic energies on disorder which in the opinion of the authors of Ref. [217] demonstrate the narrow region of coexistence of superconductivity and insulating state. In Fig. 26 we show the dependence of localization length and superconducting coherence length on disorder according to Ref. [217]. It demonstrates qualitative agreement with our general criterion of coexistence of superconductivity and localization; localization length must be larger or at least of the order of the size of the Cooper pair.

Very impressive are the data for amorphous $Si_{1-x}Au_x$ alloy [203,204,218]. In Fig. 27 [204] the data on T_c and conductivity dependence on the Au concentration x are shown. In Fig. 28, $H_{c2}(T)$ dependence for this system is shown for different alloy compositions [204]. From these data it is



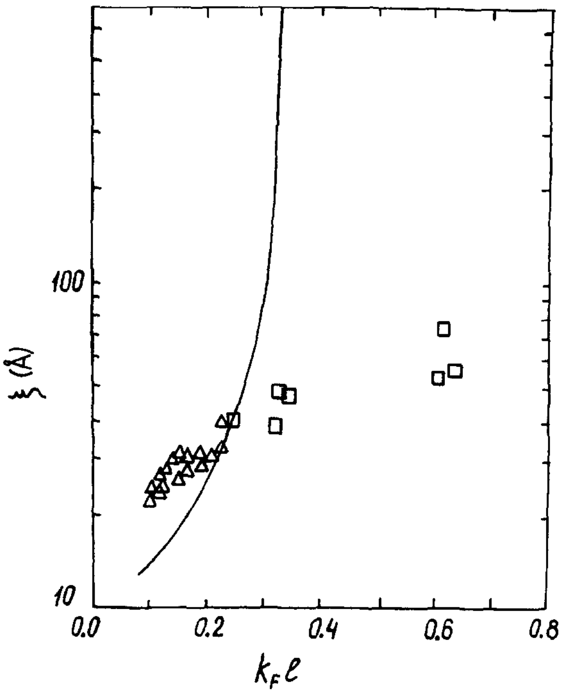
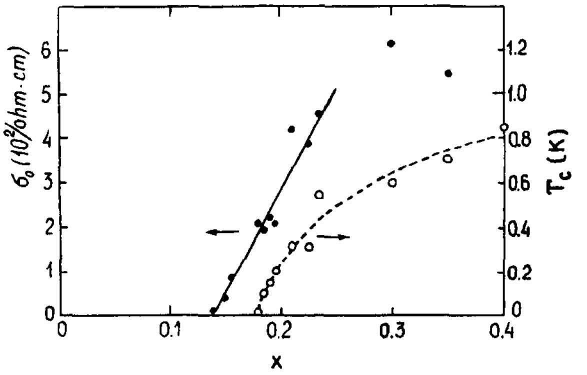


Fig. 25. The dependence of activation energy of hopping conductivity (triangles) and superconducting transition temperature T_c (squares) in amorphous films of In/InO_x on disorder parameter $p_F l/\hbar$ as determined from room-temperature conductivity and Hall measurements. Long-dashed line represents $\Delta = 1.76T_c$ following the BCS gap formula. The short-dashed line best fits the insulating data points with $(p_F l/\hbar)_c \approx 0.35$ – the critical disorder of metal–insulator transition. A narrow region of superconductivity within the insulating phase can be inferred from these data.

Fig. 26. Disorder dependence of localization length (full curve) and superconducting coherence length in amorphous In/InO_x films. Squares represent superconducting ξ for metallic films while triangles refer to insulating samples.



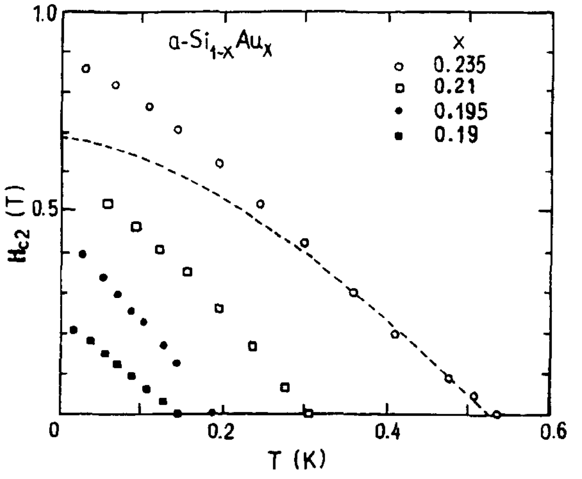


Fig. 27. Conductivity σ and T_c dependence on gold concentration in amorphous $\mathrm{Si}_{1-x}\mathrm{Au}_x$ alloy.

Fig. 28. $H_{c2}(T)$ in amorphous $\mathrm{Si}_{1-x}\mathrm{Au}_x$ alloy.

clearly seen that T_c vanishes *before* the metal–insulator transition. The metal–insulator transition itself is *continuous*, conductivity vanishes linearly with the decrease of gold concentration and the values of conductivity significantly less than the estimated "minimal metallic conductivity" are definitely observed. The system remains superconducting even for such low conductivity values. The slope of $H_{c2}(T)$ at $T = T_c$ is practically constant, irrespective of the change of conductivity (disorder) in a rather wide range. This behavior apparently cannot be explained only by the appearance of the correlation pseudogap in the density of states observed in Ref. [218], which becomes significant only very close to the metal–insulator transition. Low temperature deviation from standard convex dependence on T is also clearly seen. In Fig. 29 from Ref. [218] we show the temperature dependence of resistivity and the superconducting energy gap (determined by tunneling) of a sample with $x = 0.21$. It nicely demonstrates the superconducting transition in a system which is very close to a disorder-induced metal–insulator transition. Note that according to Ref. [218] the superconducting energy gap in this sample is substantially broadened which may indicate the growth of the statistical gap fluctuations due to the same fluctuations of the local density of states. These data are in obvious qualitative correspondence with the general theoretical picture described throughout this review.

These data show that in systems which are superconducting close to the disorder-induced metal–insulator (Anderson) transition, T_c decreases rather quickly and practically in all reliable cases vanishes before transition to the insulating state. At the same time the temperature dependence of H_{c2} is not described by the standard theory of "dirty" superconductors both with respect to the $(\mathrm{d}H_{c2}/\mathrm{d}T)_{T_c}$ behavior and at low temperatures, where the upward deviations from the standard dependence are readily observed. This confirms most of our theoretical conclusions.

Some indications of a possible superconducting state in the insulating phase of granular Al and Al–Ge were observed in Refs. [219, 220]. Obviously, the granular systems are more or less outside the scope of our review. However, we should like to mention that the strong smearing of BCS-like

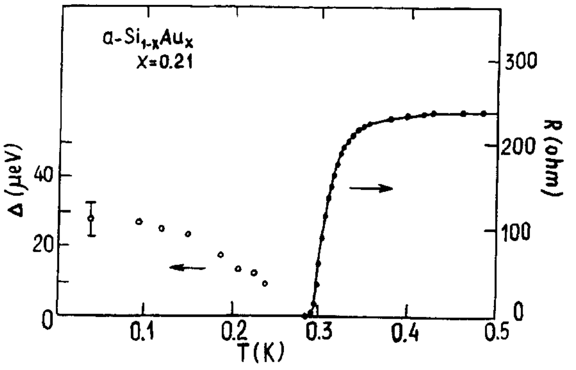


Fig. 29. Temperature dependences of the superconducting energy gap Δ and of the resistance R for amorphous $Si_{0.79}Au_{0.21}$.

density of states and the gapless regime of superconductivity was observed (via tunneling measurements) in Refs. [221, 222] close to the metal–insulator transition in these systems. This may confirm our picture of statistical fluctuation smearing of the density of states. Note that a more recent work on granular Al [223] apparently excludes the possibility of superconductivity in the insulating phase. In this work a small amount of Bi was added to granular Al in order to enhance spin–orbit scattering, which leads to an *antilocalization* effect [39]. This shifts both metal–insulator and to the *same extent* the superconducting transition, with the preservation of a narrow range of concentration on the metallic side where the material is not fully superconducting. The fact that the superconducting transition shifts with the metal–insulator transition demonstrates that its position is determined by its vicinity to the metal–insulator transition, and that it is the impending transition to the insulating state which inhibits superconductivity. Similar conclusions on superconductivity vanishing at the point of metal–insulator transition were reached for amorphous Al_xGe_{1-x} [224] and amorphous Ga–Ar mixtures [225]. This later case is particularly interesting because it has been shown that the conductivity exponent at the metal–insulator transition here is $\nu \approx 0.5$ which places this system in a different universality class than those discussed above and similar to that observed in some doped uncompensated semiconductors like Si:P [226]. Usual interpretation of this difference is based upon the importance of interaction effects in these systems [108]. Starting with the value of T_c of amorphous Ga ($T_c = 7.6$ K), T_c decreases rather slowly with decreasing Ga volume fraction v, until one enters the critical region near $v_c \approx 0.145$. Further approach to v_c leads to a rapid decrease of T_c. Taking McMillan formula Eq. (3.25) for T_c (with $\omega_{\log}/1.20 = 320$ K and $\lambda = 0.45$) and assuming negligible Coulomb repulsion μ^* for pure amorphous Ga the increase of μ^* on the approach of metal–insulator transition can be determined from the experimental data for T_c. This increase is approximately given by $\mu^* \sim (v - v_c)^{-0.33}$. From this it is easy to see that $T_c \to 0$ for $v \to v_c$, so that these data do not indicate the survival of superconductivity beyond the metal–insulator transition. These results are not surprising since we have seen the existence of strong mechanisms of T_c degradation close to disorder-induced metal–insulator transition.

The interesting new high-pressure metastable metallic phase of an amorphous alloy $Cd_{43}Sb_{57}$ exhibiting the gradual metal–insulator transition during the slow decay at room temperature and atmospheric pressure has been studied in Refs. [227, 228]. Authors claim that during this decay the system remains homogeneous while going from metallic to insulating phase. At the same time the metallic phase is superconducting with $T_c \approx 5$ K and remains so up to the metal–insulator transition. Close to it the superconducting transition becomes smeared, while incomplete transition persists even in the insulating state. While these data are reminiscent of data on quench-condensed films of Sn and Ga [229], which were interpreted as reentrant superconductivity due to sample inhomogeneities, it is stressed in Refs. [227, 228] that, in this new system situation is different and we are dealing with an intrinsically inhomogeneous state of superconductors discussed in Refs. [62, 182, 183]. From our point of view, further studies of this system are necessary in order to show unambiguously the absence of structural inhomogeneities. Also, a rather peculiar characteristic of this system is the almost complete independence of the onset temperature of the superconducting transition on disorder.

The general conclusion is that, in most cases of traditional superconducting systems, we cannot find an unambiguous demonstration of the possibility of superconductivity in an insulating state induced by disorder. At the same time, we can see a rather rich variety of data on superconductivity close to the metal–insulator transition which stimulate further studies. Some of the anomalies of the superconducting behavior discussed above can be successfully explained by theories presented in this review, while the others require further theoretical investigations.

5.2. *High-T_c superconductors*

Very soon after the discovery of high-temperature oxide superconductors [14, 15] it was recognized that localization effects have an important role to play in these systems. There are many sources of disorder in these systems and the low level of conductivity indicate from the very beginning their closeness to Anderson transition. In the field where there are hundreds of papers published on the subject it is impossible to review or even to quote all of them. A more or less complete impression about the status of high-T_c research can be obtained from Conference Proceedings [230]. Here we shall concentrate almost only on papers which deal with disordering by fast neutron irradiation which we consider probably the "purest" method to introduce disorder into the system (allowing to neglect the complicated problems associated with chemical substitutions). Also, historically, it is apparently the earliest method used to study the disorder effect in high-T_c superconductors in a controllable way [231, 232].

