

*Chapter 18***DISORDERED ELECTRON LIQUID
WITH INTERACTIONS**

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The metal–insulator transition (MIT) observed in a two-dimensional dilute electron liquid raises the question about the applicability of the scaling theory of disordered electrons, the approach pioneered by Phil Anderson and his collaborators,⁸ for the description of this transition. In this context, we review here the scaling theory of disordered electrons with electron–electron interactions. We start with the disordered Fermi liquid, and show how to adjust the microscopic Fermi-liquid theory to the presence of disorder. Then we describe the non-linear sigma model (NLSM) with interactions. This model has a direct relation with the disordered Fermi liquid, but can be more generally applicable, since it is a minimal model for disordered interacting electrons. The discussion is mostly about the general structure of the theory emphasizing the connection of the scaling parameters entering the NLSM with conservation laws. Next, we show that the MIT, as described by the NLSM with interactions, is a quantum phase transition and identify the parameters needed for the description of the kinetics and thermodynamics of the interacting liquid in the critical region of the transition. Finally, we discuss the MIT observed in Si-MOSFETs. We consider it as an example of the Anderson transition in the presence of the electron interactions. We demonstrate that the two-parameter RG equations, which treat disorder in the one-loop approximation but incorporate the full dependence on the interaction amplitudes, describe accurately the experimental data in Si-MOSFETs including the observed non-monotonic behavior of the resistance and its strong drop at low temperatures. The fact that this drop can be reproduced theoretically, together with the argument that Anderson localization should occur at strong disorder, justified the existence of the MIT within the scaling theory.

1. Disordered Fermi-Liquid: Dk^2 , ω , $T < 1/\tau_{el}$

The original Fermi-liquid theory has been formulated in terms of quasiparticles labeled with momenta \mathbf{p} . The most distinctive feature of the Fermi-liquid is the jump in the occupation number $n(\mathbf{p})$ at the Fermi-surface. Since in the presence of disorder the Fermi-surface is smeared, for some people this means the end of applicability of Fermi-liquid theory. This is, however, not completely correct. Indeed, the description in terms of plane waves is not working well for low-lying excitations with energies less than the rate of collisions with static impurities. Still, some elements of Fermi-liquid theory hold as far as rescattering of electron-hole pairs is considered. The Fermi-liquid description stops working only when the production of multiple electron-hole pairs becomes important [N.1].^a Elastic impurity scattering by itself does not generate electron-hole pairs and therefore, some elements of the Fermi-liquid description should be preserved even in the presence of disorder.

In this section, we show how to adjust the Fermi-liquid description to disordered electron systems.¹⁻³ While conventional Fermi-liquid theory has been constructed starting from single-particle excitations, in the case of a disordered Fermi-liquid, the focus shifts towards diffusing electron-hole pairs. In Landau's original microscopic theory of the clean Fermi-liquid the term $v_F \mathbf{n} \mathbf{k} / (\omega - v_F \mathbf{n} \mathbf{k})$ is used as the propagator of an electron-hole quasiparticle pair, see Chapter 2, Sec. 17 in Ref. 4. This expression describes propagation of the pair along the direction \mathbf{n} , when the momentum difference of the two quasiparticles is \mathbf{k} , and the frequency difference is ω . The combination $v_F \mathbf{n} \mathbf{k}$ originates from the energy difference of the constituents of the pair, $\delta \epsilon_k(\mathbf{p}) = \epsilon(\mathbf{p} + \mathbf{k}) - \epsilon(\mathbf{p}) \approx v_F \mathbf{n} \mathbf{k}$. The two quasiparticle poles sitting close-by make the discussed term singular. This in turn makes the two-particle vertex function $\Gamma(\omega, \mathbf{k})$ singular since it describes, among other processes, multiple rescattering of electron-hole pairs. The propagator $v_F \mathbf{n} \mathbf{k} / (\omega - v_F \mathbf{n} \mathbf{k})$ may be rewritten as the sum of a static and a dynamic part: $[-1 + \omega / (\omega - v_F \mathbf{n} \mathbf{k})]$. In fact, it is more convenient to keep explicitly only the *dynamic* part of this propagator, $\omega / (\omega - v_F \mathbf{n} \mathbf{k})$, and to delegate the static part (i.e., -1) to the amplitude of the electron-electron ($e-e$) interaction. This amplitude is denoted as Γ^k . Index k in Γ^k means that in the singular amplitude $\Gamma(\mathbf{k}, \omega)$ one first takes the limit $\omega = 0$ and only afterwards the limit $k \rightarrow 0$, i.e., $\Gamma^k = \Gamma(\mathbf{k} \rightarrow 0, \omega = 0)$. The choice to work with the *static* amplitude Γ^k is motivated by the following reasoning. Generally speaking, $\Gamma(\mathbf{k}, \omega)$ includes:

^aSee the list of Notes which follows the main text.

(i) a part irreducible with respect to particle–hole pair propagators (the diagram for such an amplitude cannot be separated into disconnected blocks by cutting two single-particle Green’s functions only), along with contributions from incoherent background, and (ii) the contributions containing rescattering of quasi-particle pairs that has been already mentioned. The contributions from the irreducible part and from incoherent background are determined by short scales. Therefore, they are robust and, apart from small corrections, not sensitive to modifications of the electron spectrum near the Fermi-energy ϵ_F . On the contrary, the terms describing rescattering of quasi-particle pairs are fragile, and they require certain care. Remarkably, the amplitude Γ^k is also insensitive to a modification of the low-energy part of the energy spectrum unless the density of states changes significantly. Indeed, in the considered order of limits the combination $\delta\epsilon/(\omega - \delta\epsilon)$ is equal to -1 for any energy spectrum of electrons. These arguments led us to conclude^{1,3} that the amplitude Γ^k is not influenced by not too strong disorder, $1/\tau_{el} \lesssim \epsilon_F$. The robustness of the static amplitude Γ^k makes it particularly convenient for the purpose of a microscopic analysis in the presence of disorder.

It is almost evident from the discussion above that disorder reveals itself most clearly in dynamics. Diagrammatically, the dynamic part of the particle–hole propagator can be obtained from a product of two Green’s functions where one is retarded (R), while the other one is advanced (A). We will refer to such a product as RA -section. After integration over the energy variable $\xi = p^2/2m^* - \mu$, and summation over the fermionic frequency ϵ_n , the RA -section generates just the dynamic part of the electron–hole propagator, $\omega_n/(\omega_n + iv_F\mathbf{n}\mathbf{k})$. [From now on, we prefer to use Matsubara frequencies for which $\omega/(\omega - v_F\mathbf{n}\mathbf{k}) \implies \omega_n/(\omega_n + iv_F\mathbf{n}\mathbf{k})$]. In the presence of disorder, the dynamic part of the propagator given by an RA -section changes its functional form. When the effective collisions of electrons with impurities are frequent enough (in the sense of inequalities given in the section head) multiple impurity scattering leads to the diffusive propagation of the quasi-particles for times $t \gg \tau_{el}$; here τ_{el} is the elastic mean free time for scattering from static impurities. Under these circumstances, the propagator of an electron–hole pair changes in such a way that its denominator acquires the diffusive form:

$$\frac{\omega_n}{\omega_n + iv_F\mathbf{n}\mathbf{k}} \implies \frac{\omega_n}{\omega_n + Dk^2}. \quad (1.1)$$

Here, $D = v_F^2\tau_{el}/d$ is the diffusion coefficient for the spatial dimension d [N.2]. With this result at hand, let us consider the two-particle amplitude $\Gamma(\mathbf{k}, \omega)$. The amplitude $\Gamma(\mathbf{k}, \omega)$ can be represented as a series in which Γ^k

and RA -sections alternate with each other [N.3]. In a sense, in disordered systems, the process of multiple rescattering is even simpler than in clean ones. The point is that in the clean Fermi-liquid, because of the angular dependence contained in \mathbf{nk} , the angular harmonics of the interaction amplitudes, denoted as Γ_l^k , come into play. On the other hand, for the slow propagation of an electron-hole pair in the presence of disorder only the zeroth harmonic, $l = 0$, remains singular. Consequently, only the zeroth harmonic of the interaction amplitude, $\Gamma_{l=0}^k$, is relevant for the processes of rescattering of diffusing electron-hole pairs. As a result, the calculation of $\Gamma(\mathbf{k}, \omega)$ reduces to a geometric series.

The two-particle amplitude $\Gamma_{l=0}^k$ can be split into parts which can be classified according their spin structure:

$$\begin{aligned} \nu a^2 \Gamma_{l=0}^k \begin{matrix} \alpha_1 \alpha_2 \\ \alpha_3 \alpha_4 \end{matrix} &= \tilde{\Gamma}_1 \delta_{\alpha_1, \alpha_3} \delta_{\alpha_2, \alpha_4} - \Gamma_2 \delta_{\alpha_1, \alpha_2} \delta_{\alpha_3, \alpha_4} \\ &= \frac{1}{2} [(2\tilde{\Gamma}_1 - \Gamma_2) \delta_{\alpha_1, \alpha_3} \delta_{\alpha_2, \alpha_4} - \Gamma_2 \vec{\sigma}_{\alpha_1, \alpha_3} \vec{\sigma}_{\alpha_2, \alpha_4}]. \end{aligned} \quad (1.2)$$

Here, ν is the single-particle density of states per one spin component at energy ϵ_F , and the factor a describes the weight (residue) of the quasi-particle part in Green's function $\mathcal{G}(i\epsilon, \mathbf{p})$ [N.4]; $\Gamma_{1,2}$ are dimensionless. The minus sign in the amplitude Γ_2 is due to the anti-commutation of the fermionic operators. The two-particle propagators can be classified in terms of the total spin of the particle-hole pairs. The combination $\tilde{\Gamma}_\rho = 2\tilde{\Gamma}_1 - \Gamma_2$ operates inside the singlet channel, $S = 0$, and controls propagation of the particle-number density $\rho(\mathbf{k}, \omega)$, while $\Gamma_\sigma = -\Gamma_2$ controls the spin density, i.e., the triplet channel, $S = 1$ [N.5]. To obtain the amplitude $\Gamma(\mathbf{k}, \omega)$, one has to sum, depending on the spin structure, a ladder of either $\tilde{\Gamma}_\rho$ or Γ_σ with RA -sections in between; see Fig. 18.1. The resulting amplitudes $\tilde{\Gamma}_\rho(\mathbf{k}, \omega)$ and $\Gamma_\sigma(\mathbf{k}, \omega)$ acquire the form:

$$\Gamma_\alpha(\mathbf{k}, \omega) = \Gamma_\alpha \frac{Dk^2 + \omega_n}{Dk^2 + (1 - \Gamma_\alpha)\omega_n}, \quad \alpha = \rho, \sigma. \quad (1.3)$$

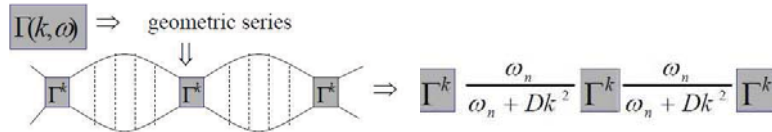


Fig. 18.1. Disordered Fermi-liquid: geometric series leading to Eq. (1.3). Dashed lines describe impurity scattering.

Note the shift of the position of the diffusion poles in $\Gamma_{\rho,\sigma}(k,\omega)$ as a result of summation of the geometric series. Eventually, this shift is the origin of the renormalization of the diffusion coefficients in the disordered electron liquid.

