Многочастичная Локализация Андерсона

Борис Альтшулер
Колумбийский Университет

Летняя школа Фонда Дмитрия Зимина “Династия”
"Актуальные проблемы теории конденсированного состояния"
4 – 14 июля 2010г.

Lecture1.

1.Introduction

>50 years of Anderson Localization

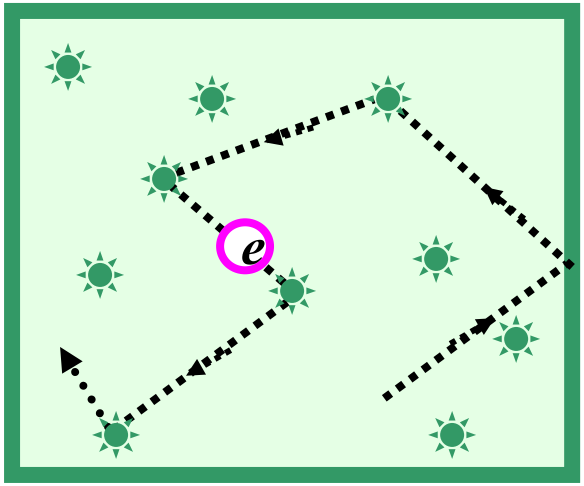
PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey

(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.



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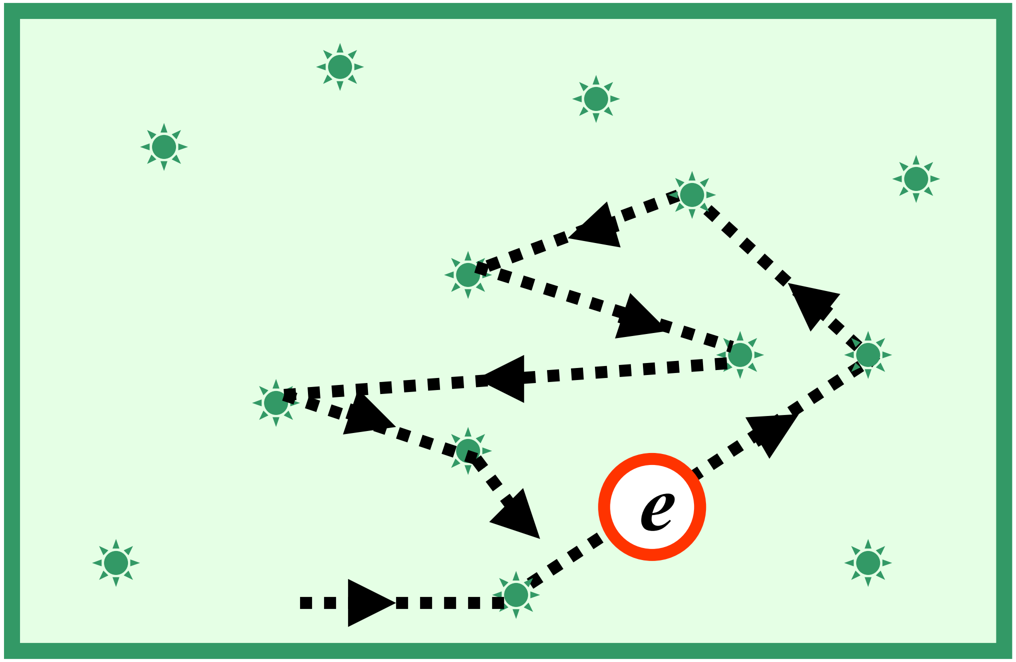

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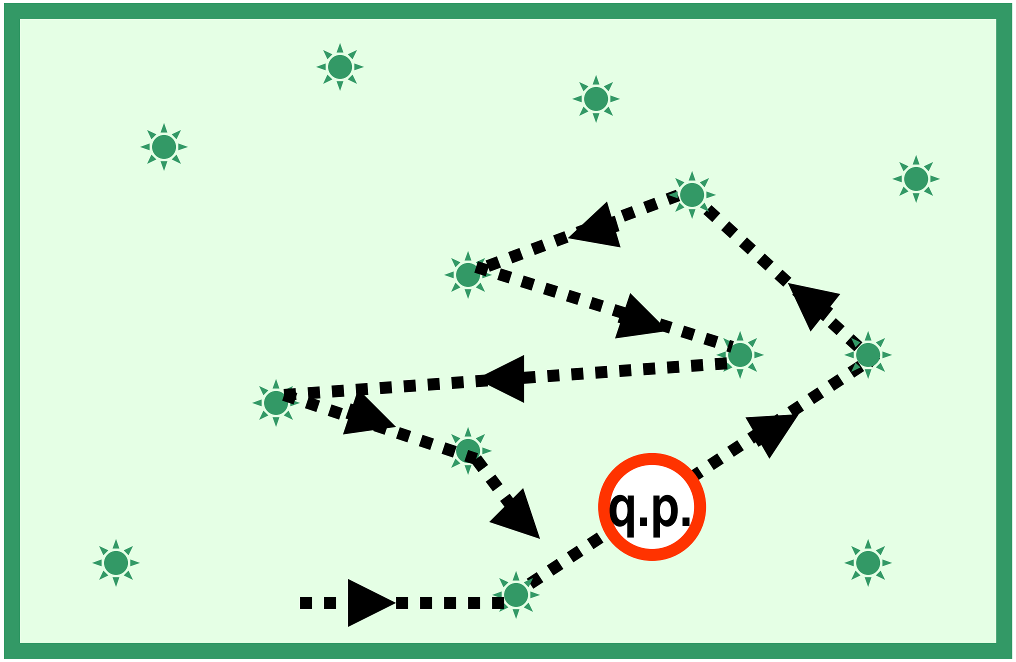


Einstein (1905):

Random walk

$\Downarrow$

always diffusion

as long as the system
has no memory

$$\langle r^2 \rangle = Dt$$

diffusion constant

Anderson(1958):

For quantum
particles

$\Downarrow$

not always!

It might be that

$$\langle r^2 \rangle \xrightarrow[t\to\infty]{} const$$

$$D = 0$$

Quantum interference $\Rightarrow$ memory

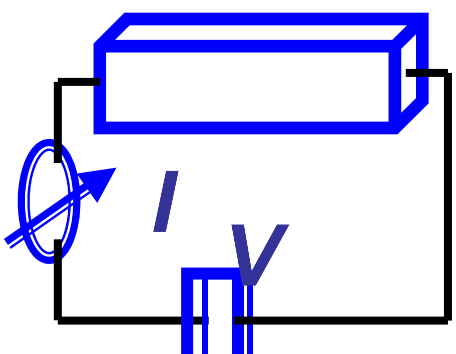
Einstein Relation (1905)

$$\sigma = e^2 D\nu \qquad \nu \equiv \frac{dn}{d\mu}$$

Conductivity

Diffusion Constant

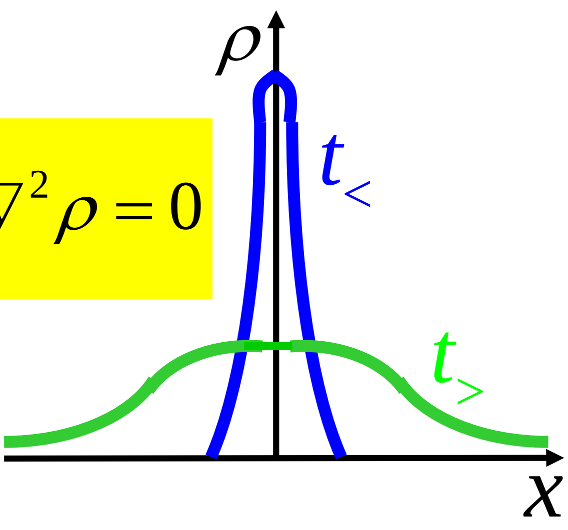
Density of states



$$G = \left(\frac{I}{V}\right)_{V=0} ;$$

$$\sigma = G\frac{L}{A}$$

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0$$



Einstein Relation (1905)

$$\sigma = e^2 D \nu \qquad \nu \equiv \frac{dn}{d\mu}$$

Conductivity

Diffusion Constant

Density of states

No diffusion - no conductivity

Localized states - insulator
Extended states - metal

Metal - insulator transition

Einstein (1905):

Random walk

⇓

always diffusion

as long as the system has no memory

$$\langle r^2 \rangle = Dt$$

diffusion constant

$$\boxed{conductivity \propto D}$$

Einstein relation

Anderson(1958):

For quantum particles

⇓

not always!

It might be that

$$\langle r^2 \rangle \xrightarrow[t\to\infty]{} const$$

$$D = 0$$

⇓

$$\boxed{conductivity = 0}$$

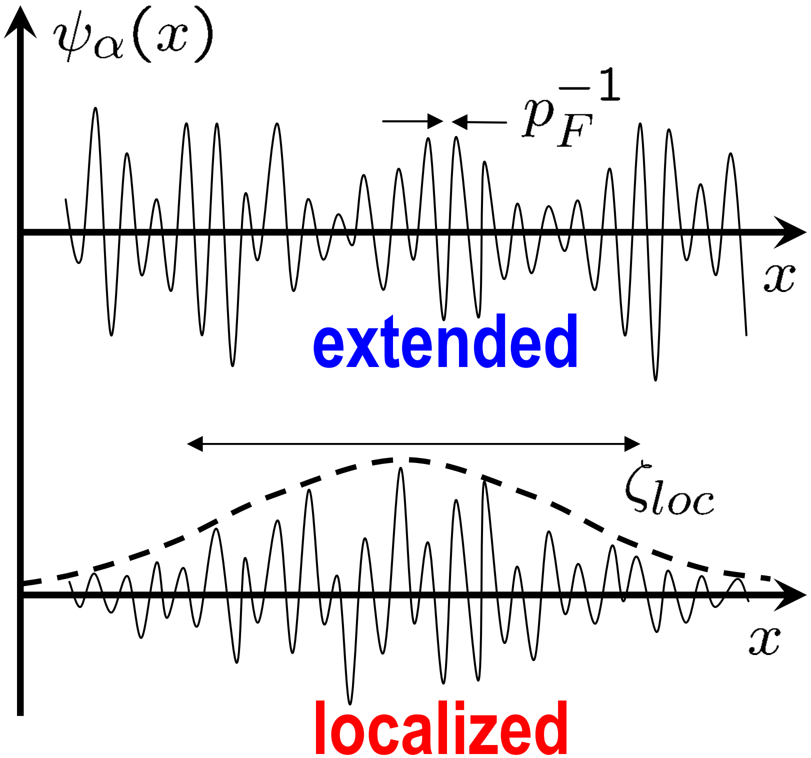
Anderson insulator

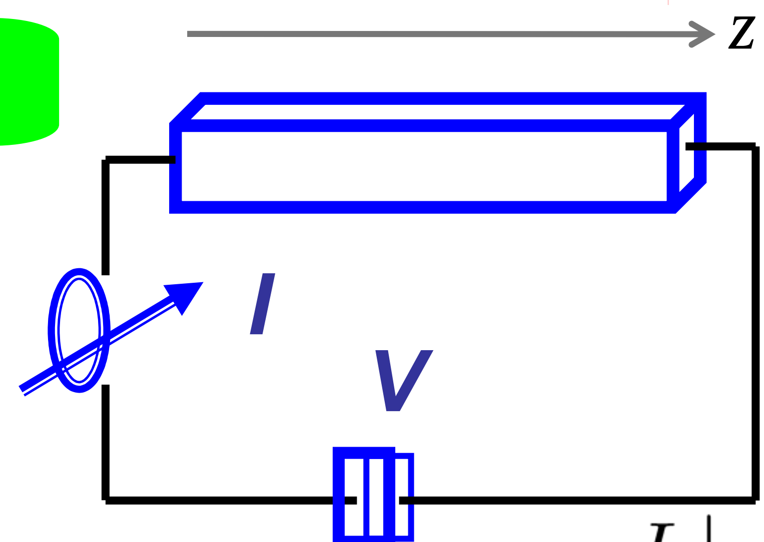
Quantum interference ⇒ memory

Localization of single-electron wave-functions:

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha\psi_\alpha(\boldsymbol{r})$$

Disorder





Conductance $G = \left.\frac{I}{V}\right|_{V\to 0}$

$$= \begin{cases} \sigma \dfrac{L_x L_y}{L_z} & \text{extended} \\ \propto \exp\left(\dfrac{-L_z}{\varsigma_{loc}}\right) & \text{localized} \end{cases}$$

Philip W. Anderson

The Nobel Prize in Physics 1977

Nobel Lecture

Nobel Lecture, December 8, 1977

Local Moments and Localized States

I was cited for work both. in the field of magnetism and in that of disordered systems, and I would like to describe here one development in each held which was specifically mentioned in that citation. The two theories I will discuss differed sharply in some ways. The theory of local moments in metals was, in a sense, easy: it was the condensation into a simple mathematical model of ideas which. were very much in the air at the time, and it had rapid and permanent acceptance because of its timeliness and its relative simplicity. What mathematical difficulty it contained has been almost fully- cleared up within the past few years.

Localization was a different matter: very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it .

Experiment

Spin Diffusion

Feher, G., Phys. Rev. 114, 1219 (1959); Feher, G. & Gere, E. A., Phys. Rev. 114, 1245 (1959).

Light

Wiersma, D.S., Bartolini, P., Lagendijk, A. & Righini R. “Localization of light in a disordered medium”, *Nature* 390, 671-673 (1997).

Scheffold, F., Lenke, R., Tweer, R. & Maret, G. “Localization or classical diffusion of light”, *Nature* 398,206-270 (1999).

Schwartz, T., Bartal, G., Fishman, S. & Segev, M. “Transport and Anderson localization in disordered two dimensional photonic lattices”. *Nature* 446, 52-55 (2007).

C.M. Aegerter, M.Störzer, S.Fiebig, W. Bührer, and G. Maret : JOSA A, 24, #10, A23, (2007)

Microwave

Dalichaouch, R., Armstrong, J.P., Schultz, S.,Platzman, P.M. & McCall, S.L. “Microwave localization by 2-dimensional random scattering”. *Nature* 354, 53, (1991).

Chabanov, A.A., Stoytchev, M. & Genack, A.Z. Statistical signatures of photon localization. *Nature* 404, 850, (2000).

Pradhan, P., Sridar, S, “Correlations due to localization in quantum eigenfunctions od disordered microwave cavities”, PRL 85, (2000)

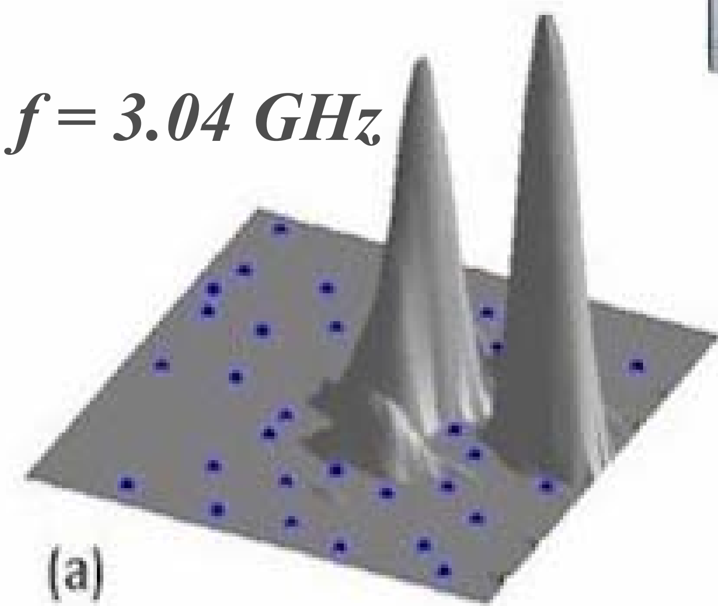
Sound

Weaver, R.L. Anderson localization of ultrasound. *Wave Motion* 12, 129-142 (1990).