There are several reasons for localization effects to be important in high-T_c oxides:

- *Two-dimensionality.* All the known high-T_c systems (with $T_c > 30$ K) are strongly anisotropic or quasi-two-dimensional conductors. We have seen above that for such systems it is natural to expect the strong enhancement of localization effects due to the special role of spatial dimensionality $d = 2$: in the purely two-dimensional case, localization appears for infinitely small disorder [31, 4, 6, 7]. The inplane conductivity scale for the metal–insulator transition in such systems as given by Eq. (2.12) or Eq. (2.93) is larger than in the isotropic case. Reasonable estimates show that the values of inplane "minimal metallic conductivity" may exceed $10^3\ \Omega^{-1}\,\mathrm{cm}^{-1}$. While due to the continuous nature of the Anderson transition there is no rigorous meaning of minimal metallic conductivity, these estimates actually define the scale of conductivity near the metal–insulator transition caused by disorder. Then it is clear that most of the real samples of high-T_c superconductors are quite close to the Anderson transition and even the very slight disordering is sufficient to transform them into Anderson insulators [171].

- "*Marginal*" *Fermi liquid.* During our discussion of interaction effects we have seen that there are serious reasons to believe that importance of localization effects in high-T_c oxides may be actually due to more fundamental reasons connected with the anomalous electronic structure and interactions in these materials. The concept of a "marginal" Fermi liquid [124] leads to extreme sensitivity of such a system to disordering and the appearance of localized states around the Fermi level at a rather weak disorder [126, 127].

On the other hand, high-T_c systems are especially promising from the point of view of the search for superconductivity in the Anderson insulator:

- High transition temperature T_c itself may guarantee the survival of superconductivity at relatively high disorder.
- Due to the small size of Cooper pairs high-T_c systems in combination with high-T_c (large gap!) we can easily satisfy the main criterion for superconductivity in the localized phase as given by Eq. (3.20).
- Being narrow band systems, as most of the conducting oxides, high T_c systems are promising due to low values of the Fermi energy E_F which leads to less effective T_c degradation due to localization enhancement of the Coulomb pseudopotential μ^* (cf. Eq. (3.61)).

Anomalous transport properties of high-T_c oxides are well known [233]. Experimentally, there are two types of resistivity behavior of good single-crystals of these systems. In the highly conducting *ab* plane of $YBa_2Cu_3O_{7-\delta}$ and other oxides, resistivity of a high-quality single-crystal always shows the notorious linear-T behavior (by "good" we mean the samples with resistivity $\rho_{ab} < 10^3\ \Omega\,\mathrm{cm}$). However, along the orthogonal c direction the situation is rather curious: most samples produce semiconductor-like behavior $\rho_c \sim 1/T$, though some relatively rare samples (apparently more pure) show metallic-like $\rho_c \sim T$ (with strong anisotropy $\rho_c/\rho_{ab} \approx 10^2$ remaining) [233, 234]. Metallic behavior in the c direction was apparently observed only in the best samples of $YBa_2Cu_3O_{7-\delta}$ and almost in no other high-T_c oxide. In Fig. 30 taken from Ref. [235] we show the temperature dependence of ρ_c in a number of high-T_c systems. It is seen that $\rho_c(T)$ changes between metallic and semiconducting behavior depending on whether the resistivity is below or above the Ioffe–Regel limit defined for the quasi-two-dimensional case by Eq. (2.12). Rather strange is the absence of any obvious correlation between the behavior of ρ_c and T_c.

This unusual behavior leads us to the idea that most of the samples of high-T_c systems which are studied in the experiment are actually already in a localized phase due to internal disorder which is always present. Surely, we realize that such a drastic assumption contradicts the usual expectations and propose it just as an alternative view open for further discussion. The attempted justification of this idea may be based upon the quasi-two-dimensional nature of these systems or on marginal Fermi liquid effects. In this case a simple conjecture on the temperature behavior of resistivity of single-crystals can be made which qualitatively explains the observations [236, 237]. In case of localized states at the Fermi level and for finite temperatures it is important to compare the localization length R_{loc} with the diffusion length due to inelastic scattering $L_\varphi \approx \sqrt{D\tau_\varphi}$, where D is the diffusion coefficient due to elastic scattering on disorder, while τ_φ is the phase coherence time determined by inelastic processes. For $T > 0$ this length L_φ effectively replaces the sample size L in all expressions of scaling theory of localization when $L \gg L_\varphi$, because on distances larger than L_φ all information on the nature of wave functions (e.g. whether they are localized or extended) is smeared out. Taking into account the usual low-temperature dependence like $\tau_\varphi \sim T^{-p}$ (where p is some integer, depending on the mechanism of inelastic scattering) this can lead to a nontrivial

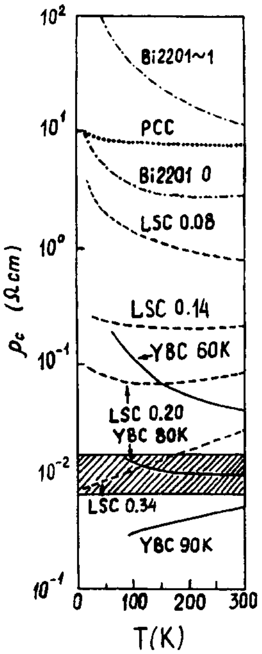


Fig. 30. Temperature dependence of ρ_c for different high-T_c cuprates. The dashed region indicates the resistivity range corresponding to Ioffe–Regel limit.

temperature dependence of conductivity, in particular to the possibility of a negative temperature coefficient of resistivity [33]. Similar expressions determine the temperature dependence of conductivity also for the localized phase unit $L_\varphi < R_{\text{loc}}$. In this case, electrons do not "feel" being localized and conductivity at high enough T will show metallic like behavior. For localization to be important we must go to low enough temperatures, so that L_φ becomes greater than R_{loc}. If disordered high-T_c superconductors are in fact Anderson insulators with a very anisotropic localization length, $R_{\text{loc}}^{ab} \gg R_{\text{loc}}^{c}$ and both localization lengths diminish as disorder grows, L_φ is also anisotropic and we can have three different types of temperature behavior of resistivity [236]:

1. Low T or strong disorder, when we have

$$L_\varphi^{ab} \approx \sqrt{D_{ab}\tau_\varphi} \gg R_{\text{loc}}^{ab}\,, \quad L_\varphi^{c} \approx \sqrt{D_c\tau_\varphi} \gg R_{\text{loc}}^{c}\,. \tag{5.1}$$

This gives semiconductor-like behavior for both directions.

2. Medium T or medium disorder, when

$$L_\varphi^{ab} < R_{\text{loc}}^{ab}\,, \quad L_\varphi^{c} > R_{\text{loc}}^{c}\,, \tag{5.2}$$

and metallic behavior is observed in the ab plane, while semiconducting temperature dependence of resistivity is observed along the c-axis.

3. High T or low disorder, when

$$L_\varphi^{ab} < R_{\text{loc}}^{ab}\,, \quad L_\varphi^{c} < R_{\text{loc}}^{c} \tag{5.3}$$

and metallic behavior is observed in both directions.

Here we do not speculate on the inelastic scattering mechanisms leading to the concrete temperature behavior in high-T_c oxides, in particular on linear T behavior in the *ab* plane or $1/T$ behavior in the *c* direction. Unfortunately too little is known on these mechanisms [233] to be able to make quantitative estimates on the different types of behavior predicted above. Of course detailed studies of such mechanisms are necessary to prove the proposed idea and to explain the temperature dependence of resistivity in high-T_c systems on its basis. However, most of the experimental data as we shall see below at least do not contradict the idea of the possibility of Anderson localization in disordered high-T_c cuprates.

Now let us consider the experiments on controllable disordering of high-temperature superconductors. Already the first experiments on low temperature ($T = 80$ K) fast neutron irradiation of ceramic samples of high-T_c systems [238–242, 244] have shown that the growth of structural disorder leads to a number of drastic changes in their physical properties:

- continuous metal–insulator transition at very slight disordering,
- rapid degradation of T_c,
- apparent coexistence of hopping conductivity and superconductivity at intermediate disorder,
- approximate independence of the slope of H_{c2} at $T \sim T_c$ on the degree of disorder,
- anomalous *exponential* growth of resistivity with defect concentration.

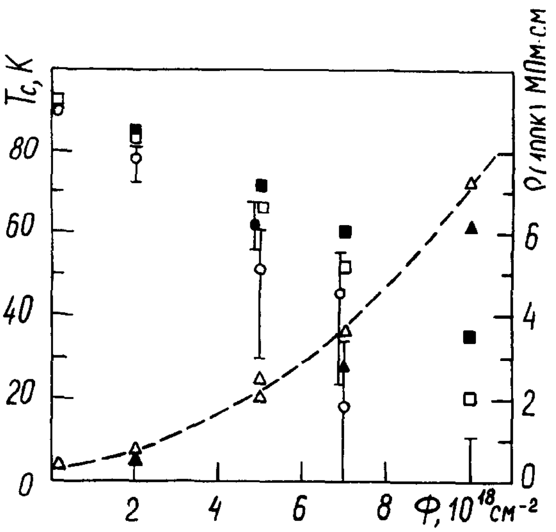
These anomalies were later confirmed on single-crystals and epitaxial films [243, 245–247], and were interpreted [171, 16] using the ideas of possible coexistence of Anderson localization and superconductivity.

In Fig. 31 we show data [171] on the dependence of the superconducting transition temperature and resistivity (at $T = 100$ K, i.e. just before the superconducting transition) on fast neutron fluence for $YBa_2Cu_3O_{6.95}$. In all high-T_c compounds, introduction of defects leads to a strong broadening of the superconducting transition. The derivative $(dH_{c2}/dT)_{T_c}$ in ceramic samples measured at the midpoint of the superconducting transition does not change as $\rho_{100\,K}$ grows by an order magnitude. In Fig. 32 [171] we show the temperature dependence of resistivity for samples of $YBa_2Cu_3O_{6.95}$ and $La_{1.83}Sr_{0.17}CuO_4$ for different degrees of disorder. In all these materials the $\rho(T)$ curves vary in the same way. In the fluence range $\Phi > 10^{19}$ cm^{-2}, where superconductivity is absent, $\rho(T)$ follows a dependence which is characteristic of conductivity via localized states [2, 3]:

$$\rho(T) = \rho_0 \exp(Q/T^{1/4}), \quad Q = 2.1[N(E_F)R_{loc}^3]^{-1/4} \tag{5.4}$$

as shown in Fig. 33 (Mott's variable-range hopping conduction).

The most striking anomaly of resistivity behavior of all high-T_c systems under disordering is nonlinear, practically *exponential* growth of resistivity at fixed temperature (e.g. $\rho(T = 100\,\mathrm{K})$) with fluence, starting from the low fluences $\Phi < 7 \times 10^{18}$ cm^{-2}, including superconducting samples [238, 171, 239–242]. These data are shown in Fig. 34 [171] for the dependence of $\rho(T = 80\,\mathrm{K})$ on Φ obtained from measurements made directly during the process of irradiation. For comparison, the similar data for $SnMo_6S_8$ are shown which do not demonstrate such an anomalous behavior, its resistivity is just proportional to Φ and saturates at large fluences. We relate this exponential growth of ρ with the increase of Φ (i.e. of defect concentration) in all high-T_c systems to localization, which already appears for very small degrees of disorder in samples with high values of T_c. As we have seen in samples with much reduced or vanishing T_c localization is observed directly via Mott's hopping in the temperature behavior of resistivity given by Eq. (5.4).