Now that we know how disorder affects the amplitudes of the e - e interactions, we may study correlation functions. As we shall see, the conservation of particle-number (i.e., charge) and spin constrains the possible form of the corresponding correlation functions. We first consider the polarization operator $\Pi(k,\omega_n)$ in the presence of disorder. We discuss here a true electron liquid, i.e., a quantum liquid with *charged* current carriers. This is the reason why we are interested in $\Pi(k,\omega_n)$, the density-density correlation function irreducible with respect to the Coulomb interaction. To obtain this irreducible part, one has to exclude from $\tilde{\Gamma}_1$ all terms that can be disconnected by cutting a single line of the Coulomb interaction. As a result of this operation, $\tilde{\Gamma}_1$ transforms into Γ_1 and correspondingly, $\tilde{\Gamma}_\rho$ transforms into $\Gamma_\rho = 2\Gamma_1 - \Gamma_2$. (The part of $\tilde{\Gamma}_\rho$ that can be disconnected by cutting a single line will be denoted as Γ_ρ^0 , while the amplitude Γ_ρ incorporates the remaining irreducible part, so that $\tilde{\Gamma}_\rho = \Gamma_\rho^0 + \Gamma_\rho$.) As we shall see, the dimensionless parameters, Γ_ρ and Γ_σ , determine the Fermi-liquid renormalizations of the disordered charged liquid. The separation of the polarization operator $\Pi(k,\omega_n)$ into static and dynamic parts is performed in the same way as for $\Gamma(k,\omega_n)$: the static part does not contain RA -section, while all the rest goes to the dynamic part. Consequently, the dynamic part of the polarization operator contains two “triangle” vertices γ^ρ separated by a ladder of the RA -sections, see Fig. 18.2. In other words, both the left and right vertices γ^ρ are irreducible with respect to RA -sections (i.e., each of them extends from an external vertex to the first RA -section) [N.6]. Collecting the static and dynamic parts of the polarization operator, one gets

$$\Pi(k,\omega_n) = \Pi_{st} - 2\nu(\gamma^\rho)^2 \left[\frac{\omega_n}{Dk^2 + (1 - \Gamma_\rho)\omega_n} \right] \quad (1.4a)$$

$$\implies \Pi_{st} \frac{Dk^2}{Dk^2 + (1 - \Gamma_\rho)\omega_n} . \quad (1.4b)$$

The transition between the two lines will be commented upon below.

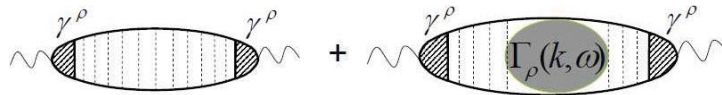


Fig. 18.2. Disordered Fermi-liquid: dynamic part of the polarization operator $\Pi(k,\omega_n)$.

In Eq. (1.4b), we have arrived at the canonical form for a correlation function of the density of any conserved quantity. This form recovers not only the static limit when $\omega_n = 0$ but, most remarkably, it vanishes in the limit $k \rightarrow 0$ when $\omega_n \neq 0$. Let us explain why such a vanishing necessarily occurs for any *conserved* quantity. The general form of a retarded correlation function is

$$\chi(\mathbf{k}, \omega) = i \int_0^\infty dt e^{i\omega t} \langle [x(t), x(0)] \rangle_{\mathbf{k}}. \quad (1.5)$$

In the limit $k \rightarrow 0$ the densities $x(t)$ and $x(0)$ transform into the quantities, $X(t)$ and $X(0)$, that are integrated over the space. In the case when $X(t)$ is conserved in time, it obviously commutes with $X(0)$ at any moment. Consequently, $\chi(k \rightarrow 0, \omega)$ should vanish at any frequency.

It remains to show how the expression (1.4b) follows from Eq. (1.4a). A specific cancellation between the static and dynamic parts of $\Pi(k, \omega_n)$ leading to the desired form of Eq. (1.4b) takes place only if the following relation is fulfilled:

$$\Pi_{st} = 2\nu (\gamma^\rho)^2 \frac{1}{(1 - \Gamma_\rho)}. \quad (1.6)$$

As is well known, the static part of the polarization operator, $\Pi_{st} = \Pi(k \rightarrow 0, \omega_n = 0)$, reduces to the thermodynamic quantity $\partial n / \partial \mu$, which is related to the compressibility and is also responsible for linear screening in the electron gas. Under the approximation of a constant density of states [N.7], $\partial n / \partial \mu$ is not sensitive to the disorder, if $1/\tau_{el} \ll \epsilon_F$. The point is that, generally speaking, μ can be measured with respect to an arbitrary energy, i.e., it can be shifted by an arbitrary value. For some quantity to be sensitive to a variation of μ , the chemical potential should be tied to a certain physical energy level which can serve as a reference point. In the discussed problem, the only special energy-level is the bottom of the conduction band. It is clear, however, that the information about moderate disorder cannot extend from ϵ_F up to the bottom of the band. Therefore, $\partial n / \partial \mu$ is not changed by disorder, unless it is very strong. Next, since the vertex γ^ρ is also connected with the derivative of the Green function with respect to the chemical potential, $\partial \mathcal{G} / \partial \mu$ [N.8], the arguments concerning insensitivity of $\partial n / \partial \mu$ to disorder remain valid for this quantity as well. Thus, we may use for $\Pi_{st} = \partial n / \partial \mu$ and γ^ρ their values known from the Fermi-liquid theory in the clean limit.

The Fermi-liquid theory connects $\partial n / \partial \mu$ with the Fermi-liquid parameter F_0^ρ as follows: $\partial n / \partial \mu = 2\nu / (1 + F_0^\rho)$, see Chapter 2, Sec. 2 in Ref. 4. Then, the relation connecting F_0^ρ with Γ_ρ yields $\Pi_{st} = 2\nu / (1 + F_0^\rho) = 2\nu(1 -$

Γ_ρ). Furthermore, it is known from identities for derivatives of the Green functions (see Chapter 2, Sec. 19 in Ref. 4 and Refs. 5 and 6) that $\gamma^\rho = (1 - \Gamma_\rho)$. Thus, the necessary relation holds with, one may say, excessive strength: $\Pi_{st}/2\nu = \gamma^\rho = (1 - \Gamma_\rho)$.

One may rewrite the expression given in Eq. (1.4b) in the more conventional form corresponding to diffusion (see also [N.2]):

$$\Pi(k, \omega_n) = \Pi_{st} \frac{D_\rho k^2}{D_\rho k^2 + \omega_n} \quad D_\rho = \frac{D}{1 - \Gamma_\rho}. \quad (1.7)$$

Here, D_ρ is the diffusion coefficient of the particle-number density ρ . The next step is to relate $\Pi(k, \omega_n)$ through the continuity equation, $\partial\rho/\partial t + \text{div}\mathbf{j} = 0$, to the current-current correlation function. Then, with the help of the Kubo formula, one can obtain the Fermi-liquid expression for the electric conductivity (e is electron charge):

$$\begin{aligned} \frac{\sigma_{\text{charge}}}{e^2} &= \lim_{k \rightarrow 0} \frac{\omega_n}{k^2} \Pi(k, \omega_n) = \\ &= \frac{\partial n}{\partial \mu} D_\rho = 2\nu D. \end{aligned} \quad (1.8)$$

The above equation is nothing else but the Einstein relation for the electric conductivity σ ; $\sigma \equiv \sigma_{\text{charge}}$. It is worth emphasizing that the product $(\partial n/\partial \mu)D_\rho$ is equal to σ/e^2 rather than $(\partial n/\partial \mu)D$. This point is very important in view of contemporary experiment in heterostructures hosting two-dimensional ($2d$) electron gas. In these systems the electron gas is often studied under conditions when $\partial n/\partial \mu$ becomes negative, i.e., $1/(1 + F_0^\rho) = (1 - \Gamma_\rho) < 0$. However, as we have observed, in σ the two negative renormalizations exactly cancel each other, so that conductivity is unquestionably positive [N.9].

It is worth mentioning that the arguments presented above about the insensitivity of $\partial n/\partial \mu$ as well as γ^ρ to disorder are not restricted to the Fermi-liquid. Under the approximation of a constant density of states, $\partial n/\partial \mu$ and γ^ρ are not changed even if one goes beyond the framework of the disordered Fermi-liquid theory; we will come back to this point later.

The scheme outlined above can be straightforwardly applied for the analysis of the spin-density correlation function.^{2,7} We will now rely on the arguments that lead us to the conclusion that the static amplitude Γ^k is not affected by moderately strong disorder. Actually, these arguments carry over to any static Fermi-liquid parameter. In the discussed case, the external vertices contain a spin operator $\sigma^x/2$ that corresponds to a probing magnetic field directed along x -axis. These vertices are renormalized by the $e-e$

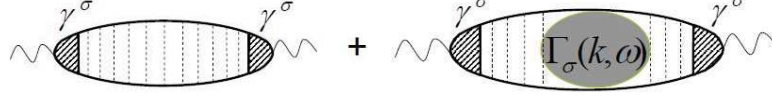


Fig. 18.3. Disordered Fermi-liquid: dynamic part of the spin-density correlation function $\chi_s(k, \omega_n)$.

interactions, and the corresponding renormalization factor is denoted below as γ^σ , see Fig. 18.3. In spite of this modification, all formulas are similar to those obtained in the case of $\Pi(k, \omega_n)$. The only needed change is to substitute in the above expressions Γ_ρ by Γ_σ . The spin susceptibility χ_s determines the static limit of the spin-density correlation function $\chi_s^{xx}(k, \omega_n)$, just like $\partial n / \partial \mu$ determines the static limit of the polarization operator. The spin susceptibility χ_s is modified by the Stoner factor equal to $(1 - \Gamma_\sigma)$. As a result, $\chi_s = \chi_s^{xx}(k \rightarrow 0, \omega_n = 0) = \chi_s^0(1 - \Gamma_\sigma) = (g\mu_B/2)^2 2\nu(1 - \Gamma_\sigma)$. The vertex γ^σ is equal to the same renormalization factor, $\gamma^\sigma = (1 - \Gamma_\sigma)$. As a result, the sum of the static and dynamic parts acquires the structure already familiar from the calculation of $\Pi(k, \omega_n)$:

$$\begin{aligned} \chi_s^{xx}(k, \omega) &= \chi_s^0(1 - \Gamma_\sigma) \frac{Dk^2}{Dk^2 + (1 - \Gamma_\sigma)\omega_n} \\ &= \chi_s \frac{D_\sigma k^2}{\omega_n + D_\sigma k^2}, \end{aligned} \quad (1.9)$$

where $D_\sigma = D/(1 - \Gamma_\sigma)$. Note that Γ_σ is connected with the standard Fermi-liquid parameter F_0^σ as follows: $\Gamma_\sigma = F_0^\sigma / (1 + F_0^\sigma)$. Usually F_0^σ is negative. Then, $(1 - \Gamma_\sigma)$ describes the Stoner enhancement of the spin susceptibility due to the e - e interaction, as well as the suppression of the spin-diffusion coefficient $D_\sigma = D/(1 - \Gamma_\sigma)$.

Since we discuss the case when spin is conserved, we may now derive the Einstein relation for the spin-density current by following the route outlined previously for the electric conductivity, see Eq. (1.8):

$$\begin{aligned} \frac{\sigma_{\text{spin}}}{(\mu_B/2)^2} &= \frac{1}{(g\mu_B/2)^2} \lim_{k \rightarrow 0} \frac{\omega_n}{k^2} \chi_s^{xx}(k, \omega_n) \\ &= 2\nu(1 - \Gamma_\sigma)D_\sigma = 2\nu D. \end{aligned} \quad (1.10)$$

Taken together, Eqs. (1.8) and (1.10) reflect the fact that both the charge and the spin are carried by the same particles.

Conclusion: The theory of the disordered Fermi-liquid focuses on diffusing electron-hole pairs. In the diffusion regime, i.e., for temperatures (frequencies) less than the elastic scattering rate, $T \lesssim 1/\tau_{el}$, diffusion modes and

not quasi-particles are the low lying propagating modes. The conservation of particle-number (i.e., charge) and the conservation of spin constrains the possible form of the corresponding correlation functions. Besides ν and D , the theory contains two dimensionless parameters, Γ_ρ and Γ_σ , which describe Fermi-liquid renormalizations in the charge- and spin-density channels, respectively.