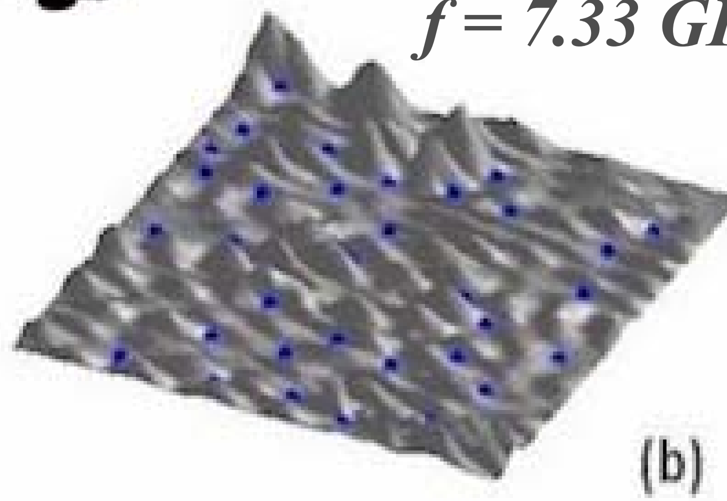
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

f = 3.04 GHz



f = 7.33 GHz

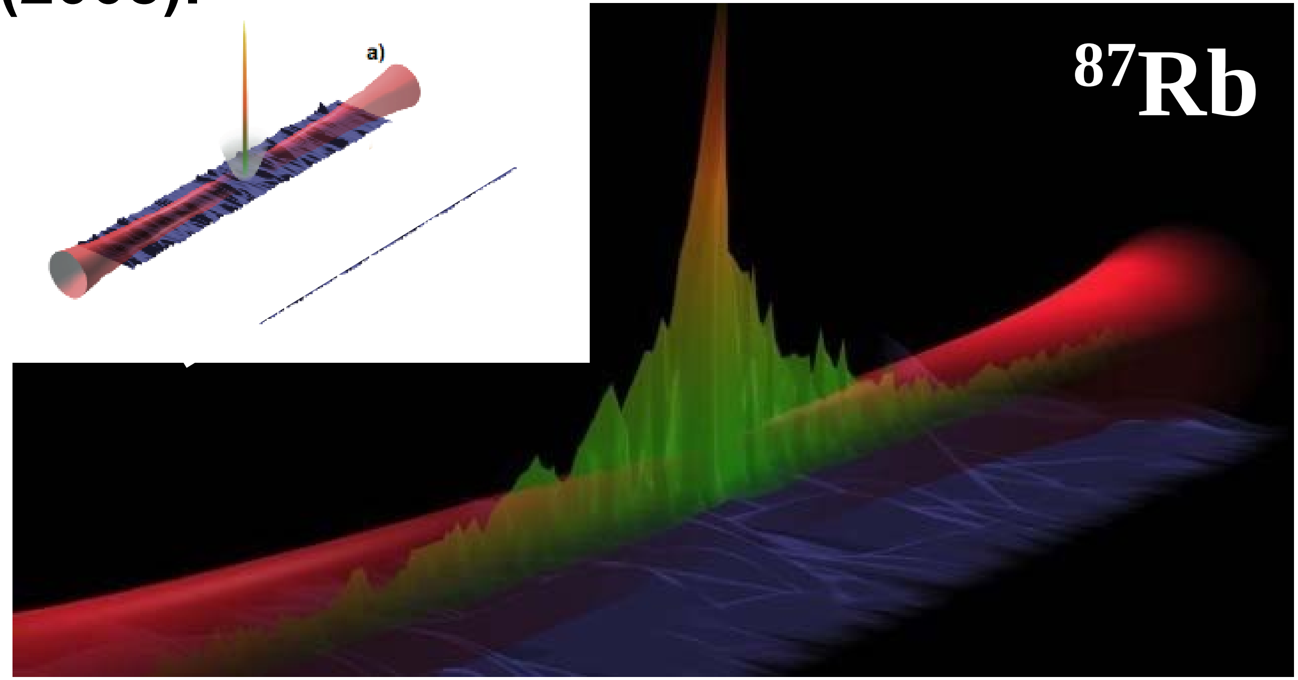


Localized State
Anderson Insulator

Extended State
Anderson Metal

Localization of cold atoms

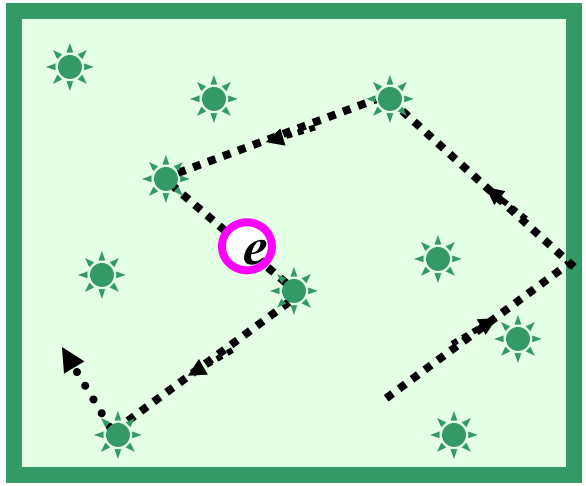
Billy et al. "Direct observation of Anderson localization of matter waves in a controlled disorder". Nature 453, 891- 894 (2008).



Roati et al. "Anderson localization of a non-interacting Bose-Einstein condensate". Nature 453, 895-898 (2008).

Q: What about electrons ?

A: Yes,… but electrons interact with each other



Scattering centers, e.g., impurities

Models of disorder:

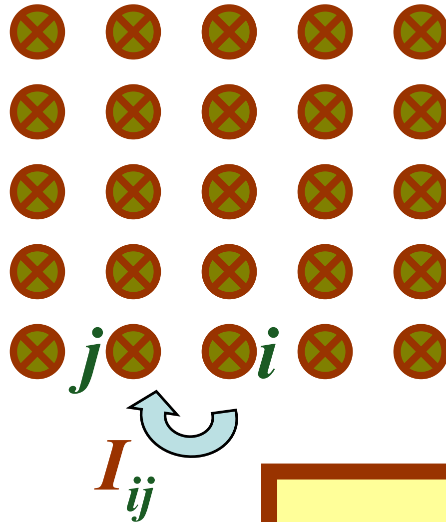
Randomly located impurities

White noise potential

Lattice models

- **Anderson model**
- Lifshits model

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$$I_c = f(d) * W$$

$I < I_c$

Insulator

All eigenstates are localized

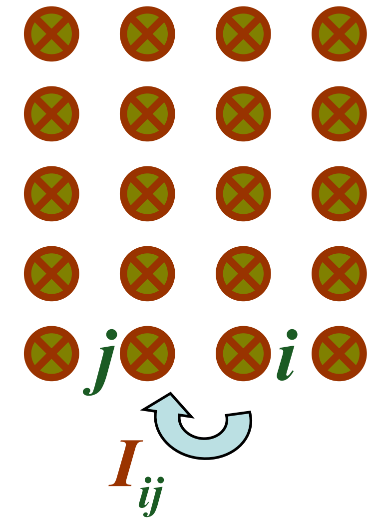
Localization length ξ

$I > I_c$

Metal

There appear states extended all over the whole system

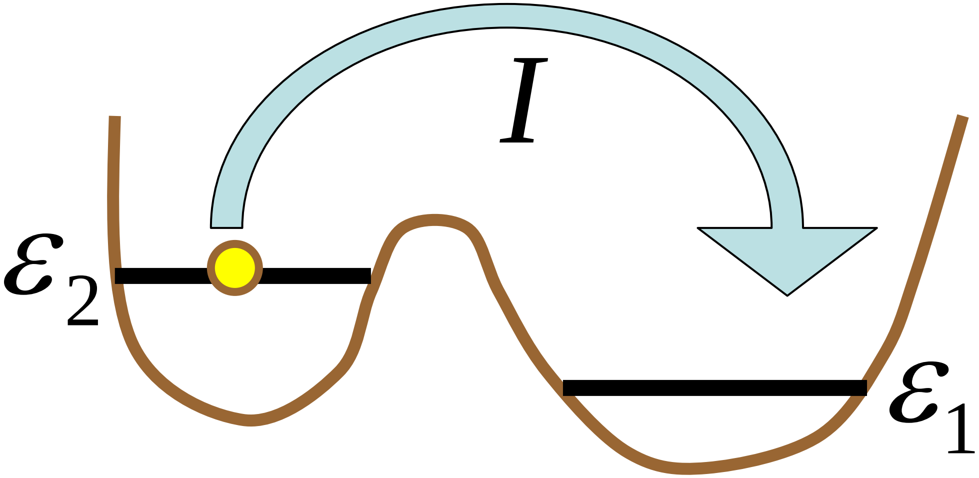
Q: Why arbitrary weak hopping I is not sufficient for the existence of the diffusion ?



Einstein (1905): Marcovian (no memory) process → diffusion

Quantum mechanics is not marcovian
There is memory in quantum propagation !

Why ?



Hamiltonian

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{\left(\varepsilon_2 - \varepsilon_1\right)^2 + I^2}$$

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{array}{ll} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{array}$$

von Neumann & Wigner “noncrossing rule”

Level repulsion

v. Neumann J. & Wigner E. 1929 Phys. Zeit. v.30, p.467

What about the eigenfunctions ?

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

What about the eigenfunctions ?

$$\phi_1, \varepsilon_1; \phi_2, \varepsilon_2 \quad \Leftarrow \quad \psi_1, E_1; \psi_2, E_2$$

$$\varepsilon_2 - \varepsilon_1 >> I$$

$$\psi_{1,2} = \varphi_{1,2} + O\left(\frac{I}{\varepsilon_2 - \varepsilon_1}\right)\varphi_{2,1}$$

Off-resonance

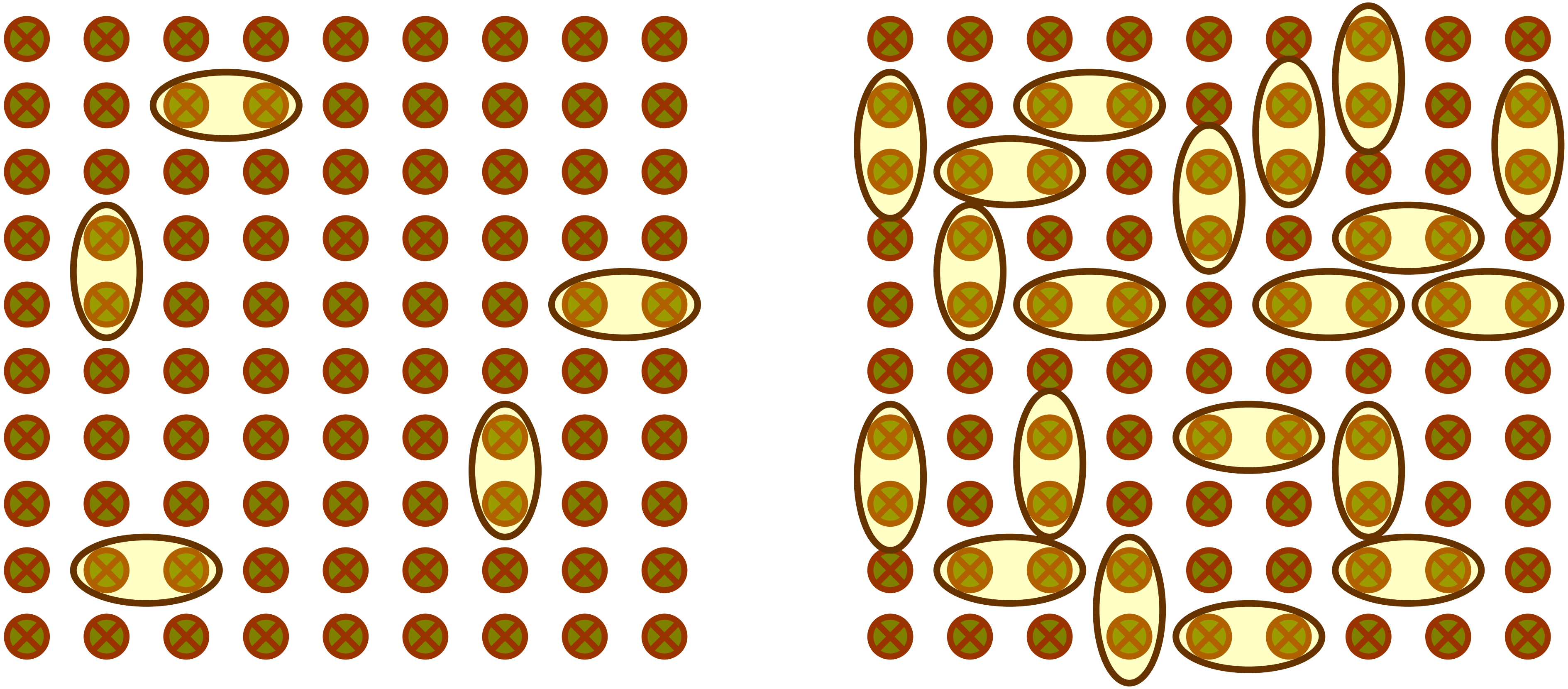
Eigenfunctions are close to the original on-site wave functions

$$\varepsilon_2 - \varepsilon_1 << I$$

$$\psi_{1,2} \approx \varphi_{1,2} \pm \varphi_{2,1}$$

Resonance

In both eigenstates the probability is equally shared between the sites



Anderson insulator

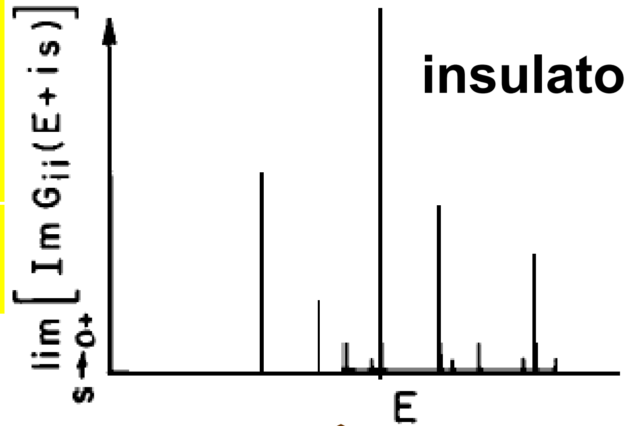
Few isolated resonances

Anderson metal

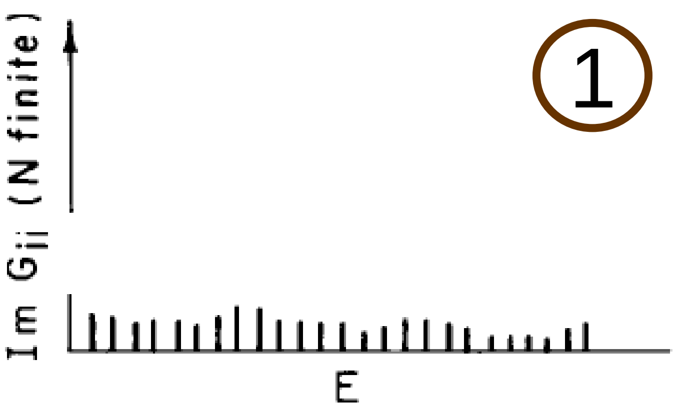
There are many resonances and they overlap