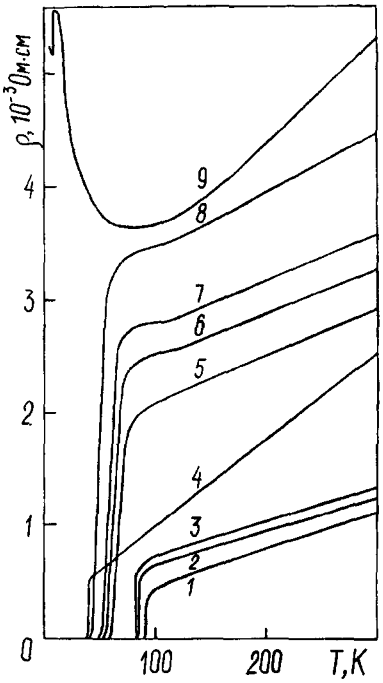


Fig. 31. Dependence of the superconducting transition temperature and resistivity (at $T = 100$ K) on neutron fluence for ceramic $YBa_2Cu_3O_{6.95}$. Different notations correspond to different methods of measurement and also evolution after annealing at 300 K.

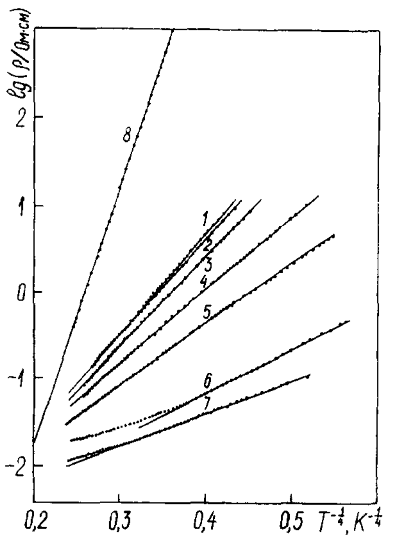
Fig. 32. Temperature dependence of resistivity ρ for ceramic samples of $YBa_2Cu_3O_{6.95}$ (curves 1–3 and 5–8) and $La_{1.83}Sr_{0.17}CuO_4$ (curves 4, 9) irradiated at $T = 80$ K with different fluences: 1, $\Phi = 0$; 3, 6, 8, $\Phi = 2.5$ and $7 \times 10^{18}\,\mathrm{cm}^{-2}$ plus annealing for 2 h at 300 K; 2, 5, 7, irradiated with $\Phi = 2.5$ and $7 \times 10^{18}\,\mathrm{cm}^{-2}$ plus annealing for two weeks at 300 K; 4, $\Phi = 0$; 9, $\Phi = 5 \times 10^{18}\,\mathrm{cm}^{-2}$ plus annealing for 2 h at 300 K.

From these results it follows that the electronic system of high-T_c superconductors is very close to the Anderson transition. The observed variation of ρ as a function of both fluence and of temperature can be described by the following empirical formula [238]:

$$\rho(T, \Phi) = (a + cT)\exp(b\Phi/T^{1/4})\,. \tag{5.5}$$

Identifying the exponential factors in Eqs. (5.4) and (5.5) it is possible to obtain a fluence dependence of localization length (cf. Ref. [171] and below).

Detailed neutron diffraction studies of structural changes in irradiated samples were also performed [238, 171, 248]. These investigations have shown definitely that there is no oxygen loss in $YBa_2Cu_3O_{6.95}$ during low temperature irradiation. Only some partial rearrangement of oxygens between positions O(4) and O(5) in the elementary cell occur as radiation-induced defects are introduced. In addition, in all high-T_c compounds the Debye–Waller factors grow and the lattice parameters a, b, c increase slightly [171, 248]. The growth of Debye–Waller factors reflect significant atomic shifts, both static and dynamic, from their regular positions, which induce a random potential. This disorder is pretty small from the structural point of view, the lattice is only



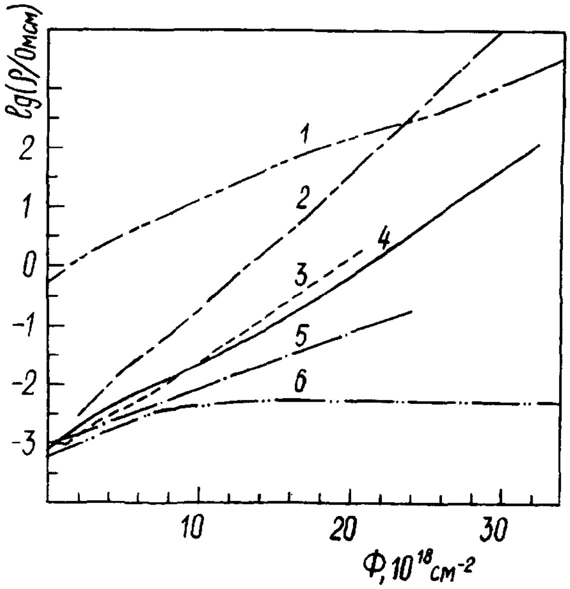


Fig. 33. Dependence of $\ln\rho$ on $T^{-1/4}$ for $YBa_2Cu_3O_{6.95}$ irradiated with a fluence of $\Phi = 1.2\times 10^{19}\,cm^{-2}$ at $T = 80$ K (curve 1), and after 20 min annealing at $T = 150$ K (2); 200 K (3); 250 K (4); 300 K (5) and two weeks annealing at $T = 300$ K (7). Similar dependence for $La_{1.83}Sr_{0.17}CuO_4$ for $\Phi = 2\times 10^{19}\,cm^{-2}$ annealed for 2 h at 300 K (6) and for La_2CuO_4 for $\Phi = 2\times 10^{19}\,cm^{-2}$ annealed for 2 h at 300 K (8).

Fig. 34. Dependence of $\ln\rho$ on fluence Φ during irradiation at $T = 80$ K: 1 – La_2CuO_4; 2 – $YBa_2Cu_3O_{6.95}$; 3 – single crystalline ρ_{ab} in $YBa_2Cu_3O_{6.95}$; 4 – $La_{1.83}Sr_{0.17}CuO_4$; 5 – Bi–Sr–Ca–Cu–O; 6 – $SnMo_6Se_8$.

slightly distorted. However, we have seen that this small disorder is sufficient to induce a metal–insulator transition and complete degradation of superconductivity. The absence of oxygen loss implies that there is no significant change in concentration of carriers and we really have a disorder-induced metal–insulator transition. This is also confirmed by other methods [244, 249]. In Fig. 35 we show the data [244, 245] regarding temperature dependence of the Hall concentration of ceramic samples of irradiated and oxygen deficient $YBa_2Cu_3O_{7-\delta}$. It is seen that disordering weakens the anomalous temperature dependence of Hall effect, but Hall concentration n_H at low T practically does not show significant difference with the data on oxygen deficient samples, where n_H drops by several orders. This also confirms the picture of a disorder-induced metal–insulator transition in radiation disordering experiments. Similar Hall data were obtained on epitaxial films [246] and single-crystals [247].

Qualitatively identical resistivity behavior was also obtained in the experiments on radiation disordering of single-crystals [243, 245] and epitaxial films [246]. Electrical resistivities of $YBa_2Cu_3O_{7-\delta}$ single crystals were measured at $T = 80$ K directly during irradiation by fast neutrons. The data are shown in Fig. 36. We can see that ρ_{ab} increases exponentially with Φ (defect concentration) starting from the smallest doses, while ρ_c grows more slowly and only for $\Phi > 10^{19}\,cm^{-2}$ they grow at the same rate. At large fluences, both ρ_{ab} and ρ_c demonstrate [250] Mott's hopping $\ln\rho_{ab,c} \sim T^{-1/4}$. Similar data of Ref. [246] show $\ln\rho \sim T^{-1/2}$ characteristic of the

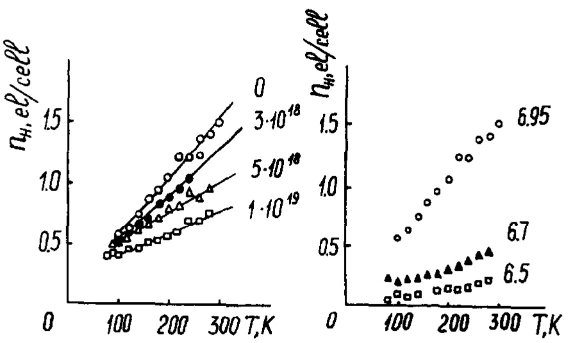
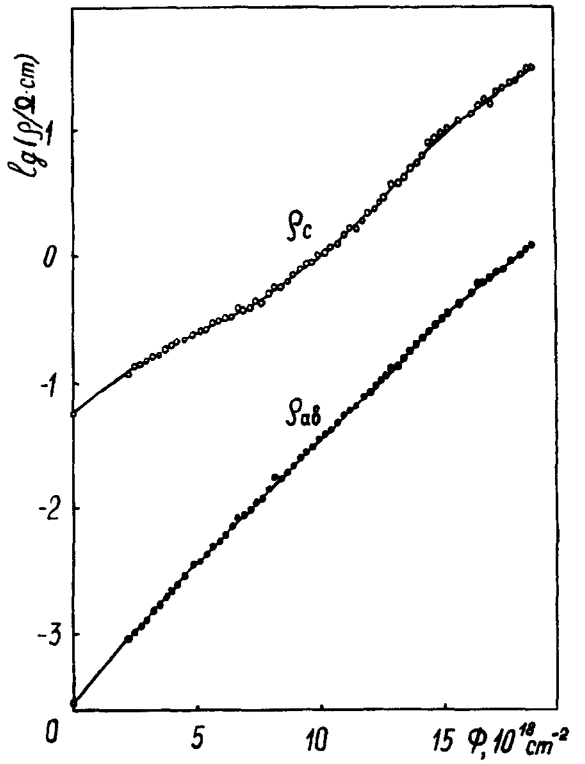


Fig. 35. Temperature dependence of Hall concentration for the irradiated (left) and oxygen deficient (right) ceramic samples of $YBa_2Cu_3O_{7-\delta}$.



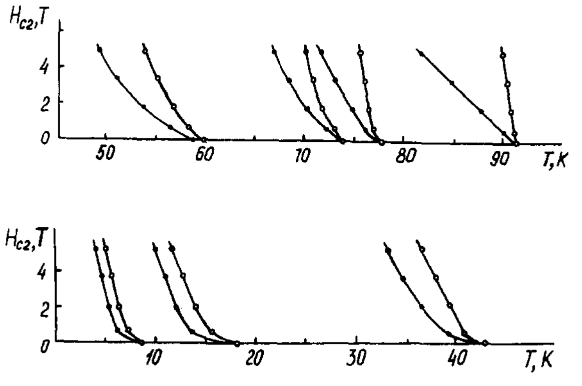


Fig. 36. Fluence dependence of ρ_{ab} and ρ_c at $T = 80$ K during fast neutron irradiation.