2. Beyond Fermi-Liquid Theory: Non-Linear Sigma Model and Renormalized Fermi-Liquid Theory

Let us explain why the theory of the disordered Fermi-liquid discussed above is incomplete. Obviously, the expression for the diffusion coefficient, $D = v_F^2 \tau_{el} / d$, has to be modified by the interference (weak localization) corrections, which in $d = 2$ are logarithmic.^{8,9} This, by itself, does not affect the described above structure of the Fermi-liquid, and could easily be repaired. However, there is a number of other effects, which demand certain care. *Up to now*, averaging over disorder both in the polarization operator and the spin-density correlation function has been performed in a very particular fashion. Namely, in the ladders given in Figs. 18.2 and 18.3, the interaction amplitudes and disorder-averaged propagators appear in separate blocks. In fact, matrix elements determining amplitudes of the e - e interaction are seriously modified by disorder, especially for states that are close in energy. Two examples showing how it happens *after averaging* over disorder are presented in Fig. 18.4. One can see from these examples that ladder-diagram propagators describing diffusion of electron-hole pairs play a special role in modifying (renormalizing) the interaction amplitudes. Such propagators, see Fig. 18.5, contain a diffusion pole and are, therefore, called diffusion modes or just “diffusons”. Technically, diffusion modes participating in the processes similar to those shown in Fig. 18.4 have to be integrated over their momentum \mathbf{q} within the interval determined by $1/\tau_{el} > Dq^2 \gtrsim T$.

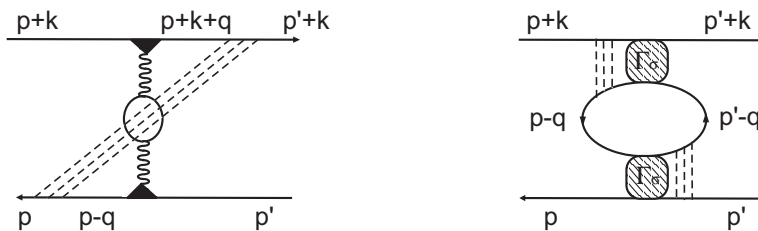


Fig. 18.4. Examples of the e - e interaction amplitudes modified by disorder.

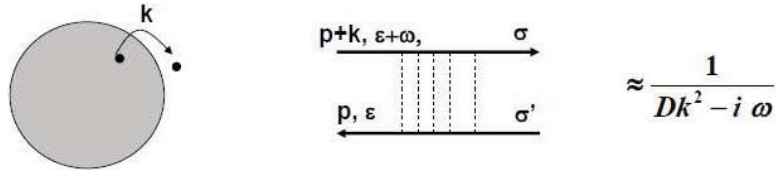


Fig. 18.5. Diffuson: disorder-averaged propagator of an electron-hole pair. These propagators capture the diffusive evolution of the quasiparticles at large times and length scales.

The scattering rate $1/\tau_{el}$ acts as a high-energy cutoff because only states with energy/frequency less than $1/\tau_{el}$ are relevant in the diffusive regime. On the other hand, temperature always enters as a low-energy cutoff in the effects related to the $e-e$ interactions, because it determines smearing of the *energy* distribution of electrons [N.10]. (We emphasize energy, because momentum-smearing of single-particle states is already irrelevant when we use description in terms of diffusons.) As a result of the outlined integrations over the momenta [N.11], amplitudes of the $e-e$ interaction acquire corrections that are non-analytic in temperature.^{1,2,7}

As is well known,^{10,11} the electric conductivity and, correspondingly, the diffusion constant D also acquire corrections (that are non-analytic in temperature) due to the combined action of the $e-e$ interaction and disorder in the diffusive regime [N.12]. Two diagrams illustrating the origin of the effect are shown in Fig. 18.6.

In addition, there are corrections to conductivity due to the interference processes determined by “cooperon” modes. Diagrammatically, cooperons are described by a disorder-averaged particle–particle propagator with small total momentum of the scattering particles, see Fig. 18.7. These propagators also contain a diffusion pole [N.13]. In $d = 2$, *all* corrections both to the electric conductivity and interaction amplitudes are logarithmically divergent in

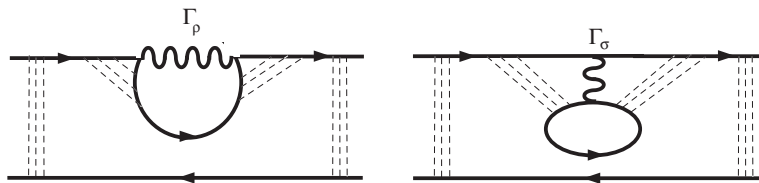


Fig. 18.6. Diagrams illustrating the origin of corrections to the diffusion constant D due to combined action of the $e-e$ interactions and disorder.

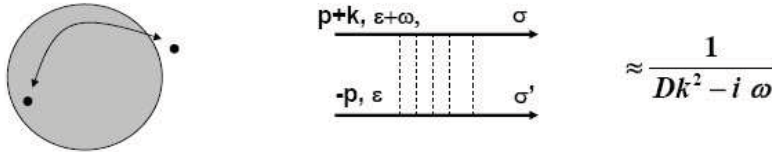


Fig. 18.7. Cooperon: disorder-averaged propagator of a particle–particle pair with small total momentum. The cooperons capture the effects of quantum interference which lead to the weak-localization corrections.

temperature, i.e., $\propto \ln 1/T\tau_{el}$ [N.14]. In higher dimensions, the problem becomes logarithmic near the metal–insulator transition, and it still contains several running parameters. The corrections divergent in temperature signal the breakdown of perturbation theory, and the need for a resummation of the divergent terms.

We reached an important point: Diffusing electrons dwell long in each other’s vicinity, becoming more correlated at low enough energies. As a result, the e – e scattering amplitudes Γ_ρ and Γ_σ characterizing the interaction of diffusion modes acquire corrections, which are the more significant the stronger disorder is. Conversely, resistivity — which is a measure of the effective strength of disorder — in its turn also gets corrections which depend on the value of the interaction amplitudes. We see that one needs a scheme that can account for both effects in a self-consistent fashion. Such a scheme is the renormalization group (RG) analysis of the problem. The RG-theory applied to the dirty Fermi-liquid system is able to capture scale dependences originating from the interplay of disorder and interactions to all orders in the interaction amplitudes, making it a highly effective analytical tool for understanding the physics of the metal–insulator transition in disordered electron systems. Pedagogical reviews of the RG-theory can be found in Refs. 3, 12–14. For more recent advances, see Refs. 15 and 16.

The RG-analysis of the disordered electron liquid is best described by the matrix non-linear sigma model (NLSM).^{17–20} In matrix terms, the disorder-averaged N -replica partition function of the interacting electrons reads as follows^{1–3}:

$$\langle Z_N \rangle = \int dQ e^{-S[Q]}, \quad (2.1)$$

$$S[Q] = \frac{\pi}{8} \int d^d r \nu \text{Tr} [D(\nabla \hat{Q})^2 - 4z(\hat{\epsilon} \hat{Q}) - \frac{\pi}{16} \int d^d r \nu \{ \hat{Q}(\hat{\Gamma}_\rho^0 + \hat{\Gamma}_\rho) \hat{Q} + \hat{Q} \hat{\Gamma}_\sigma \hat{Q} + \hat{Q} \hat{\Gamma}_c \hat{Q} \}. \quad (2.2)$$

Here, the functional integration has to be performed over an auxiliary matrix field \hat{Q} within the manifold limited by the constraints: $\hat{Q}^2 = 1$, $\hat{Q} = \hat{Q}^\dagger$, and $\text{Tr } \hat{Q} = 0$. These constraints make the problem non-linear as well as very non-trivial. The components of \hat{Q} are defined as $Q_{n_1, n_2}^{ij, \alpha\beta}$, where n_1, n_2 are the Matsubara fermionic energy indices with $\epsilon_n = (2n + 1)\pi T$; i, j are the replica indices, and α, β include the spin and quaternion indices. The quaternion indices are needed to incorporate both the diffuson and cooperon modes. The trace is taken over all these variables. Eventually, the replica limit, $N \rightarrow 0$, should be performed [N.15]. The frequency matrix $\hat{\epsilon} = \epsilon_n \delta_{nm} \delta_{ij} \delta_{\alpha\beta}$. All the interaction terms are restricted by the energy and momentum conservation laws. The interaction terms are written symbolically omitting such important details as the Pauli matrices acting in the spaces of spin and quaternion degrees of freedoms; the description of the matrix structure can be found in Refs. 2 and 3. Note that the ρ -term is split into two pieces: The part that can be disconnected by cutting a single line of the Coulomb interaction is denoted as $\hat{\Gamma}_\rho^0$, while the term $\hat{\Gamma}_\rho$ incorporates the irreducible part; see previous discussion in connection with the polarization operator. The additional term $\hat{\Gamma}_c$ describes the interaction in the Cooper channel. The interaction amplitudes $\Gamma_{\alpha=\rho, \sigma, c}$ are dimensionless, but elements of the forms $(\hat{Q} \hat{\Gamma}_a \hat{Q})$ contain a factor of $2\pi T$ which appears as a result of the discrete Fourier transform from the Matsubara time to frequency [N.16]. Last point to be commented is the parameter z introduced in front of the frequency matrix in the action $S[Q]$. For free electrons, $z = 1$; also in the course of the Fermi-liquid analysis of a disordered electron liquid we have not met it so far. As we shall see soon, this parameter is absolutely needed to make the RG-procedure compatible with the charge- and spin-conservation laws. Moreover, since z determines the relative scaling of the frequency with respect to the length scale,²¹ it plays a central role for both kinetic and thermodynamic quantities in the critical region of the metal-insulator transition.

The equilibrium (i.e., saddle-point) value of the matrix \hat{Q} , usually denoted as $\hat{\Lambda}$, is fixed by the frequency term ($\hat{\epsilon} \hat{Q}$) in the above action; $\Lambda_{n, m}^{ij, \alpha\beta} = \text{sign} n \delta_{nm} \delta_{ij} \delta_{\alpha\beta}$. It is clear that for small ϵ_n the strength of the fixation of \hat{Q} along the equilibrium position is weak and, correspondingly, fluctuations are strong [N.17]. The fluctuations of the Q -field are nothing else but diffusons and cooperons. Their propagators, $\mathcal{D}(k, \omega_n) = 1/(Dk^2 + z\omega_n)$, can be obtained by expansion of the first two terms in $S[Q]$ up to quadratic order in $\delta\hat{Q} = (\hat{Q} - \hat{\Lambda})$. These two terms yield a diffusion-like singularity in the propagators $\mathcal{D}(k, \omega_n)$ [N.18]. Furthermore, with the use of quadratic

expansion of the $\hat{\Gamma}$ -terms in $\delta\hat{Q}$, one can reproduce the scattering amplitudes $\Gamma_{\rho,\sigma}(k, \omega)$ given by Eq. (1.3) and illustrated in Fig. 18.1; see [N.19].

The functional $S[Q]$ describes disordered interacting electrons with energies less than $1/\tau_{el}$. The coefficients in the action $S[Q]$ incorporate the Fermi-liquid renormalizations of the clean liquid as the input parameters. One may look on this from the RG point of view: “integrating out” the high-energy states till the energy interval $\sim 1/\tau_{el}$ around the Fermi-level leads to the Fermi-liquid renormalizations. The next step is integrating out the interval starting from $1/\tau_{el}$ down to temperature, which should result in the “true” RG-descriptions.

One may conclude that the Fermi-liquid description of the disordered electron liquid is given by the quadratic expansion of the action $S[Q]$ in deviations of \hat{Q} from its equilibrium value. As to the renormalization corrections, they are determined by non-quadratic (i.e., anharmonic) terms in the action. The disorder-averaged interaction amplitudes, the diffusion coefficient D as well as the parameter z , are all scale-dependent at low energies $\lesssim 1/\tau_{el}$. Note that splitting into independent channels occurs only on the level of the quadratic form of the action. During the course of the RG-procedure different channels mix [N.20].