Anderson's recipe:

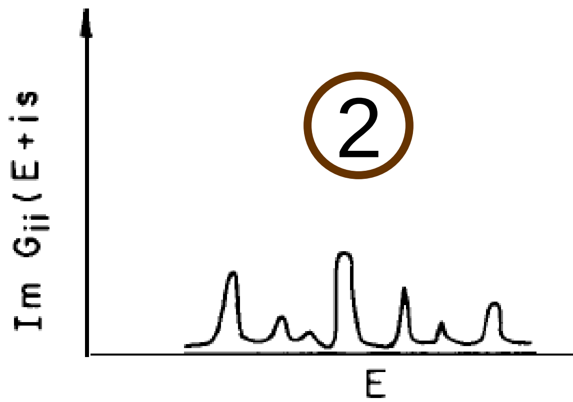
1. take discrete spectrum E_μ of H_0
2. Add an infinitesimal Im part $i\eta$ to E_μ
3. Evaluate $Im\Sigma_\mu$ imaginary part of the renormalized energy



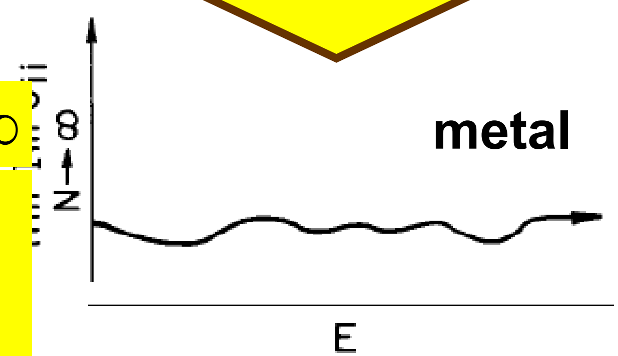
1



2

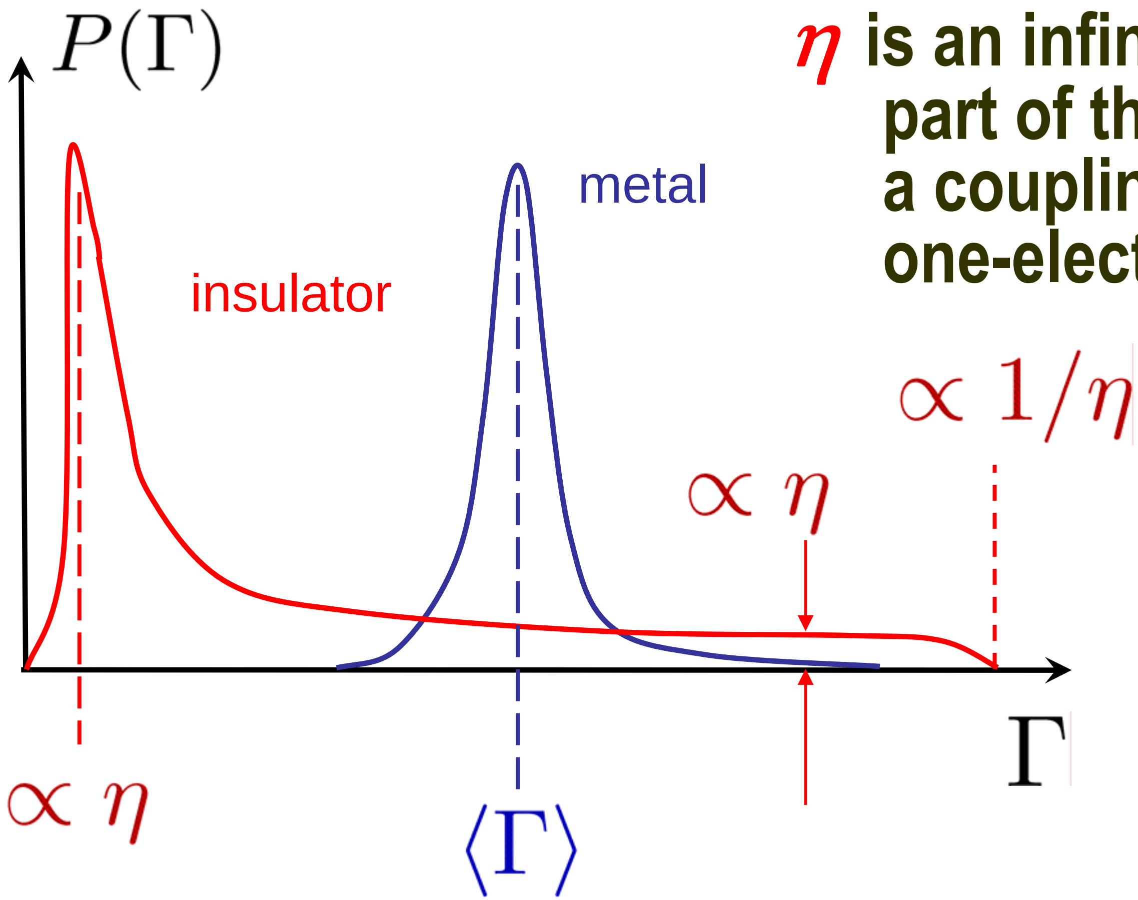


4 *limits* 1) $N \to \infty$ 2) $\eta \to 0$



4. take limit $\eta \to 0$ but only after $N \to \infty$!
5. "What we really need to know is the *probability distribution* of $Im\Sigma$, *not* its average…" P.W. Anderson Nobel Lecture

Probability Distribution of $\Gamma = Im\ \Sigma$



η **is an infinitesimal width (*Im* part of the self-energy due to a coupling with a bath) of one-electron eigenstates**

Look for:

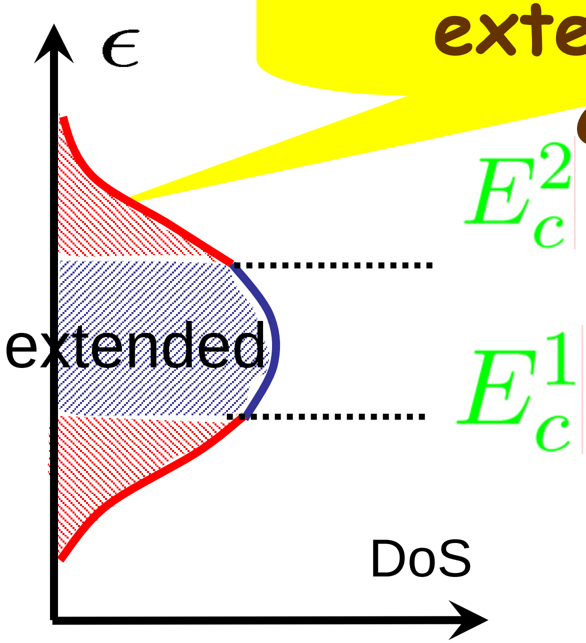
$$\lim_{\eta \to +0} \lim_{V \to \infty} P(\Gamma > 0) = \begin{cases} > 0; & metal \\ 0; & insulator \end{cases}$$

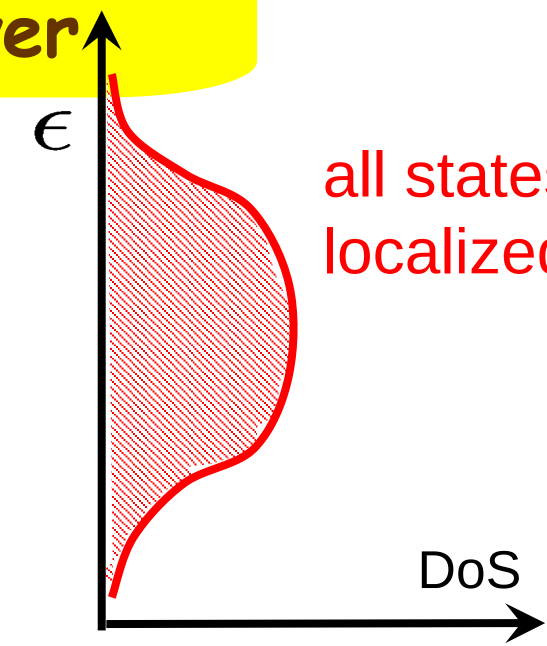
Anderson Transition

$I > I_c$

$I < I_c$

localized and extended never coexist!





all states are localized

E_c - mobility edges (one particle)

$I > I_c$

extended

$I < I_c$

localized

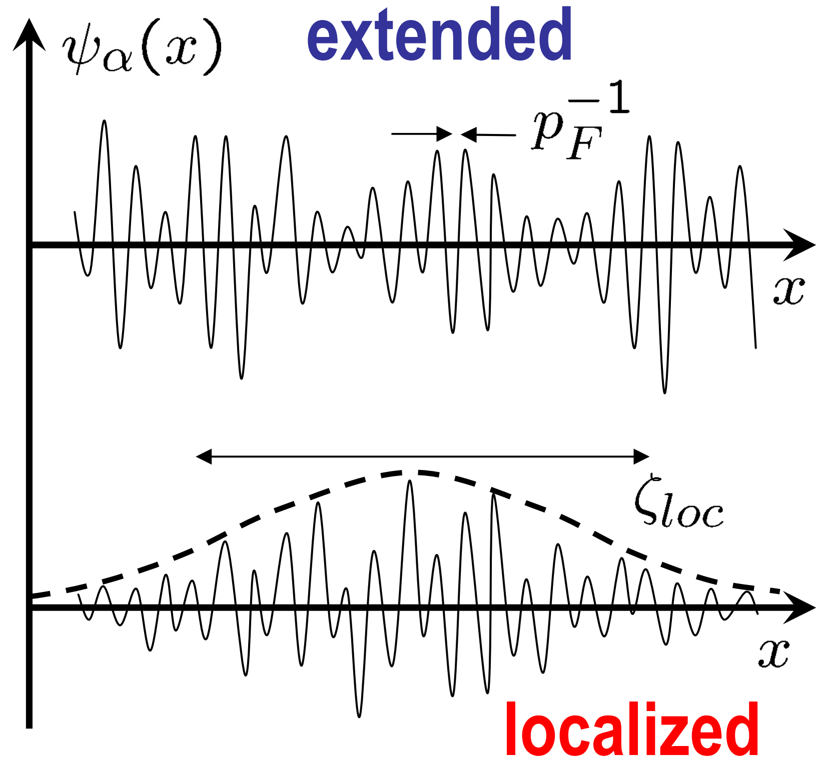
$$I_c = f(d) \times W \qquad f(1) = f(2) = 0$$

Strong disorder localized

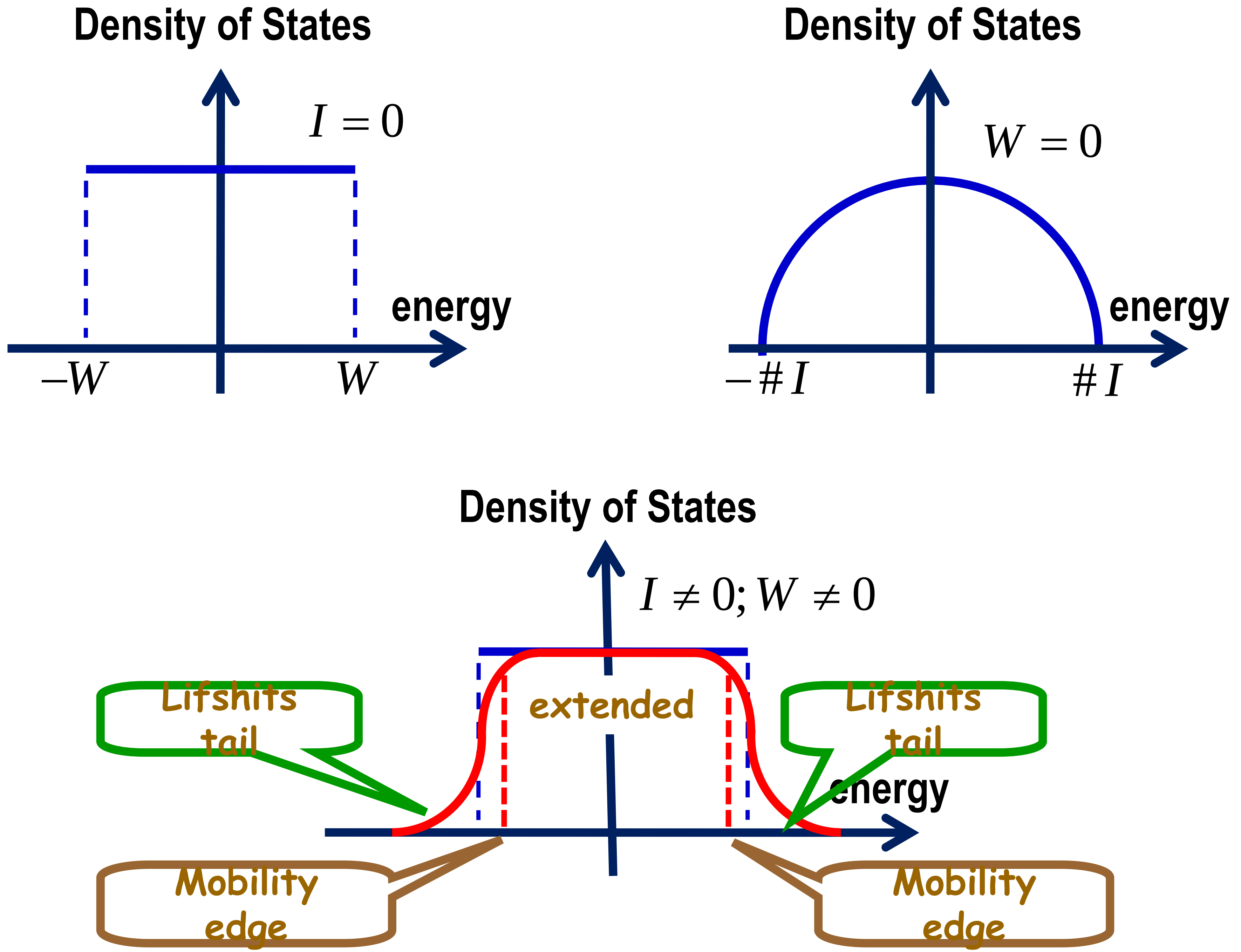
Weak disorder extended

?

Eigenfunctions



Q: Does anything interesting happen with the spectrum ?