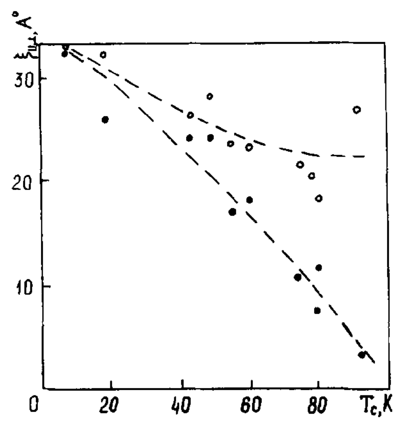
Fig. 37. Temperature dependence of $H_{c2}^{\parallel}$ (upper curves) and $H_{c2}^{\perp}$ (lower curves) for the single-crystals of $YBa_2Cu_3O_{7-\delta}$ with different degrees of disorder.

Coulomb gap. We do not know the reasons for this discrepancy between single-crystalline and epitaxial films data (note that another method of disordering by 1 MeV Ne^+ ions was used in Ref. [246]). Anisotropy ρ_c/ρ_{ab} at $T = 80$ K drops rapidly (to the values ~ 30 for $\Phi = 10^{19}\,\mathrm{cm}^{-2}$) and then practically remains unchanged and the "residual" anisotropy is of the order of the room-temperature value as in initial samples. This means that the temperature dependence of anisotropy weakens in the disordered samples. Note that, unfortunately, only the single-crystals with "semiconducting" temperature dependence of resistivity along the c axis were investigated up to now.

The upper critical fields of $\mathrm{YBa_2Cu_3O_{7-\delta}}$ single-crystals (determined from standard resistivity measurements) for different degrees of disorder are shown in Fig. 37 [245]. Temperature dependence of H_{c2} in disordered samples is essentially nonlinear, especially for samples with low T_c. The temperature derivative of $H_{c2}^{\perp}$ (field along the c axis) estimated from high-field regions increases with disorder. However, a similar derivative of $H_{c2}^{\parallel}$ (field along ab plane) drops in the beginning and then does not change. Anisotropy of H_{c2} decreases with disorder and in samples with $T_c \sim 10$ K the ratio of $(H_{c2}^{\parallel})'/(H_{c2}^{\perp})'$ is close to unity. According to Eq. (3.118) this means the complete isotropisation of the Cooper pairs. This is illustrated by Fig. 38 [251]. The remaining anisotropy of resistivity may be connected with some kind of planar defects in the system.

In a recent paper [252] Osofsky et al. presented the unique data on the temperature dependence of the upper critical field of high-temperature superconductor $\mathrm{Bi_2Sr_2CuO_y}$ in a wide temperature interval from $T_c \approx 19$ K to $T \approx 0.005T_c$, which has shown a rather anomalous dependence with positive curvature at any temperature. The authors of Ref. [252] have noted that this type of behavior is difficult to explain within any known theory. It is sharply different from the standard behaviour the BCS-model. It was demonstrated in Refs. [253, 254] that the observed dependence of $H_{c2}(T)$ can be satisfactorily explained by localization effects in a two-dimensional (quasi-two-dimensional) model in the limit of sufficiently strong disorder. Measurements of H_{c2} in Ref. [252] were performed on epitaxially grown films of $\mathrm{Bi_2Sr_2CuO_y}$: however it is quite possible that the films were still disordered enough, which can be guessed from the rather wide (~ 7 K) superconducting transition. Unfortunately, the relevant data, in particular, regarding the conductivity of the films studied are absent. This gives us some ground to try to interpret the data obtained in Ref. [252] in the framework of a very strong disorder, the effects of which are obviously enhanced by the quasi-two-dimensional nature of high-temperature superconductors.

The general discussion of the temperature dependence of the upper critical field in the two-dimensional and quasi-two-dimensional cases with strong localization effects was presented above in Section 3.3.1. Note that we mainly analyzed there the case of a magnetic field perpendicular to the highly conducting planes, which is precisely the case of Ref. [252]. We have seen [81] that the anomalies of the upper critical field due to the frequency dependence of the diffusion coefficient appear only for temperatures $T \ll \mathrm{e}^{-1/\lambda}/\tau$. For higher temperatures we obtained the usual behavior of "dirty" superconductors. Also we have noted [81] that superconductivity survives in a system with finite localization length if $T_c \gg \lambda\,\mathrm{e}^{-1/\lambda}/\tau$, which is equivalent to our criteria for the smallness of Cooper pair size compared with localization length. This latter length is exponentially large in two-dimensional systems with small disorder ($\lambda \ll 1$). The most interesting (for our aims) limit of a relatively strong disorder is defined by $T_c \ll \mathrm{e}^{-1/\lambda}/\tau$, so that in fact we are dealing with a pretty narrow region of λs when $\lambda\,\mathrm{e}^{-1/\lambda}/\tau \ll T_c \ll \mathrm{e}^{-1/\lambda}/\tau$. In this case we have seen that the upper critical



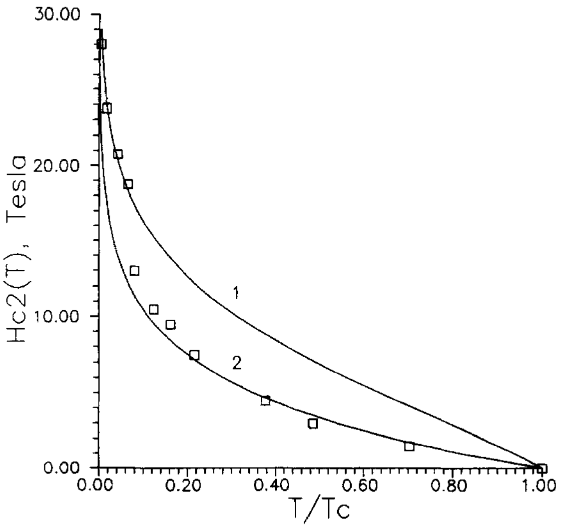


Fig. 38. The dependence of coherence lengths determined from H_{c2} behavior under disordering on the critical temperature T_c: $\xi_{\parallel}$ – open circles; $\xi_{\perp}$ – filled circles.

Fig. 39. Temperature dependence of the upper critical field: theoretical curve (1) is given for the case of $e^{-1/\lambda}/T_c\tau = 2$, $\lambda = 0.18$, while curve (2) is for $e^{-1/\lambda}/T_c\tau = 20$, $\lambda = 0.032$. Squares represent the experimental data for $Bi_2Sr_2CuO_y$.

field is practically defined by Eq. (3.141):

$$\ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T}\right) = \left(1 + 4\pi\frac{D_0}{\phi_0}\frac{\tau H_{c2}}{e^{-1/\lambda}}\right)\ln\left(\frac{\gamma}{2\pi}\frac{e^{-1/\lambda}}{\tau T_c}\left(1 + 4\pi\frac{D_0}{\phi_0}\frac{\tau H_{c2}}{e^{-1/\lambda}}\right)\right) \qquad (5.6)$$

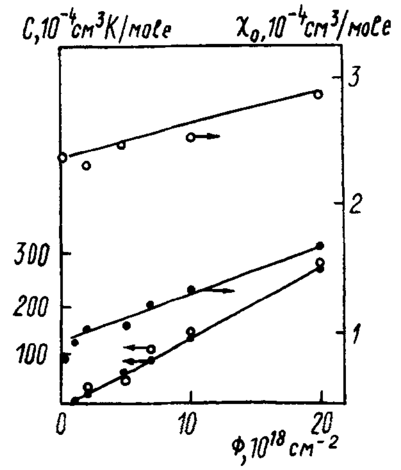
($\gamma = 1.781$) from which we can directly obtain the $T(H_{c2})$-dependence. The appropriate behavior of the upper critical field for two sets of parameters is shown in Fig. 39. The curve of $H_{c2}(T)$ demonstrates positive curvature and H_{c2} diverges for $T \to 0$. We have seen that this weak (logarithmic) divergence is connected with our neglect of the magnetic field influence upon diffusion. Taking this influence into account we can suppress this divergence of H_{c2} as $T \to 0$. This is the main effect of broken time invariance and it is clear that it is important only for extremely low temperatures [81]. In the following we neglect it. For the quasi-two-dimensional case on the dielectric side of Anderson's transition, but not very close to it, the behavior of diffusion coefficient is quite close to that of purely two-dimensional case, so that the upper critical field can be analyzed within the two-dimensional approach. Close to the transition (e.g. over interplane transfer integral) both for metallic and insulating sides and for parameters satisfying the inequality $\lambda e^{-1/\lambda}/\tau \ll T_c \ll e^{-1/\lambda}/\tau$, the temperature dependence of H_{c2} is in fact again very close to those in the purely two-dimensional case considered above [81]. Some deviations appear only in a very narrow region of very low temperatures [81].

In Fig. 39 we also show the experimental data for H_{c2} from Ref. [252]. Theoretical curve (1) is given for the parameters which lead to rather good agreement with experiment in the low temperature region. The curve (2) corresponds to parameters giving good agreement in a wide temperature region except the lowest temperatures. The cyclotron mass m was always assumed to be equal that of the free electron. In general we observe satisfactory agreement between theory and experiment. Unfortunately, the values of the ratio $e^{-1/\lambda}/T_c\tau$ for the second curve, while corresponding to quite reasonable values of λ, lead to nonrealistic (too small) values of $T_c\tau$, which are rather doubtful for a system with relatively high T_c. For the first curve, the situation is much better though the electron damping on the scale of T_c is still very large which corresponds to strong disorder. Note however, that the detailed discussion of these parameters is actually impossible without the knowledge of additional characteristics of the films studied in Ref. [252]. In particular, it is quite interesting to have an independent estimate of λ. We also want to stress the relatively approximate nature of these parameters due to our two-dimensional idealization. More serious comparison should be done using the expressions of Ref. [81] for the quasi-two-dimensional case, which again requires the additional information on the system, in particular, the data on the anisotropy of electronic properties.

In our opinion, the relatively good agreement of experimental data of Ref. [252] with theoretical dependences obtained for the two-dimensional (quasi-two-dimensional) case of a disordered system with Anderson localization illustrates the importance of localization effects for the physics of high-temperature superconductors. However, we must note that similar anomalies of the temperature dependence of the upper critical field were also observed in Ref. [255] for the single crystals of the overdoped $Tl_2Ba_2CuO_{6+\delta}$ which authors claim to be extremely clean, so that apparently no explanation based upon strong localization effects can be used. Similar data were recently obtained for thin films of underdoped $YBa_2(Cu_{0.97}Zu_{0.03})_3O_{7-\delta}$ with pretty low transition temperatures [256]. These films again seem to be disordered enough to call localization effects as a possible explanation of the unusual positive curvature of $H_{c2}(T)$ dependence for all temperatures.

Under irradiation, localized moment contribution appears in the magnetic susceptibility of high-T_c oxides [238, 171]. In the temperature range from T_c to 300 K, $\chi(T)$ is satisfactorily described by a Curie–Weiss type dependence: $\chi(T) = \chi_0 + C/(T - \Theta)$. The value of χ_0 and the Curie constant C as a function of fluence Φ are given in Fig. 40. The value of C is proportional to the fluence. Note that the threefold larger slope of $C(\Phi)$ in $YBa_2Cu_3O_{6.95}$ as compared with $La_{1.83}Sr_{0.17}CuO_4$ is an evidence that this Curie-law temperature dependence is associated with localized moments forming on Cu (there are three times more copper in an elementary cell of Y compound than in an La compound).