The parameter z gives the frequency renormalization in the propagators of the diffusion and cooperon modes. To some extent, z is similar to $(1 - \partial\Sigma/\partial\epsilon)$ in the single-particle Green function $\mathcal{G}(i\epsilon, \mathbf{p})$. There is an important difference, however. According to the Migdal theorem, the combination $(1 - \partial\Sigma/\partial\epsilon)^{-1} = a$ is proportional to the jump of the occupation numbers $n(\mathbf{p})$ at the Fermi-surface; see Chapter 2, Sec. 10 in Ref. 4. This fact constrains $(1 - \partial\Sigma/\partial\epsilon)$ to be larger than 1. On the contrary, the frequency renormalization factor in the two-particle propagators is not constrained, and z may be both smaller and larger than 1. It is known that $z < 1$ in systems with magnetic impurities,²¹ in a spin-polarized system,¹⁵ or in the presence of the spin-orbit scattering.²² Only in the generic case of a purely potential impurity scattering when the spin degrees of freedom are not constrained, $z > 1$.

With the frequency renormalization parameter z being included, the action $S[Q]$ preserves its form in the course of RG-transformations. We, thereby, may come back to the discussion of a density-correlation function of a conserved quantity. The analysis includes a few steps: one has to find (i) the RG-modified static part of the correlation function, (ii) the renormalized triangle vertex, and then (iii) with the use of the quadratic expansion of already renormalized action $S[Q]$ to find the dynamic part of the correlation function. Performing all these steps, one will get the expressions similar to

those given in Eqs. (1.4b) and (1.9):

$$\begin{aligned}\chi_a(k, \omega) &= \chi_a^{\text{static}} \frac{Dk^2}{Dk^2 + (z - \Gamma_a) \omega_n} \\ &= \chi_a^{\text{static}} \frac{D_a k^2}{\omega_n + D_a k^2}, \quad \alpha = \rho, \sigma.\end{aligned}\quad (2.3)$$

Here, the coefficients of diffusion $D_a = D/(z - \Gamma_a)$. For $\chi_a(k, \omega)$ to acquire this form, a relation similar to the discussed above in Eq. (1.6) has to be fulfilled for the renormalized values of χ_a^{static} , γ^a and $(z - \Gamma_a)$:

$$\frac{\chi_a^{\text{static}}}{\chi_a^0} = \frac{(\gamma^a)^2}{(z - \Gamma_a)}. \quad (2.4)$$

Let us consider how it works for the polarization operator and spin-density correlation function. As we have already explained, the static limit of the polarization operator as well as γ^ρ are not changed by disorder. Therefore, the amplitude Γ_ρ and the parameter z are renormalized in such a way that

$$z - \Gamma_\rho = \frac{1}{1 + F_0}. \quad (2.5)$$

This, by the way, implies that the relation $\sigma/e^2 = 2\nu D$, see Eq. (1.8), still holds even in the course of the renormalizations. In the case of the spin-density correlation function, the RG-calculation yields^{2,7}:

$$\frac{\chi_\sigma^{\text{static}}}{\chi_\sigma^0} = \gamma^\sigma = z - \Gamma_\sigma. \quad (2.6)$$

One observes that the condition of Eq. (2.4) is indeed fulfilled, although these relations carry more information than would be needed for one relation. One may notice, however, that together these relations make the charge and spin conductivities equal to each other:

$$\frac{\sigma_{\text{spin}}}{(\mu_B/2)^2} = 2\nu(z - \Gamma_\sigma)D_\sigma = 2\nu D. \quad (2.7)$$

As we have already mentioned, this is a manifestation of the fact that charge and spin are transported by the same carriers.

Finally, let us turn back to the amplitude Γ_ρ^0 which carries information about the screened Coulomb interaction. In view of the singular behavior of the Fourier component $V_C(k)$ at small momenta, this part of the interaction is equal to

$$\Gamma_\rho^0 = 2\nu \frac{(\gamma^\rho)^2}{\Pi_{st}} = \frac{1}{1 + F_0}. \quad (2.8)$$

$$\Gamma_{\rho}^0 = \begin{array}{c} \gamma^{\rho} \quad \gamma^{\rho} \\ \blacktriangleleft \text{---} \text{---} \text{---} \blacktriangleright + \blacktriangleleft \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \blacktriangleright \\ \text{---} \text{---} \text{---} \end{array} + \dots$$

Fig. 18.8. The screened Coulomb interaction. The triangular vertices γ^{ρ} are attached to the ending points of the interaction line.

Here $(\gamma^{\rho})^2$ originates from attaching triangular vertices γ^{ρ} to the ending points of the screened Coulomb interaction, see Fig. 18.8. Now, Eq. (2.5) can be rewritten as

$$z = \Gamma_{\rho}^0 + \Gamma_{\rho}. \quad (2.9)$$

Thus, in $S[Q]$ the interaction amplitude in the density-channel can be substituted by z .^{1,21} This implies that for unitary class systems where only fluctuations in the density channel are important (e.g., when magnetic scattering is present or in the case of spin-polarized electrons) the theory, apart from D , contains only one scaling parameter [N.21]. In other words, the theory of the electron gas interacting via the Coulomb interaction displays a high degree of universality. Pruisken and his coauthors connected this universality to a global symmetry of the problem which they called \mathcal{F} -invariance²³ and which is intimately related to gauge invariance.

Now that the structure of the theory has been established, it is useful to regroup its parameters by combining the frequency renormalization parameter z together with ν .² Then, z acquires the physical meaning of a parameter renormalizing the density of states of the diffusion modes, while $D_Q = D/z$ can be interpreted as the diffusion coefficient of the diffusion-mode ‘‘quasi-particles’’:

$$\nu \implies z\nu, \quad D \implies D_Q = D/z. \quad (2.10)$$

It is natural to link $z\nu$ to the coefficient determining the specific heat c_V .^{24,25} Furthermore, the Einstein relation, the renormalized susceptibilities, as well as the diffusion coefficients describing the evolution of the charge- and spin-densities at large scales, all acquire the form of the Fermi-liquid theory albeit with the renormalized coefficients equal to $(1 - \Gamma_a/z)$:

$$\sigma/e^2 = 2(\nu z)D_Q; \quad (2.11)$$

$$D_a = \frac{D_Q}{(1 - \Gamma_a/z)}, \quad \alpha = \rho, \sigma; \quad (2.12)$$

$$\chi_a^{\text{static}} = z\nu(1 - \Gamma_a/z)(\chi_a^0/\nu), \quad C_V/T = z\nu. \quad (2.13)$$

Here, χ_a^0/ν are factors that does not depend on the e - e interaction and ν [N.22]. Finally, notice that as a result of regrouping the interaction amplitudes appear always as Γ_a/z .²

Conclusion: The effective model that adequately describes the problem of electrons diffusing in the field of impurities is the NLSM with interactions. This model provides a compact but comprehensive description for disordered interacting electrons that is fully compatible with the constraints imposed by conservation laws. Parameters characterizing various properties of the disordered electron liquid preserve the structure of the Fermi-liquid theory although with renormalized coefficients determined by the RG-procedure.

3. Scaling Theory of the Metal–Insulator Transition in $d = 2 + \epsilon$; Role of the Parameter z

In this section, we show how the scaling parameters D , Γ_σ , and z , together describe the transport and thermodynamic properties of the disordered electron liquid near the metal–insulator transition (MIT).

Let us start with the key points of the RG-analysis in $d = 2 + \epsilon$. As we have already mentioned, diffusion modes participating in the renormalization procedure have to be integrated over momenta, see Figs. 18.4 and 18.6 as examples. Each momentum integration involving diffusion propagators generates a factor $1/D_Q$ which eventually gives rise to the dimensionless parameter

$$\rho = \frac{r_d(\kappa)}{2\pi^2\hbar/e^2} \propto \frac{e^2}{\hbar\sigma} \kappa^{d-2}. \quad (3.1)$$

Here, ρ is equal to the resistance r_d of a d -dimensional cube of side length $\sim 2\pi/\kappa$ measured in units of $2\pi^2\hbar/e^2$ [N.23]; κ is the momentum cutoff which decreases during the renormalization [N.24].

It follows from the structure of the action $S[Q]$, when written with the help of Eqs. (2.9) and (2.10), that the RG-procedure can be performed in terms of the dimensionless resistance ρ and the reduced interaction amplitudes

$$\gamma_2 = -\Gamma_\sigma/z = \Gamma_2/z, \quad (3.2)$$

$$\gamma_c = \Gamma_c/z. \quad (3.3)$$

With these variables, the set of the RG equations takes the general form^{2,3}:

$$d \ln \rho / dy = -\frac{\epsilon}{2} + \rho \beta_\rho(\rho; \gamma_2, \gamma_c; \epsilon), \quad (3.4a)$$

$$d\gamma_2/dy = \rho\beta_{\gamma_2}(\rho; \gamma_2, \gamma_c; \epsilon), \quad (3.4b)$$

and also

$$d\gamma_c/dy = -\gamma_c^2 + \rho\beta_{\gamma_c}(\rho; \gamma_2, \gamma_c; \epsilon). \quad (3.5)$$

The parameter z is described by an additional equation:

$$d \ln z/dy = \rho\beta_z(\rho; \gamma_2, \gamma_c; \epsilon). \quad (3.6)$$

Observe that β_z , as well as β_ρ and $\beta_{\gamma_2, \gamma_c}$ are all independent of z . In the above set of RG equations, the logarithmic variable $y = \ln 1/[\max(D\kappa^2/z, \omega_n)\tau_{el}]$ has been used. This choice of the logarithmic variable is convenient because it allows us to take T as a natural lower cutoff; the upper cutoff is $1/\tau_{el}$. The explicit factor ϵ in the equation determining ρ originates from κ^{d-2} entering the definition of the RG-charge ρ .

The β -functions in the above equations are multiplied by a factor ρ to emphasize that the sought-after corrections appear as a result of disorder [N.25]. The complete form of the β -functions is unknown. The general approach, however, is to expand the functions in a power series in ρ as $\beta(\rho; \gamma_2, \gamma_c) = \beta_1(\gamma_2, \gamma_c) + \rho\beta_2(\gamma_2, \gamma_c) + \dots$, such that for each power of ρ the full dependence on γ_2 and γ_c is retained. This is possible, in principle, because the maximal number of allowed interaction amplitudes (extended by ladders) is limited by the number of momentum integrations involving the diffusive propagators. Since each integration gives a factor of $1/D \sim \rho$, for a given order in ρ the number of (extended) interaction vertices is finite [N.26].

For a repulsive interaction in the Cooper channel, the amplitude γ_c scales rapidly to a ρ -dependent fixed point, which is determined by the competition of two terms in Eq. (3.5). In the following, we replace γ_c in the β -functions describing ρ , γ_2 and z by its fixed-point value $\gamma_c(\rho)$. As a result, the RG-evolution near the MIT can be described by only Eqs. (3.4a) and (3.4b) together with Eq. (3.6) for z .

To illustrate the scheme of finding the temperature or frequency behavior of the conductivity in the critical region of the MIT,²¹ let us discuss an electron system in the presence of magnetic impurities. Then, Cooperons and fluctuations of the electron spin density are not effective because of a strong spin scattering. In this case, the parameter ρ representing the resistance of a d -dimensional cubic sample is described by a separate equation:

$$d \ln \rho/dy = -\frac{\epsilon}{2} + \rho\beta_\rho(\rho; \epsilon). \quad (3.7)$$

In the discussed case, corrections appearing as a result of the interplay of the e - e interaction and disorder^{10,11} lead to an increase of the resistance

as temperature decreases ($\beta_\rho > 0$, thus favoring localization). Therefore, the geometric factor ϵ competes with these corrections for $d > 2$. As a result, there is an unstable fixed point, $\rho = \rho_c$, which determines the critical behavior of the conductivity in the critical region of the MIT.