Density of States
$I = 0$
energy
$-W$
W
Density of States
$W = 0$
energy
$-\#I$
$\#I$
Density of States
$I \neq 0; W \neq 0$
extended
Lifshits tail
Lifshits tail
energy
Mobility edge
Mobility edge

Density of States is not singular at the Anderson transition

This applies only to the **average** Density of States **!**

Fluctuations **?**

Lecture1.

2. Spectral statistics and Localization

RANDOM MATRIX THEORY

Spectral statistics

$N \times N$ — ***ensemble of Hermitian matrices with random matrix element*** — $N \to \infty$

E_α - spectrum (set of eigenvalues)

$\delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle$ - mean level spacing

$\langle \ldots\ldots \rangle$ - ensemble averaging

$s \equiv \dfrac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

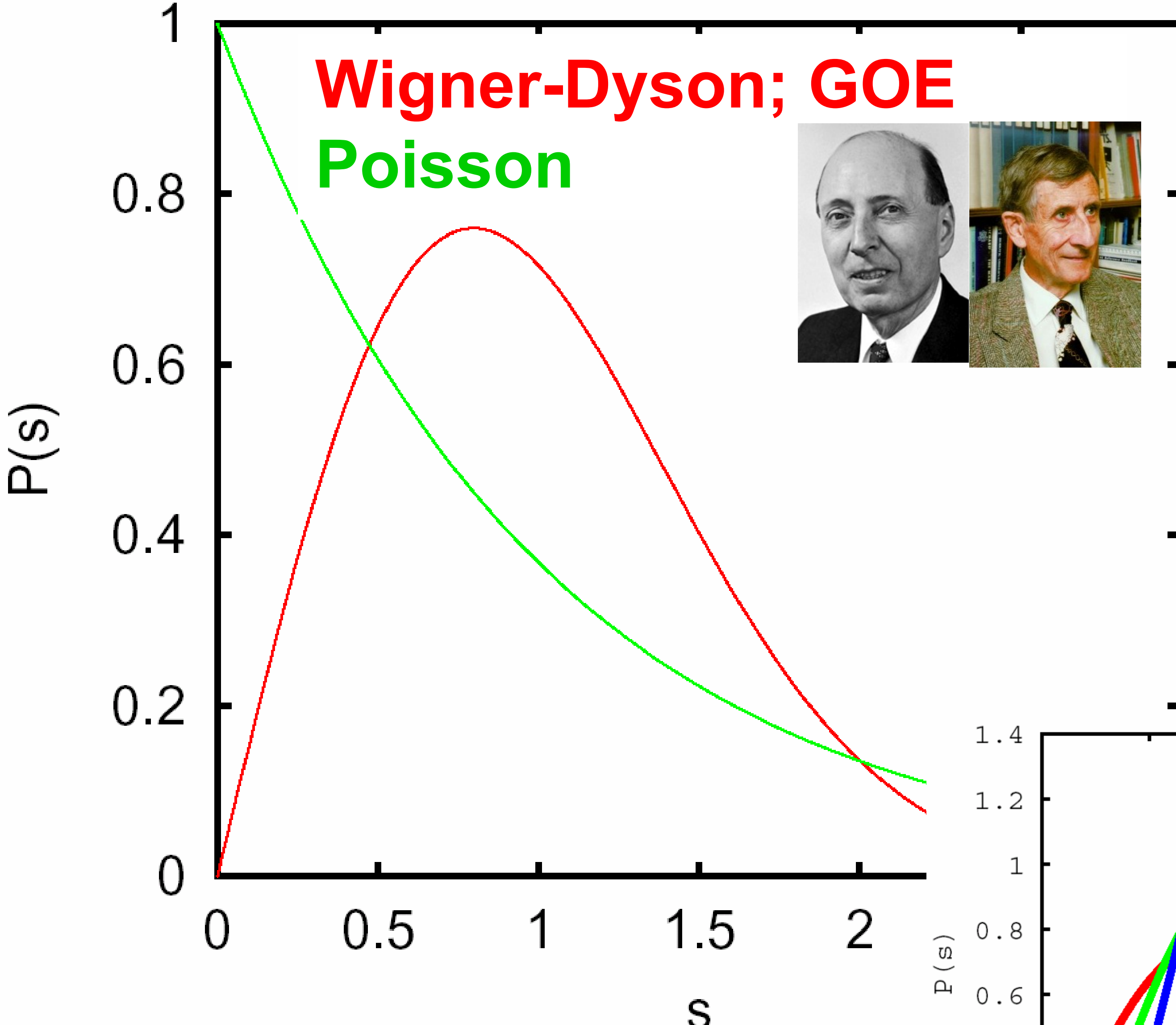
$P(s)$ - distribution function of nearest neighbors spacing between

Spectral Rigidity

Level repulsion

$$P(s=0)=0$$

$$P(s<<1) \propto s^\beta \qquad \beta=1,2,4$$

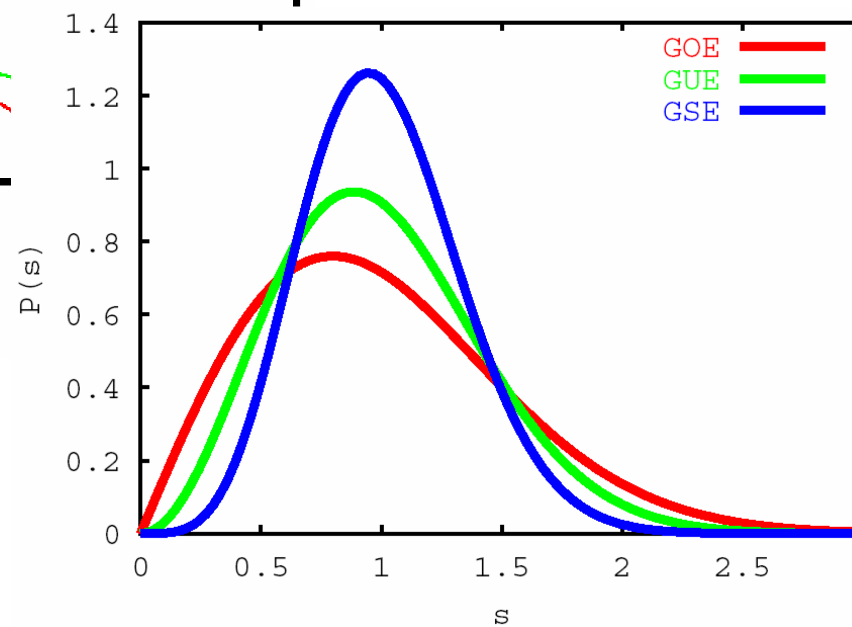


Gaussian
Orthogonal
Ensemble

Orthogonal
$\beta=1$

Unitary
$\beta=2$

Simplectic
$\beta=4$



Poisson – completely uncorrelated levels

RANDOM MATRICES

$N \times N$ matrices with random matrix elements. $N \to \infty$

Dyson Ensembles

Matrix elements	Ensemble	β	realization
real	orthogonal	1	T-inv potential
complex	unitary	2	broken T-invariance (e.g., by magnetic field)
2×2 matrices	simplectic	4	T-inv, but with spin-orbital coupling

Reason for $P(s) \to 0$ when $s \to 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{\left(H_{22} - H_{11}\right)^2 + \left|H_{12}\right|^2}$$

small

small

small

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If H_{12} is real (orthogonal ensemble), then for s to be small two statistically independent variables ($(H_{22}$- $H_{11})$ and H_{12}) should be small and thus $P(s) \propto s$ $\quad \beta = 1$

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{pmatrix}$$

$$P(E_2 - E_1) = \iint d(H_{11} - H_{22})\, dH_{12}\, \delta\left(E_2 - E_1 - \sqrt{(H_{22} - H_{11})^2 + |H_{12}|^2}\right) \times$$
$$\times\, p(H_{11} - H_{22})\, p(H_{12})$$

Distribution function of the diagonal matrix elements

Distribution function of the off-diagonal matrix elements

Distribution function of the spacing $P(s)$

Reason for $P(s) \to 0$ when $s \to 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{\underbrace{(H_{22} - H_{11})}_{\text{small}}^2 + \underbrace{|H_{12}|}_{\text{small}}^2}$$

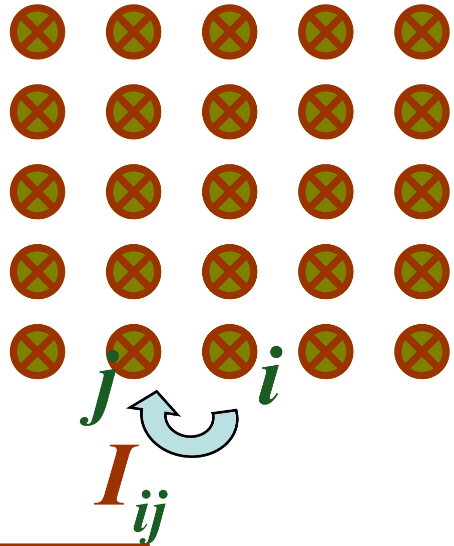
small

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If H_{12} is real (orthogonal ensemble), then for s to be small two statistically independent variables ($(H_{22}$- $H_{11})$ and H_{12}) should be small and thus $P(s) \propto s$ $\quad \beta = 1$

3. Complex H_{12} (unitary ensemble) ⟹ both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent ⟹ three independent random variables should be small ⟹ $P(s) \propto s^2$ $\quad \beta = 2$

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

Is there much in common between Random Matrices and Hamiltonians with random potential ?

Q: What are the spectral statistics of a finite size Anderson model ?

Anderson Transition

Strong disorder

$$I < I_c$$

Insulator

All eigenstates are *localized*

Localization length ξ

The eigenstates, which are localized at different places will not repel each other

↓

Poisson spectral statistics

Weak disorder

$$I > I_c$$

Metal

There appear states *extended* all over the whole system

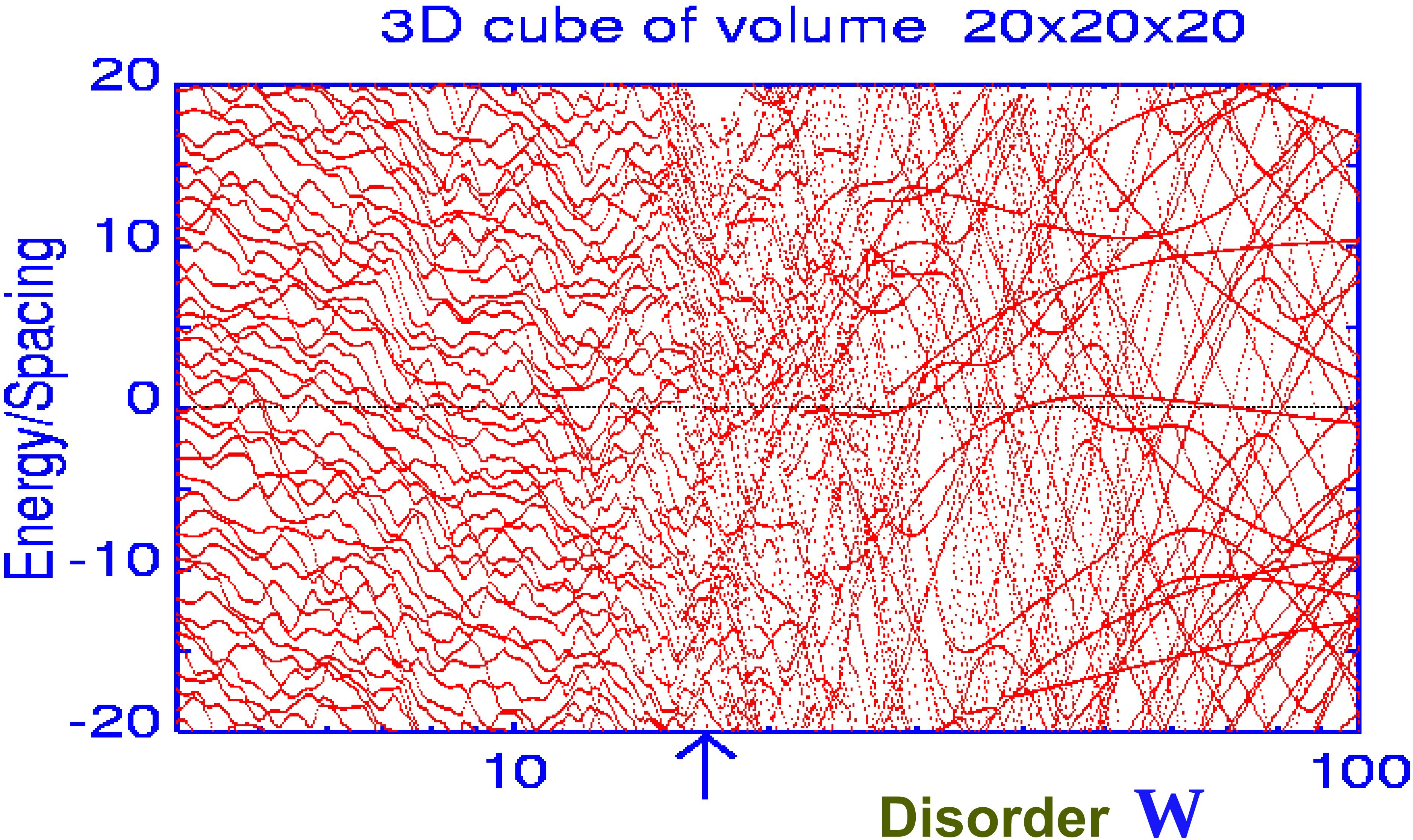
Any two extended eigenstates repel each other

↓

Wigner – Dyson spectral statistics

Zharekeschev & Kramer.