The data presented above show that electronic properties of high-T_c systems are quite different under disordering from that of traditional superconductors [214, 215] or even some closely related metallic oxides [247, 257]. We associate these anomalies with the closeness of the Anderson transition and believe that real samples of high-T_c systems which always possess some noticeable disorder may well be already in the state of the Anderson insulator. However, we must stress that it is quite difficult to decide from the experiments described above the precise position of the Anderson transition on disorder scale. Some additional information on this problem may be obtained from experiments on NMR relaxation in disordered state, using the approach proposed rather long ago by Warren [258] and later quantified theoretically in Refs. [259, 260]. The study of NMR relaxation rate on [89] Y nuclei in radiationally disordered $YBa_2Cu_3O_{6.95}$ (which is opposite to Cu nuclei demonstrate Korringa behavior) [261, 262] has shown the anomalies (a *maximum* in the so-called Warren's enhancement factor) which according to Ref. [260] may indicate the Anderson transition somewhere in the fluence interval $\Phi = (1\text{–}2) \times 10^{19}\ \mathrm{cm}^{-2}$. Unfortunately, the number of samples in these experiments was too limited to place the transition point



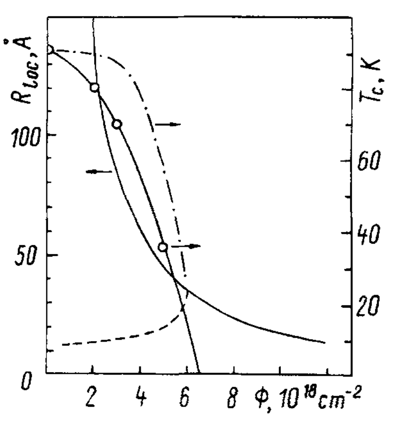


Fig. 40. Dependence of the Curie constant C and the temperature-independent part χ_0 of magnetic susceptibility on neutron fluence Φ for $La_{1.83}Sr_{0.17}CuO_4$ (filled circles) and $YBa_2Cu_3O_{6.95}$ (open circles).

Fig. 41. Dependence of T_c on fluence for $YBa_2Cu_3O_{6.95}$ (circles). The solid curve is the localization length calculated from hopping conductivity. Dashed curve defines the minimum localization length at which superconductivity can exist at given T_c. Dashed–dotted curve is the theoretical fit using expressions described in the text.

more precisely, while superconductivity disappears exactly in this interval. In this sense we still have no direct proof of coexistence of superconductivity and localization in disordered high-T_c oxides. However, the method used in Refs. [261, 262] seems to be very promising. Note that Knight shift data of Refs. [261, 262] strongly indicate Coulomb gap opening at the Fermi level of strongly disordered oxides. Independently, this conclusion was reached in tunneling experiments of Ref. [263] on a number of oxides disordered by doping.

Using the experimental data on electrical resistivity of disordered samples of $YBa_2Cu_3O_{6.95}$ and the relations given by Eq. (5.4) and Eq. (5.5) (assuming that exponentials there are identical) we can calculate the change of localization length R_{loc} as a function of fluence [171, 239–242]. This dependence is shown on Fig. 41 along with the fluence dependence of T_c. It is clearly seen that superconductivity is destroyed when localization length R_{loc} becomes smaller than ~ 30 Å, i.e. it becomes of the order or smaller than a typical size of the Cooper pair in this system (cf. Fig. 38) in complete accordance with our basic criterion of Eq. (3.20). We can estimate the minimal value of R_{loc} for which superconductivity can still exist in a system of localized electrons via Eq. (3.20) [171] taking the free-electron value of $N(E_F) \approx 5 \times 10^{33}\ (\text{erg cm}^3)^{-1}$ (for carrier concentration of $\sim 6 \times 10^{21}\ \text{cm}^{-3}$) and the gap value $\Delta \sim 5T_c$, corresponding to very strong coupling [232]. We obtain the result shown in Fig. 41. In any case we can see that criterion of Eq. (3.20) ceases to be fulfilled for $\Phi \sim (5\text{–}7) \times 10^{18}\ \text{cm}^{-2}$ in remarkably good agreement with the experiment.

In the absence of accepted pairing mechanism for high temperature superconductors it is very difficult to speculate on the reasons for T_c degradation in these systems. If we assume that the main mechanism of T_c degradation is connected with the growth of Coulomb effects during disordering, as discussed above in this review, we can try to use appropriate expressions to describe the experimental data. Assuming superconductivity in the localized phase we can use Eq. (3.68), estimating R_{loc} as above from empirical relation (5.5) and (5.4) (or directly expressing the parameters entering Eq. (3.68) via experimental dependence of resistivity on fluence as described by Eq. (5.5) [171]). The results of such a fit (with the assumption of $\mu \approx 1$) are also shown in Fig. 41. The agreement is also rather satisfactory, the more rapid degradation of T_c for small degrees of disorder can be related to additional contributions to Coulomb repulsion within Cooper pairs neglected in the derivation of Eq. (3.68). Surely, we do not claim that this is a real explanation of T_c degradation in disordered high temperature superconductors. However, note its relation to localized moment formation under disordering which leads to the usual Abrikosov–Gorkov mechanism of depairing due to spin-flip scattering on magnetic impurities. According to Mott [159] (cf. also Refs. [160, 7]) the appearance of localized moments may be related to the presence of localized states (single occupied states below the Fermi level as briefly discussed above). We can then estimate the value of the effective magnetic moment (in Bohr magnetons) in a unit cell as [171].

$$\mu R_{\text{loc}}^{-3}\Omega_0 = p_{\text{theor}}^2 , \tag{5.7}$$

where Ω_0 is the volume of a unit cell. For large degrees of disorder ($\Phi = 2\times 10^{19}\,\text{cm}^{-2}$) and $R_{\text{loc}} \approx 8$ Å with $\mu \approx 1$ we obtain $p_{\text{theor}}^2 = 0.66$ for $YBa_2Cu_3O_{6.95}$ in full agreement with experiment. However, for smaller fluences p_{theor} is considerably smaller than the experimental value. Note, though, that the estimate of Eq. (5.7) is valid only for small enough values of R_{loc}, i.e. when the Fermi level is well inside the localized region. On the other hand, the accuracy with which the Curie constant is determined in weakly disordered samples is considerably less than that in the strongly disordered case. Of course, the other mechanisms of local moment formation, which were discussed above and can become operational even before the metal–insulator transition can be important here.

Of course, plenty of work on localization effects in high-T_c oxides use disorder induced by different types of chemical substitutions in these systems. Of these we shall rather arbitrarily quote Refs. [264–268], which provide data quite similar, though not necessarily identical, to those described above on different types of systems and obtained by different experimental methods. We note that the effects of chemical disorder are always complicated by the inevitable changes of carrier concentration due to doping effects. Still all these data indicate that superconductivity in high-T_c systems is realized close to disorder induced metal–insulator transition, so that these systems provide us with plenty of possibilities to study experimentally the general problems discussed in our review. More details can be found in the extensive review paper [269].

Special attention should be paid to a recent study of angle resolved photoemission in Co doped single-crystals of $Bi_2Sr_2CaCu_2O_{8+y}$ [270]. Doping $Bi_2Sr_2CaCu_2O_{8+y}$ with Co causes superconducting–insulator transition, Co doping decreases T_c and causes increase in residual resistivity. The changes in the temperature behavior of resistivity from metallic to insulating phase correlate with the disappearance of the dispersing band-like states in angle-resolved photoemission. Authors believe that Anderson localization caused by the impurity potential of the doped Co atoms provides a consistent explanation of all experimental features and T_c reduction is not caused by magnetic impurity pairbreaking effects but by spatial localization of carriers with the superconducting ground state being formed out of spatially localized carriers. Similar data were also obtained for some exceptional (apparently strongly disordered) samples of undoped $Bi_2Sr_2CaCu_2O_{8+y}$ [271]. Of course, it will be very interesting to perform similar type of experiments on neutron irradiated samples where we deal with pure disorder.

Finally, we must stress that in our opinion these data on rather strongly disordered samples of high-temperature superconductors more or less definitely exclude the possibility of d-wave pairing in these systems. As is well known (and also can be deduced from our discussion in Appendix C), d-wave pairing is much more sensitive to disordering and is completely suppressed roughly speaking at the disorders measured by the energy scale $1/\tau \sim T_{c0}$, which is at least an order of magnitude smaller than the disorder necessary to induce the metal–insulator transition which can be estimated as $1/\tau \sim E_F$. This apparently excludes the possibility to observe any manifestations of localization effects in d-wave superconductors, though these are clearly observed in high-T_c systems. Of course, these qualitative conclusions deserve further studies within the specific models of microscopic mechanisms of high-temperature superconductivity.

We shall limit ourselves to the discussion of localization effects in high temperature superconductors. Our conclusion is that these effects are extremely important in these systems and some of the anomalies can be successfully described by theoretical ideas formulated in this review. We must stress that much additional work is needed both theoretical and experimental to clarify the general picture of disorder effects in high-T_c superconductors and we can expect that future progress, especially with the quality of samples, may provide some new and exciting results.

6. Conclusions

We conclude our review by trying to recapitulate the basic unsolved problems. From the theoretical point of view probably the main problem is to formulate the theory of superconducting pairing in strongly disordered system along the lines of the general theory of interacting Fermi systems. This problem is obviously connected with the general theory of metal–insulator transition in such an approach, which as we mentioned during our brief discussion above is rather far from its final form. Nevertheless, there were several attempts to analyze the superconducting transition within this framework [272–275, 115]. In all cases, the authors limited themselves to certain universality classes within the general renormalization group approach of the interaction theory of the metal–insulator transition. Ref. [272] dealt only with the two-dimensional problem, while Refs. [273–275, 115] also considered the bulk case. These papers have demonstrated a large variety of possible behavior of superconductivity under disordering, from disorder-induced (triplet) superconductivity [274] to a complete destruction of it close to [272, 273] or even long before the metal–insulator transition [275]. Our point of view is that at the moment it is rather difficult to make any *general* conclusions from the results of these approaches. In particular, we do not believe that the present status of these theories is sufficient to prove or disprove the general possibility of superconductivity in Anderson insulators. However, it is obvious that further theoretical progress in the problem of T_c behavior under disordering will be largely possible only within this general approach. In this sense, our simplified discussion of Coulomb effects and other mechanisms of T_c degradation in this review is only of qualitative nature. Still, more general approaches apparently do not change our qualitative conclusions. These problems become even more complicated if we address ourselves to the case of high temperature superconductors, where we do not know precisely the nature of the pairing interaction in a regular system.

Concerning the semiphenomenological approach to the theory of superconductivity close to the Anderson transition we must stress the necessity of further investigation of the region of strong statistical fluctuations with the aim of a more detailed study of their influence upon different physical properties, like e.g. the upper critical field, density of states, nuclear relaxation, etc. Obviously, all of them may be significantly changed in comparison with predictions of what we called the statistical mean-field theory. Especially important are further studies of rather exotic predictions of random fluxes in the ground state [201].

Despite our explicit limitation to a discussion of superconductivity in *bulk* disordered superconductors we have to mention the extremely interesting problem of universal conductivity at the superconductor–insulator transition at $T = 0$ in two-dimensional systems which attracted much attention recently [276–278, 19]. It is argued that the transition between the insulating and superconducting phases of a disordered two-dimensional system at zero temperature is of continuous quantum nature, but the system behaves like a normal metal right at the transition, i.e. the conductivity has a finite, nonzero value. This value is *universal* and, apparently, equal to $(2e)^2/h$ (with $2e$ being the Cooper pair charge). There is strong experimental evidence [216, 279–282, 19] that a variety of systems (metallic films, high-T_c films, etc.) show the onset of superconductivity to occur when their sheet resistance falls below a value close to $h/4e^2 \approx 6.45\ \mathrm{k}\Omega$. The theoretical analysis here is based upon boson (Cooper pairs) approach to superconductivity and the main conclusion is that in contrast to the case of localization of fermions in two dimensions, bosons exhibit a superconductor to insulator transition (as disorder grows) with the value of conductivity at the critical point being independent of microscopic details. A major theoretical problem arises to describe a crossover to such behavior e.g. in the quasi-two-dimensional case of BCS superconductivity as interplane coupling goes to zero.