Then, as it follows from Eq. (3.1), in the vicinity of the transition,

$$\sigma(\kappa)/e^2 \propto \kappa^{d-2}. \quad (3.8)$$

In the $3d$ case, for example, on the metallic side of the transition the critical behavior develops when $\kappa \gg \sigma(T=0)/e^2$. At non-zero temperature, in the critical regime of the MIT the process of renormalization ceases at a scale when

$$D(\kappa)\kappa^2/z(\kappa) \sim T \quad \xrightarrow[\text{using Eq. (2.11)}]{} \quad \kappa^d/\nu \sim zT. \quad (3.9)$$

For the electric conductivity measured at external frequency $\omega \gg T$, the renormalization is cut off by ω rather than T . The above relations are a consequence of (i) the form of the diffusion propagator $\mathcal{D}(k, \omega_n) = 1/(Dk^2 + z\omega_n)$, and (ii) the definition of the RG-parameter g which exhibits a fixed point. In addition, these relations take into account the result discussed in the previous section that (iii) all the renormalizations in between σ and D are canceled out: $\sigma/e^2 = 2\nu D$.

Thus, in order to find the temperature or frequency behavior of σ at the MIT, one has to connect the momentum and energy scales in the critical region, $\kappa \sim (z \max[\omega, T])^{1/d}$. However, z itself is a scaling parameter, see Eq. (3.6). Therefore, one needs to know the critical behavior of the parameter z at the transition, which is determined by the value of $\rho\beta_z$ at the critical point:

$$\sigma(\omega, T) \sim (z \max[\omega, T])^{\frac{d-2}{d}} \sim (\max[\omega, T])^{\frac{d-2}{d}(1+\tilde{\zeta})}, \quad (3.10)$$

$$\tilde{\zeta} = -(\rho\beta_z)_{\text{critical point}}. \quad (3.11)$$

For free electrons z is not renormalized, and at zero temperature $\sigma(\omega) \sim \omega^{1/3}$ for $d = 3$.²⁶ The e - e interaction modifies this critical behavior of the conductivity through the critical exponent $\tilde{\zeta}$. If $\omega \lesssim T$, the renormalization procedure is cut off by the temperature

$$\sigma(T) \sim (zT)^{\frac{d-2}{d}} \sim T^{\frac{d-2}{d}(1+\tilde{\zeta})}. \quad (3.12)$$

The scaling behavior described above suggests that the interplay between frequency and temperature can be described by a single function

$$\sigma(T, \omega)_{\text{critical}} = T^a f(\hbar\omega/k_bT), \quad (3.13)$$

where $f(x) \rightarrow \text{const}$ when $x \rightarrow 0$, and $f(x) \propto x^a$ when $x \rightarrow \infty$, with $a = \frac{d-2}{d}(1 + \tilde{\zeta})$. This is a typical behavior near a quantum phase transition for which the MIT is, perhaps, a primary example.

To get an idea about the value of the critical exponent $\tilde{\zeta}$, let us find it in the lowest order in ϵ . The equation describing resistance in the case of magnetic impurities in the lowest orders in ρ and ϵ is

$$d \ln t / dy = -\frac{\epsilon}{2} + \rho, \quad \rho_c = \frac{\epsilon}{2}. \quad (3.14)$$

Furthermore,

$$d \ln z / dy = -\frac{1}{2}\rho. \quad (3.15)$$

At $d = 3$, this estimate yields $\tilde{\zeta} = 1/4$ for the MIT in the presence of spin scattering and correspondingly, for $\epsilon = 1$, one gets $a = \frac{1}{3}(1 + \tilde{\zeta}) \approx 0.4$.

Although we used for the purpose of illustration the case of magnetic scattering (a system where only fluctuations in the density channel are important), the conclusion of the above discussion is quite general: the frequency or temperature behavior of the conductivity in the critical region is determined by the right-hand side of Eq. (3.6) at the fixed point of the transition [N27]. Notice that, although the ϵ -expansion has been applied to estimate the value of $\tilde{\zeta}$, the form of the combination $a = \frac{d-2}{d}(1 + \tilde{\zeta})$ is determined by the general structure of the theory only and does not rely on the ϵ -expansion [N28].

Experimentally, the dependence of σ on the temperature in the critical region can be determined with a limited accuracy only [N29]. In Ref. 27, it was shown that in a persistent photoconductor where the carrier concentration can be controlled very neatly, $\sigma(T) \sim T^{1/2}$ at the transition, i.e., $a = 1/2$; this corresponds to $\tilde{\zeta} = 1/2$. The direct measurements of $\sigma(\omega)$, are unfortunately, very rare. In Refs. 28 and 29, the temperature and frequency dependences were studied simultaneously in amorphous niobium-silicon alloys (Nb:Si) with compositions near the MIT. The measurements observed a one-to-one correspondence between the T - and ω -dependent conductivity thus confirming the above picture of the MIT as a quantum phase transition. The critical exponent a has also been found to be equal to $1/2$ for this system, i.e., $\sigma(T, \omega)_{\text{critical}} = T^{1/2} f(\hbar\omega/k_b T)$, see Eq. (3.13). The same scaling behavior should hold for the whole universality class which the discussed system represents. In measurements on the magnetic-field-induced MIT in GaAs and InSb semiconductors (representing a different universality class compared to the discussed measurements on Nb:Si) the critical behavior $\sigma(T) \sim T^{1/3}$ has been observed.³⁰ It may be worth mentioning that for

this universality class $\tilde{\zeta}$ is indeed equal to zero in the lowest order in the ϵ -expansion.³¹

Finally, let us note another important consequence of the fact that at the critical point of the MIT (determined by the fixed point of the set of Eqs. (3.4) and (3.5)) the only scaling parameter which continues to evolve is z . Since this parameter is directly related to the renormalization of the effective density of states of the diffusion modes, it follows immediately from Eq. (2.13) that the critical temperature dependence of thermodynamic quantities at the MIT is also described by the same critical exponent $\tilde{\zeta}$; see Eq. (3.11) for the definition of $\tilde{\zeta}$.

The content of this section may look like a simple dimensional analysis. In fact, it heavily relies on the structure of the theory based on the NLSM with the interaction terms, which was established in the previous section. As it was pointed out there, this low-energy field theory adequately describes the interacting electrons in the diffusive regime. In this context, the parameter z plays a special role. Since this parameter is responsible for the frequency renormalization, it is of particular importance in connection with the conservation laws of the particle-number and spin. Furthermore the law of number conservation allows to obtain the Einstein relation for the electron liquid in the appropriate form. Only with the information about the structure of the theory at hand, the critical behavior near the MIT can be found by a straightforward dimensional analysis.

Conclusion: The metal-insulator transition in a system of diffusing electrons is an example of a quantum phase transition³² with a temperature-frequency scaling controlled by the parameter z . Precisely the same parameter describes the scaling behavior of both the conductivity and the thermodynamics in the critical region of the transition. The structure of the theory is very general and not related to the ϵ -expansion which can be used for the calculation of $\tilde{\zeta}$.

4. Tunneling Density of States

The tunneling density of states (TDOS) or, as it is also called, the single-particle density of states, $\nu(\varepsilon)$, exhibits a rather pronounced critical behavior at the MIT.³³ This quantity can be obtained by measuring the differential conductance $G_j(V)$ of a tunneling junction at a finite voltage bias V : $G_j(V) \propto \nu(\varepsilon = V)$. In the early semi-phenomenological scaling theory of the MIT by McMillan,³⁴ the TDOS has been treated as a parameter which enters into the relation connecting the length and energy (or frequency) scales

and gives rise to a critical exponent which has been replaced by $\tilde{\zeta}$ in the full microscopic theory. As we have already discussed, the parameter which connects these scales is z rather than the TDOS, see Eq. (3.9). Moreover, the TDOS stands actually outside the RG-scheme. Let us explain why. The TDOS is defined as

$$\nu(\varepsilon) = -\frac{1}{\pi} \text{Im} \int \mathcal{G}^R(\varepsilon, \mathbf{p}) \frac{d\mathbf{p}}{(2\pi)^d}. \quad (4.1)$$

As such, this quantity is not gauge invariant. It can be changed by a time-dependent gauge transformation. Thereby, it cannot enter the RG-scheme which operates only with truly gauge-invariant quantities. In the case of the TDOS, it is the external electrode with respect to which the measurement of the tunneling current is performed, that makes the TDOS a physically meaningful quantity [N30].

The combined effect of Coulomb interaction and disorder leads to a strong suppression of the TDOS. This observation allowed to explain the so-called zero-bias anomaly in the tunneling spectra of disordered systems.^{10,35} Compared to other effects related to the interplay of the e - e interaction and disorder, corrections to the TDOS are the strongest. In particular, in two-dimensions the correction obtained in the lowest order in ρ appears to be log-squared rather than just logarithmic^{11,36}:

$$\nu(\varepsilon) = \nu \left[1 - \frac{\rho}{4} \ln(1/|\varepsilon|\tau_{el}) \ln(\tau_{el}\omega_0^2/|\varepsilon|) \right]. \quad (4.2)$$

Here $\omega_0 = D\kappa_{scr}^2$, while κ_{scr} is the inverse of Thomas–Fermi screening radius.

To go beyond the perturbative correction, it is useful to apply the \hat{Q} -matrix technique. With this technique, $\nu(\varepsilon)$ can be expressed as an averaged product of two matrices:

$$\nu(\varepsilon) = \nu \left\langle \hat{\Lambda} \hat{Q} \right\rangle_{\varepsilon\varepsilon}. \quad (4.3)$$

Thus, by measuring the TDOS, one may study how an ε -component of the matrix \hat{Q} fluctuates around its equilibrium position. Now compare with the physics of phonons: the quantity which measures the fluctuations of ions with respect to their equilibrium position is the Debye–Waller factor. It has been noted already in the early studies,^{1,31} that the calculation of $\nu(\varepsilon)$ is indeed very similar to the calculation of the Debye–Waller factor, and can be reduced to a Gaussian integration. [By means of the \hat{Q} -technique, the right-hand-side of Eq. (4.3) can be expressed as $\nu(\varepsilon) = \nu \langle \exp W \rangle_{\varepsilon\varepsilon}$, where W is a matrix field that describes diffusion modes in the presence of the e - e interactions.] In fact, the formal similarity with the Debye–Waller factor reflects the physical essence of the TDOS. Measurement of a tunneling

current is a kind of “*Mössbauer-type*” experiment which determines the effect of the zero-point fluctuations of the electromagnetic field on the probability of tunneling.

As it was first pointed out in Refs. 1 and 31, it follows from the structure of the discussed quantity that the perturbative correction to the TDOS should be exponentiated [N31]:

$$\nu(\varepsilon) = \nu \exp\left[-\frac{\rho}{4} \ln(1/|\varepsilon|\tau_{el}) \ln(\tau_{el}\omega_0^2/|\varepsilon|)\right]. \quad (4.4)$$

Examples of calculations of the critical exponent of the TDOS at the MIT using the ϵ -expansion are given in Ref. 31. The presence of the double-logarithmic corrections to $\nu(\varepsilon)$ when $d = 2$ leads to the fact that the ϵ -expansion of the critical exponent of the TDOS starts from a constant. The reason is that at $d = 2 + \epsilon$ in the exponent of Eq. (4.4), the factor $1/\epsilon$ replaces one of the two logs and cancels a factor ϵ coming from the charge $\rho_c \propto \epsilon$. It is worth noting that the situation discussed above is specific for the long-range nature of the Coulomb interaction. In a model description when the dynamically screened Coulomb interaction $V_C(k, \omega_n)$ is replaced by a constant, double-logarithmic corrections do not arise.

Interestingly, the log-squared corrections cancel out when calculating any other physical quantities, except the TDOS. This occurs for the following reason. The discussed corrections accumulate from the momentum integration over the region of momenta that are much smaller than those typical for diffusion, $k \ll (\omega_n/D)^{1/2}$. Therefore, this integration does not involve the diffusion propagators but only the Coulomb interaction $V_C(k, \omega_n)$ [N32]. As a result of such an integration, the dynamically screened Coulomb interaction starts to depend effectively only on the frequency. However, as it was pointed out in Refs. 37, 3 and 38, any interaction of this kind, i.e., a purely time-dependent e - e interaction, can be completely eliminated by means of a time-dependent gauge transformation which can be performed exactly. [By a standard procedure, the four-fermion term $\psi^\dagger(\tau)\psi(\tau)V_{ee}(\tau - \tau')\psi^\dagger(\tau')\psi(\tau')$ can be decoupled by a time-dependent potential acting on the fermions, $\varphi(\tau)\psi^\dagger(\tau)\psi(\tau)$, which subsequently can be integrated out.] This is the reason why the corrections originating from the unscreened singularity of the Coulomb interaction at very small momenta cannot manifest themselves in transport or thermodynamic quantities: They cannot appear in gauge-invariant quantities. In fact, the physics of this observation is very close to the arguments presented in Sec. 1 about the insensitivity of the corrections induced by disorder and the e - e interactions to the variation of the chemical potential because of the absence of a reference

point. The only difference is that now the variation of the potential is time-dependent.