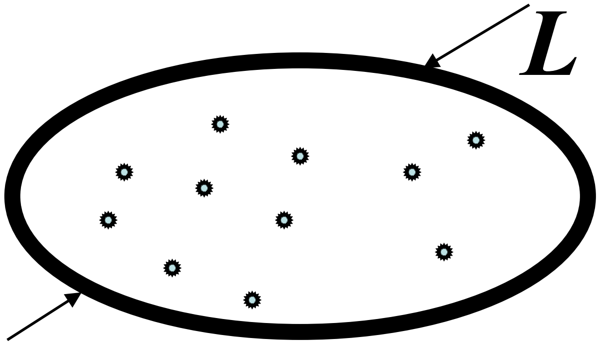
Exact diagonalization of the Anderson model

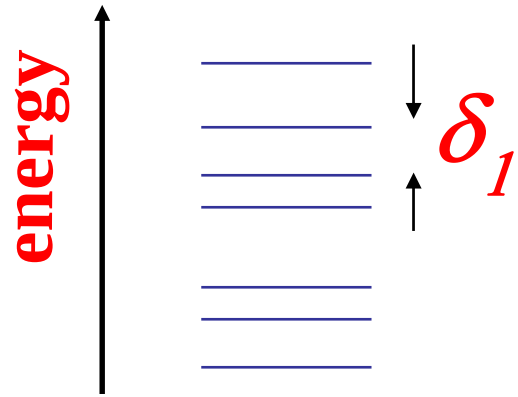


Energy scales in the localization problem. (*Thouless, 1972*)

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L is the system size;

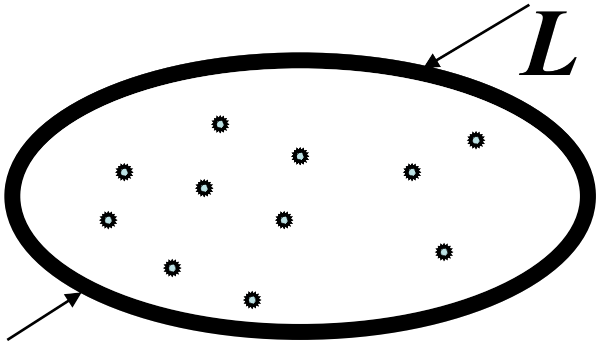
d is the number of dimensions

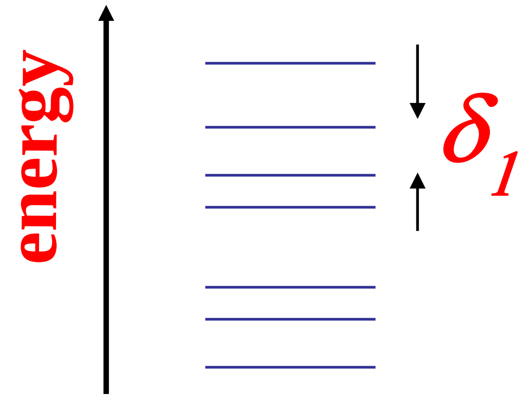
This scale exists in the Random Matrix theory

Energy scales in the localization problem. (*Thouless, 1972*)

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L is the system size;

d is the number of dimensions

This energy scale exists in the Random Matrix theory.

This is the only energy scale in the RM theory

Thouless Conductance and One-particle Spectral Statistics

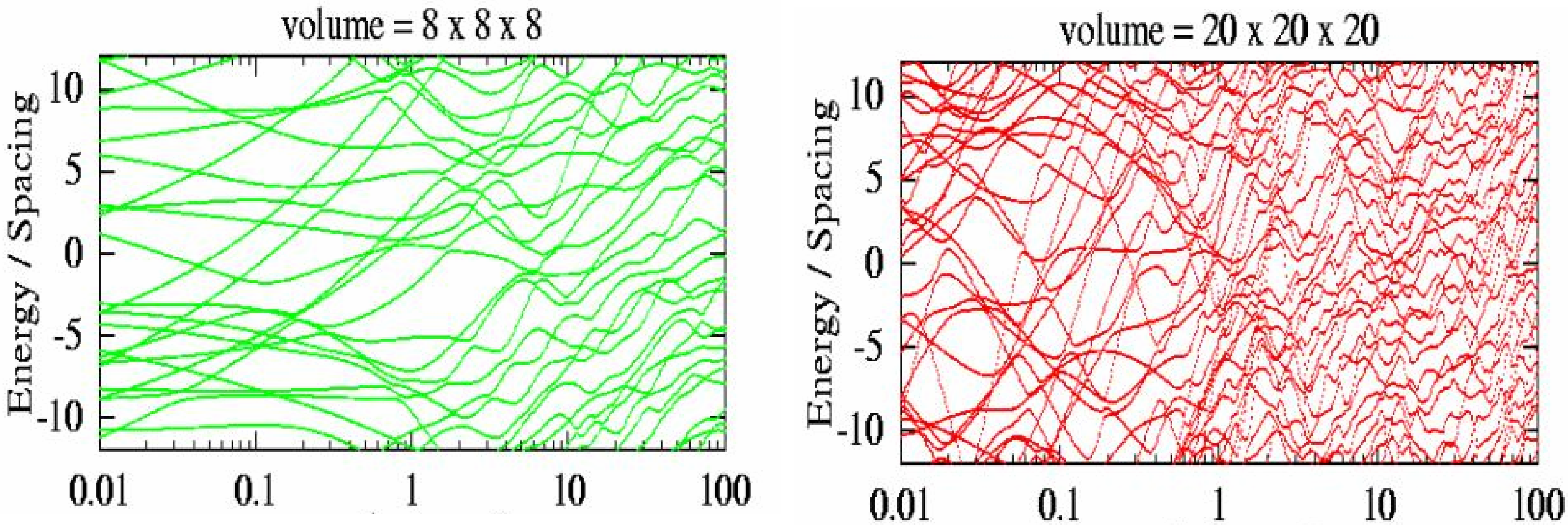
Localized states
Insulator

Poisson spectral statistics

Extended states
Metal

Wigner-Dyson spectral statistics

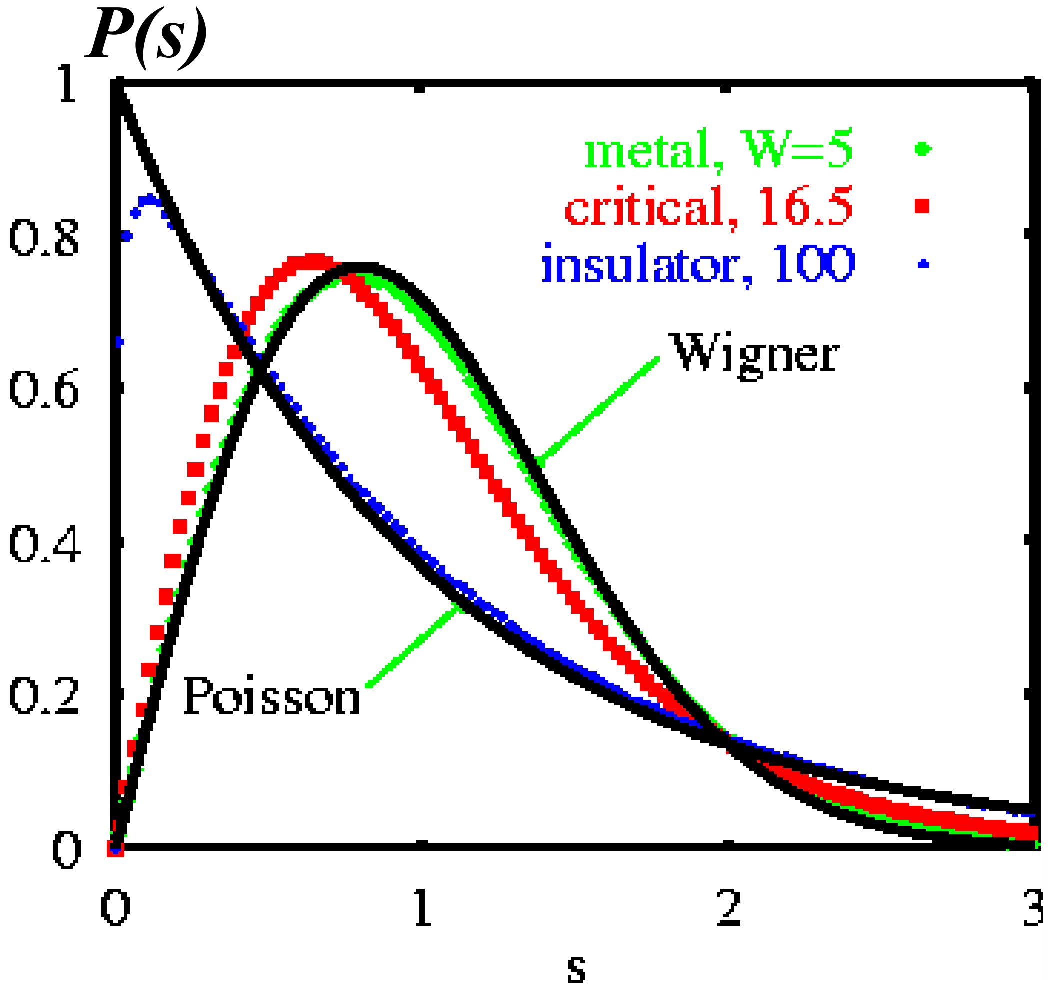
Transition at $g\sim 1$.
Is it sharp?

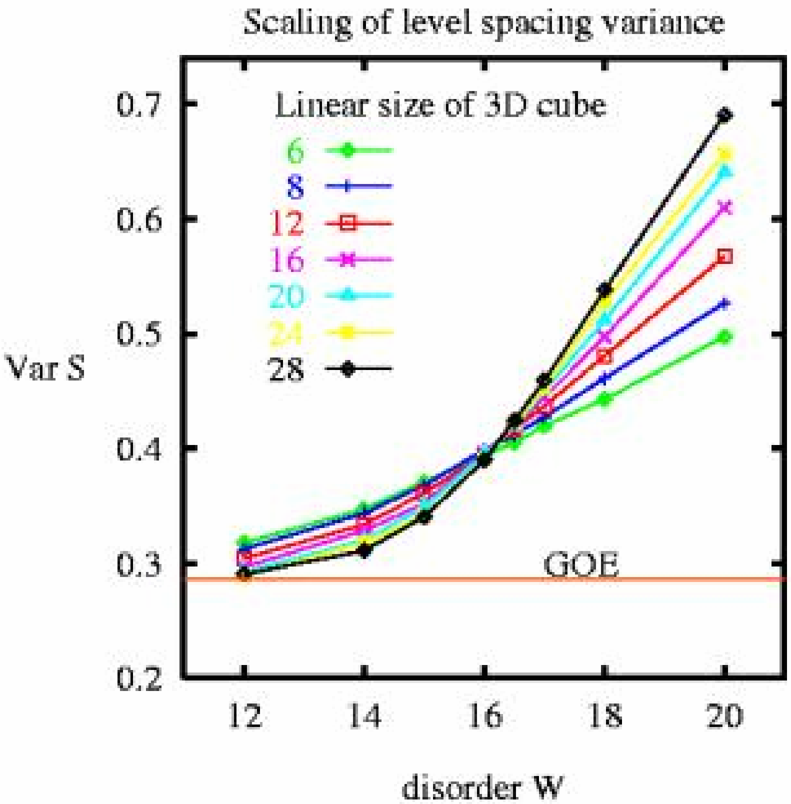


Conductance g

The bigger the system the sharper the transition

Anderson transition in terms of pure level statistics





Lecture1.

3. Quantum Chaos, Integrability and Localization

Finite size quantum physical systems

Nuclei
Atoms
Molecules
.
.
.

Quantum
Dots

ATOMS Main goal is to classify the eigenstates in terms of the quantum numbers

NUCLEI For the nuclear excitations this program does not work

Wigner: Study spectral statistics of a particular quantum system – a given nucleus

ATOMS Main goal is to classify the eigenstates in terms of the quantum numbers

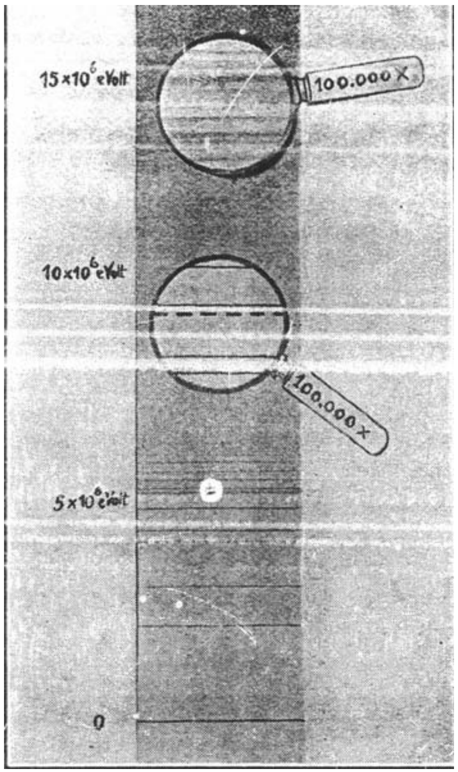
NUCLEI For the nuclear excitations this program does not work

Wigner: Study spectral **statistics** of a **particular** quantum system – a given nucleus

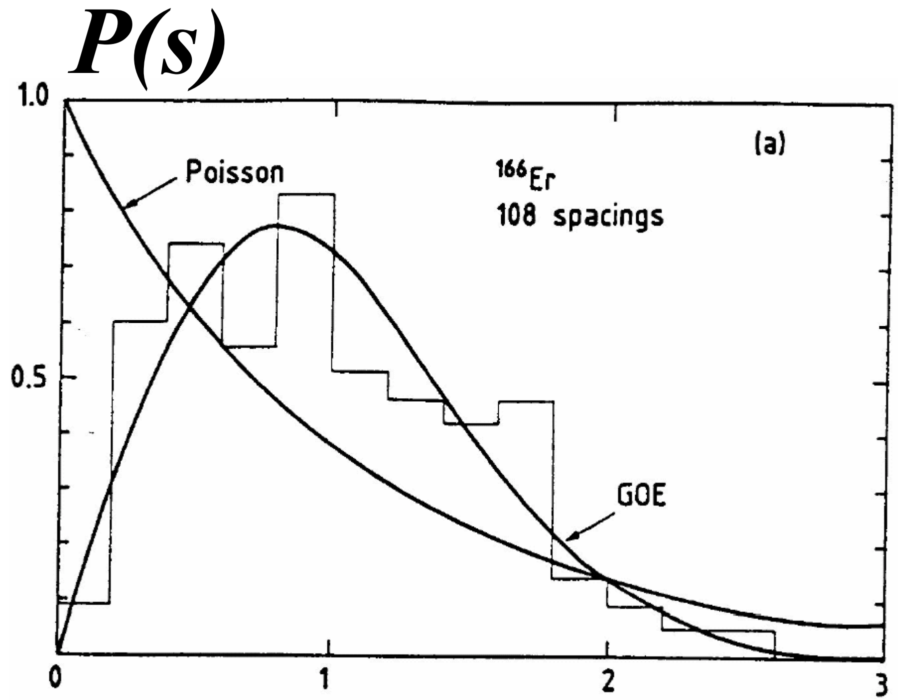
Spectra: $\{E_\alpha\}$

Random Matrices	Atomic Nuclei
• *Ensemble* • *Ensemble averaging*	•*Spectral averaging (over α)* •*Particular quantum system*

Nevertheless Statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics

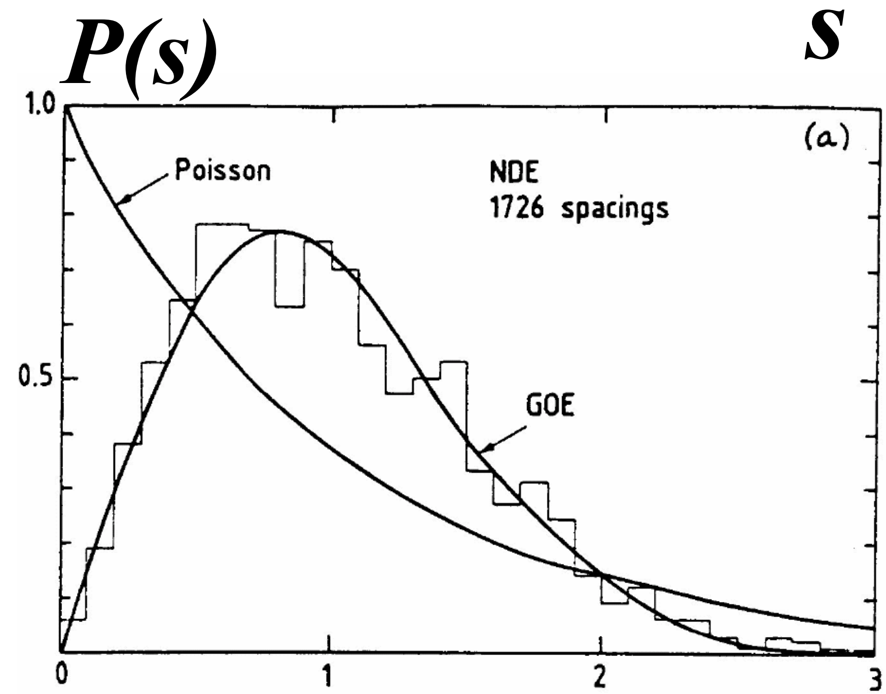


N. Bohr, Nature 137 (1936) 344.



Particular nucleus

^{166}Er



Spectra of several nuclei combined (after spacing) rescaling by the mean level

Q: *Why the random matrix theory (RMT) works so well for nuclear spectra* ?