So we are not short of theoretical problems in this important field of research. As to the experiments, certainly much is still to be done for unambiguous demonstration of the exotic possibility of the superconductivity of Anderson insulators.

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Appendix A. Spectral densities and criterion for localization

A convenient formalism to consider general properties of a disordered system is based upon exact eigenstate representation for an electron in a random field created by disorder. These eigenstates

$\phi_\nu(\boldsymbol{r})$ are formally defined by the Schroedinger equation

$$H\phi_\nu(\boldsymbol{r}) = \varepsilon_\nu \phi_\nu(\boldsymbol{r}) \,, \tag{A.1}$$

where H is the one-particle Hamiltonian of the disordered system under consideration, ε_ν are exact eigenvalues of electron energy in a random potential. Obviously, $\phi_\nu(\boldsymbol{r})$ and ε_ν are dependent on locations of scatterers $\boldsymbol{R}_n$ for a given realization of random field.

Let us define two-particle spectral densities [59, 7]:

$$\langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^{\mathrm{F}} = \frac{1}{N(E)}\left\langle \sum_{\nu\nu'} \phi_\nu^*(\boldsymbol{r})\phi_{\nu'}(\boldsymbol{r})\phi_{\nu'}^*(\boldsymbol{r}')\phi_\nu(\boldsymbol{r}')\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'})\right\rangle \,, \tag{A.2}$$

$$\langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^{H} = \frac{1}{N(E)}\left\langle \sum_{\nu\nu'} |\phi_\nu(\boldsymbol{r})|^2|\phi_{\nu'}(\boldsymbol{r}')|^2\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'})\right\rangle \,, \tag{A.3}$$

where angular brackets denote averaging over disorder and

$$N(E) = \left\langle \sum_{\nu} |\phi_\nu(\boldsymbol{r})|^2\delta(E-\varepsilon_\nu)\right\rangle \tag{A.4}$$

is one-electron (average) density of states. Obviously Eq. (A.3) is just a correlation function of *local* densities of states in a disordered system. Spectral density given by Eq. (A.2) determines electronic transport [59]. The following general properties are easily verified using the completeness and orthonormality of functions $\phi(\boldsymbol{r})$:

$$\int d\boldsymbol{r}\langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^{\mathrm{F}} = \delta(\omega) \,, \qquad \int d\omega\langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^{\mathrm{F}} = \delta(\boldsymbol{r}-\boldsymbol{r}') \tag{A.5}$$

or for the Fourier-components:

$$\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_{\boldsymbol{q}=0} = \delta(\omega) \,, \qquad \int d\omega\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_{\boldsymbol{q}} = 1 \tag{A.6}$$

and $\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_{\boldsymbol{q}} \geq 0$. From general definitions given in Eqs. (A.2) and (A.3) it is clear that

$$\langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r})\rangle\rangle^{\mathrm{F}} = \langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r})\rangle\rangle^{H} \,, \tag{A.7}$$

i.e. these spectral densities coincide for $\boldsymbol{r} = \boldsymbol{r}'$.

Terms with $\varepsilon_\nu = \varepsilon_{\nu'}$ are in general present in Eqs. (A.2) and (A.3). However, if these states are extended the appropriate wave-functions $\phi_\nu(\boldsymbol{r})$ are normalized on the total volume Ω of the system and these contributions to Eqs. (A.2) and (A.3) are proportional to Ω^{-1} and vanish as $\Omega \to \infty$. Things change if states are localized. In this case states are normalized on a finite volume of the order of $\sim R_{\mathrm{loc}}^d$. This leads to the appearance of a $\delta(\omega)$-contribution to spectral densities:

$$\langle\langle \rho_E(\boldsymbol{r})\rho_{E+\omega}(\boldsymbol{r}')\rangle\rangle^{\mathrm{F},H} = A_E(\boldsymbol{r}-\boldsymbol{r}')\delta(\omega) + \rho_E^{\mathrm{F},H}(\boldsymbol{r}-\boldsymbol{r}'\omega) \tag{A.8}$$

or in momentum representation:

$$\langle\langle \rho_E\rho_{E+\omega}\rangle\rangle_{\boldsymbol{q}}^{\mathrm{F},H} = A_E(\boldsymbol{q})\delta(\omega) + \rho_E^{\mathrm{F},H}(\boldsymbol{q}\omega) \,, \tag{A.9}$$

where the second term is regular in ω. This singular behavior was proposed as a general criterion for localization [59]. It is easy to show that

$$A_E(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{N(E)}\left\langle \sum_{\nu} \delta(E-\varepsilon_\nu)|\phi_\nu(\boldsymbol{r})|^2|\phi_\nu(\boldsymbol{r}')|^2 \right\rangle \qquad \text{(A.10)}$$

$$A_E = A_E(\boldsymbol{r}-\boldsymbol{r}')|_{\boldsymbol{r}=\boldsymbol{r}'} \sim R_{\mathrm{loc}}^{-d}\,.$$

$A_E(\boldsymbol{r}-\boldsymbol{r}')$ represents the so-called inverse participation ratio [27, 87]. Roughly speaking, its value at $\boldsymbol{r}=\boldsymbol{r}'$ is inversely proportional to the number of atomic orbitals which effectively form the quantum state ν.

These $\delta(\omega)$-singularities in spectral densities signal nonergodic behavior of the system in a localized state. This leads to a difference between the so-called adiabatic and isothermal response functions [100, 60, 7]. The intimate connection between localization and nonergodic behavior was already noted in the first paper by Anderson [1].

From general properties given by Eqs. (A.5) and (A.6) for $\boldsymbol{q}\to\boldsymbol{0}$ in the localization region we have [59]:

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_{\boldsymbol{q}}^{\mathrm{F}} \approx [1-R_{\mathrm{loc}}^2 q^2]\delta(\omega) + \cdots\,, \qquad \text{(A.11)}$$

where

$$R_{\mathrm{loc}}^2 = \frac{1}{2dN(E)}\int \mathrm{d}^d r\, r^2 \left\langle \sum_{\nu} \delta(E-\varepsilon_\nu)|\phi_\nu(\boldsymbol{r})|^2|\phi_\nu(0)|^2 \right\rangle \qquad \text{(A.12)}$$

defines the localization length. Delocalization leads to a smearing of the $\delta(\omega)$-singularity for finite q.

Spectral densities of Eqs. (A.2) and (A.3) can be expressed via two-particle Green's functions [7]. Using nonaveraged retarded and advanced Green's functions

$$G^{\mathrm{R}}(\boldsymbol{r}\boldsymbol{r}'E) = G^{\mathrm{A}*}(\boldsymbol{r}\boldsymbol{r}'E) = \sum_{\nu} \frac{\phi_\nu(\boldsymbol{r})\phi_\nu^*(\boldsymbol{r}')}{E-\varepsilon_\nu+\mathrm{i}\delta}\,, \qquad \text{(A.13)}$$

we immediately get from Eqs. (A.2) and (A.3)

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E'}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F}} = \frac{1}{2\pi^2 N(E)}\mathrm{Re}\{\langle G^{\mathrm{R}}(\boldsymbol{r}\boldsymbol{r}'E')G^{\mathrm{A}}(\boldsymbol{r}'\boldsymbol{r}E)\rangle - \langle G^{\mathrm{R,A}}(\boldsymbol{r}\boldsymbol{r}'E')G^{\mathrm{R,A}}(\boldsymbol{r}'\boldsymbol{r}E)\rangle\}\,, \qquad \text{(A.14)}$$

$$\langle\!\langle \rho_E(\boldsymbol{r})\rho_{E'}(\boldsymbol{r}')\rangle\!\rangle^{H} = \frac{1}{2\pi^2 N(E)}\mathrm{Re}\{\langle G^{\mathrm{R}}(\boldsymbol{r}\boldsymbol{r}E')G^{\mathrm{A}}(\boldsymbol{r}'\boldsymbol{r}'E)\rangle - \langle G^{\mathrm{R,A}}(\boldsymbol{r}\boldsymbol{r}E')G^{\mathrm{R,A}}(\boldsymbol{r}'\boldsymbol{r}'E)\rangle\}\,. \qquad \text{(A.15)}$$

In the momentum representation, Eq. (A.14) is equivalent to

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_{\boldsymbol{q}}^{\mathrm{F}} = \frac{1}{\pi N(E)}\mathrm{Im}\{\Phi_E^{\mathrm{RA}}(\omega\boldsymbol{q}) - \Phi_E^{\mathrm{RR}}(\omega\boldsymbol{q})\}\,, \qquad \text{(A.16)}$$

where

$$\Phi_E^{\mathrm{RA(R)}}(\boldsymbol{q}\omega) = -\frac{1}{2\pi\mathrm{i}}\sum_{\boldsymbol{p}\boldsymbol{p}'} \langle G^{\mathrm{R}}(\boldsymbol{p}+\boldsymbol{p}'_+E+\omega)G^{\mathrm{A(R)}}(\boldsymbol{p}'_-\boldsymbol{p}_-E)\rangle \qquad \text{(A.17)}$$

and $\boldsymbol{p}_{+-} = \boldsymbol{p} \pm 1/2\boldsymbol{q}$. It can be shown [53, 54] that $\Phi_E^{\mathrm{RR(AA)}}(\omega\boldsymbol{q})$ are nonsingular for small ω and $\boldsymbol{q}$. Accordingly, $\delta(\omega)$-singularity signalling localization can appear only from the first term in Eq. (A.16).