Conclusion: The Coulomb interaction, $V_C(k, \omega_n)$, with a momentum transfer much smaller than those typical for diffusion, can contribute only to a quantity for which the condition of measurement makes it possible to detect the effect of the long-range time-dependent fluctuations of the electric potential. An example of such a quantity is the TDOS. By fabricating a counter-electrode of the tunneling junction, one creates a reference point which allows to study the effects of time-dependent long-range fluctuations of the electric potential which do not contribute to other physical quantities [N33].

5. The Anderson Transition in the Presence of Interactions in a Two-Dimensional System

Here we will apply the two-parameter scaling theory, in which ρ and γ_2 are the flowing parameters, for the discussion of the $2d$ -MIT. We use the data obtained in Si-MOSFETs for comparison with the theory. The MIT in a $2d$ electron gas, which does not occur for free electrons,⁸ has been observed experimentally in dilute electron systems.^{39,40} Obviously, this fact indicates that the e - e interactions are of crucial importance. The unexpected discovery of the $2d$ -MIT generated renewed interest in disordered electron systems with interactions (see the review articles⁴¹⁻⁴³ and references therein).

In Fig. 18.9, the data of Pudalov *et al.*⁴⁴ is presented which demonstrates clearly the existence of the MIT in $2d$; different curves here correspond to different electron densities. In the metallic phase the resistance $\rho(T)$ drops noticeably as the temperature is lowered. [This drop is suppressed when a relatively weak in-plane magnetic field is applied.⁴⁵ The sensitivity to an in-plane magnetic field highlights the importance of the spin degrees of freedom for the MIT. Therefore, the spin related modes should be one of the ingredients of the theory of the transition.] A highly non-trivial feature revealed by the data shown in Fig. 18.9, is the non-monotonicity of $\rho(T)$ on the metallic side of the transition. This non-monotonic behavior is of the principle importance, because it points towards a competition between different mechanisms determining resistance [N34]. In the theory of the MIT developed by the author together with Alex Punnoose,^{16,46} there is a competition between the charge-density diffusion modes and cooperons, on the one hand, and the fluctuations of the spin- (and valley-) degrees of freedom, on the other hand. The former favor

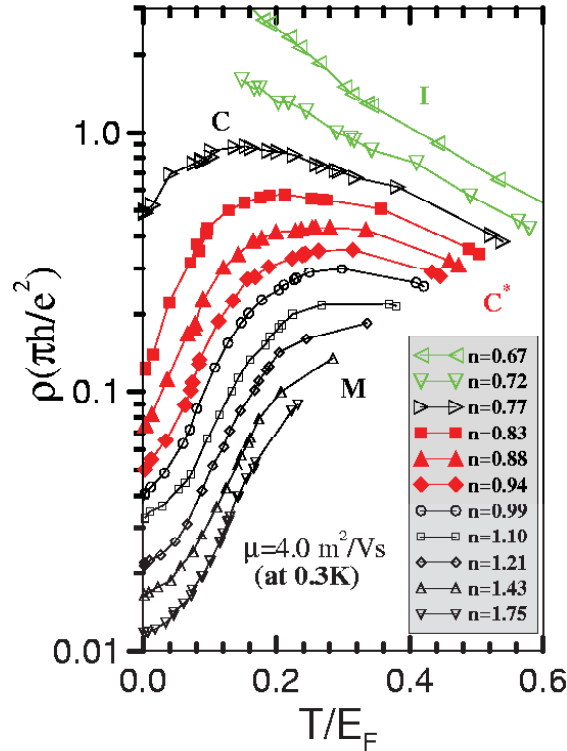


Fig. 18.9. Resistivity of a high mobility Si-MOSFET sample for various densities as a function of temperature (adapted from Ref. 44). The data clearly demonstrates the existence of the metal-insulator transition. The electron densities are defined in units of 10^{11} cm^{-2} . Note that resistance is measured in units of $2\pi^2\hbar/e^2$. The three non-monotonous curves below the transition (shown in red color and denoted as C*) are fitted in Fig. 18.10. Reprinted (Fig. 1) with permission from *Phys. Rev. Lett.* **88**, 016802 (2002). © American Physical Society.

localization, while the latter act against it, thus stabilizing the metallic state.

The critical resistance at which the transition occurs in high-mobility Si-MOSFETs has been shown experimentally to be universal,⁴² suggesting the applicability of the RG-description of the MIT of the kind discussed above in Sec. 3 [N35]. But can we use the disordered Fermi-liquid (or at least the NLSM with interactions) as a starting platform in a system with r_s of the order of 10? Measurements of the Shubnikov-de Haas oscillations and the Hall coefficient in Si-MOSFETs observe no anomalies in the properties characterizing the electron liquid on the metallic side of the MIT (at least, when applied magnetic fields are not too high). The Fermi-liquid renormalizations

extracted as a result of these measurements were significant but not giant [N36]. We, therefore, have all reasons to apply the approach based on the RG-analysis of the NLSM discussed above for the analysis of the MIT in this material. Generally speaking, we believe that the NLSM description can be applicable even without the prerequisites of the Fermi-liquid. This is because the diffusion modes are more robust than the single-particle excitations and, hence, the NLSM with interactions, as a *minimal model*, can be valid even in the absence of the Fermi-liquid background.

For the discussion of Si-MOSFETs, the only necessary modification is due to the fact that Si-MOSFET is a multi-valley system. The conduction band of an n -(001) silicon inversion layer has two almost degenerate valleys. In the following we consider the number of equivalent valleys to be equal to n_v . Because inter-valley scattering requires a large change of the momentum, we assume that the interactions couple electrons in different valleys but do not mix them. This implies that inter-valley scattering processes, including those due to the disorder, are neglected. This assumption is appropriate for samples with high-mobility [N37]. In this limit, the RG equations describing the evolution of the resistance and the scattering amplitude γ_2 in $2d$ have the form⁴⁶:

$$\frac{d \ln \rho}{dy} = \rho [n_v + 1 - (4n_v^2 - 1)\Phi(\gamma_2)], \quad (5.1a)$$

$$\frac{d\gamma_2}{dy} = \rho \frac{(1 + \gamma_2)^2}{2}. \quad (5.1b)$$

The equations above are obtained in the lowest order in ρ (the one-loop order), but they incorporate the full dependence on the e - e amplitudes. Here, the amplitude γ_2 acts inside spin-valley “triplet channels.” (The definition of γ_2 is given in Eq. (3.2); note that for repulsive interactions $\gamma_2 > 0$.) In the first equation, $\Phi(\gamma_2) = \frac{1+\gamma_2}{\gamma_2} \ln(1 + \gamma_2) - 1$; the factor $(4n_v^2 - 1)$ corresponds to the number of spin-valley “triplet” channels, while the factor n_v corresponds to the weak-localization (cooperon) corrections. The factor of one entering the square brackets in Eq. (5.1a) is the contribution of the long-ranged Coulomb singlet-amplitude (after dynamic screening) and is, therefore, universal. Furthermore, it should be emphasized that the factor of one appearing in Eq. (5.1b) for γ_2 also originates from the Coulomb singlet-amplitude combined with scattering induced by disorder, see [N20]. Consequently, setting the initial value of γ_2 to zero does not imply the absence of interactions.

The following salient features should be noted: While the amplitude γ_2 increases monotonically as the temperature is reduced, the resistance, as a

result, has a characteristic non-monotonic form changing from insulating behavior ($d\rho/dT < 0$) at high temperatures to metallic behavior ($d\rho/dT > 0$) at low temperatures. The change in slope occurs at a maximum value ρ_{\max} at a temperature $T = T_{\max}$, neither of them is universal. The corresponding value of the amplitude γ_2 is, however, universal at the one-loop order, depending only on n_v ; for $n_v = 1$, it is 2.08, whereas for $n_v = 2$, it has the considerably lower value 0.45. Next, it follows from the general form of Eqs. (5.1a) and (5.1b), that $\rho(T)/\rho_{\max}$ and $\gamma_2(T)$ can be presented as the universal functions $R(\eta_T)$ and $\tilde{\gamma}_2(\eta_T)$ when the argument η_T is introduced^{1-3,46}:

$$R(\eta_T) \equiv \rho(T)/\rho_{\max} \quad \tilde{\gamma}_2(\eta_T) \equiv \gamma_2(T), \quad (5.2)$$

$$\eta_T = \rho_{\max} \ln(T/T_{\max}).$$

The non-monotonic function $R(\eta)$ together with the fit of the resistance curves obtained for two samples of different origin are presented in Fig. 18.10. After re-scaling, the data at various densities is described by a *single* curve. The drop of $\rho(T)$ by a factor of five and the subsequent flattening of the curve at low T are captured in the correct temperature interval. The full

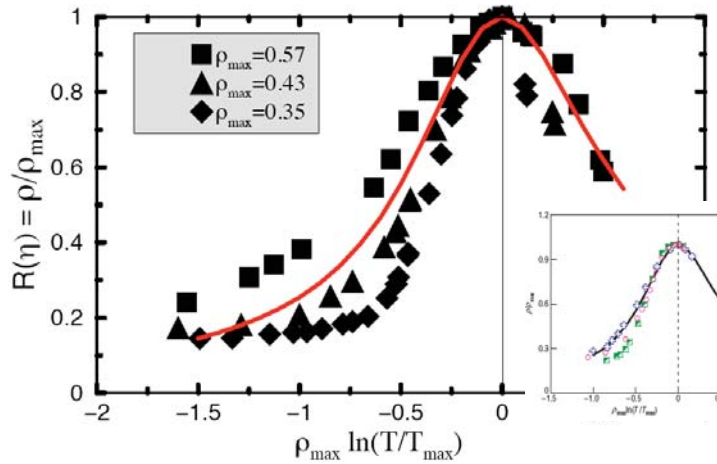


Fig. 18.10. RG-fitting of the resistivity for two different Si-MOSFETs. Main panel⁴⁶: the data corresponding to $n = (0.83, 0.88, 0.94) \times 10^{11} \text{ cm}^{-2}$ in Fig. 18.9 are scaled according to Eq. (5.2). The solid line (in red) is the solution of the RG equations (5.1a) and (5.1b) with $n_v = 2$; no adjustable parameters have been used in the fit of the data. Inset⁴⁹: the same for a sample from a different wafer. Main figure reprinted (Fig. 2) with permission from *Phys. Rev. Lett.* **88**, 016802 (2002). © American Physical Society. Inset reprinted (Fig. 4a) with permission from *Nature Phys.* **707** (2007). © Macmillan Publishers Ltd.

temperature dependence of the resistance is completely controlled by its value ρ_{\max} at the maximum; there are no other free (or fitting) parameters.

We can draw an important conclusion: We proved theoretically the existence of the MIT in $2d$. Since (i) the one-loop approximation gives a *drop* of the resistance at low temperatures for a moderate strength of disorder (i.e., in the region of the applicability of this approximation), and (ii) the Anderson localization at strong disorder is indisputable; it is therefore *logically unavoidable that the MIT should exist* in-between. Owing to the drop in the resistance, the reliability of the obtained RG equations is improved. Therefore, the conclusion about the existence of the MIT can be justified even within the one-loop approximation. Thus, the anti-localization effect of the e - e interactions fundamentally alters the common point of view that electrons in $2d$ are “eventually” (i.e., at $T = 0$) localized.