Original answer: *These are systems with a large number of degrees of freedom, and therefore the "complexity" is high*

Q: *Why the random matrix theory (RMT) works so well for nuclear spectra* ?

Original answer:

These are systems with a **large number of degrees of freedom**, and therefore the "complexity" is high

Later it became clear that

there exist very "simple" systems with as many as 2 degrees of freedom (d=2), which demonstrate RMT - like spectral statistics

Classical Dynamical Systems with d degrees of freedom

Integrable Systems

The variables can be separated and the problem reduces to d one-dimensional problems

$\Rightarrow$

d integrals of motion

Classical ($\hbar$ =0) Dynamical Systems with d degrees of freedom

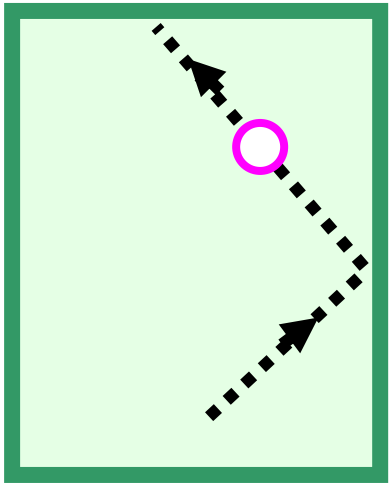
Integrable Systems

The variables can be separated and the problem reduces to d one-dimensional problems

⇒ d integrals of motion

Examples

1. A ball inside rectangular billiard; $d=2$



- Vertical motion can be separated from the horizontal one
- Vertical and horizontal components of the momentum, are both integrals of motion

Classical ($\hbar$ =0) Dynamical Systems with d degrees of freedom

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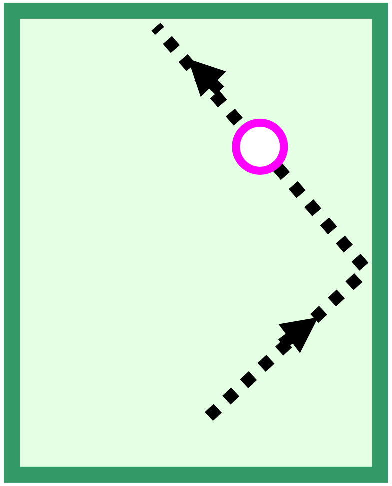
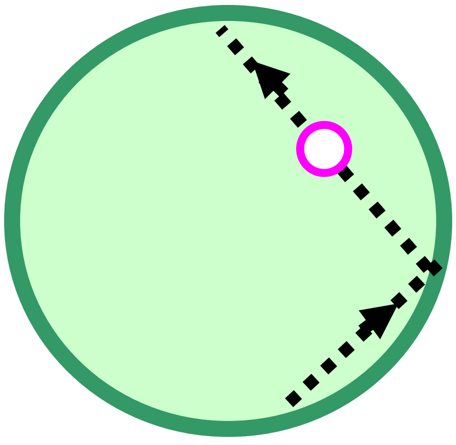


- Vertical motion can be separated from the horizontal one
- Vertical and horizontal components of the momentum, are both integrals of motion

2. Circular billiard; $d=2$



- Radial motion can be separated from the angular one
- Angular momentum and energy are the integrals of motion

Classical Dynamical Systems with d degrees of freedom

Integrable Systems

The variables can be separated ⇨ d one-dimensional problems ⇨ d integrals of motion

Rectangular and circular billiard, Kepler problem, . . . ,
1d Hubbard model and other exactly solvable models, . .

Classical Dynamical Systems with *d* degrees of freedom

Integrable Systems

The variables can be separated ⇨ *d* one-dimensional problems ⇨ *d* integrals of motion

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Chaotic Systems

The variables can not be separated ⇨ there is only one integral of motion - energy

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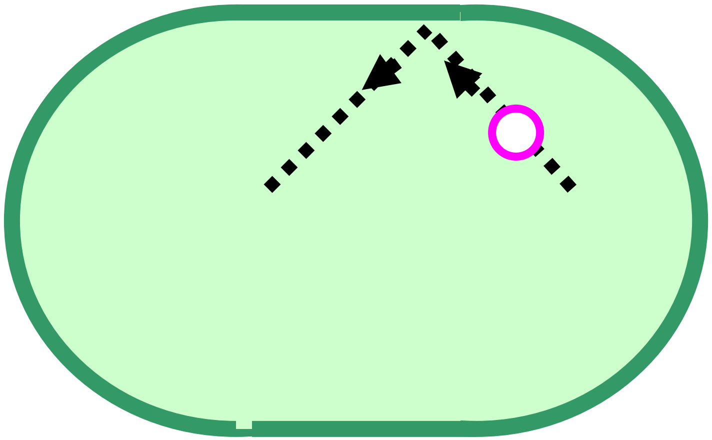

Stadium

Classical Dynamical Systems with d degrees of freedom

Integrable Systems

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Chaotic Systems

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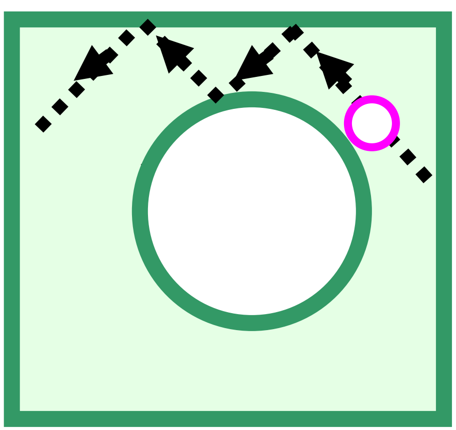
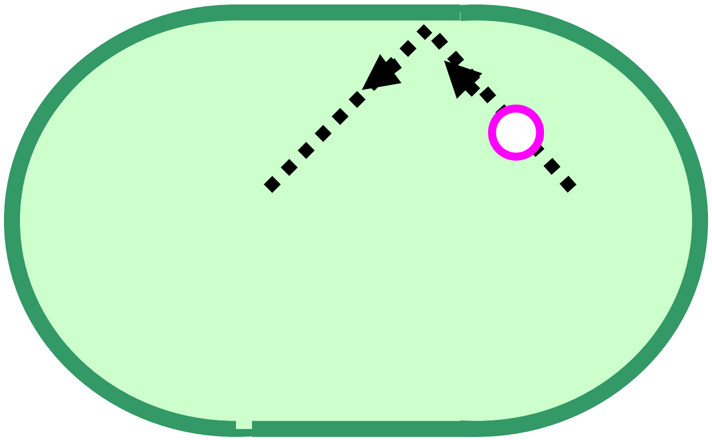


Sinai billiard



Stadium

Classical Dynamical Systems with d degrees of freedom

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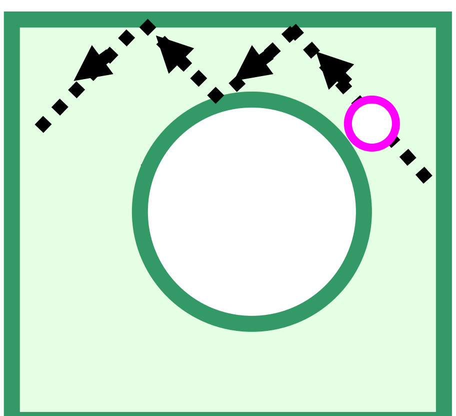

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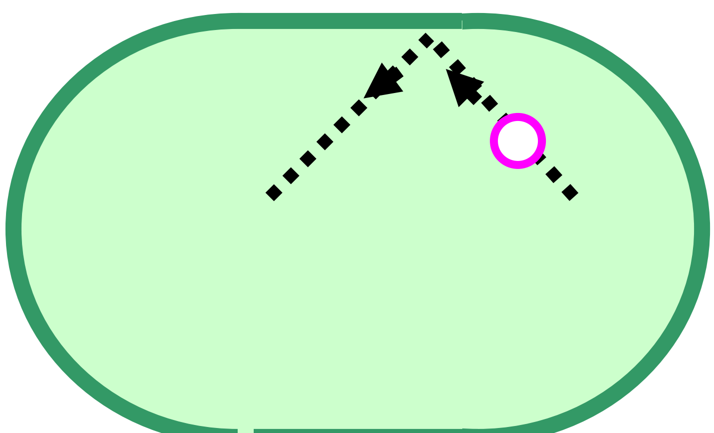
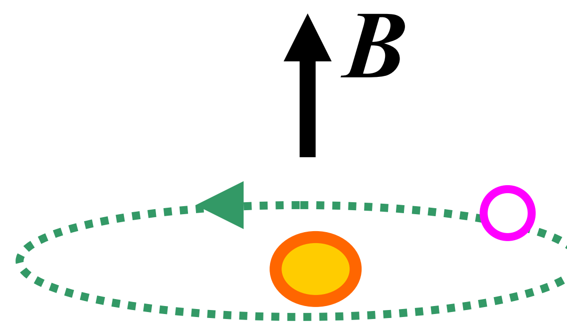


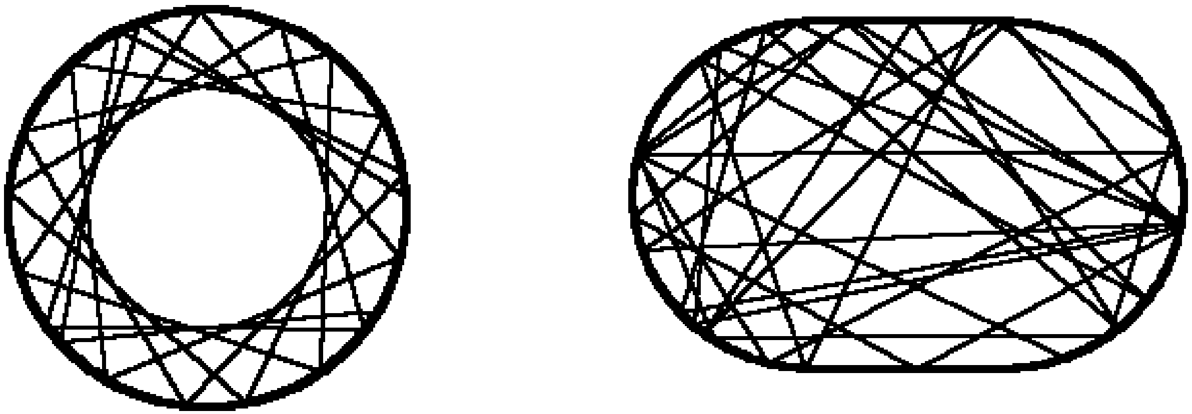
Stadium



Kepler problem in magnetic field

Classical Chaos
$\hbar = 0$

- *Nonlinearities*
- *Exponential dependence on the original conditions (Lyapunov exponents)*
- *Ergodicity*



Quantum description of any System with a finite number of the degrees of freedom is a linear problem – Shrodinger equation

Q: What does it mean Quantum Chaos ?

$\hbar \neq 0$ Bohigas – Giannoni – Schmit conjecture

VOLUME 52 2 JANUARY 1984 NUMBER 1

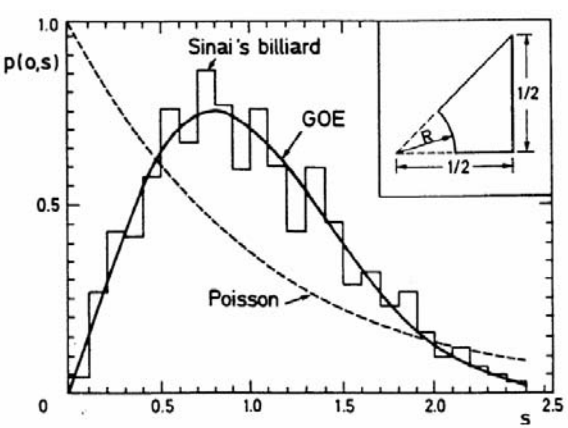
Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

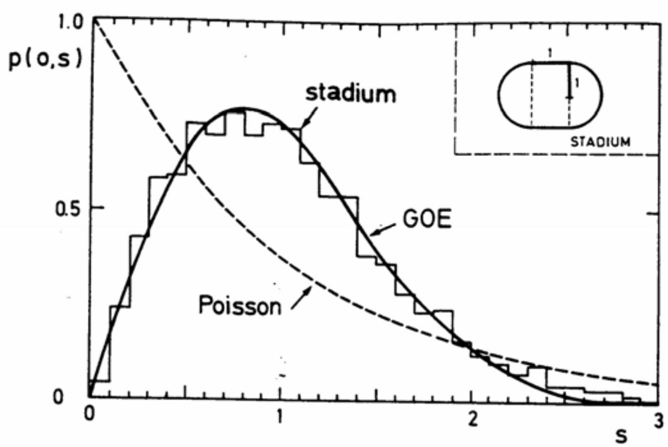
O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France

(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai's billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal-invariant systems whose classical analogs are K systems show the same fluctuation properties as predicted by GOE