Appendix B. Linearized gap equation in disordered system

Let us consider the derivative of the linearized gap equation Eq. (3.55) used to determine T_c [157, 158]. Equation for Gorkov's anomalous Green's function in an inhomogeneous disordered system (before any averaging procedure) at $T = T_c$ takes the following form:

$$(\varepsilon_n^2 + \hat{\varepsilon}_r^2)F(\boldsymbol{rr}'\varepsilon_n) = -\,T_c \sum_m V(\boldsymbol{rr}'\varepsilon_n - \varepsilon_m)F(\boldsymbol{rr}'\varepsilon_m)\,, \tag{B.1}$$

where $\varepsilon_n = (2n+1)\pi T_c$ and $V(\boldsymbol{rr}'\varepsilon_n - \varepsilon_m)$ is an effective interelectron potential, $\hat{\varepsilon}_r$ is a one electron energy operator (energy zero is at the Fermi energy). Define

$$\Delta(\boldsymbol{rr}') = -\,2\hat{\varepsilon}_r \coth\frac{\hat{\varepsilon}_r}{2T_c}\, T_c \sum_n F(\boldsymbol{rr}'\varepsilon_n) \tag{B.2}$$

and assume the following relation between $\Delta(\boldsymbol{rr}')$ and $F(\boldsymbol{rr}'\varepsilon_n)$:

$$F(\boldsymbol{rr}'\varepsilon_n) = \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_r^2}\, T_c \sum_m V(\boldsymbol{rr}'\varepsilon_n - \varepsilon_m)\frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_r^2}\,\hat{Q}(\boldsymbol{rr}'\varepsilon_m)\Delta(\boldsymbol{rr}')\,, \tag{B.3}$$

where $\hat{Q}$ is some unknown operator. Then after substitution of Eq. (B.3) into Eq. (B.2) we get a BCS-like equation for T_c:

$$\Delta(\boldsymbol{rr}') = -\,\hat{U}(\boldsymbol{rr}')\,\frac{\tanh(\hat{\varepsilon}_r)/2T_c}{2\hat{\varepsilon}_r}\,\Delta(\boldsymbol{rr}')\,, \tag{B.4}$$

where the operator of "effective" interaction is defined by

$$\begin{aligned}\hat{U}(\boldsymbol{rr}') = {}& 2\hat{\varepsilon}_r \coth\left(\frac{\hat{\varepsilon}_r}{2T_c}\right) T_c \sum_n \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_r^2}\, T_c \sum_m V(\boldsymbol{rr}'\varepsilon_n - \varepsilon_m)\\ &\times \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_r^2}\,\hat{Q}(\boldsymbol{rr}'\varepsilon_m)2\hat{\varepsilon}_r \coth(\hat{\varepsilon}_r/2T_c)\,.\end{aligned} \tag{B.5}$$

From Eqs. (B.1)–(B.3) we obtain the following equation for $\hat{Q}$ (we drop $\boldsymbol{rr}'$ for brevity):

$$\begin{aligned}\hat{Q}(\varepsilon_n) = {}& 1 - T_c \sum_m V(\varepsilon_n - \varepsilon_m)\frac{1}{\varepsilon_m^2 + \hat{\varepsilon}^2}\,\hat{Q}(\varepsilon_m)\\ &+ 2\hat{\varepsilon}\coth\left(\frac{\hat{\varepsilon}}{2T_c}\right) T_c \sum_{n'} \frac{1}{\varepsilon_{n'}^2 + \hat{\varepsilon}^2}\, T_c \sum_m V(\varepsilon_{n'} - \varepsilon_m)\frac{1}{\varepsilon_m^2 + \hat{\varepsilon}^2}\,\hat{Q}(\varepsilon_m)\,.\end{aligned} \tag{B.6}$$

In case of weak coupling in the lowest order over interaction in Eq. (B.6) we can leave only the first term $\hat{Q}(\varepsilon_n) = 1$. Then Eq. (B.5) reduces to

$$\hat{U}(\boldsymbol{r}\boldsymbol{r}') = 2\hat{\varepsilon}_r \coth\left(\frac{\hat{\varepsilon}_r}{2T_c}\right) T_c \sum_n \frac{1}{\varepsilon_n^2 + \hat{\varepsilon}_r^2}\, T_c \sum_m V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m) \frac{1}{\varepsilon_m^2 + \hat{\varepsilon}_r^2}\, 2\hat{\varepsilon}_r \coth\left(\frac{\hat{\varepsilon}_r}{2T_c}\right) \tag{B.7}$$

and Eq. (B.4) completely determines T_c.

Using the usual definition of the superconducting gap

$$\Delta(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = T_c \sum_m V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m) F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_m) = -(\varepsilon_n^2 + \hat{\varepsilon}_r^2) F(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) \tag{B.8}$$

it is easy to get

$$\Delta(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) = \hat{Q}(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)\Delta(\boldsymbol{r}\boldsymbol{r}') \tag{B.9}$$

so that $\Delta(\boldsymbol{r}\boldsymbol{r}')$ represents the energy gap in the absence of frequency dispersion, while $\hat{Q}$ describes the frequency dependence of the energy gap.

Cooper pairing takes place in states which are time-reversed, thus in the exact eigenstate representation of an electron in disordered system we have

$$\Delta(\boldsymbol{r}\boldsymbol{r}') = \sum_\nu \Delta_\nu \phi_\nu^*(\boldsymbol{r}')\phi_\nu(\boldsymbol{r}) \tag{B.10}$$

and Eq. (B.4) gives

$$\Delta_\nu = -\sum_{\nu\nu'} \frac{1}{2\varepsilon_{\nu'}} \tanh\frac{\varepsilon_{\nu'}}{2T_c}\, U_{\nu\nu'}\Delta_{\nu'}\,, \tag{B.11}$$

where the kernel

$$U_{\nu\nu'} = \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}'\, \phi_\nu^*(\boldsymbol{r})\phi_{\nu'}^*(\boldsymbol{r}')\hat{U}(\boldsymbol{r}\boldsymbol{r}')\phi_\nu(\boldsymbol{r}')\phi_\nu(\boldsymbol{r}) \tag{B.12}$$

has the form of a "Fock" matrix element of an effective interaction. From Eq. (B.7) we have

$$U_{\nu\nu'} = T_c^2 \frac{2\varepsilon_\nu\varepsilon_{\nu'}}{\tanh(\varepsilon_\nu/2T_c)\tanh(\varepsilon_{\nu'}/2T_c)} \sum_n \sum_m \frac{1}{\varepsilon_n^2 + \varepsilon_\nu^2} \frac{1}{\varepsilon_m^2 + \varepsilon_{\nu'}^2}$$
$$\times \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}'\, \phi_\nu^*(\boldsymbol{r})\phi_{\nu'}^*(\boldsymbol{r}') V(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n - \varepsilon_m)\phi_\nu(\boldsymbol{r}')\phi_{\nu'}(\boldsymbol{r})\,. \tag{B.13}$$

It is convenient to rewrite Eq. (B.11) introducing summation over states belonging to some surface of constant energy with subsequent integration over energies:

$$\Delta_\nu = -\int_{-\infty}^{\infty} \mathrm{d}E' \frac{1}{2E'} \tanh\left(\frac{E'}{2T_c}\right) \sum_{\nu'\in E'} \mathcal{N}(E') U_{\nu\nu'(E)}\Delta_{\nu'(E')}\,, \tag{B.14}$$

where $\mathcal{N}(E) = \sum_\nu \delta(E - \varepsilon_\nu)$.

Consider now averaging of the gap equation. Define

$$\Delta(E) = \frac{1}{N(E)}\left\langle \sum_{\nu} \Delta_{\nu}\delta(E - \varepsilon_{\nu})\right\rangle , \tag{B.15}$$

i.e. the gap averaged over disorder and a surface of constant energy $E = \varepsilon_{\nu}$. Here as usual we denote $N(E) = \langle \mathcal{N}(E)\rangle$. Suppose now that $\Delta_{\nu} = \Delta(\varepsilon_{\nu}) = \Delta(E = \varepsilon_{\nu})$, i.e. that Δ_{ν} depends only on energy $E = \varepsilon_{\nu}$, but not on the quantum numbers ν. This is similar to the usual assumption of $\Delta(\boldsymbol{p})$ depending only on $|\boldsymbol{p}|$ in a homogeneous and isotropic system [152].

After the usual decoupling used e.g. in transforming Eq. (3.12) into (3.13), i.e. assuming the self-averaging of the gap, we obtain the following linearized gap equation determining T_{c}:

$$\Delta(E) = -\int_{-\infty}^{\infty} \mathrm{d}E'\, K(E,E')\frac{1}{2E'}\tanh\left(\frac{E'}{2T_{\mathrm{c}}}\right)\Delta(E') , \tag{B.16}$$

where

$$\begin{aligned} K(E,E') &= \frac{1}{N(E)}\left\langle \sum_{\nu\nu'} U_{\nu\nu'}\delta(E - \varepsilon_{\nu})\delta(E' - \varepsilon_{\nu'})\right\rangle \\ &= T_{\mathrm{c}}^{2}\sum_{n}\sum_{m}\left[\frac{2E}{\tanh(E/2T_{\mathrm{c}})}\frac{1}{\varepsilon_n^2 + E^2}\right]\left[\frac{2E'}{\tanh(E'/2T_{\mathrm{c}})}\frac{1}{\varepsilon_m^2 + E'^2}\right] \\ &\quad \times \int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{r}'\, V(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n - \varepsilon_m)\, \langle\!\langle \rho_E(\boldsymbol{r})\rho_{E'}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F}} , \end{aligned} \tag{B.17}$$

where we have again introduced the Gorkov–Berezinskii spectral density defined in Eq. (A.2). Effective interaction can be written as

$$V(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n - \varepsilon_m) = V_{\mathrm{p}}(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n - \varepsilon_m) + V_{\mathrm{C}}(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n - \varepsilon_m) , \tag{B.18}$$

i.e. as the sum of some kind of Boson-exchange attractive interaction V_{p} and Coulomb repulsion V_{C}, which leads to

$$K(E,E') = K_{\mathrm{p}}(E,E') + K_{\mathrm{C}}(E,E') . \tag{B.19}$$

Assuming $V_{\mathrm{C}}(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n - \varepsilon_m) = v(\boldsymbol{r} - \boldsymbol{r}')$, i.e. static approximation for Coulomb repulsion, we obtain

$$K_{\mathrm{C}}(E,E') = \int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{r}'\, v(\boldsymbol{r} - \boldsymbol{r}')\, \langle\!\langle \rho_E(\boldsymbol{r})\rho_{E'}(\boldsymbol{r}')\rangle\!\rangle^{\mathrm{F}} \tag{B.20}$$

which coincides with Eq. (3.33) used above in our analysis of Coulomb repulsion within Cooper pairs in disordered systems. Above we have used the approximation of Eq. (3.52) to model K_{p} due to electron–phonon pairing mechanism (or similar model for some kind of excitonic pairing). In this case Eq. (B.16) reduces to Eq. (3.55).

Note that $V_{\mathrm{C}}(\boldsymbol{r} - \boldsymbol{r}'\varepsilon_n - \varepsilon_m)$ may be taken also as a dynamically screened Coulomb interaction. Then we must use the appropriate expressions for the dielectric function $\varepsilon(\boldsymbol{q}\omega_m)$ which may be found using the self-consistent theory of localization [98, 60]. Then after some tedious calculations we can get the expressions for $K_{\mathrm{C}}(E,E')$ which for small $|E - E'|$ practically coincide with those used by us above for the case of static short-range interactions [158].

Appendix C. Localization and *d*-wave pairing

There is a growing body of experimental evidence in high-T_c superconductors that indicate that the pairing state is of $d_{x^2-y^2}$ symmetry [283, 284]. In superconductors with an anisotropic order parameter, both magnetic and non-magnetic impurities are pair breaking. For d-wave symmetry, the effect of non-magnetic impurities is equivalent to magnetic impurities in s-wave superconductors [285, 286]. Effectively, this means that superconductivity in such systems cannot persist until disorder becomes high enough to transform the system into an Anderson insulator. The situation is different for the so-called extended s wave symmetry. This corresponds to an order parameter with uniform sign which could, in particular, vanish at certain directions in momentum space [287]. Point impurities are not pair breaking in this case, but they are "pair-weakening": for small impurity concentration, T_c decreases linearly with disorder, but the critical impurity concentration is formally infinite, i.e. Anderson's theorem works after essential isotropisation of the gap [288].