Let us turn now to the interaction amplitude $\gamma_2(T)$. Since this amplitude is related to the spin degrees of freedom, the information about the dependence of this amplitude on the temperature can be extracted from the in-plane magnetoconductance. This is because the fluctuations of the spin-density lead — with participation of γ_2 — to finite temperature corrections to the resistivity.⁴⁷ The spin-splitting induced by the in-plane magnetic field reduces spin-density fluctuations and leads, in this way, to a temperature dependent magnetoconductance. Hence, the magnetoconductance contains information about the value of the amplitude γ_2 and its evolution with temperature. In order to extract the value of γ_2 , it is important, however, to perform measurements in weak magnetic fields, such that $g\mu_B(1 + \gamma_2)B/k_B T \ll 1$. Weak magnetic field is needed in order not to drive the electron liquid, which at large r_s is very “fragile”, into some other state. As long as electrons are in the diffusive regime, $k_B T < h/\tau_{el}$, i.e., the temperature is less than the scattering rate on the static impurities, the expression for magnetoconductivity in the limit $b = g\mu_B B/k_B T \ll 1$ is given⁴⁷⁻⁴⁹ as:

$$\Delta\sigma = -(e^2/\pi h) K_v C_{ee}(\gamma_2, \rho) b^2, \quad (5.3)$$

where in a system with n_v degenerate valleys, $K_v = n_v^2$. In the case when the resistance ρ is not too high, the coefficient determining the magnetoconductance, C_{ee} , is explicitly related to the amplitude γ_2 as follows: $C_{ee} = 0.091\gamma_2(\gamma_2 + 1)$. The experimental details and the results of the comparison with theory can be found in Refs. 49 and 50 where, for the first time, the scaling of the interaction amplitude was established. Not too close to the MIT, the extracted values of γ_2 are close to those predicted by the theory. Remarkably, the parameter γ_2 at $T = T_{\max}$ was found to correspond to 0.45 as predicted by theory for $n_v = 2$.

In spite of this success, Eqs. (5.1a) and (5.1b) have a limited applicability. Obviously, the *single-curve* solution $R(\eta)$ cannot provide the description of the MIT. To approach the critical region of the MIT, the disorder has to be treated beyond the lowest order in ρ , while adequately retaining the effects of the interaction.

An internally consistent theory of the MIT [38] which goes beyond the one-loop calculations was developed in Ref. 16 using the number of identical valleys as a large parameter, $n_v \rightarrow \infty$. The valley degrees of freedom are akin to flavors in standard field-theoretic models. Generally, closed loops play a special role in the diagrammatic RG-analysis in the limit when the number of flavors N is taken to be very large.⁵¹ This is because each closed loop involves a sum over all the flavors, generating a large factor N per loop. It is then typical to send a coupling constant λ to zero in the limit $N \rightarrow \infty$ keeping λN finite. For interacting spin-1/2 electrons in the presence of n_v valleys ($N = 2n_v$), the screening makes the bare values of the interaction amplitude γ_2 to scale as $1/(2n_v)$. Furthermore, the increase in the number of conducting channels results in the resistance ρ to scale as $1/n_v$. It is, therefore, natural to introduce the amplitudes $\Theta = 2n_v\gamma_2$ together with the resistance parameter $t = n_v\rho$; the parameter t is thus the resistance per valley, $t = 1/[(2\pi)^2\nu D]$. Both quantities Θ and t remain finite in the large- n_v limit.

Following the large- n_v approximation scheme outlined above, the RG equations at order t^2 (i.e., in the two-loop approximation) have been derived. The obtained equations describe the competition between the e - e interactions and disorder in $2d$. The resulting resistance-interaction (t - Θ) flow diagram is plotted in Fig. 18.11. The arrows indicate the direction of the flow as the temperature is lowered. The quantum critical point, which corresponds to the fixed point of the equations describing the evolution of t and Θ , is marked by the circle. This quantum critical point separates the metallic phase, which is stabilized by electronic interactions, from the insulating phase where disorder prevails over the electronic interactions. The attractive (“horizontal”) separatrix separate the metallic phase from the insulating phase. Crossing the separatrix by changing the initial values of t and Θ (e.g., by changing the carrier density) leads to the MIT.

In Ref. 49, the two-parameter scaling theory has been verified experimentally. In Fig. 18.12, the experimentally obtained flow diagram is presented. In this plot, the coefficient C_{ee} effectively represents the interaction amplitude in the spin-density channel. The authors used the fact that the coefficient C_{ee} reflects the strength of spin-related interactions of the diffusion

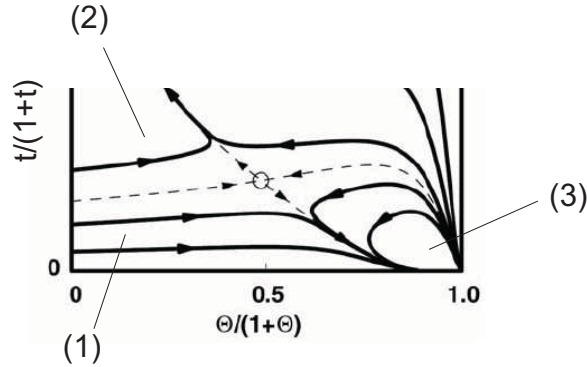


Fig. 18.11. The resistance-interaction (t - Θ) flow diagram obtained in the two-loop calculations.¹⁶ The arrows indicate the direction of the flow as the temperature is lowered. The quantum critical point is marked by the circle. Area (1) is the metallic phase, which is stabilized by the interaction. Area (2) is the insulating phase where disorder prevails. Area (3) is the region of strong spin correlations. The attractive separatrices separate the metallic phase from the insulating phase.

modes at *any* value of the resistance. Therefore, one may get much broader insight into the MIT by studying the temperature dependence of the coefficient C_{ee} even without knowing exact relation connecting C_{ee} with γ_2 . This procedure has been applied for the first time in Ref. 49, where the coefficient C_{ee} has been determined by fitting the $\Delta\sigma(B, T)$ traces to Eq. (5.3). Because the traces are taken at different temperatures, one obtains the RG-evolution of C_{ee} as a function of temperature.

We see that the flow diagram presented in Fig. 18.12 confirms all the qualitative features of the theoretical predictions, including the quantum critical point and the non-monotonic behavior of the resistance as a function of T on the metallic side of the transition. At not too high resistance, but still within the diffusive region, the data presented in this flow diagram can be accurately described by the RG theory without any fitting parameters, see the inset in Fig. 18.10. Most important, however, is that the possibility of presenting the data as a flow diagram gives a very strong argument in favor of the applicability of the two-parameter scaling theory in Si-MOSFETs.

So far, we described scaling in terms of two parameters, leaving aside the parameter z . Being related to the frequency renormalization of the diffusion modes, this parameter determines the transport and thermodynamic quantities in the critical region of the MIT. In the limit $N \rightarrow \infty$, the equation for z reads as follows: $d \ln z / dy = \beta_z(t, \Theta) = t\Theta$. Consequently, in the case discussed in this section, and unlike in the case of magnetic impurities

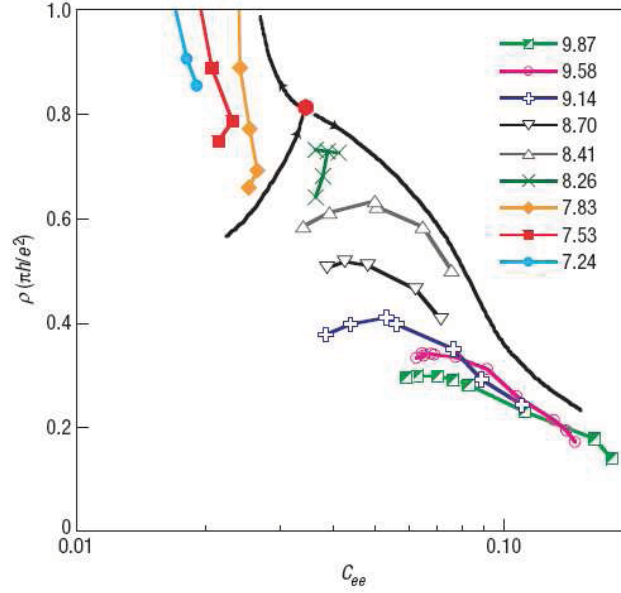


Fig. 18.12. The disorder–interaction flow diagram of the 2d electrons in Si-MOSFET.⁴⁹ The circle indicates the location of the quantum critical point from which the three separatrices emanate. The arrows shown on the separatrices indicate the direction of the flow as the temperature is lowered. The electron densities are indicated in units of 10^{10} cm^{-2} . The interaction amplitude in the spin-density channel is represented by the coefficient C_{ee} which was extracted from the magnetoresistance. Reprinted (Fig. 3) with permission from *Nature Phys.* **3**, 707 (2007). © Macmillan Publishers Ltd.

discussed in Sec. 3, z diverges in the vicinity of the MIT: $z \sim T^{\tilde{\zeta}}$ where $\tilde{\zeta} = -(t\Theta)_{\text{critical point}} < 0$. As we already mentioned, z can be interpreted as the parameter renormalizing the density of states of the diffusion modes and as such it controls the thermodynamic quantities. Hence, in the critical regime of the 2d-MIT, the specific heat C_V/T diverges (owing to the softening of the diffusion modes induced by z). Furthermore, a similar divergence is also expected in the Pauli spin susceptibility¹⁶:

$$C_V/T = \nu z(T) \propto T^\zeta, \quad \chi_{spin}/\chi_a^0 = z(T)(1 + \gamma_2) \propto T^\zeta. \quad (5.4)$$

Since the interaction parameter Θ is finite at the critical point, the divergence in the Pauli spin susceptibility is not related to any Stoner-like magnetic instability.

Before we conclude this section, let us touch upon a delicate point: How general is the discussed theory of the 2d-MIT? It is applicable only within

the diffusive regime at low enough temperatures when $k_B T \ll \hbar/\tau_{el} \lesssim \epsilon_F$. Under these conditions, the charge and spin perturbations of the degenerate electron gas propagate diffusively. The boundary of the diffusive regime can be determined from the measurements of the magnetoconductivity described above. [The point is that the relation $\Delta\sigma \propto b^2$, where $b = g\mu_B B/k_B T \ll 1$, holds only inside the diffusive regime, $k_B T \ll \hbar/\tau_{el}$, while in the ballistic regime $\Delta\sigma \propto (T\tau_{el})b^2$.] In Si-MOSFETs, the diffusive regime extends up to a few Kelvin in a density range around the critical density of the MIT. The corresponding Fermi-temperature is of the order of 10 K and, therefore, at temperatures convenient for measurements electrons are already degenerate. In addition, the $2d$ electron gas in Si-MOSFETs (which in fact is a *moderately* high-mobility system) is *unique* in the sense that the scattering is mostly short-range in character, so that the *MIT occurs in the diffusive regime*. On the contrary, in *true* high-mobility systems like GaAs/AlGaAs or n-SiGe heterostructures the single particle scattering rate typically differs by a factor of ten compared to the transport scattering rate. Therefore, it is difficult to access the diffusive regime, because the smoothness of the disorder drives the system directly from the ballistic to the insulating phase. Furthermore, the presence of two equivalent valleys strongly enhances the anti-localization effect of the $e-e$ interaction and disorder making MOSFETs ideal systems to study the MIT in $2d$ [N39].

We may thereby conclude that Si-MOSFETs provide an ideal playground to study the properties of a $2d$ disordered electron liquid and, in particular, the Anderson localization in the presence of the $e-e$ interactions. Within the region of its applicability, the RG-theory gives not only a qualitative but also a quantitative description of the experimental data in these systems, see Figs. 18.10–18.12.

Still, a question arises — How can the one-loop theory work so well up to very high resistances [N40], in spite of the fact that the system is placed under such extreme conditions that r_s is as large as 10? In our opinion, it is maybe *not in spite but because* of these extreme conditions. The point is that at large r_s an electron is mostly trapped inside a temporary potential minimum created by other electrons as a result of Wigner-crystal like short range order. Naturally, the kinetic energy of the electrons temporarily trapped by the strong Coulomb interaction is larger than that of free electrons at the same density. Therefore disorder is actually small compared to both the kinetic and the interaction energies, even though the resistance is not small.