Chaotic classical analog

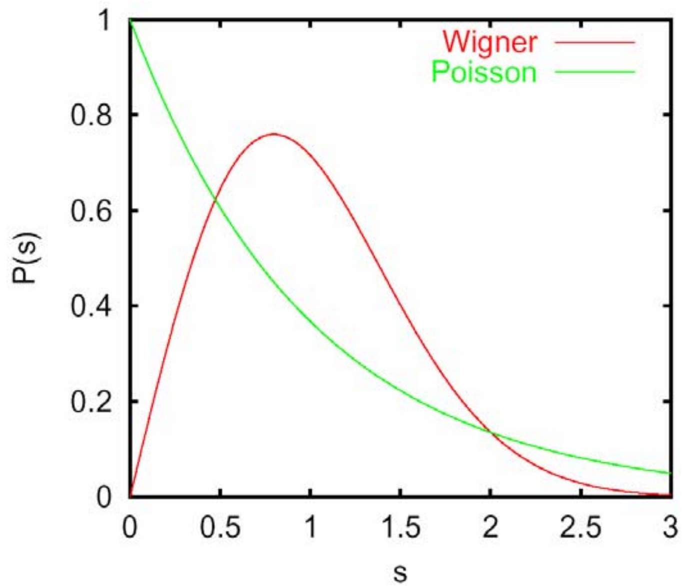
Wigner- Dyson spectral statistics

No quantum numbers except energy

Classical Quantum

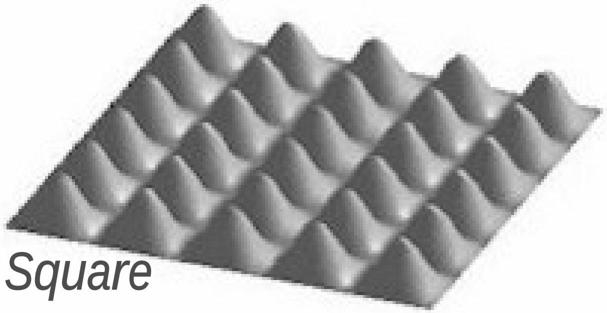
Integrable ? **Poisson**

Chaotic ? **Wigner-Dyson**

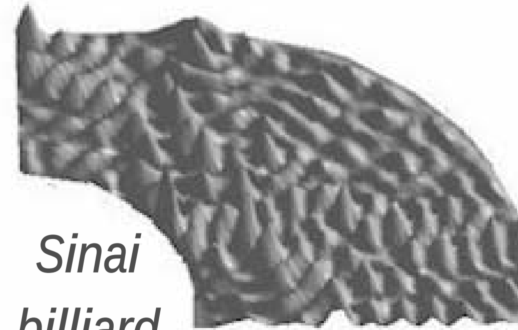


Integrable

Chaotic

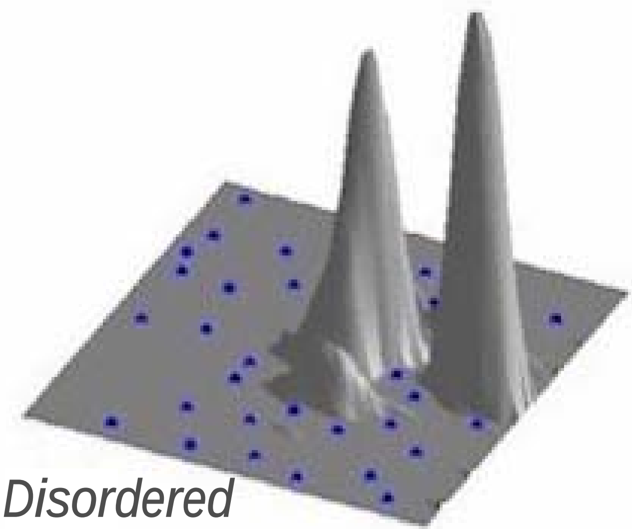


Square billiard

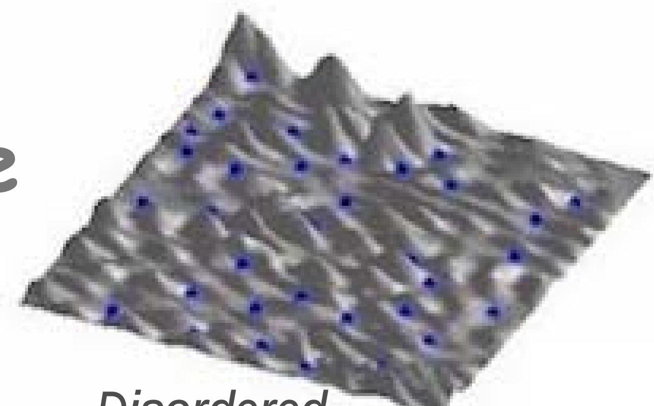


Sinai billiard

All chaotic systems resemble each other.



Disordered localized



Disordered extended

All integrable systems are integrable in their own way

Lecture1.

4. Localization beyond real space

VOLUME 49 23 AUGUST 1982 NUMBER 8

Chaos, Quantum Recurrences, and Anderson Localization

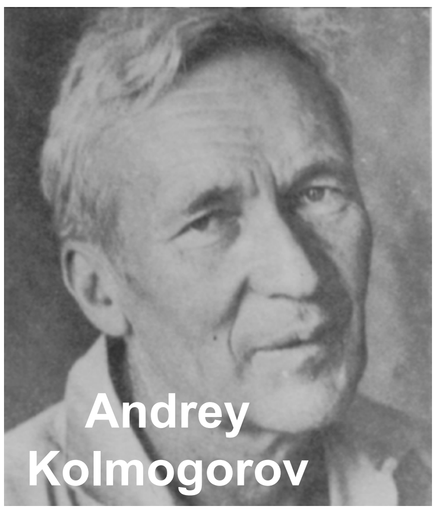
Shmuel Fishman, D. R. Grempel, and R. E. Prange
Department of Physics and Center for Theoretical Physics, University of Maryland, College Park, Maryland 20742
(Received 6 April 1982)

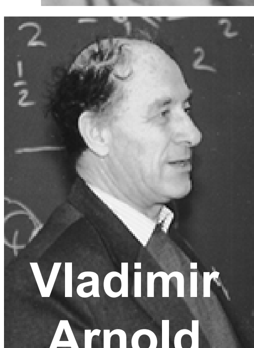
A periodically kicked quantum rotator is related to the Anderson problem of conduction in a one-dimensional disordered lattice. Classically the second model is always chaotic, while the first is chaotic for some values of the parameters. With use of the Anderson-model result that all states are localized, it is concluded that the *local* quasienergy spectrum of the rotator problem is discrete and that its wave function is almost periodic in time. This allows one to understand on physical grounds some numerical results recently obtained in the context of the rotator problem.

Localization in the angular momentum space

Kolmogorov – Arnold – Moser (KAM) theory

A.N. Kolmogorov,
Dokl. Akad. Nauk SSSR, 1954.
Proc. 1954 Int. Congress of Mathematics, North-Holland, 1957





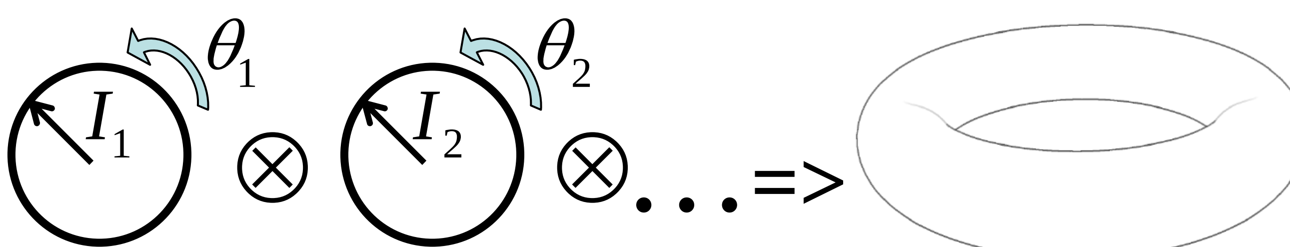
$$\hbar = 0$$

Integrable classical Hamiltonian $\hat{H}_0$, $d>1$:

Separation of variables: d sets of action-angle variables

$$I_1, \theta_1 = 2\pi\omega_1 t; \ldots, I_2, \theta_2 = 2\pi\omega_2 t; ..$$

Quasiperiodic motion:
set of the frequencies, $\omega_1, \omega_2, .., \omega_d$ which are in general incommensurate. Actions I_i are integrals of motion $\partial I_i / \partial t = 0$



tori

Integrable dynamics:

Each classical trajectory is quasiperiodic and confined to a particular torus, which is determined by a set of the integrals of motion

space	Number of dimensions
real space	d
phase space: (x,p)	$2d$
energy shell	$2d-1$
tori	d

Each torus has measure zero on the energy shell !

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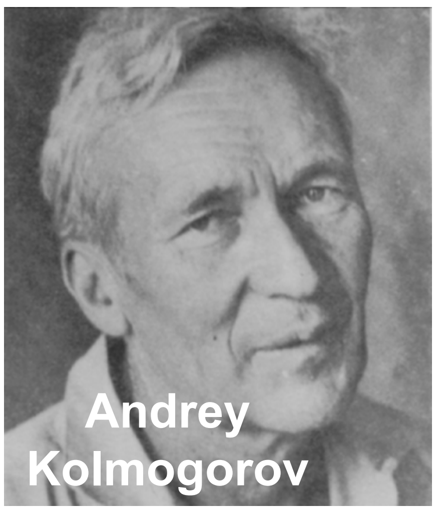

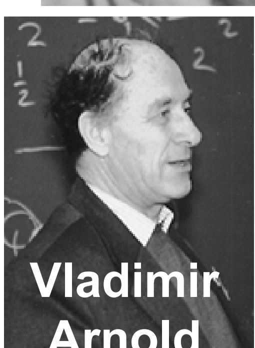

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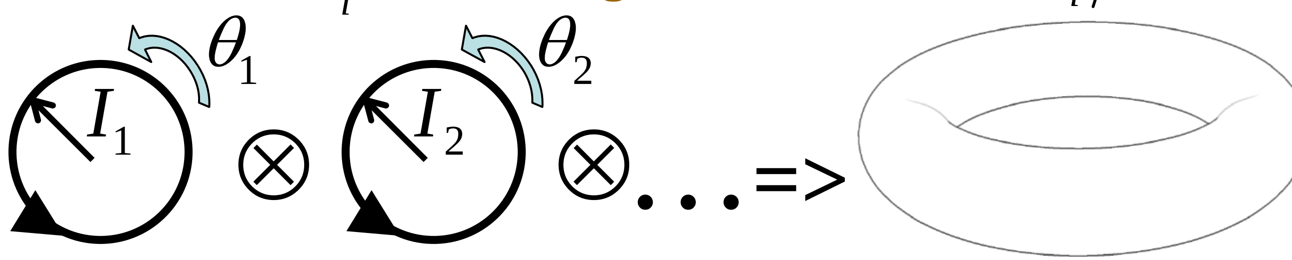

Q: Will an arbitrary weak perturbation $\hat{V}$ of the integrable Hamiltonian $\hat{H}_0$ destroy the tori and make the motion ergodic (when each point at the energy shell will be reached sooner or later) **?**

A: Most of the tori survive weak and smooth enough perturbations

KAM theorem

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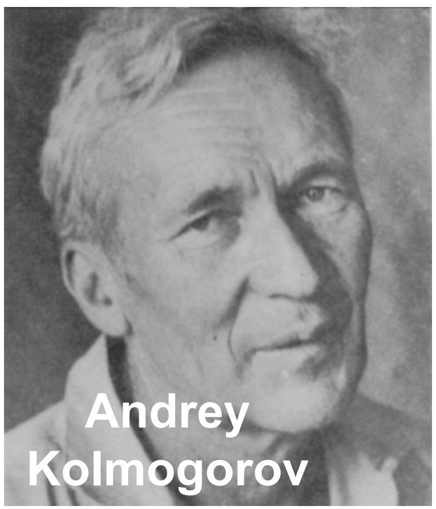
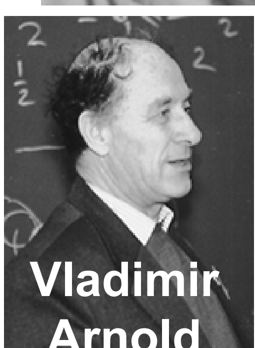
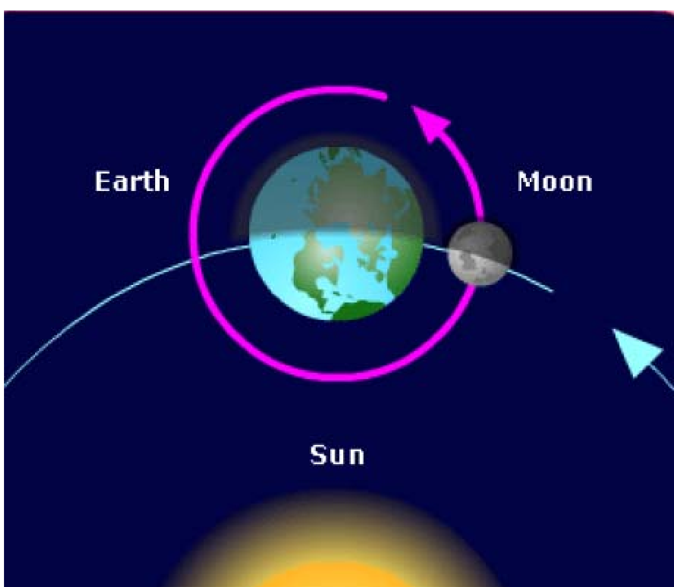


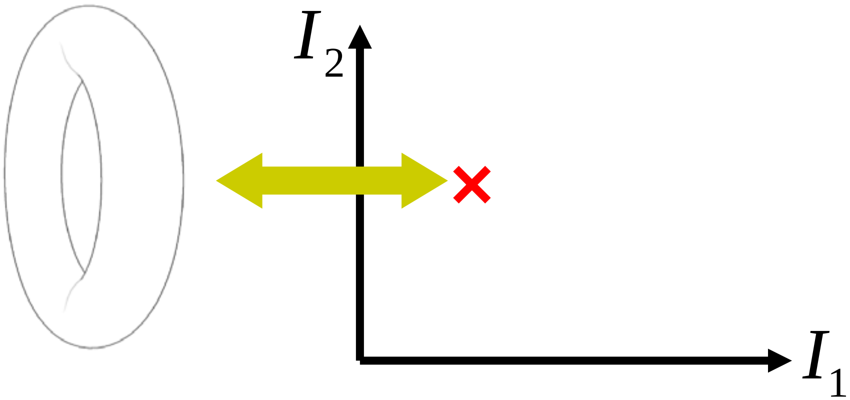


?