We shall present now some of the relevant equations along the lines of our discussion of the Anderson theorem in the main body of the review. Here, we partly follow Ref. [289]. We shall consider d-wave pairing on a two-dimensional lattice induced by the following interaction Hamiltonian:

$$H_{\text{int}} = -g\sum_{\boldsymbol{r}} \hat{\Delta}_{\boldsymbol{r}}^{\dagger}\hat{\Delta}_{\boldsymbol{r}} \tag{C.1}$$

where $\boldsymbol{r}$ denotes lattice sites. This Hamiltonian corresponds to an instantaneous anisotropic attractive interaction with an implicit cutoff at a characteristic energy $\langle\omega\rangle$. In order to model $d_{x^2-y^2}$ symmetry we choose $\hat{\Delta}_{\boldsymbol{r}}^{\dagger}$ in the following form:

$$\hat{\Delta}_{\boldsymbol{r}}^{\dagger} = \frac{1}{\sqrt{2}}\sum_{\delta} \varepsilon_{\delta}(c_{\boldsymbol{r}\uparrow}^{\dagger}c_{\boldsymbol{r}+\delta\downarrow}^{\dagger} - c_{\boldsymbol{r}\downarrow}^{\dagger}c_{\boldsymbol{r}+\delta\uparrow}^{\dagger}) \tag{C.2}$$

with $\delta = \pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2$ being the lattice vectors, and $\varepsilon_{\pm e_1} = -\varepsilon_{\pm e_2} = 1$.

Next we can perform the analysis similar to that used in deriving Eqs. (3.7)–(3.16) and find that now we again have Eq. (3.16) determining the critical temperature T_c with the kernel $K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n)$ in the exact eigenstates representation taking the following form:

$$\begin{aligned} K(\boldsymbol{r}\boldsymbol{r}'\varepsilon_n) &= gT\left\langle \sum_{\mu\nu\delta\delta'} \varepsilon_{\delta}\varepsilon_{\delta'} \frac{\phi_{\mu}^{*}(\boldsymbol{r})\phi_{\nu}^{*}(\boldsymbol{r}+\boldsymbol{\delta})\phi_{\nu}(\boldsymbol{r}')\phi_{\mu}(\boldsymbol{r}'+\boldsymbol{\delta})}{(\varepsilon_{\nu}-\mathrm{i}\varepsilon_n)(\varepsilon_{\mu}+\mathrm{i}\varepsilon_n)} \right\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}E\, N(E) \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\langle\langle d_E(\boldsymbol{r}) d_{E+\omega}(\boldsymbol{r}')\rangle\rangle}{(\mathrm{i}\varepsilon_n + E)(E+\omega-\mathrm{i}\varepsilon_n)}, \end{aligned} \tag{C.3}$$

where we have introduced the spectral density:

$$\langle\langle d_E(\boldsymbol{r}) d_{E+\omega}(\boldsymbol{r}')\rangle\rangle = \frac{1}{N(E)}\left\langle \sum_{\mu\nu}\sum_{\delta\delta'} \varepsilon_{\delta}\varepsilon_{\delta'}\phi_{\nu}^{*}(\boldsymbol{r}+\boldsymbol{\delta})\phi_{\mu}(\boldsymbol{r})\phi_{\mu}^{*}(\boldsymbol{r}')\phi_{\nu}(\boldsymbol{r}'+\boldsymbol{\delta}')\delta(E-\varepsilon_{\nu})\delta(E+\omega-\varepsilon_{\nu'}) \right\rangle . \tag{C.4}$$

Now we can rewrite Eq. (3.16) for T_c as

$$1 = gT_c \int_{-\infty}^{\infty} \mathrm{d}E\, N(E) \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{\varepsilon_n} \frac{g(\omega)}{(E + \mathrm{i}\varepsilon_n)(E + \omega - \mathrm{i}\varepsilon_n)}, \tag{C.5}$$

where

$$g(\omega) = \int \mathrm{d}\boldsymbol{r}' \langle\!\langle d_E(\boldsymbol{r}) d_{E+\omega}(\boldsymbol{r}')\rangle\!\rangle = \langle\!\langle d_E d_{E+\omega}\rangle\!\rangle_{\boldsymbol{q}=\boldsymbol{0}} . \tag{C.6}$$

No sum rules similar to that given by Eqs. (A.5), (A.6) exist for the spectral density of Eq. (C.4). However, it can be easily expressed via the Green's functions and we obtain the following relations similar to those obtained in Appendix A:

$$\langle\!\langle d_E d_{E+\omega}\rangle\!\rangle^{\boldsymbol{q}} = \frac{1}{\pi N(E)} \operatorname{Im}\{\Phi_{Ed}^{\mathrm{RA}}(\omega \boldsymbol{q}) - \Phi_{Ed}^{\mathrm{RR}}(\omega \boldsymbol{q})\} , \tag{C.7}$$

where

$$\Phi_{Ed}^{\mathrm{RA(R)}}(\boldsymbol{q}\omega) = -\frac{1}{2\pi \mathrm{i}} \sum_{\boldsymbol{p}\boldsymbol{p}'} \gamma_{\boldsymbol{p}}^{d} \langle G^{\mathrm{R}}(\boldsymbol{p}_{+}\boldsymbol{p}'_{+} E + \omega) G^{\mathrm{A(R)}}(\boldsymbol{p}'_{-}\boldsymbol{p}_{-} E)\rangle \gamma_{\boldsymbol{p}'}^{d} \tag{C.8}$$

with the vertices $\gamma_p^d = \cos p_x - \cos p_y$ for d-wave. If from now on we ignore the lattice effects then $\gamma_p^d = \cos 2\theta_p$, which corresponds to a gap function $\Delta(\boldsymbol{k}) = \Delta(T) \cos 2\theta_p$, where θ_p is the polar angle in the plane [287]. Similar expressions will determine T_c for the case of anisotropic s-wave pairing with the vertices γ_p^d replaced by appropriate angle-dependent expressions [287].

Now we can write as usual:

$$g(\omega) = \frac{1}{\pi N(E)} \operatorname{Im}\{\Phi_{Ed}^{\mathrm{RA}}(\omega \boldsymbol{q} = 0)\} = \frac{1}{\pi N(E)} \operatorname{Im}\left\{ -\frac{1}{2\pi \mathrm{i}} \sum_{\boldsymbol{p}\boldsymbol{p}'} \cos 2\theta_p\, \Phi_{\boldsymbol{p}\boldsymbol{p}'}^{\mathrm{RA}}(E\omega \boldsymbol{q} = 0) \cos 2\theta_{p'} \right\} . \tag{C.9}$$

Here $\Phi_{\boldsymbol{p}\boldsymbol{p}'}^{\mathrm{RA}}(E\omega\boldsymbol{q} = 0)$ obeys the $\boldsymbol{q} = 0$ limit of the Bethe–Salpeter equation (Eq. (2.44)) which is easily transformed to the following kinetic equation [53]:

$$\left(\omega - \frac{\mathrm{i}}{\tau}\right) \Phi_{\boldsymbol{p}\boldsymbol{p}'}^{\mathrm{RA}}(E\omega) = -\Delta G_{\boldsymbol{p}} \left[\delta(\boldsymbol{p} - \boldsymbol{p}') + \sum_{\boldsymbol{p}''} U_{\boldsymbol{p}\boldsymbol{p}''}^{E}(\omega) \Phi_{\boldsymbol{p}''\boldsymbol{p}'}^{\mathrm{RA}}(E\omega) \right] \tag{C.10}$$

with $\Delta G_{\boldsymbol{p}} \equiv G^{\mathrm{R}}(\boldsymbol{p}E + \omega) - G^{\mathrm{A}}(\boldsymbol{p}E)$. If we replace in (C.10) the irreducible vertex by the bare vertex $U_0 = \rho V^2$, we obtain finally

$$g(\omega) = \frac{1}{4\pi} \frac{\tau}{1 + (\omega\tau)^2} \tag{C.11}$$

with the usual scattering rate $1/\tau = 2\pi\rho V^2 N(E)$. Inserting (C.11) in (C.5) and following the standard analysis [9] we obtain the well-known expression for the critical temperature variation [286] $\ln(T_{c0}/T_c) = \Psi(1/2 + 1/4\pi\tau T_c) - \Psi(1/2)$ which is similar to the case of magnetic impurity scattering in superconductors. However, here the normal potential scattering rate is operational leading to very fast degradation of T_c – superconducting state is completely destroyed for

$1/\tau > 1.76T_{c0}$. Actually, this result does not depend on the spatial dimensionality of the system, i.e. the same dependence works in three dimensions.

Effectively, this makes it impossible to reach the Anderson transition before superconductivity is destroyed: critical disorder for metal–insulator transition is determined by $1/\tau \sim E_F \gg T_c$. The only hope seems to be to analyze the quasi-two-dimensional case, where this critical disorder can be reduced due to a small enough interplane transfer integral w as in Eqs. (2.90), (2.91). Localization appears for $w < w_c = \sqrt{2}/\tau \exp(-\pi E_F \tau)$ and take as an estimate some $1/\tau \approx T_{c0}$, so that superconductivity is still possible, we can arrive at the following criterion of coexistence of localization and superconductivity:

$$w < T_{c0} \exp(-\pi E_F/T_{c0}) . \tag{C.12}$$

In a typical situation even for high-temperature superconductors we have $T_{c0} < 0.1E_F$ and the inequality in Eq. (C.12) can be satisfied only for extremely anisotropic systems with $w \ll T_{c0}$. Most known superconductors apparently fail in this respect. This probably makes d-wave pairing irrelevant for the main body of our review. It is then quite difficult to reconcile the existing data on the closeness of e.g. radiationally disordered high-T_c systems to the disorder-induced metal–insulator transition and all the evidence for d-wave pairing in these systems. However, this reasoning does not apply to the case of anisotropic s-wave pairing, where Anderson theorem effectively works for large degrees of disorder [287]. In this respect the experiments on disordering in high-T_c systems can become crucial in solving the problem of the nature of pairing (and thus of its microscopic mechanisms) in these systems.

Still, even in the case of d-wave pairing localization effects may become important and interesting, but for a quite different problem – that of localization of BCS-quasi-particles within the superconducting gap at relatively small disorder [290–293]. It is known that while in the pure d-wave superconductor, density of states close to the Fermi level is linear in energy $N(E) \sim E$ due to the gap nodes at the Fermi surface, the impurity scattering makes it finite at $E = 0$ [285]. In this sense the system becomes similar to the *normal* metal and we can calculate [290] the low lying quasi-particle contribution to conductivity $\sigma(\omega \to 0)$. This conductivity equals

$$\sigma \approx \frac{e^2}{2\pi\hbar} \frac{\xi_0}{a} , \tag{C.13}$$

where $\xi_0 = v_F/\pi\Delta_0$ is the superconducting coherence length and a is the lattice spacing (we assume $T = 0$). The surprising thing is that σ is *independent of the scattering rate* $1/\tau$, i.e. of disorder. For the two-dimensional case (applicable probably for high-T_c systems) we know that all states are localized with localization controlled by dimensionless conductance which now is equal to $g = \sigma/(e^2/2\pi\hbar) = \xi_0/a$. The value of g may be small enough in high temperature superconductors due to the small values of ξ_0, which are typically only slightly larger than the lattice constant. This can make localization effects important with BCS-quasi-particles forming a mobility gap in the vicinity of the Fermi level, leading to anomalies in the low temperature behavior of microwave conductivity and the penetration depth of a d-wave superconductor [290].

These results were first obtained [290] for point-like impurity scattering, later it was shown in Ref. [291] that the finite range of the impurity potential can lead to the nonuniversal disorder-dependent behavior of conductivity which becomes proportional to the normal state scattering rate. Situation was further complicated by the claim made in Refs. [292, 293] that the more rigorous analysis leads to the density of states of the impure d-wave superconductor behaving as $N(E) \sim |E|^{\alpha}$ with $\alpha > 0$, but dependent on the type of disorder. The renormalization group for the conductivity then apparently leads to some kind of a fixed point of intermediate nature, suggesting the finite conductivity in two-dimensions. All these aspects of disorder and localization for d-wave superconductors deserve further intensive studies.

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Michael V. SADOVSKII

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