Summary: We demonstrated that the two-parameter RG-theory of the disordered electron liquid reviewed here captures both quantitatively (for

moderate disorder) and qualitatively (for larger disorder) the physics of the disordered liquid in the diffusive regime. The possibility of presenting the data as a flow diagram is a strong argument in favor of the applicability of the two-parameter scaling theory in Si-MOSFETs. Finally, we showed that the existence of the MIT in $2d$ in Si-MOSFETs can be justified theoretically by combining the RG-analysis in the one-loop approximation with the fact of the existence of Anderson localization at strong disorder: The one-loop approximation gives a noticeable drop of the resistance at low temperatures for a moderate strength of disorder. Since, on the other hand, at very strong disorder the Anderson localization is unavoidable, it follows that in between the MIT should exist.

6. Notes

- N.1 For example, when the rate of inelastic e - e collisions exceeds characteristic excitation energies, which are of the order of the temperature T or frequency ϵ .
- N.2 In the diffusion coefficient D , both v_F and τ_{el} incorporate the Fermi-liquid renormalizations.
- N.3 While the static part of the amplitude $\Gamma(\mathbf{k}, \omega)$ is equal to Γ^k , the dynamic part contains at least one RA -section. The amplitude Γ^k can be formally defined as the part of the two-particle amplitude $\Gamma(\mathbf{k}, \omega)$ which does not contain any RA -sections.
- N.4 With the use of the effective mass m^* in the quasiparticle spectrum ϵ_p and redefining the interaction amplitudes, the explicit dependence on the residue a drops out from Fermi-liquid theory. This is the reason for attaching a^2 to the matrix $\Gamma_{l=0}^k$ in Eq. (1.2).
- N.5 In the textbook notations, $2\tilde{\Gamma}_1 - \Gamma_2 = B_{l=0}$ and $\Gamma_2 = -C_{l=0}$; see Eqs. (18.7) and (18.9) of Chapter 2, Sec. 18 in Ref. 4.
- N.6 Also, for obvious reasons, γ^ρ does not contain any terms that can be disconnected by cutting a line of the Coulomb interaction.
- N.7 Constant density of states is a usual approximation for Fermi-liquid theory, and it is in particular valid for a two-dimensional electron gas.
- N.8 This is a standard Ward identity; see Chapter 2, Sec. 19 in Ref. 4.
- N.9 Since we have touched a rather confusing question about negativity of $\partial n/\partial\mu$, it is worth mentioning that the stability of a liquid with charged carriers is determined by the combination $[V_C(k) + \partial\mu/\partial n] > 0$, rather than $\partial\mu/\partial n$ alone; $V_C(k)$, which is the Fourier component of the Coulomb interaction, stabilizes such a liquid.

- N.10 In the Matsubara technique, temperature enters as the low-energy cutoff because of the discreteness of the fermionic frequencies, $\epsilon_n = (2n + 1)\pi T$.
- N.11 On the contrary, in the process of rescattering discussed in Sec. 1, see Figs. 18.1–18.3, there are no integrations over the momenta of the diffusion propagators.
- N.12 In the ballistic region, $T > 1/\tau_{el}$, non-analytic temperature corrections due to the interplay of interaction and disorder also exist; they are linear in T . For studies of electric conductivity in the ballistic regime see Refs. 52–54. In our opinion, the effects in the ballistic and diffusive regions have little in common.
- N.13 The term “cooperon” reflects relevance of these modes to the same channel in which the Cooper instability develops.
- N.14 For our purposes, the difference between the temperature and the rate of de-coherence, $1/\tau_\varphi$, can be ignored.
- N.15 An alternative to the replica description of the effects of the e – e interaction of disordered electrons exists; namely, the Keldysh formalism; see Refs. 55 and 56.
- N.16 The factor $2\pi T$ which appears in $\hat{\Gamma}$ as a result of the Fourier transform from the Matsubara time to frequency has the same origin as $1/\text{Length}$ appearing in the transitions from spatial coordinates to the wave vectors.
- N.17 To get some intuition, one may look on the NLSM as a sort of the Heisenberg functional used in the theory of magnetism. Then, the frequency term in $S[Q]$ is equivalent to interaction with the external magnetic field which determines the direction of the spontaneous magnetization. Furthermore, the fluctuations of the magnetization, i.e., magnons, are the counterparts of the diffusons and cooperons in the discussed problems.
- N.18 Magnetic impurities, external magnetic field, or spin-orbit scattering induce additional terms in the action $S[Q]$, see e.g., Ref. 20. These terms make some of the diffusion modes gapped. Then a possible strategy is to preserve the general form of $S[Q]$ as given by Eq. (2.2), but to reduce the auxiliary matrix field \hat{Q} to such a manifold that only singular diffusion modes remain, while all gapped modes will be excluded. Systems with different sets of singular fluctuation propagators (i.e., when the \hat{Q} -fields are elements of different manifolds) belong to different universality classes. When

only the fluctuations of density are important, such a system belongs to the so-called unitary class. The richest case is the orthogonal class in which fluctuations of the charge- and spin- densities as well as different kind of cooperons are relevant. A system where spin-orbit interaction reduces singular fluctuations to the charge-density mode and the singlet cooperon belongs to the symplectic class.

- N.19 In Fig. 18.1, which illustrates Eq. (1.3), the intermediate sections are equal to $\frac{\omega_n}{\omega_n + Dk^2}$. Here, the denominator is determined by the diffusion propagator $\mathcal{D}(k, \omega_n)$, while ω_n in the numerator appears as a result of summation over a fermionic frequency within the interval available for this propagator.
- N.20 For example, in the diagram presented on the left side of Fig. 18.4, the amplitude Γ_ρ^0 is converted into Γ_σ as a result of scattering induced by disorder.
- N.21 Naturally, the number of the e - e interaction terms involved in the action $S[Q]$ is different for different classes. The most general form of the e - e interaction presented in Eq. (2.2) is needed for the orthogonal class systems, while for the unitary class only the ρ -term remains.
- N.22 For the polarization operator, $\Pi_{st}^0/\nu = 2$. Similarly, in the case of the spin susceptibility, $\chi_\sigma^0/\nu = 2(g_L\mu_B/2)^2$.
- N.23 Compared to the standard definition of the quantum resistance, this unit contains an additional factor π .
- N.24 Decreasing κ corresponds to enlarging blocks in the real-space renormalization procedure.
- N.25 The right hand side of Eq. (3.5) starts from the γ_c^2 -term which describes the rescattering in the Cooper channel. This is the only term in the RG equations that does not contain ρ . In the case of attraction, $\gamma_c < 0$, this term is responsible for the superconducting instability at low temperatures. Then, there is a competition between the two terms, and β_{γ_c} describes the *suppression* of the temperature of the superconducting transition by disorder. In amorphous films superconductivity can be totally suppressed by a moderate amount of disorder.⁵⁷
- N.26 The statement about the maximal number of the (extended) interaction vertices at a given number of momentum integrations demands a certain clarification. As it has been explained above, the rescattering of the electron-hole pairs described by the ladder diagrams (see Figs. 18.1–18.3) is not accompanied by integrations over momenta of the diffusion propagators. Therefore, extending vertices by ladders, as

- described in Eq. (1.3), does not generate any additional factors ρ . Owing to this fact, in the given order of ρ , the full dependences on the interaction amplitudes can be obtained by means of ladder extensions.
- N.27 Connection of the frequency dependent conductivity in the critical region of the MIT with the dielectric constant on the insulating side of the transition was discussed in Refs. 31 and 3.
- N.28 In our previous works, the combination $a = \frac{d-2}{d}(1 + \tilde{\zeta})$ was written as $\frac{d-2}{d-\zeta}$.
- N.29 For the analysis of the critical behavior, the data should be taken outside the region of perturbation corrections, $\sigma(T) - \sigma(T = 0) \gtrsim \sigma(T = 0)$ but, on the other hand, one should remain within the quantum transport region, $\sigma(T) < \sigma_{\min}$. (The Mott minimal conductivity σ_{\min} is a conditional boundary separating regions where transport is dominated by classical or quantum mechanisms.) In practice, these inequalities leave a limited window of $\sigma(T)$ appropriate for the analysis.
- N.30 In short, the tunneling conductance is determined by the Fourier transform of a *product of two* Green functions of electrons located on the opposite sides of the tunneling junction. Each of them is not gauge invariant by itself, while the product is. Therefore, it is the presence of the counter-electrode, with respect to which the measurement is performed, that makes the TDOS a physically meaningful quantity.
- N.31 This result was re-derived by many authors and in a different ways, see e.g., Refs. 58, 59 and 56.
- N.32 In this region of momenta, the dynamically screened Coulomb interaction $V_C(k, \omega_n)$ is proportional to $(Dk^2 + \omega_n)/Dk^2$ i.e., it is singular despite of screening. This singularity is the origin of the log-squared corrections to the TDOS.
- N.33 The other quantity that is sensitive to this kind of fluctuations is the thermal conductivity. The coordinate-dependent temperature invalidates the arguments about the absence of the energy reference level discussed in the main text.⁶⁰
- N.34 Therefore, any “universal” theory of the MIT in dilute electron systems that emphasizes only *one* aspect of the discussed systems — most often it is a very large r_s — cannot describe the observed non-monotonic $\rho(T)$. An electron liquid characterized by a very strong Coulomb interaction *alone* is, in a sense, as featureless (and universal) as the free electron gas. Such a featureless description cannot provide a non-monotonic $\rho(T)$.

- N.35 The universality has been confirmed by comparing the data obtained in samples from different wafers, see Fig. 3 in Ref. 42. Although the critical density at the MIT is sample dependent, the critical resistance has been found to be the same.
- N.36 The g_L -factor is about 1.5 times larger than for free-electrons, i.e., $g_L/g_L^0 = \frac{1}{1+F_0^2} = 1 - \Gamma_\sigma \approx 1.5$. The effective mass is about 3 times larger than the band mass, $m^*/m_b \approx 3$.
- N.37 This is, actually, a crucial point. That is where the high mobility becomes important in the case of the Si-MOSFET. It was shown in Ref. 61 that in this device the ratio τ_v/τ_{el} monotonically increases as the electron density decreases; here τ_v is the time of the inter-valley scattering. High mobility allows to reach low densities such that for the temperature interval we are interested in the inter-valley scattering is negligible (i.e., the two distinct valleys are well defined).
- N.38 The problematic feature of the scaling given by Eqs. (5.1a) and (5.1b) is that the amplitude γ_2 diverges at a finite temperature T^* and thereafter the RG-theory becomes uncontrolled.^{2,7} Fortunately, the scale T^* decreases very rapidly with n_v ; it was found in Ref. 46 that $\ln \ln(1/\tau_{el}T^*) \sim (2n_v)^2$. This observation makes the problem of the divergence of γ_2 irrelevant for all practical purposes, even for $n_v = 2$ which corresponds to Si-MOSFETs. At $n_v \rightarrow \infty$, the theory becomes internally consistent: $T^* \rightarrow 0$. Still, a delicate issue is the nature of the ground state of a system with finite n_v . For discussions of this question, see e.g., Refs. 62, 63, 3, 25, 13, and 64.
- N.39 The measurements in Ref. 61 confirm our original idea⁴⁶ that the difference between high- and low-mobility MOSFET samples is in the strength of the inter-valley scattering rather than in r_s , which anyway is not too large even in the best Si-MOSFET samples.
- N.40 For $n_v = 2$, $R(\eta)$ describes quantitatively the temperature dependence of the resistance of high-mobility Si-MOSFETs in the region of ρ up to $\rho \sim 0.5$,⁴⁶ which is not so far from the critical region.

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Personal Note

I am pleased to contribute this article to the volume celebrating 50 years of Anderson localization. Anderson's contributions to Science influenced my scientific work, especially in the beginning of my career as a many-body physicist. I would like to mention in particular his papers on the Kondo problem. These papers gave a very impressive example of mapping one problem onto another, an approach that in a general sense has also been applied here.

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