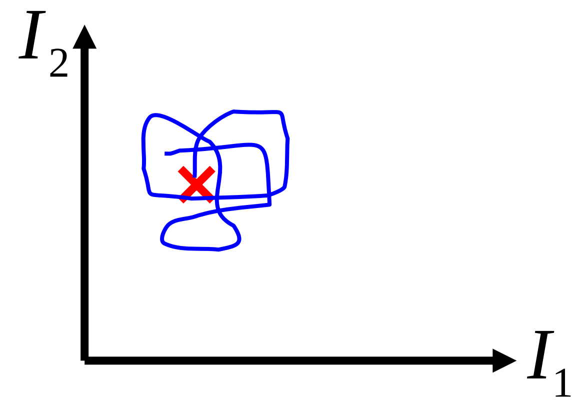


KAM theorem:

Most of the tori survive weak and smooth enough perturbations



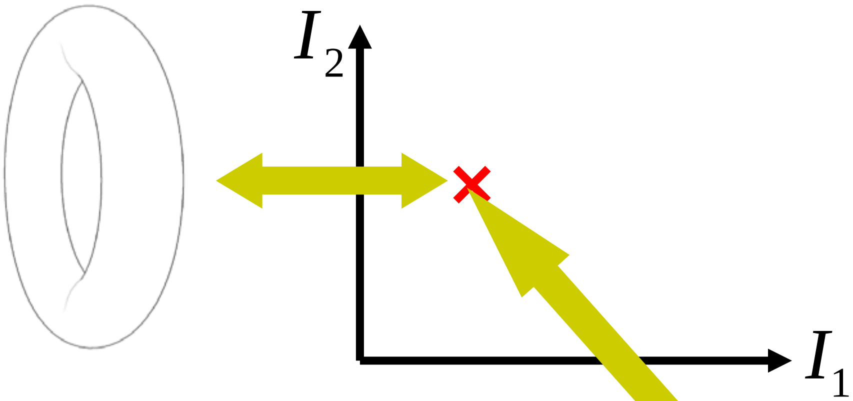
$$\hat{V} \neq 0$$



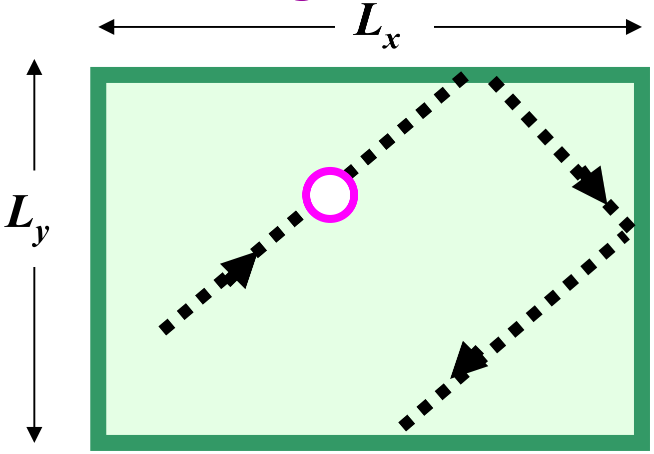
Each point in the space of the integrals of motion corresponds to a torus and vice versa

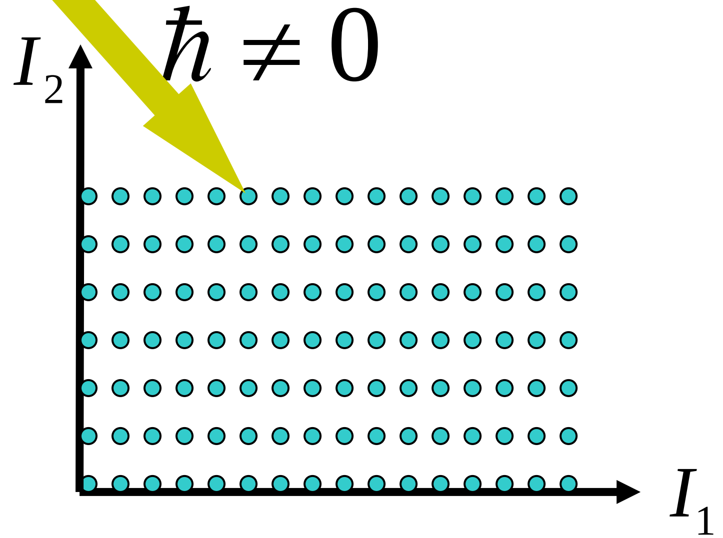
Finite motion.
Localization in the space of the integrals of motion ?

KAM theorem: Most of the tori survive weak and smooth enough perturbations



Rectangular billiard

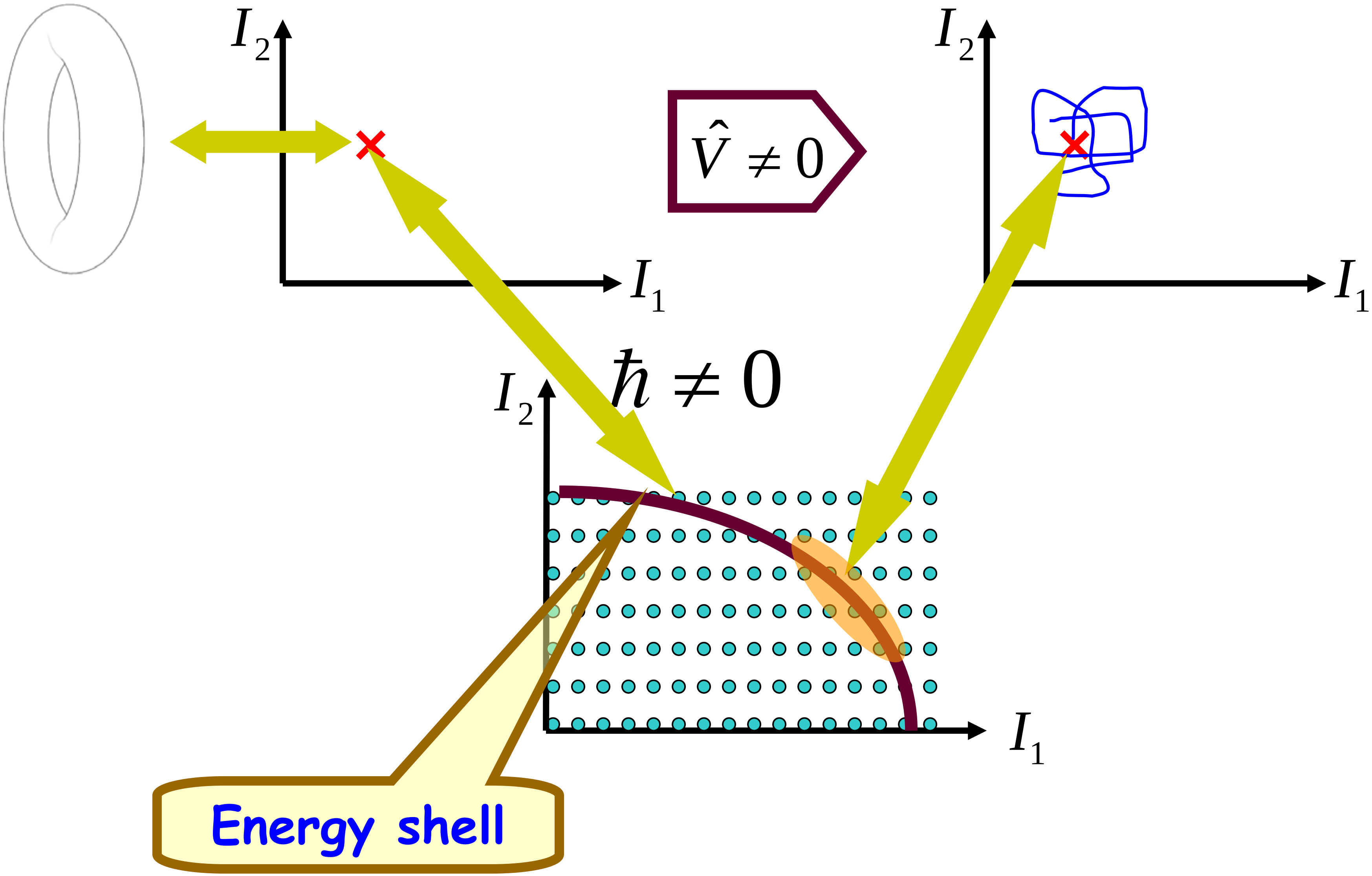




Two integrals of motion

$$p_x = \frac{\pi n}{L_x}; \quad p_y = \frac{\pi m}{L_x}$$

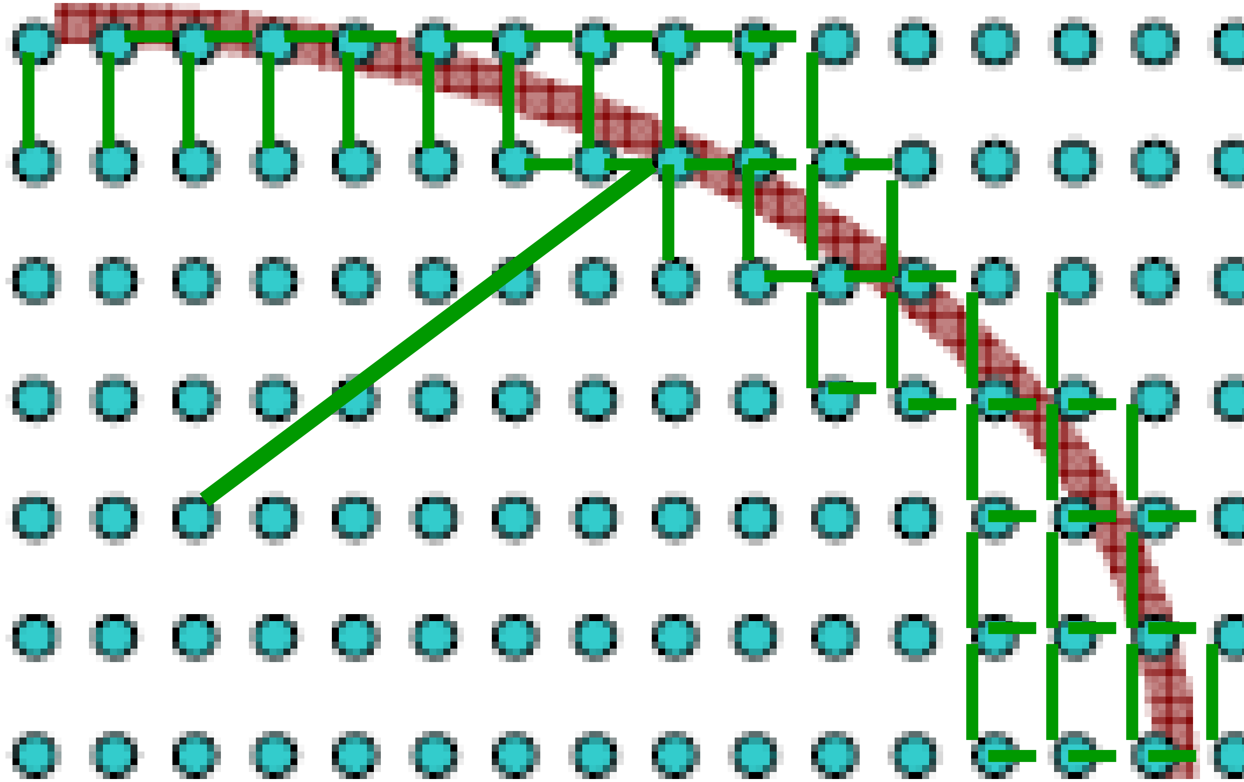
KAM theorem: **Most of the tori survive weak and smooth enough perturbations**



$$\hat{V}_{\mu,\nu}$$

μ ○—○ ν

Matrix element of the perturbation



$$\left|\mu\right\rangle = \left|\vec{I}^{(\mu)}\right\rangle$$

$$\vec{I}^{(\mu)} = \left\{I_1^{(\mu)}, \ldots, I_d^{(\mu)}\right\}$$

One can speak about localization provided that the perturbation is somewhat local in the space of quantum numbers of the original Hamiltonian

AL hops are local – one can distinguish "near" and "far"

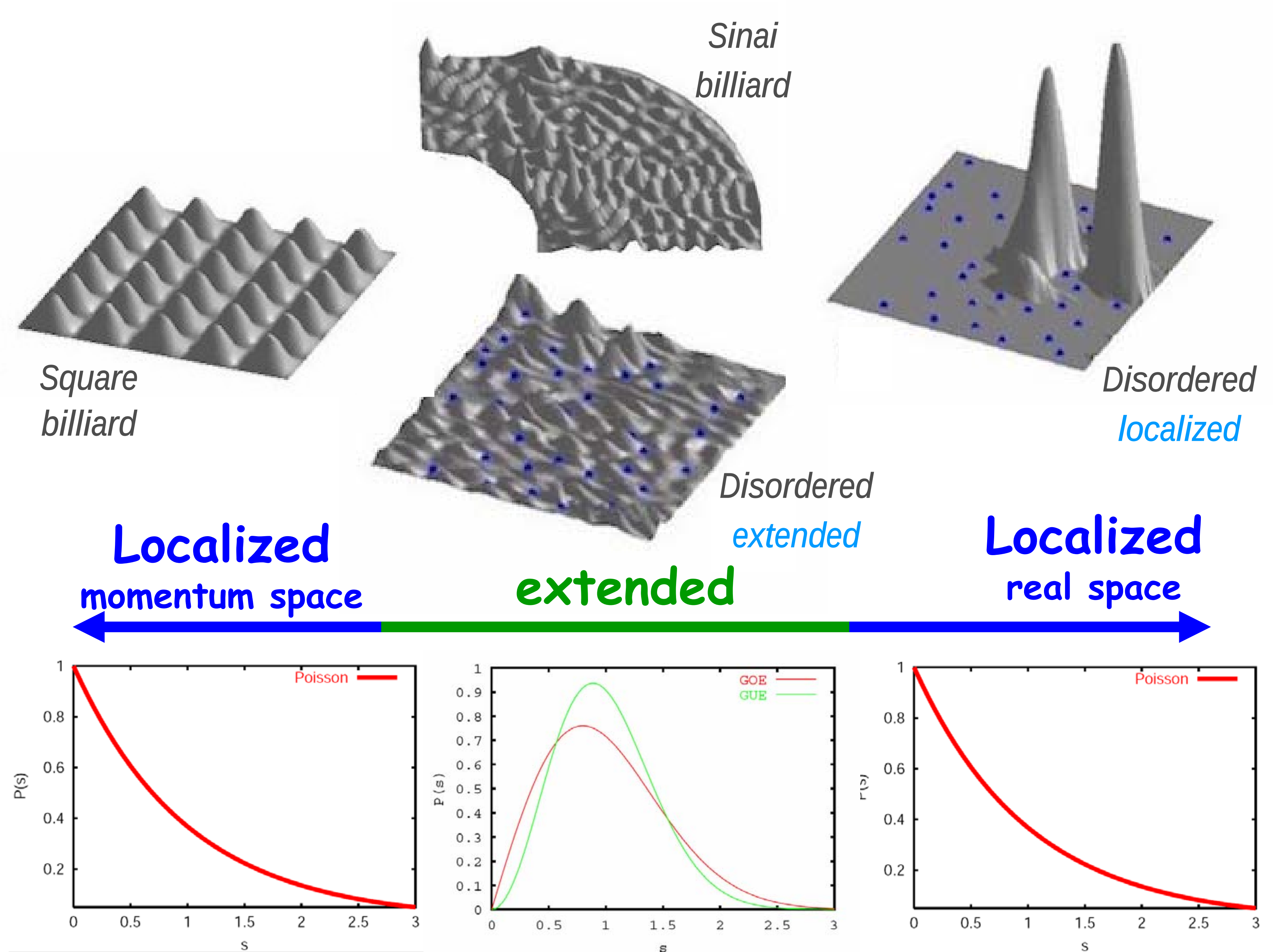
KAM perturbation is smooth enough

Consider an integrable system.
Each state is characterized by a set of quantum numbers.

It can be viewed as a point in the space of quantum numbers. The whole set of the states forms a lattice in this space.

A perturbation that violates the integrability provides matrix elements of the hopping between different sites (Anderson model !?)

Weak enough hopping:
    Localization - Poisson
Strong hopping:
    transition to Wigner-Dyson



Sinai
billiard
Square
billiard
Disordered
localized
Disordered
extended
Localized
momentum space
extended
Localized
real space
Poisson
P(s)
s
GOE
GUE
P (s)
s
Poisson
s

~~Strong disorder~~	~~localized~~
~~Weak disorder~~	~~extended~~

Strong disorder	localized
Moderate disorder	extended
No disorder chaotic	extended
No disorder integrable	localized
Too weak disorder int.	localized